Systematic Trading from First Principles

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"$y = X\beta + \epsilon$, the rest is commentary."

About Me

Point of This Talk

Outline

Securities Markets

Spot Transactions

The point of trading is to obtain an asset by giving up money, or obtain money by giving up an asset.

If I give $q > 0$ units of some asset A, and you give me \$$p$, then:

- I have **sold** q units of A to you at $\frac{\$p}{q}$
- You have **bought** q units of A from me for $\frac{\$p}{q}$

Buying and selling are collectively called 'trading'.

Suppose I own some amount of A and some amount of money. If we let s be $+1$ for buying and -1 for selling, then the result of any trade is to add qs to the amount of A I own, and add $-qps$ to the amount of money I have.

Securities Markets and Exchanges

The **market** is the collective activity of all traders. When we don't care who we trade with, we can just 'trade with the market'.

A **securities market** for some asset A, open at a time t, is any standardised way for traders to reach agreements to buy or sell A at a specified **settlement time** $T \geq t$.

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- t ('spot', e.g. blockchain)
- $t+1, t+2, \ldots$ ('clearing', e.g. equities)
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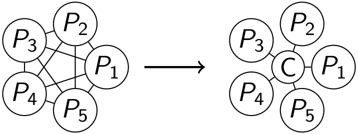
If you agree to give something to someone, you have an **obligation**. If someone agrees to give you something, you have a **right**.

Counterparty Risk

If I have an agreement with P_1 to buy 10 units for $\$p_1$ at T, and an agreement with P_2 to sell 10 units at $\$p_2$ at T, and no further rights/obligations, am I guaranteed to meet my obligations?

Centralisation

A **securities exchange** is a centralised venue serving a securities market for **exchange participants** (e.g. ASX, NYSE, TSE, HKEX, LME). Agreements not made through an exchange are often called OTC (over-the-counter).



Centralisation generally reduces **search costs** and **counterparty risk**.

Netting

Centralisation allows for **netting** of rights and obligations.
For any settlement time T, I only need to keep track of the difference between money owed to and by me, and units owed to and by me.
The quantity of A owned by me, plus the quantity owed to me, minus the quantity owed by me to others, is known as my **net position** in A.
If this is positive, I have a **long position**. If it is negative, I have a **short position**. If it is zero, I am **flat**.

Collateralisation

At certain intermediate times t' ($t \leq t' \leq T$), participants may be required to physically give ('post') something to the exchange to **collateralise** their obligations.

- Money ('margin')
- Assets ('locate'/'borrow')

If an agreement made on the exchange gives you rights to money or assets at T, this is typically as good as posting actual money or assets for an obligation at $T' \geq T$.
Some amount of interest may be charged to make up for the difference between the size of our obligations and the size of our collateral. For cash, this is according to an **interest rate**; for other assets, it is according to a **borrow rate**/**short rate**.

Summary

- **Trading** is swapping money and assets
- A **market** is whatever you use to trade
- A **securities market** is a standardised way to agree to trades
- Agreements consist of **rights** and **obligations**
- Finding a **counterparty** may involve **search cost**
- Agreements between two parties are subject to **counterparty risk**
- A **securities exchange** is a centralised trading venue
- After trades are agreed to on an exchange, they will be **settled** in some standardised way
- The net quantity of A that I have some claim to can either be positive (**long position**), negative (**short position**), or zero (**flat**).
- Traders may be obligated to post assets ('locate') or money ('margin')

Trading

Setup

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Suppose that at each time t we have cash holdings of $\$c_t$ and net holdings of a_t units of some asset A.
Suppose also that trades $(s_t, q_t, \$p_t)$ take place at a finite set of distinct times

$$\tau = \{t_1, \ldots t_n\} \subset T = [t_-, t_+],$$

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Suppose further that p_t is a right-continuous function $\mathbb{R} \to \mathbb{R}$ with left-limits.

For instance, we could take $p_t = p_{\max(\tau \cap (-\infty, t])}$ for $t \geq \min \tau$ and $p_t = x$ otherwise for some arbitrary x. This is known as the last traded price.

Accounting

For any time-varying quantity x_t, let x_t^+ and x_t^- denote the right- and left-limits respectively.
Furthermore, define a signed measure x_ω such that for any interval $T' = [t'_-, t'_+]$ we have

$$x_{T'} := x_{t_+}^+ - x_{t_-}^- .$$

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Then we have

$$\begin{aligned} a_{T'} &= \sum_{t \in \tau} s_t q_t &&= \int_{t \in T} da, \\ c_{T'} &= \sum_{t \in \tau} -p_t(s_t q_t) &&= \int_{t \in T} -p_t da, \\ p_{T'} &= p_{t'_+} - p_{t'_-}^- . &&= \int_{t \in T} dp. \end{aligned}$$

Cash Holdings

It can be shown (see appendix) that the cashflow over the entire interval $T = [t_-, t_+]$ is

$$\$c_T = \int_{t\in T} -\$p_t da = \$p_{t_-} a_{t_-} - \$p_{t_+} a_{t_+} + \$ \int_{t\in T} a_t^- dp.$$

This is similar in spirit to integration by parts:

$$\int_a^b f \frac{dg}{dx} dx = f(b)g(b) - f(a)g(a) - \int_a^b g \frac{df}{dx} dx.$$

Then we have

$$\$(c_{t_+} + p_{t_+} a_{t_+}) - (c_{t_-} + p_{t_-} a_{t_-}) = \$ \int_{t\in T} a_t^- dp.$$

The quantity $\$v_t = \$p_t a_t$ is known as the **dollar value** of our A holdings **marked** to the price $\$p_t$.

Portfolio Valuation

Suppose now that we trade multiple assets, such that p_t, a_t and v_t are vector-valued, with v_t the elementwise product of p_t and a_t.
A collection of assets held in quantities a_t is known as a **portfolio**.

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We can write

$$(c_{t_+} + p_{t_+} \cdot a_{t_+}) - (c_{t_-} + p_{t_-} \cdot a_{t_-}) = \int_{t \in T} a_t^- \cdot dp,$$

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$$\Pi_t = c_t + p_t \cdot a_t = c_t + \sum v_t.$$

We call $\$\Pi_t$ the **value** of our portfolio **marked** to p_t.

Profitability

The quantity $\$\Pi_{t_+} - \$\Pi_{t_-}$ is our **net P&L** (profit and loss) over the interval T, marked to p_t. Then we have

$$\Pi_T = \int_{t \in T} a_t^- \cdot dp.$$

we can write

$$\Pi_{[t_i, t_{i+1}]} = \int_{[t_i, t_{i+1}]} a_t^- \cdot dp = \int_{t \in [t_i, t_{i+1}]} v_t^- \cdot \frac{dp}{p_t^-},$$

where the quotient $\frac{dp}{p_t^-}$ is computed elementwise.

Leverage

Suppose we can always make any trade we like at time t with price $\$p_t$.
Then we can freely convert a portfolio with value $\$\Pi_t$ to that much in cash.
Conversely, we can convert $\$\Pi_t$ worth of cash into any portfolio with that value.
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If we begin with a portfolio worth $\$\Pi_{t_1}$ and make a sequence of trades of the form (s_t, q_t, p_t) that result in a portfolio worth $\$\Pi_{t_n}$, then we could instead begin with a portfolio worth $L\$\Pi_{t_1}$ and make trades (s_t, Lq_t, p_t) to arrive at a portfolio worth $L\$\Pi_{t_n}$. The ratio L is known as the **leverage ratio**.

Return on Capital

Because of collateralisation requirements, portfolio management uses up cash.

Consider long-only spot-settled trading. If we were to turn our portfolio into cash, or convert cash into an identical portfolio, we would receive/require $\$\Pi_t$.[1]

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The **return on capital** is defined as the increase in P&L per dollar added to initial portfolio value, i.e.

$$R_T = \frac{L \int_T d\Pi - \int_T d\Pi}{N} = \frac{\int_T d\Pi}{\Pi_{t_-}}.$$

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Log Returns

If we define

$$\ell_t = \log \Pi_t$$

for any t'_-, t'_+, then we have

$$R_T = \exp(\ell l_T) - 1,$$

and for any measurable set ω we can define

$$R_\omega = \exp(\ell_\omega) - 1 \approx \ell_\omega + O(\ell_\omega^2) \text{ (for small } \ell_\omega).$$

We call ℓ_T the **log-return** over the interval T.

Properties of Returns and Log-Returns

Let $w_t := \frac{1}{\Pi_t} v_t$ be the **weight vector**.
For an interval $T' = (t_i, t_{i+1}]$, we have a_t^- equal to a constant over T', and

$$R_{T'} = w_{t_i}^+ \cdot r_{T'},$$

where the elementwise quotient

$$r_{T'} = \frac{p_{t_{i+1}} - p_{t_i}}{p_{t_i}}$$

is known as the **asset returns** vector over T'. In contrast, $\ell_{T'}$ is not linear in $r_{T'}$.
For a disjoint collection of measurable sets $\omega_1, \ldots \omega_n$ whose union is Ω, we have

$$\ell_\Omega = \sum_{i=1}^{n} \ell_{\omega_i},$$

$$R_\Omega = \left(\prod_{i=1}^{n} (1 + R_{\omega_i}) \right) - 1 \approx \sum_{i=1}^{n} R_{\omega_i} + O\left(\sum_{i=1}^{n} \sum_{j=1}^{n} |R_{\omega_i} R_{\omega_j}| \right).$$

Summary

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Market Microstructure

Trade Formation

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The most common type of matching engine design is a **limit-order book** (sometimes called a double auction), which can operate in either a **continuous** or **batched** fashion.

Limit Order Book

At any point in time, market participants can create a request ('**limit order**') of the form $(s, q, \$p)$ to trade up to q units in direction $s = \pm 1$ at any price $\$(p - sm), m \geq 0$.
The value $\$m$ is known as the **price improvement**.
They are then said to be "**bid** for $\$p$" $(s = +1)$ or "**ask**ing/**offer**ing at $\$p$" $(s = -1)$.

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All limit orders active at time t are collected into a **limit-order book** $\mathcal{L}_t$.
By convention, $\mathcal{L}_t$ is right-continuous with left limits.
Users can add, cancel and modify orders, subject to exchange-specific rules.

Order Matching

Whenever $(+1, q_1, \$p_1), (-1, q_2, \$p_2) \in \mathcal{L}_t$ with $p_2 \leq p_1$, both orders could be at least partly satisfied by trading up to $q_{\text{max}} = \min(q_1, q_2)$ units with one another at a price $\$p \in [\$p_2, \$p_1]$. If such a pair exists the book is said to be **in cross**.

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The ability to quickly find matches for a large number of units at a reasonable price is known as **liquidity**, and is another major benefit of centralisation.

Supply and Demand Curves

We can partition $\mathcal{L}_t$ into $\mathcal{L}_t = \mathcal{B}_t \cup \mathcal{A}_t$, with $\mathcal{B}_t$ the bid orders and $\mathcal{A}_t$ the ask orders.

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Now define the functions

$$Q_t(+1, \$p) = \sum_{\substack{(+1, q', \$p') \in \mathcal{B}_t \\ \$p \leq \$p'}} q'$$

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The functions $Q_t(-1, \$p)$ and $Q_t(+1, \$p)$ are known as the **supply curve** and **demand curve** respectively. The function $M_t(\$p)$ represents the **matchable quantity** at $\$p$. The book $\mathcal{L}_t$ is in cross if and only if there exists some $\$p$ with $M_t(\$p) > 0$.

Batch Matching

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1. Prior to the **match time** t^*, users can typically add, modify and cancel limit orders.
2. At each time $t \leq t^*$, an **indicative price** $\$p_t^*$ will be selected such that $M_t(\$p_t^*)$ is maximal. Tiebreaking will depend on exchange rules.
3. Finally, at the match time t^*, some subset of the crossed limit orders will be matched at $\$p^*$ for a total quantity $M_{t^*}(\$p_{t^*}^*)$. After the match, the book will no longer be crossed.

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3. Finally, at the match time t^*, some subset of the crossed limit orders will be matched at $\$p^*$ for a total quantity $M_{t^*}(\$p_{t^*}^*)$. After the match, the book will no longer be crossed.

Maximising $M_t(\$p_t^*)$ is equivalent to maximising the sum of qm across all orders, where q is the quantity filled and m is the price improvement.
It is common to use this matching style at the beginning or end of a trading day or lunch break, or when there is some kind of market instability such as following a large price move or company announcement.
Sometimes t^* is referred to as a **liquidity event** because of the large volume traded, and the relative insensitivity of $\$p_{t^*}^*$ to individual orders.

Batch Matching Properties

The following monotonicity properties typically hold:

- $\$p_t^*$ nondecreasing in $\mathcal{B}_t$ and nonincreasing in $\mathcal{A}_t$
- For each $\$p$, $M_t(\$p)$ nondecreasing in $\mathcal{L}_t$
- For individual orders $(s, q, \$p)$, we will have $\$p_t^*$ nondecreasing in $\$p$ and sq.
- For individual orders $(s, q, \$p)$ and each $\$p'$, we will have $M_t(\$p')$ nondecreasing in q and nondecreasing in $\$sp$.

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Price Priority

Because $M_t(\$p')$ is nondecreasing in sp, the matching will be designed to obey **price priority**.
If we have two orders $(s_1, q_1, \$p_1), (s_2, q_2, \$p_2)$ with $\$s_1 p_1 > \$s_2 p_2$, then the second order cannot be matched unless the first is completely filled.

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Time Priority

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The order book is often visible to all participants. Traders may be incentivised to wait until immediately before the match time to post orders. If everyone does this, the matching engine may be overloaded, and p_t^* will change very rapidly leading up to t^*.

The match time is typically chosen at random in some short interval in order to disincentivise this behaviour. To further encourage early submission of orders, many exchanges also implement a time priority rule.

Time Priority

If two orders exist at the same price $\$p$, the one that reached the matching engine later cannot be matched unless the earlier order is completely filled.

Tick Size

Time priority would not have much effect if we could just insert the later order at a price $\$p + s\epsilon$ for some very small $\epsilon > 0$.

To avoid this, prices must be integer multiples of some small increment $\$\delta$, known as the **tick size**.

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The only orders involved in the match will be the arriving order and some set of orders $\mathcal{M}_t$ in the opposite direction.

The arriving order is known as the **active** or **aggressive** order, and the pre-existing orders are known as **passive**.

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The per-unit price achieved by the active trader will be

$$\$p_t^* = \frac{\sum_{(s,q,\$p)\in\mathcal{M}_t} \$pq}{\sum_{(s,q,\$p)\in\mathcal{M}_t} q}.$$

Instantaneous Price Impact

If we aggressively trade a very large quantity, we will exhaust all passive orders we would most prefer to trade with and $\mathcal{M}_t$ will need to include orders at worse price levels. This is sometimes known as **walking the book**.

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The unit price of the match will be given by

$$\$p_t^*(sq) = \frac{1}{q} \int_0^q \$P_t(sq')dq'.$$

We call the sensitivity of $\$p_t^*$ to sq the **instantaneous price impact**.

Bid-Ask Spread

We call the prices

$$\begin{aligned} \$b_t &= \lim_{q\to 0^+} \$p_t^*(-q) && = \max_{(+1,q,\$p)\in\mathcal{B}_t} \$p \\ \$a_t &= \lim_{q\to 0^+} \$p_t^*(q) && = \min_{(-1,q,\$p)\in\mathcal{A}_t} \$p \end{aligned}$$

the **bid price** and **ask price** respectively. All bid orders have price at most $\$b_t$ and all ask orders have price at least $\$a_t$.
The interval $[\$b_t, \$a_t]$ is known as the **spread**, and $\$a_t - \$b_t$ is the **width** of the spread. If $\$a_t - \$b_t = \$\delta$, we say that the market for the asset is **large-tick** or **tick-constrained**.

Price Proxies

Note that $\$p_t^*(0)$ is not yet defined. So long as we choose some price $\$m_t$ satisfying $\$m_t \in [\$b_t, \$a_t]$, setting $\$p_t^*(0) := \$m_t$ will make $\$p_t^*(\cdot)$ nondecreasing.

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Some simple choices for m_t include:

- $\frac{1}{2}b_t + \$\frac{1}{2}a_t$ (arithmetic **midprice**)
- $\$\sqrt{b_t a_t}$ (**geometric midprice**)
- $\$(1 - I_t)b_t + \$I_t a_t$ (depth-$\$d$ **weighted midprice**)
- $\$b_t^{1-I_t} a_t^{I_t}$ (depth-$\$d$ **geometrically weighted midprice**)

We call I_t the **book imbalance**.

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We call I_t the **book imbalance**.

A popular choice for this is

$$I_t = \frac{Q_t(+1, \$b_t)}{Q_t(+1, \$b_t) + Q_t(-1, \$a_t)}.$$

Alternative price proxies are described in the appendix.

Persistent Price Impact

We call the difference $\$\lambda_t(sq) = \$p_t^*(sq) - \$m_t$ the **instantaneous price impact curve** of trading q units in direction s.
Buy orders will have nonnegative instantaneous price impact, while sell orders will have nonpositive instantaneous price impact.

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The realised instantaneous price impact is given by

$$\$\lambda_t = \$p_t - \$m_t,$$

while the realised persistent price impact is given by

$$\$\nu_t = \$m_t - \$m_t^{\emptyset},$$

where $\$m_t^{\emptyset}$ is the path the microprice process would have taken had we not interacted at all with the matching engine.

P&L with transaction costs

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\$\Pi_T &= \$\int_{t\in T} a_t^{-}(dm + dn + d\lambda) \\
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Attempts to make ν_t and da covary negatively are very hard to pull off and usually considered manipulative.

But if we only use passive execution, it is guaranteed that λ_t and da will covary negatively, and this term will change from a loss to a profit. Trying to make money solely from this term is known as **market making** or **liquidity provision**.

Downsides of Market Making

Assuming $a_{t_-} = a_{t_+} = 0$,

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However, it is still possible to make money on both terms. This is particularly true if a_t changes relatively slowly (**low-frequency trading**).
Whether this results in better performance overall is a different question.

Market Making Strategy

A highly simplified model of optimal market making is given by Avellaneda and Stoikov (2008). The market maker maintains two limit orders at any time in opposite directions, of the form

$$\left(s, 1, \$\left(m_t - \gamma q - \frac{1}{2} s \varsigma)\right)\right),$$

where γ is a risk-aversion parameter[2] and ς is the difference between the two prices.
In general, to avoid large a_t, (which would make our P&L very sensitive to price changes), we want the amount we buy to match the amount we sell.

[2] Defined differently in the paper

Market Impact Modeling

Market Data & Market Prices

In order to inform trading activity, market participants receive certain data about the orders and trades on the exchange.

Portfolio Management

Uncertainty

An **event** is a set of possible worlds distinguished from other possible worlds by some common feature.
For instance, there is a set of possible worlds where Australia wins its next game of cricket against the UK.
Suppose Ω is the set of all events.[3] At each time t', we can construct a set $\mathcal{F}_{t'} \subseteq \Omega$ containing all events whose outcome is known at time t'.[4]
An **Arrow-Debreu security** A_ω for $\omega \in \mathcal{F}_t$ is one which we can exchange at time t for \$1 if ω occurs and \$0 otherwise. Assume there exists t such that $\omega \in \mathcal{F}_t$.
Let $\$\mathbb{Q}_{t'}(\omega)$ be the market price of ω at time $t' \leq t$. Assume unlimited liquidity, no fees, and no collateral requirements or interest payments.
The market is **arbitrage-free** if there is no fixed set of trades that guarantees positive P&L.
Assuming this is the case, we can deduce properties that $\mathbb{Q}_{t'}(\cdot)$ must satisfy.

$\mathbb{Q}_{t'}(\omega) \in [0, 1]$

Decision-Making

Compounding

Risk

Portfolio Selection

Capital Asset Pricing Model

Factor Models

α

Statistical Arbitrage

Options Trading

Appendix

Proof Sketch for c_T Identity

$$\begin{aligned}
c_T = \sum_{t\in\tau} -p_t(s_t q_t) \quad &= \sum_{i=1}^{n} -p_{t_i}(a^+_{t_i} - a^-_{t_i}) = -\sum_{i=1}^{n} p_{t_i} a^+_{t_i} + \sum_{i=1}^{n} p_{t_i} a^-_{t_i} \\
&= -\sum_{i=1}^{n-1} p_{t_i} a^-_{t_{i+1}} - p_{t_n} a^+_{t_n} + \sum_{i=1}^{n-1} p_{t_{i+1}} a^-_{t_{i+1}} + p_{t_1} a^-_{t_1} \\
&= p_{t_1} a^-_{t_1} + -p_{t_n} a^+_{t_n} + \sum_{i=2}^{n} (p_{t_i} - p_{t_{i-1}}) a^-_{t_i} \\
&= p_{t_1} a^-_{t_1} - p_{t_n} a^+_{t_n} + \int_{t\in[t_1,t_n]} a^-_t \, dp \\
&= p_{t_-} a_{t_-} - p_{t_+} a_{t_+} + \int_{t\in T} a^-_t \, dp.
\end{aligned}$$

Annualised Returns

The **annualised log-return** over ω is $\ell_\omega \frac{1 \text{ year}}{\lambda_\omega}$, where λ_ω is the duration (lebesgue measure) of ω in units of time.

The **geometrically annualised return** over ω is $(1 + R_\omega)^{\frac{1 \text{ year}}{\lambda_\omega}} - 1 = \exp\left(\ell_\omega \frac{1 \text{ year}}{\lambda_\omega}\right) - 1.$

The **arithmetically annualised return** over ω is $R_\omega \frac{1 \text{ year}}{\lambda_\omega}$.

Alternative Price Proxies

More generally, we can define the depth-$\$d$ imbalance,

$$I_t(\$d) = \frac{Q_t(+1, \$b_t - \$d)}{Q_t(+1, \$b_t - \$d) + Q_t(-1, \$a_t + \$d)}.$$

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We can also define an **exponentially weighted imbalance**,

$$I_t(\alpha) =$$

Some microprices with more theoretical backing are discussed in the appendix.

Stuff I missed