Notes: An Effective Theory of $X^+ - H$ Scattering

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1 Intro

At low energies, the $X^+ - H$ system can be considered a system of two particles — the X^+ particle does not "resolve" the $p - e$ structure of the H atom. However, as the X^+ particle approaches the H atom, it induces a dipole, which increases the attraction between the two "particles". Induced dipole attraction goes like $-1/r^4$. So, what happens when we model the $X^+ - H$ system as a two-particle system that interacts via an attractive $1/r^4$ potential?

Specifically,

- If the bound-state spectrum of the $X^+ - H$ system is a series of low-energy states, shouldn't we be able to describe that spectrum with an effective theory consisting of an attractive $1/r^4$ interaction and an arbitrary short-range interaction?
- If I tune my system to reproduce the shallowest bound state, will I get the rest of the spectrum?
- Are the $X^+ - H$ bound states $1/r^4$ states? Specifically the ground state... or is it associated with / affected by some other length scale?

2 $\pi^+ - H$

2.1 Local Regulator

There are different ways we can choose to regulate the $1/r^4$ interaction and define the short-distance pieces. We will start with a purely local system where we can keep the number of states fixed (is this cheating?).

Our interaction is

$$V(r) = \left[1 - e^{-(r/R)^2}\right]^4 \left(-\frac{C_4}{r^4}\right) + g_{\rm LO} e^{-(r/R)^4} \ , \tag{1}$$

where we fix $C_4 = \lim_{r\to\infty} \alpha(r)/2 = 9/4$ and vary $g_{\rm LO}$ to reproduce the shallowest bound state for each value of R.

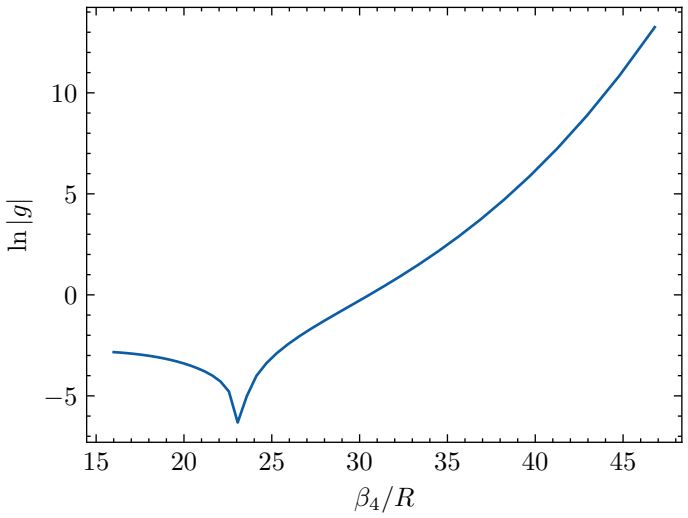

Figure 1: The running of the counterterm strength, g, as the short-distance cutoff, R, goes to zero. $B_2^{(6)}$ was kept fixed as R was varied. g is negative to the left of the "dip". $\beta_4 \equiv (2\mu C_4)^{1/2}$.

Before we continue, it is worth discussing what we're comparing to. Lazuaskas and Carbonell already did a two-body calculation (before surpassing it with a proper three-body calculation) using the Mott-Massey potential

$$V_{\rm MM}(r) = -\frac{\alpha(r)}{2r^4} \ , \tag{2}$$

where

$$\alpha(r) = \frac{9}{2} - \frac{2}{3}e^{-2r}\left(r^5 + \frac{9}{2}r^4 + 9r^3 + \frac{27}{2}r^2 + \frac{27}{2}r + \frac{27}{4}\right) \ . \tag{3}$$

The shallowest bound state (sixth excited state) lies at $B_2^{(6)} = 1.20\times 10^{-4}$ a.u. This is the state we will tune with $g_{\rm LO}$. Then we will compare the rest of the spectrum to the spectrum found with $V_{\rm MM}$.

The Renormalization Group (RG) flow is shown Figure 1. This is, of course, just one of an infinite number of branches, each with a unique number of bound states. There are 7 states in this branch.

Only $B_2^{(6)}$ is fixed in this calculation. The other binding energies have significant R dependence. The hypothesis here is that they will stabilize near the $V_{\rm MM}$ results. From Figure 2, two things are expected:

- Disagreement grows with energy. The further away the state is from the sixth excited state, the worse the agreeement. This is expected from an effective theory.

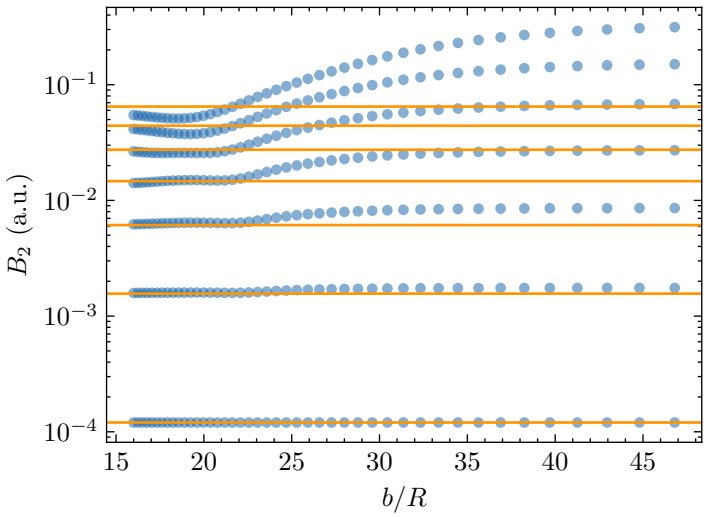

Figure 2: Binding energies as R goes to zero. Blue circles represent the effective theory calculation. Orange lines represent the spectrum found with V_{MM}. The shallowest state's agreement is by design.

- The asymptotic values of $B_2^{(n)}$ for $n < 4$ are very far from the V_{MM} results.

In light of the second point, it is worth looking at the energy dependence of the deviations, shown in Figure 3. Here we see a nearly linear dependence on $B_{2,\text{MM}}$. The ground state seem to deviate from the linear trend.

It is important to note that the values shown in Figure 3 are extrapolated. The convergence of each state (other than the sixth excited state) is shown in Figure 4. A simple fit was done to extract an asymptotic value where the curves are assumed to follow the form

$$E_2(R) = C_1 + C_2 e^{-C_3 R} \ , \tag{4}$$

where C_m are fit. C_1 values are shown in Figure 4.

2.2 Why Doesn't This Work Better?

The electron binding energy is approximately 0.5 a.u. If that is the relevant breakdown scale (?), then the states we are trying to predict with our effective theory are easily considered "low-energy". We ought to be doing better than this at leading order (LO).

Below, in Figures 5 and 6, the effectiver potential (without a counterterm) is shown in comparison to the Mott-Massey potential. Figure 5 shows the absolute comparison while Figure 6. For Figure 5, $R = 1.5$ a.u. This is approximately where the predictions match the MM spectrum as seen in Figure 2. For Figure 6,

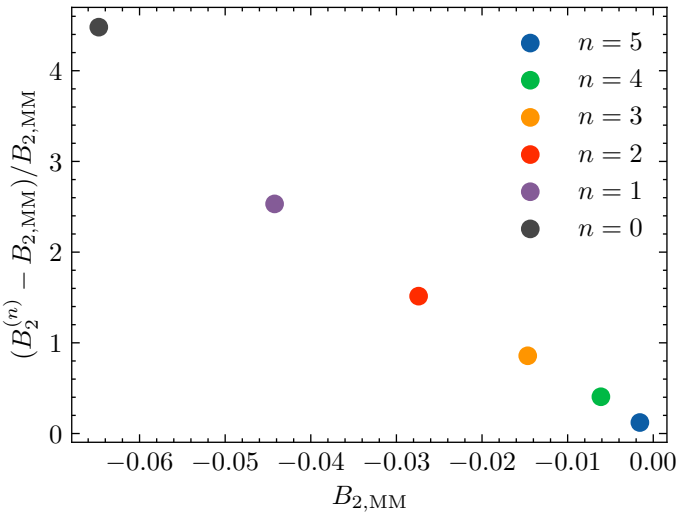

Figure 3: Relative differences of the binding energies as the states get deepeer. The zero crossing for $n = 6$ (blue circle) is again by design.

a range of R values is shown. Beyond $r \approx$ 5 a.u., is seems that the potentials are nearly identical. (Perhaps this tells us something about the breakdown scale?)

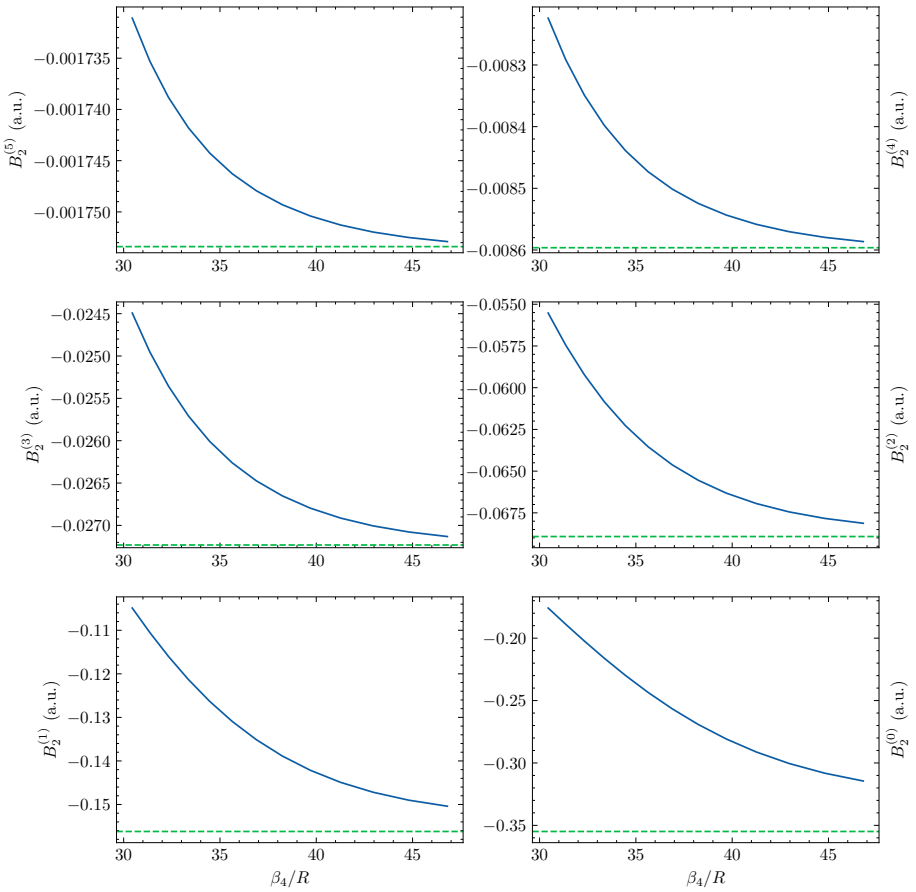

Figure 4: Convergence of deeper states as R goes to zero. Blue lines represent the effective theory calculation. Green lines represent the extracted asymptotic values.

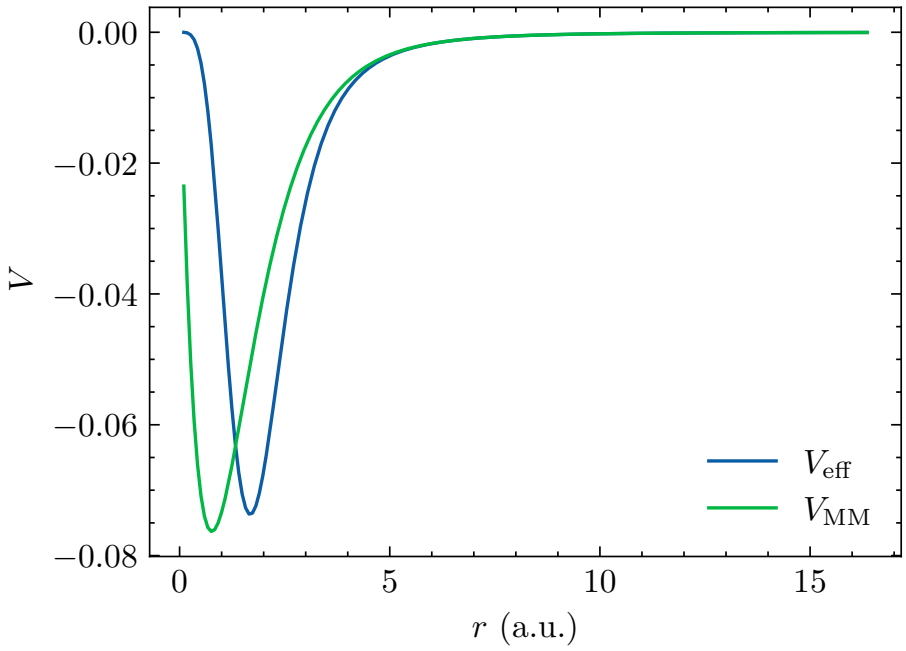

Figure 5: Comparison of the regulated $1/r^4$ and Mott-Massey potentials at $R = 1.5$.

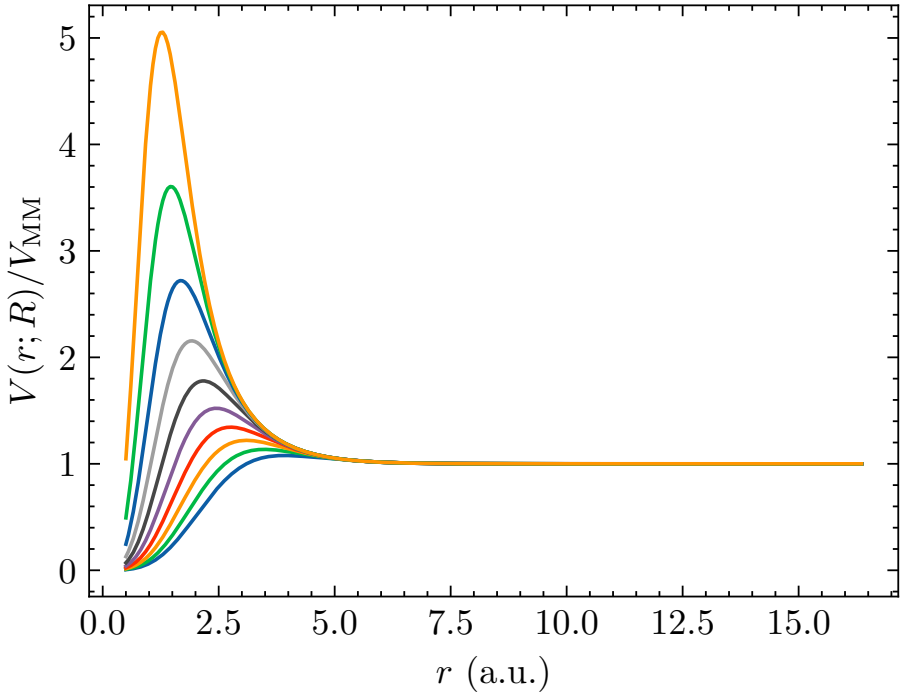

Figure 6: Relative comparison of the regulated $1/r^4$ and Mott-Massey potentials for a range of R values.