Factor Models of Returns

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"$y = X\beta + \epsilon$, the rest is commentary."

About Me

Outline

Securities Markets

Spot Transactions

The point of trading is to obtain an asset by giving up money, or obtain money by giving up an asset.

If I give you q units of some asset A, and you give me $\$p$, then:

- I have **sold** q units of A to you at $\frac{\$p}{q}$
- You have **bought** q units of A from me for $\frac{\$p}{q}$

Buying and selling are collectively called ‘trading’.

Suppose I own some amount of A and some amount of money. If we let s be $+1$ for buying and -1 for selling, then the result of any trade is to add qs to the amount of A I own, and add $-qps$ to the amount of money I have.

Securities Markets and Exchanges

The **market** is the collective activity of all traders. When we don't care who we trade with, we can just 'trade with the market'.

A **securities market** for some asset A, open at a time t, is any standardised way for traders to reach agreements to buy or sell A at a specified **settlement time** $T \geq t$.

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- t ('spot', e.g. blockchain)
- $t+1, t+2, \dots$ ('clearing', e.g. equities)
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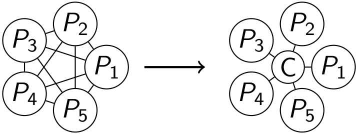
If you agree to give something to someone, you have an **obligation**. If someone agrees to give you something, you have a **right**.

Counterparty Risk

If I have an agreement with P_1 to buy 10 units for $\$p_1$ at T, and an agreement with P_2 to sell 10 units at $\$p_2$ at T, and no further rights/obligations, am I guaranteed to meet my obligations?

Centralisation

A **securities exchange** is a centralised venue serving a securities market for **exchange participants** (e.g. ASX, NYSE, TSE, HKEX, LME). Agreements not made through an exchange are often called OTC (over-the-counter).



Centralisation generally reduces **search costs** and **counterparty risk**.

Settlement and Clearing

Netting

Centralisation allows for **netting** of rights and obligations.
For any settlement time T, I only need to keep track of the difference between money owed to and by me, and units owed to and by me.
The quantity of A owned by me, plus the quantity owed to me, minus the quantity owed by me to others, is known as my **net position** in A. If this is positive, I have a **long position**. If it is negative, I have a **short position**. If it is zero, I am **flat**.

Collateralisation

At certain intermediate times t' ($t \leq t' \leq T$), participants may be required to physically give ('post') something to the exchange to **collateralise** their obligations.

- Money ('margin')
- Assets ('locate'/'borrow')

If an agreement made on the exchange gives you rights to money or

Summary

- **Trading** is swapping money and assets
- A **market** is whatever you use to trade
- A **securities market** is a standardised way to agree to trades
- Agreements consist of **rights** and **obligations**
- Finding a **counterparty** may involve **search cost**
- Agreements between two parties are subject to **counterparty risk**
- A **securities exchange** is a centralised trading venue
- After trades are agreed to on an exchange, they will be **settled** in some standardised way
- The net quantity of A that I have some claim to can either be positive (**long position**), negative (**short position**), or zero (**flat**).
- Traders may be obligated to post assets ('locate') or money ('margin')

Trading

Setup

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Suppose that at each time t we have cash holdings of $\$c_t$ and net holdings of a_t units of some asset A.

Suppose also that trades $(s_t, q_t, \$p_t)$ take place at a finite set of distinct times

$$\tau = \{t_1, \ldots t_n\} \subset T = [t_-, t_+],$$

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where $t_- < t_1 < \dots < t_n < t_+$.

Suppose further that p_t is a right-continuous function $\mathbb{R} \to \mathbb{R}$ with left-limits.

For instance, we could take $p_t = p_{\max(\tau \cap (-\infty, t])}$ for $t \geq \min \tau$ and $p_t = x$ otherwise for some arbitrary x.

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Define measures $\mathfrak{a}_\omega, \mathfrak{c}_\omega, \mathfrak{p}_\omega$ such that for any interval $T' = [t'_-, t'_+]$ we have

$$\mathfrak{a}_{T'} := a^+_{t'_+} - a^-_{t'_-}, \mathfrak{c}_{T'} := c^+_{t'_+} - c^-_{t'_-}, \mathfrak{p}_{T'} := p^+_{t'_+} - p^-_{t'_-}.$$

Then we have

$$\begin{aligned}
a^+_{t'_+} - a^-_{t'_-} &= \sum_{t\in\tau} s_t q_t && = \int_{t\in T} d\mathfrak{a}, \\
c^+_{t'_+} - c^-_{t'_-} &= \sum_{t\in\tau} -p_t(s_t q_t) && = \int_{t\in T} -p_t d\mathfrak{a}, \\
p^+_{t'_+} - p^-_{t'_-} &= p_{t'_+} - p^-_{t'_-}. && = \int_{t\in T} d\mathfrak{p}.
\end{aligned}$$

Cash Holdings

It can be shown (see appendix) that the cashflow over the entire interval $T = [t_-, t_+]$ is

$$c_{t_+} - c_{t_-} = \int_{t \in T} -p_t d\mathfrak{a} = p_{t_-} a_{t_-} - p_{t_+} a_{t_+} + \int_{t \in T} a_t^- d\mathfrak{p}.$$

This is similar in spirit to integration by parts:

$$\int_a^b f \frac{dg}{dx} dx = f(b)g(b) - f(a)g(a) - \int_a^b g \frac{df}{dx} dx.$$

Then we have

$$(c_{t_+} + p_{t_+} a_{t_+}) - (c_{t_-} + p_{t_-} a_{t_-}) = \int_{t \in T} a_t^- dp.$$

The quantity $\$v_t = \$p_t a_t$ is known as the **dollar value** of our A holdings **marked** to the price $\$p_t$.

Portfolio Valuation

Suppose now that we trade multiple assets, such that p_t, a_t and v_t are vector-valued, with v_t the elementwise product of p_t and a_t.
A collection of assets held in quantities a_t is known as a **portfolio**.

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We can write

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where p is now a vector-valued measure. Let

$$\Pi_t = c_t + p_t \cdot a_t = c_t + \sum v_t.$$

We call \$$\Pi_t$ the **value** of our portfolio **marked** to p_t.

Profitability

The quantity $\$\Pi_{t_+} - \$\Pi_{t_-}$ is our **net P&L** (profit and loss) over the interval T, marked to p_t. Then we have

$$\Pi_{t_+} - \Pi_{t_-} = \int_{t \in T} a_t^- \cdot dp.$$

Introducing a measure

$$\pi_{T'} := \Pi_{t'_+}^+ - \Pi_{t'_-}^-,$$

we can write

$$\Pi_{t_{i+1}} - \Pi_{t_i} = \int_{t \in [t_i, t_{i+1}]} d\pi = a_t^- \cdot dp = \int_{t \in [t_i, t_{i+1}]} v_t^- \cdot \frac{dp}{p_t^-},$$

where $\frac{dp}{p_t^-}$ is the elementwise quotient.

Leverage

Suppose we can always make any trade we like at time t with price $\$p_t$.
Then we can freely convert a portfolio with value $\$\Pi_t$ to that much in cash.
Conversely, we can convert $\$\Pi_t$ worth of cash into any portfolio with that value.
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If we begin with a portfolio worth $\$\Pi_{t_1}$ and make a sequence of trades of the form (s_t, q_t, p_t) that result in a portfolio worth $\$\Pi_{t_n}$, then we could instead begin with a portfolio worth $L\$\Pi_{t_1}$ and make trades (s_t, Lq_t, p_t) to arrive at a portfolio worth $L\$\Pi_{t_n}$. The ratio L is known as the **leverage ratio**.

Returns

Because we typically need money to collateralise some fraction of unsettled trades and short borrow, and hold assets instead of cash, portfolio management requires capital that cannot be used elsewhere.

In the case of long-only spot-settled trading, if we were to turn our portfolio into cash, or convert cash into an identical portfolio, we would receive/require $\$\Pi_t$.

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If our initial portfolio value were $\$\Pi_{t\_} + \$N$ instead of $\Pi_{t\_}$, and we could simply scale up trade sizes at the same prices, then setting

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means our new P&L would simply be $L \int_T d\pi$.
The increase in P&L per dollar added to initial portfolio value is then

$$R = \frac{L \int_T d\pi - \int_T d\pi}{} = \frac{\int_T d\pi}{}$$

Log Returns

If we define

$$\ell_t = \log \Pi_t$$

and a measure $\mathfrak{l}_\omega$ satisfying

$$\mathfrak{l}_{[t'_-, t'_+]} = \ell^+_{t'_+} - \ell^-_{t'_-}$$

for any t'_-, t'_+, then we have

$$R_T = \exp(\mathfrak{l}_T) - 1,$$

and for any measurable set ω we can define

$$R_\omega = \exp(\mathfrak{l}_\omega) - 1 \approx \mathfrak{l}_\omega + O(\mathfrak{l}_\omega^2) \text{ (for small } \mathfrak{l}_\omega).$$

We call $\mathfrak{l}_T$ the **log-return** over the interval T.

Properties of Returns and Log-Returns

Let $w_t := \frac{1}{\Pi_t} v_t$ be the **weight vector**.
For an interval $T' = (t_i, t_{i+1}]$, we have a_t^- equal to a constant over T', and

$$R_{T'} = w_{t_i}^+ \cdot r_{T'},$$

where

$$r_{T'} = \frac{p_{t_{i+1}} - p_{t_i}}{p_{t_i}}$$

is known as the **asset returns** vector over T'. In contrast, $\ell_{T'}$ is not linear in $r_{T'}$.
For a disjoint collection of measurable sets $\omega_1, \ldots \omega_n$ whose union is Ω, we have

$$\ell_\Omega = \sum_{i=1}^{n} \ell_{\omega_i},$$

$$R_\Omega = \left(\prod_{i=1}^{n} (1 + R_{\omega_i}) \right) - 1 \approx \sum_{i=1}^{n} R_{\omega_i} + O\left(\sum_{i=1}^{n} \sum_{j=1}^{n} |R_{\omega_i} R_{\omega_j}| \right).$$

Summary

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Market Microstructure

Trade Formation

In practice, the trades we can make at a time t and a price p_t are limited by our ability to find a willing counterparty.

On an electronic exchange, trades are formed by interacting with the exchange's **matching engine**.

The most common type of matching engine design is a **limit-order book** (sometimes called a double auction), which can operate in either a **continuous** or **batched** fashion.

Limit Order Book

At any point in time, market participants can create a request ('**limit order**') of the form (s, q, p) to trade up to q units in direction $s = \pm 1$ at a price $\$p$ (or better).
They are then said to be "**bid** for $\$p$" $(s = +1)$ or "**ask**ing/**offer**ing at $\$p$" $(s = -1)$.

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Whenever the book contains orders $(+1, q_1, \$p_1), (-1, q_2, \$p_2)$ with $p_2 \leq p_1$, both orders could be satisfied by trading $q = \min(q_1, q_2)$ units with one another at a price $\$p \in [\$p_2, \$p_1]$. If such a pair exists the book is said to be **crossed**.

If these orders are **matched**, q units will trade at \$$p$ and the orders will become $(+1, p_1, q_1 - q), (-1, p_2, q_2 - q)$. Order(s) with no quantity remaining will be removed from the book.

Matching Mechanism

Batch Matching

Continuous Matching

Continuous matching is triggered as soon as the book becomes crossed. Orders are matched in such a way as to uncross the book.

Market Data & Market Prices

In order to inform trading activity, market participants receive certain data about the orders and trades on the exchange.

Transaction Costs

Portfolio Management

Uncertainty

Decision-Making

Risk

Decision Theory

Capital Asset Pricing Model

Factor Models

Statistical Arbitrage

Options Trading

Appendix

Proof Sketch for $\mathfrak{c}_T$ Identity

$$
\begin{aligned}
c_{t_+} - c_{t_-} = \sum_{t\in\tau} -p_t(s_t q_t) \quad &= \sum_{i=1}^{n} -p_{t_i}(a_{t_i}^+ - a_{t_i}^-) = -\sum_{i=1}^{n} p_{t_i} a_{t_i}^+ + \sum_{i=1}^{n} p_{t_i} a_{t_i}^- \\
&= -\sum_{i=1}^{n-1} p_{t_i} a_{t_{i+1}}^- - p_{t_n} a_{t_n}^+ + \sum_{i=1}^{n-1} p_{t_{i+1}} a_{t_{i+1}}^- + p_{t_1} a_{t_1}^- \\
&= p_{t_1} a_{t_1}^- + -p_{t_n} a_{t_n}^+ + \sum_{i=2}^{n} (p_{t_i} - p_{t_{i-1}}) a_{t_i}^- \\
&= p_{t_1} a_{t_1}^- - p_{t_n} a_{t_n}^+ + \int_{t\in[t_1,t_n]} a_t^- \, d\mathfrak{p} \\
&= p_{t_-} a_{t_-} - p_{t_+} a_{t_+} + \int_{t\in T} a_t^- \, d\mathfrak{p}.
\end{aligned}
$$

Annualised Returns

The **annualised log-return** over ω is $\ell_\omega \frac{1 \text{ year}}{\lambda_\omega}$, where λ_ω is the duration (lebesgue measure) of ω in units of time.

The **geometrically annualised return** over ω is $(1 + R_\omega)^{\frac{1 \text{ year}}{\lambda_\omega}} - 1 = \exp\left(\ell_\omega \frac{1 \text{ year}}{\lambda_\omega}\right) - 1$.

The **arithmetically annualised return** over ω is $R_\omega \frac{1 \text{ year}}{\lambda_\omega}$.

Stuff I missed