

# POINT PROCESS MODELLING OF A LIMIT ORDER BOOK

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BACHELOR OF ADVANCED MATHEMATICS WITH HONOURS



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## Acknowledgements

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## Abstract

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This is the abstract





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# CHAPTER 1

## Introduction

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On an average day in 2023, the US stock market saw around \$500 billion dollars worth of shares traded on various exchanges and other venues, almost 2% of the annual GDP [7]. Modern securities markets facilitate the exchange of shares and other financial assets at extremely high frequency, as a result of aggressive investment in specialised networking hardware, custom-made computer architectures, and high-throughput machine learning systems. In facilitating capital flows for the global economy, financial markets have at the same time managed to claim an increasing fraction of resources and attention, with economic consequences that are not yet fully understood [1].

Despite many mysteries and open questions about the origins and dynamics of market phenomena, the financial sector itself has readily adapted to the increasing scale and complexity of the markets in which it operates. The practical design of exchange rules and trading systems has in large part been an empirical endeavour on the part of market participants, operators, and regulators. Many phenomena have been observed to emerge in an apparently decentralised fashion from the application of exploitative heuristics and predictive algorithms that interact with exchanges and aggregate to produce desirable outcomes. While users of trading strategies aim to maintain acceptable risk levels while generating profits over the long term, market operators and regulators are tasked with the design of incentive mechanisms that exploit this self-interested behaviour to improve market outcomes, including reduced transaction costs, fast and accurate incorporation of external information (such as economic news or earnings reports), and adherence to various concepts of fairness, propriety, and legality.

In this thesis, I describe and extend prior work from the empirical market microstructure literature, making extensive use of the state-dependent Hawkes process model for event arrivals. I begin with a conceptual overview of the trading mechanism, and formalise the mathematical tools that will be used to construct and describe variations on the basic Hawkes process model. I continue with a summary of existing literature on point processes as applied to market data, including both mathematical foundations and empirical findings. Next, I explore techniques to reduce the computational burden of parametric inference for point processes on large datasets. Finally, empirical applications and findings are discussed, including applications of generative modeling to a variety of open problems in market microstructure.

## 1.1 Limit Order Books

Intraday trading allows participants to respond to exogenous news and endogenous market events in a manner that maintains acceptable levels of risk and generates profits over the long term. This activity has a significant influence on the formation of market prices and plays a key role in reducing transaction costs while increasing the speed at which large institutions can control their exposure to various financial risks and opportunities.

Securities exchanges facilitate automated matching of buyers and sellers at prices favourable to both. Understanding the dynamics of this exchange process at a high degree of resolution can provide insights into the design of automated *matching engines* that produce desirable market behaviour, as well as insights into the design of *trading strategies* that exploit the dynamics of the exchange process to generate profits.

A matching engine is tasked with receiving and acting on various messages from market participants indicating their intent to buy or sell a particular security with particular conditions. As a result of this process, trades may be formed that match buyers and sellers at a mutually agreeable price and quantity. Trade reports are broadcast to relevant participants and may be used for the purposes of risk management and forecasting, as well as for the ultimate transfer of the assets that have been traded (which often occurs after trading hours).

A typical matching engine permits two kinds of incoming messages, known as *order insertion* and *order cancellation*, and maintains an internal state consisting of a single data structure, known as a *limit order book*. An order insertion message indicates a participant's willingness to buy (or sell) some quantity of a particular asset at or below (respectively, above) a particular price, and results in the addition of an *order* to the limit order book  $\mathcal{L}$ . Conversely, an order cancellation message results in the removal of a particular order from the limit order book, either in part (by reducing the remaining volume associated with the order) or in full (by removing the order entirely from the book). **It might be good to include a statistic about roughly how large cancellation rates are, to highlight the importance of this message type.**

Formally, a limit order book can be defined as a set of tuples ("orders") of the form

$$(\text{side, price, time, size}) = (s, p, t, q) \in \{-1, 1\} \times P \times T \times Q.$$

Each published order represents an intention to buy (or sell) some quantity of an asset at a maximum (respectively, minimum) price.

For example, Figure 1.1 shows the structure of a limit order submitted at 9:52am, expressing an intention to sell up to two units of a particular security for a price at least as favourable as \$84.10 per unit.

**Describe tick sizes, lot sizes.**

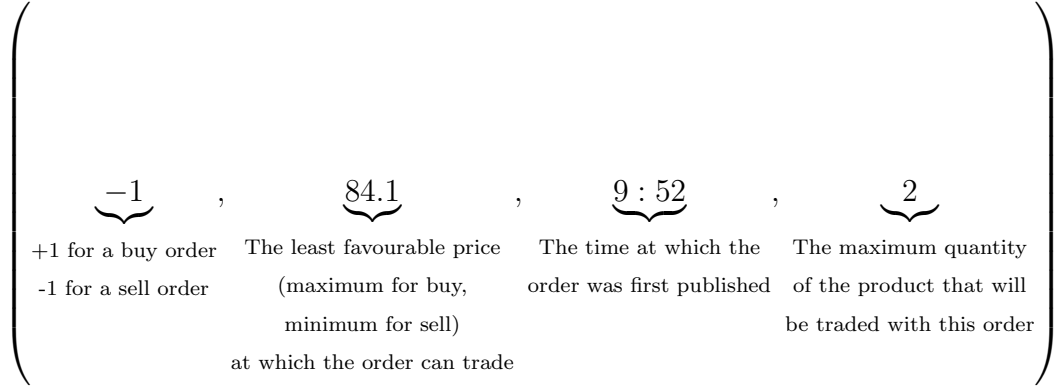


Figure 1.1: Components of an example limit order

## 1.2 The Matching Algorithm

### 1.2.1 The Bid and Ask

In order to describe the process by which changes to  $\mathcal{L}$  results in trades, I will consider partitioning  $\mathcal{L}$  into a *bid side*,

$$\mathcal{L}^+ = \{(1, p, t, q) \in \mathcal{L}\},$$

and an *ask side*,

$$\mathcal{L}^- = \{(-1, p, t, q) \in \mathcal{L}\}.$$

The most competitive prices on each side of the book are known as the *bid price* and *ask price*, given by

$$\begin{aligned} \text{bid}_{\mathcal{L}} &= \max_{\{p:(1,p,t,q) \in \mathcal{L}\}} p, \\ \text{ask}_{\mathcal{L}} &= \min_{\{p:(-1,p,t,q) \in \mathcal{L}\}} p. \end{aligned}$$

These prices are known as the *top levels* of the order book.

Whenever an order insertion message is received whose addition would result in  $\text{bid}_{\mathcal{L}} \geq \text{ask}_{\mathcal{L}}$ , the exchange will attempt to form trades with the existing orders on the opposing side of the book, such that as much volume as possible is matched. Existing orders on the opposing side of the book will be removed or depleted to equal the volume of the incoming order, and volume that cannot be matched at a price agreeable to the incoming order will finally be added to  $\mathcal{L}$ . As a result of this process, the invariant

$$\text{bid}_{\mathcal{L}} < \text{ask}_{\mathcal{L}}$$

is maintained after the processing of each book event.

Because existing orders in  $\mathcal{L}$  may be matched against incoming orders, the total volume posted to the book may be depleted over time, even in the absence of cancellations.

A book whose bid price exceeds the fair value of the product will be depleted on the bid side by traders seeking to sell the product at a premium to its true value. Conversely, an ask price below the fair value will be forced up by participants hoping

to buy at a discount. It is therefore common to regard the bid and ask as lower and upper bounds respectively on the consensus fair price of the product.

Motivated by this, the *midprice* is a naive point estimate for the consensus fair price, defined by

$$\text{mid}_{\mathcal{L}} = \frac{1}{2}(\text{bid}_{\mathcal{L}} + \text{ask}_{\mathcal{L}}).$$

Many other proxies for the consensus fair price exist that make greater use of information contained in  $\mathcal{L}$ , and it is common to use these as prediction targets in the construction of trading signals. **I might discuss them later**

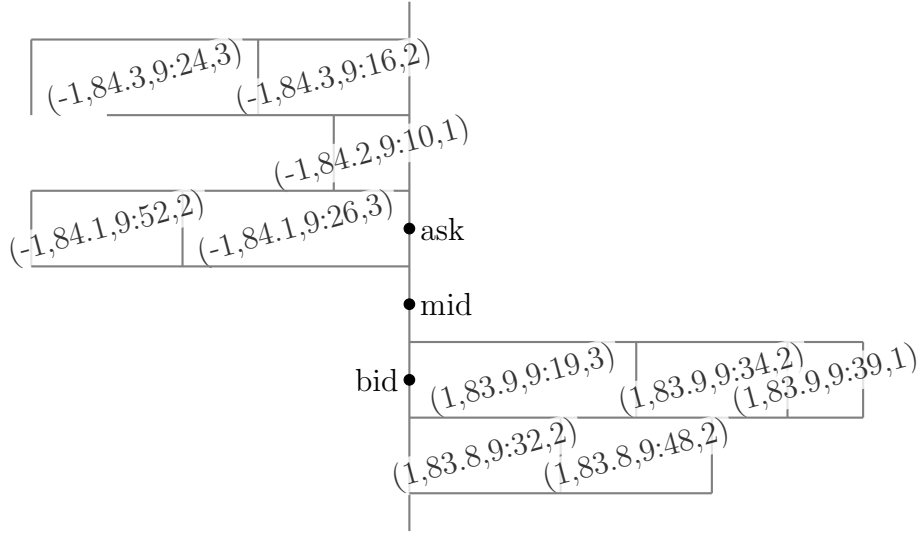


Figure 1.2: An example order book, arranged by order price and time

### 1.2.2 Liquidity

At any point in time, the contents of the limit order book represent trading opportunities presented to all market participants. The abundance of these opportunities, also known as *liquidity*, represents a positive externality insofar as it allows impatient traders (*liquidity takers*) to buy or sell products precisely under those circumstances where it is favourable to them. Conversely, order publishers (*liquidity providers*) must adhere to the terms of trades formed against a posted order, regardless of whether it is in their interests at the time the trade occurs.

One common measure for market liquidity is the *bid-ask spread*, defined as

$$\text{spread}_{\mathcal{L}} = \text{ask}_{\mathcal{L}} - \text{bid}_{\mathcal{L}}.$$

This can be described in units of price, but may also be described as a percentage of the midprice or as a multiple of the tick size (in which case we say the market is  $n$  ticks wide at some point in time). Notably, if  $\text{mid}_{\mathcal{L}}$  is taken to represent the fair value of the product, then the *half-spread*, i.e.  $\frac{1}{2}\text{spread}_{\mathcal{L}}$ , represents the premium paid to liquidity providers by participants trading against the posted bid or ask. There are many other aspects of liquidity, including the quantity posted at each level, and the impact (instantaneous and permanent) of trades on the order book, **which I may discuss later**.

## Adverse selection: large orders predict price moves

### 1.2.3 Queue Priority

When an order is submitted to the matching engine resulting in a trade, there may be some ambiguity about which existing orders it is to be matched against. For instance, consider the book displayed in figure 1.2, with a bid price of 83.9 and an ask price of 84.1, and suppose that a new buy order  $(1, 84.2, 9 : 53, 2)$  arrives. Because the buy order has a price of 84.2, its addition to the book would raise the bid price to 84.2, exceeding the ask price of 84.1. Consequently, it needs to be matched against any of the sell orders with a price at least as favourable as 84.2, i.e.  $(-1, 84.1, 9 : 26, 3)$ ,  $(-1, 84.1, 9 : 52, 2)$ , and  $(-1, 84.2, 9 : 10, 1)$ . It is not initially clear whether we should match two units against the first order, or against the second order, or match one unit against each of any two orders.

To eliminate this ambiguity, *queue priority* rules are set out that describe how to match incoming orders against opposing orders. In almost all cases, these rules obey *price priority*, where an existing order can only participate in a match if every order on the same side with a more competitive price has already been matched in full. This incentivises participants to submit orders with maximally competitive prices to increase the probability of a match, decreasing the bid-ask spread.

Beyond this, the two most common matching rules are known as *time priority* and *pro-rata*, which govern how matches are assigned in the case where orders have identical prices. Time priority requires that an order cannot participate in a match until all orders at the same price that have an earlier submission time have been fully matched. This incentivises early submission of orders and disincentivises cancellations. Pro-rata matching, on the other hand, seeks to allocate matches among orders with the same price in approximate proportion to the number of units in each order.

The ES and MES contracts traded on the Chicago Mercantile Exchange, which I will consider in [which chapter](#), follow a pro-rata matching rule set out in [citation](#). In order to determine a match from orders at a given price level, the size of each existing order is divided by the sum of all order sizes at that level, and this fraction is multiplied by the size of the incoming order and rounded down to the nearest integer. If the allocated trade quantity is less than two, it is rounded down to zero. After matching has taken place in this fashion, there may still be unmatched units in the incoming order, in which case they are matched against the remaining orders on the level according to time priority, and then if there are still unmatched units in the incoming order they may be matched against the next best price level or added to the order book.

## 1.3 Queueing Models of the Order Book

Changes in the price of a product over time may be viewed as emergent from individual order book events. For instance, changes to either the bid or ask price must result either from the addition of orders to the book or the depletion of a level due to cancellations or trades. From this perspective, the midprice is seen to follow a jump process governed by the evolution of the order book, which in turn is governed by the arrival of insertions or cancellations.



Analysing price changes from the perspective of event arrivals was originally (double check originality) proposed in cite the sante fe paper , where the arrival of order insertion and cancellation are taken to be governed by a Poisson process model. Despite the simplicity of their approach, they show that it replicates known empirical findings apparently unrelated to the model specification, such as the concavity of price impact (the expected change in midprice) as a function of trade size. This has come to be known as the Santa Fe model for limit order book evolution.

Chapter 5 of Bouchaud et. al. (2018) proposes queueing models for the top levels of the order book in a one-tick wide market. Under the assumption of Poisson processes for arrivals of insertion and cancellation requests at these levels (assuming identical rate parameters for the bid and ask), they Describe Bouchaud et al work [3]

Describe queue imbalance. It will be referred to later on

Evolution of the limit order book over time → motivate the emphasis on modeling arrival times

Price evolution as a jump process (what is a jump process) “Swishchuk and Huffman (2020) construct a compound Hawkes process” modeling price changes with a jump process where jump sizes are a markov chain. Jump process is controlled by one-dimensional point process. “Coinciding with the first preprint version of the present paper, Wu et al. (2019) develop a queue-reactive Hawkes process based on (2.4). In their model,  $X$  is endogenous and carries information about queue lengths in the LOB, while the multi-dimensional counting process driven by the intensity (2.4) models events pertaining to these queues. Wu et al. (2019) estimate their model on German bond (Bund) and index (DAX) futures LOB data.” “Subsequently, Mounjid et al. (2019) generalise the queue-reactive Hawkes process to a more general point process framework that allows for non-linearity and quadratic Hawkes structure. Mounjid et al. (2019) additionally establish ergodicity for the model and also derive functional limit theorems for its long-term behaviour. They apply the model to evaluate and rank equities market makers on Euronext Paris.”

Importance of arrival time modeling

<https://www.amazon.com/Point-Processes-Queues-Martingale-Statistics/dp/0387905>

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## CHAPTER 2

### Point Process Models

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With the goal of modelling the arrival time process, I will now provide an introductory overview of point processes, and then highlight some key variants that are relevant to order book modeling.

This section will make extensive use of theoretical concepts described in the appendix **Make sure to update this if there ends up being more than one appendix**. For an overview of measure theory, probability spaces, and stochastic processes, please refer to the relevant sections.

#### 2.1 Overview of Point Processes

Given a probability space  $(\Omega, \Sigma, \mathbb{P})$ , and a measurable space  $(T, \Sigma_T)$  representing times, a point process is any increasing sequence of random times

$$\mathcal{T} : \Omega \rightarrow T^{\mathbb{N}},$$

meaning that  $\mathcal{T}(\omega)_n$  is increasing in  $n$  for any  $\omega \in \Omega$ .

Concretely,  $T$  may be chosen to be  $\mathbb{R}_{\geq 0}$ , and equipped with the Borel  $\sigma$ -algebra  $B(\mathbb{R}_{\geq 0})$ .

The times in the point process are often referred to as *event times*, with the implication that the sequence represents the times at which some event of interest occurs (for instance, the arrival of messages sent to a matching engine).

For any point process, there exists a corresponding càdlàg stochastic process  $N_{\mathcal{T}} : \Omega \times T \rightarrow \mathbb{N}$  known as the *counting process*, that gives the number of events having occurred before or at a given time. This is defined as

$$N_{\mathcal{T}}(\omega, t) = |\{i \in \mathbb{N} | \mathcal{T}(\omega)_i \leq t\}|.$$

#### **Overlay barcode plot for trade times with corresponding counting process**

All **(or perhaps many)** of the point processes considered in this thesis will additionally be adapted with respect to some filtration  $\mathcal{F}$  indexed by  $T$ . **In what sense?**

Furthermore, I will only consider *nonexplosive point processes*, defined as those point processes  $\mathcal{T}$  for which

$$\lim_{n \rightarrow \infty} \mathcal{T}(\omega)_n = \infty, \mathbb{P}\text{-a.s.}$$

For any bounded interval of time  $(t_{\min}, t_{\max})$ , a nonexplosive point process will almost surely contain only a finite set of times in that interval, i.e.

$$(t_{\min}, t_{\max}) \cap \mathcal{T}(\omega) \text{ finite, } \mathbb{P}\text{-a.s.}$$

### 2.1.1 The Counting Measure

For any  $\omega \in \Omega$ , we can define the *counting measure*  $\Lambda_\omega$  of the point process as

$$\Lambda_\omega : \Sigma_T \rightarrow \mathbb{N}$$

$$\Lambda_\omega(S) := |S \cap \mathcal{T}(\omega)|.$$

For any finite or countable collection of disjoint sets  $A_n \in \Sigma$ , we have

$$\Lambda_\omega \left( \bigcup_n A_n \right) = \left| \bigcup_n (A_n \cap \mathcal{T}(\omega)) \right|.$$

Since subsets of disjoint sets are also disjoint, the terms in the union on the right-hand side will be disjoint, and so

$$\Lambda_\omega \left( \bigcup_n A_n \right) = \sum_n |A_n \cap \mathcal{T}(\omega)| = \sum_n \Lambda_\omega(A_n).$$

Thus  $\Lambda_\omega$  is a measure on  $(T, \Sigma_T)$ .

### 2.1.2 The Expectation Measure

Taking the expectation of the counting measure with respect to some measure  $\mathbb{P}$  on a measurable space  $(\Omega, \Sigma)$  gives the *expectation measure*,

$$\Lambda_{\mathbb{P}}(S) := \mathbb{E}_{\mathbb{P}}[\Lambda_\omega(S)] = \int_{\Omega} \Lambda_\omega(S) d\mathbb{P}(\omega).$$

Since  $\Lambda_\omega(S)$  is a non-negative function of  $\omega$ , it follows that

$$\Lambda_{\mathbb{P}} : \Sigma_T \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Furthermore, for any finite or countable sequence of  $A_n \in B(\mathbb{R})$ , we have

$$\Lambda_{\mathbb{P}} \left( \bigcup_n A_n \right) = \int_{\Omega} \Lambda_\omega \left( \bigcup_n A_n \right) d\mathbb{P}(\omega) = \int_{\Omega} \sum_n \Lambda_\omega(A_n) d\mathbb{P}(\omega),$$

by the countable additivity of  $\Lambda_\omega$ . Thus  $\Lambda_{\mathbb{P}}$  is a measure on  $(T, \Sigma_T)$ .

To handle the sum inside the integral, we can write each term as a sum of indicator functions:

$$\Lambda_\omega(A_n) = |A_n \cap \mathcal{T}(\omega)| = \sum_i 1_{A_n}(\mathcal{T}(\omega)_i).$$

Then, since  $A_n$  are measurable sets, the indicator functions are each measurable functions of  $\omega$ . Because finite sums and pointwise limits of measurable functions

are measurable, we have that  $\Lambda_\omega(A_n)$  is measurable. Finally, since each term in the sum over  $n$  is a nonnegative measurable function, the integral commutes with the sum, and hence  $\Lambda_\mathbb{P}$  is countably additive. Therefore the expectation measure is also a valid measure.

### 2.1.3 Intensity of a Point Process

For a point process  $\mathcal{T}$ , the counting measure of a half-open interval  $(t, t + \epsilon]$  can be written in terms of the counting process  $N_\mathcal{T}$  in the form

$$\Lambda_\omega((t, t + \epsilon]) = N_\mathcal{T}(\omega, t + \epsilon) - N_\mathcal{T}(\omega, t).$$

Integrating out  $\omega$  with respect to  $\mathbb{P}$  then gives us the identity

$$\Lambda_\mathbb{P}((t, t + \epsilon]) = \mathbb{E}_\mathbb{P}[N_\mathcal{T}(t + \epsilon)] - \mathbb{E}_\mathbb{P}[N_\mathcal{T}(t)],$$

allowing us to write the expectation measure in terms of a finite difference of the first moment of  $N_\mathcal{T}$ .

If  $\mathbb{E}_\mathbb{P}[N_\mathcal{T}(t)]$  is differentiable at  $t$ , we can further say that

$$\lim_{\epsilon \rightarrow 0} \frac{\Lambda_\mathbb{P}((t, t + \epsilon])}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{\mathbb{E}_\mathbb{P}[N_\mathcal{T}(t + \epsilon)] - \mathbb{E}_\mathbb{P}[N_\mathcal{T}(t)]}{\epsilon} = \frac{d\mathbb{E}_\mathbb{P}[N_\mathcal{T}(t)]}{dt}.$$

For brevity, I will write

$$\lambda_\mathbb{P}(t) := \frac{d\mathbb{E}_\mathbb{P}[N_\mathcal{T}(t)]}{dt}.$$

Assuming further that  $\mathbb{E}_\mathbb{P}[N_\mathcal{T}(t)]$  is differentiable on some open interval  $(t, t + \epsilon')$ , **Prove this is Radon-Nikodym derivative of  $\Lambda_\mathbb{P}$  in the open interval.** I will therefore refer to this quantity as the *expectation density*.

It then follows from the fundamental theorem of calculus that

$$\int_t^{t+\epsilon} \lambda_\mathbb{P}(s) ds = \lambda_\mathbb{P}(t) \cdot \epsilon + o(\epsilon), \quad \epsilon \rightarrow 0.$$

So **since  $\lambda_\mathbb{P}$  is a radon-nikodym derivative** we can write **Why?**

$$\Lambda_\mathbb{P}((t, t + \epsilon]) = \lambda_\mathbb{P}(t) \cdot \epsilon + o(\epsilon), \quad \epsilon \rightarrow 0$$

Therefore the expected number of events occurring in a small interval  $(t, t + \epsilon]$  is approximately proportional to its length.

If we instead take the expectations above conditioned on  $\mathcal{F}_t$ , we can define a stochastic process

$$\lambda(\omega, t) := \lim_{s \rightarrow t^-} \lambda_{\mathbb{P}|\mathcal{F}_s}(\omega, t)$$

as the left limit of the expectation density as the filtration index approaches  $t$ . This is commonly known as the *intensity* or *arrival rate* of the point process, and represents the expected number of events that will arrive in the next  $\epsilon$  units of time, for very small  $\epsilon$ , based on the information contained in the filtration up to time  $t$ .

**Show that  $\mathbb{E}_{\mathbb{P}|\mathcal{F}_s}[\lambda(\omega, t)] = \lambda_{\mathbb{P}|\mathcal{F}_s}(t)$  for  $s < t$**

Add a note somewhere to check the whole document for correct usage of left vs right limit of  $\lambda$ . This is very important.

Existence and uniqueness shown in <https://projecteuclid.org/journals/annals-of-probability/volume-24/issue-3/Stability-of-nonlinear-Hawkes-processes/10.1214/aop/10657251> under restrictions on the kernel

#### 2.1.4 Residuals and the Compensator Process

Given some point process  $\mathcal{T}$  with intensity  $\lambda$ , we may define the stochastic process

$$\bar{\Lambda}(\omega, t) = \int_{t_{\min}}^t \lambda(\omega, s) ds,$$

known as the *compensator process* of  $\mathcal{T}$ . Since  $\lambda$  is adapted to some filtration  $\mathcal{F}$ , it follows **why** that  $\bar{\Lambda}$  will be similarly adapted to  $\mathcal{F}$ .

Since  $\lambda$  is positive **what if its zero?**, any realisation of  $\bar{\Lambda}$  must be increasing in  $t$ . The random sequence  $\tau(\omega) = \{\bar{\Lambda}(\omega, \mathcal{T}(\omega)_i)\}_{i \in \mathbb{N}}$  will therefore be increasing for each  $\omega$ , making it a point process. Theorem 7.4.1 from Daley and Vere-Jones (2003) [5] demonstrates that regardless of the initial point process  $\mathcal{T}$ , the transformed process  $\tau$  is a Poisson process with unit rate, implying that the quantities

$$r_i(\omega) := \int_{\mathcal{T}(\omega)_i}^{\mathcal{T}(\omega)_{i+1}} \lambda(\omega, t) dt = \tau(\omega)_{i+1} - \tau(\omega)_i$$

for each  $i \in \mathbb{N}$  are independent and follow an  $\text{Exp}(1)$  distribution.

These are commonly known as the *residuals* of the point process, and serve as a common diagnostic tool for evaluating model quality. Deviation from  $\text{Exp}(1)$  or serial dependence between residuals are common signs of model misspecification or misestimation, and may give qualitative hints about how to improve the fit of a model to empirical data.

#### 2.1.5 Marked and Multivariate Point Processes

A point process  $\mathcal{T}$  may be *marked*, in which case it is associated with one or more random sequences of marks

$$\mathcal{X} : \Omega \rightarrow X^{\mathbb{N}},$$

drawn from a set  $X$ , that represent additional information about each event. These may be adapted to the filtration  $\mathcal{F}$  **In what sense?**

A common special case of this is when  $X$  is partitioned into a set  $E$  of *event types*, and define a random sequence  $\mathcal{E} \in E^{\mathbb{N}}$  whose  $i$ th entry is the event type of  $\mathcal{X}_i$ . In this case we may refer to  $(\mathcal{T}, \mathcal{E})$  as a *multivariate point process*.

Each part  $e$  of the partition then has an associated point process

$$\mathcal{T}_e = \mathcal{T}_{\{n \in \mathbb{N} : \mathcal{X}_n \in e\}},$$

formed by the subsequence of times where the corresponding element of  $\mathcal{X}$  is in  $e$ . These point processes will then have their own counting functions, intensities, and other characteristics. It is common to arrange these quantities in vector form, with one entry for each event type. For instance, the counting function of a multivariate point process will be a function  $N_{\mathcal{T}} : \Omega \times T \rightarrow \mathbb{N}^{|E|}$ .

On the other hand, given a collection of point processes adapted with respect to a common filtration, we can form a single marked point process by interleaving the sequences of event times. The counting process, counting measure, expectation measure, and intensity of the combined process will be the sum of those for the individual point processes. **Can i relate this to composite point processes**

**Barcode plot and counting process for buys and sells, and the combined process**

### 2.1.6 Composite Point Processes

Given point processes  $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n$  whose counting processes are adapted to a common filtration  $\mathcal{F}$ , we may construct a *composite point process* whose set of event times is the union of the event times in each  $\mathcal{T}$ . **Define these in a way that leads into hawkes processes. They should be adapted to the natural filtration on the overall point process (NOT the individual point processes, this would preclude interaction between them)**

## 2.2 Point Process Models of Book Updates

**For each extension: - show some example realisations for various parameter choices: barcode and intensity plot**

### 2.2.1 Poisson Processes

The simplest modeling assumption we can make is to assume that the intensity of the process is a constant  $\nu \in \mathbb{R}$ . A point process  $T$  whose intensity  $\lambda$  is given by some positive constant  $\nu$  is known as a Poisson process. Equivalently, if  $\mathcal{T}$  has counting measures  $\Lambda_\omega$ , we say that it is a Poisson process with rate  $\nu$  if and only if the following two properties hold:

1. For any finite collection of disjoint measurable sets  $A_n \in \Sigma_T$ , their counting measures  $\Lambda_\omega(A_n)$  are independent random variables.
2. For any measurable set  $A \in \Sigma_T$ , we have

$$\Lambda_{\mathbb{P}}(A) = \nu \mu_{\text{Lebesgue}}(A) \propto \mu_{\text{Lebesgue}}(A).$$

**Why is this the same as saying that the intensity is constant?** The random variables  $\mathcal{T}(\omega)_{i+1} - \mathcal{T}(\omega)_i$  for each  $i \in \mathbb{N}$  will then be independently  $\text{Exp}(\nu)$  distributed, with a mean of  $\frac{1}{\nu}$ . **Is there a quick proof of this**

By Markov's inequality, we know that

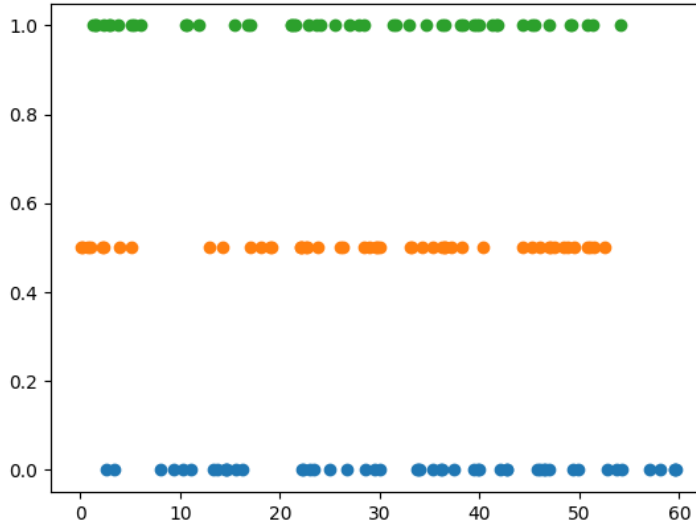
$$\begin{aligned} \mathbb{P}(\mathcal{T}(\omega)_k \leq k^2) &= \mathbb{P}(-\mathcal{T}(\omega)_k \geq -k^2) \\ &\leq \frac{\mathbb{E}_{\mathbb{P}}[\mathcal{T}(\omega)_k]}{k^2} = \frac{\mathbb{E}_{\mathbb{P}}[\sum_{j=0}^{k-1} \mathcal{T}(\omega)_{j+1} - \mathcal{T}(\omega)_j]}{k^2} = \frac{k}{k^2\nu} = \frac{1}{k\nu}, \end{aligned}$$

from which it follows that

$$\lim_{n \rightarrow \infty} \mathcal{T}(\omega)_n = \infty, \mathbb{P}\text{-a.s.},$$

and hence the Poisson process is non-explosive.

In the multivariate case with  $n$  event types, we have a vector  $\nu \in \mathbb{R}_{>0}^n$  containing the constant intensities of each individual point process.



I think I should change this to be separate plots. Can show the intensity function for each; e.g. for inhomogenous it's the same for every realisation, but for hawkes process the intensity function is itself random.

### 2.2.2 Inhomogeneous Poisson Process

An inhomogeneous Poisson process is a generalisation of the Poisson process for which the second condition is replaced by the weaker requirement that every measurable set  $A \in \Sigma_T$  with finite Lebesgue measure  $\mu_{\text{Lebesgue}}(A)$  has a finite expectation measure, and every  $A$  satisfying  $\mu_{\text{Lebesgue}}(A) = 0$  also satisfies  $\Lambda_{\mathbb{P}}(A) = 0$ .

By the Radon-Nikodym theorem, since  $\Lambda_{\mathbb{P}}$  is absolutely continuous with respect to  $\mu_{\text{Lebesgue}}$ , the expectation density  $\lambda_{\mathbb{P}}$  exists. Furthermore, because the counting measure on  $(t, t + \epsilon]$  is independent from the counting measure on any disjoint set, we have that  $\lambda_{\mathbb{P}}(\omega, t)$  is adapted to the same filtration  $\mathcal{F}$  as the counting process, from which it follows that the intensity process is simply  $\lambda = \lambda_{\mathbb{P}}$  and is therefore constant with respect to  $\omega$ .

Since  $\lambda$  is deterministic, the compensator process  $\bar{\Lambda}$  will also be deterministic (i.e. a function of  $t$  only), which allows us to simulate an inhomogenous point process by taking the random sequence  $\{\bar{\Lambda}^{-1}(\tau(\omega)_i)\}_{i \in \mathbb{N}}$  for some Poisson process  $\tau$ , so long as the inverse function  $\bar{\Lambda}^{-1}$  exists (i.e.  $\bar{\Lambda}$  is strictly increasing).

Deterministic variation in  $\lambda$  as a function of  $t$  is useful for modeling deterministic seasonality in the point process, such as having more events near the start and end of each trading session, or near known events such as news releases.

Show U-shape from empirical data Show U-shape from quadratic IPP

### 2.2.3 Hawkes Processes

In the case of the homogenous and inhomogenous Poisson processes, the intensity function is deterministic, i.e. constant in  $\omega$ . While such an assumption is appropriate for modeling the arrival of independent events, it is common in financial

markets for events to “cluster” together in a non-deterministic fashion, with the intensity dependent on the recent historical behaviour of the point process. One explicit test for this phenomenon is to examine the autocorrelation of residuals of an inhomogenous Poisson process model, **which will be covered in some section of the applications chapter. Show ACF of inter-arrival times**

The increase in intensity seen after a series of related events in quick succession is known as *self-excitation*<sup>1</sup>, and requires us to introduce a stochastic component to the intensity of our model that will depend on the realised history of the point process up until  $t$ . By analogy with time series literature, the dependence of forecasts on recent history is referred to as *autoregressive*.

A popular class of autoregressive models are known as *Hawkes Processes*, first introduced by **Original hawkes process paper <https://academic.oup.com/biomet/article-abstract/58/1/83/224809?redirectedFrom=fulltext> Discuss history of their application to financial datasets**

Univariate Hawkes process models require that the intensity process have the functional form

$$\lambda(\omega, t) = \nu + \int_{t_{\min}}^t k(t-s) d\Lambda_{\omega}(s),$$

where  $\nu \in \mathbb{R}_{>0}$  represents a constant arrival rate,  $k$  is a kernel function encoding the impact of each event on the future intensity, and  $t_{\min}$  is the earliest time events can occur, such as the opening time of a trading session. I will assume that  $k$  takes positive values for nonnegative arguments, and is zero for negative arguments.

**Existence and nonexplosiveness of such processes is established... where?**

In the case of a multivariate point process  $(\mathcal{T}, \mathcal{E})$ , we can generalise this functional form to

$$\lambda_e(\omega, t) = \nu_e + \sum_{e'} \int_{t_{\min}}^t k_{e',e}(t-s) d\Lambda_{\omega}^{e'}(s),$$

where  $k_{e',e}$  is the kernel associated with the pair of event types  $(e', e) \in E^2$ . For convenience, we can arrange the  $\lambda_e(\omega, t)$  and  $\Lambda_{\omega}$  into  $|E|$ -dimensional vector-valued functions  $\lambda(\omega, t)$  and  $\Lambda_{\omega}$ . Similarly, the base rates  $\nu_e$  can be arranged into a vector  $\nu$ , and the functions  $k_{e',e}$  into a matrix-valued function  $k(t-s)$ . Then finally we have

$$\lambda(\omega, t) = \nu + \int_{t_{\min}}^t k(t-s)^T d\Lambda_{\omega}(s).$$

Let  $K(t) = \int_0^t k(u) du$ .

**Since  $\lambda$  can't take on negative values, the kernel function must be positive. Show also that it must have finite L1 norm**

It is possible to generate samples from a multivariate Hawkes process  $(\mathcal{T}, \mathcal{E})$  in the following manner. Firstly, draw samples from a multivariate Poisson process  $\mathcal{T}'$  with rate  $\nu$ , and let  $\tau_0$  be the resulting sequence of pairs  $(t, e) \in T \times E$ , ordered by time. For each  $e$ , we have the function  $K_{e_i,e}$  and its inverse function  $K_{e_i,e}^{-1}$ . Draw

---

<sup>1</sup>In the case of multivariate point processes, interactions between different event types are often called *cross-exciting*. The opposite behaviour, where recent occurrences temporarily decrease the intensity, is known as *self-inhibition* or *cross-inhibition*.



samples from an independent Poisson process  $\pi_i^e$  with unit rate until one of the times exceeds  $K_{e_i,e}(\infty)$ . Following this, let

$$g_i^e = \{(K_{e_i,e}^{-1}(t) + \tau'_{i,i}, e) | t \in \pi_i^e\},$$

where  $\tau'_{i,i}$  is the time component of the pair  $\tau_{i,i}$ . Lastly, let

$$\tau_{i+1} = \tau_i \cup \bigcup_{e \in E} g_i^e.$$

Then the sequence of pairs  $\{\tau_{i,i}\}_{i \in \mathbb{N}}$  will be distributed in accordance with the desired multivariate Hawkes process. **Proof? or source? Immigration-birth interpretation [13] (aka Watson-Galton models). Introduce the term *parent event* Stability criterion. I think it's  $\det K < 1$ . but how to prove. have  $\bar{\lambda} = (I - K)^{-1}\nu$ .**

For computational simplicity, it is often convenient to use a kernel of the form

$$k_{e',e}(u) = \sum_i \alpha_{e',e,i} \exp(-\beta_{e',e,i}u),$$

for some real numbers  $\alpha_{e',e,i}$  and positive real numbers  $\beta_{e',e,i}$ . Suppose that  $k_{e',e}$  has some different functional form, but is still continuous. The Weierstrass approximation theorem guarantees that every continuous function on some closed interval can be uniformly approximated as closely desired by polynomials. Since polynomials in  $\exp(-u)$  with no constant term can be written in the above form, and this is a continuous bijective function of  $u$ , it follows that any continuous function on a closed interval can be approximated arbitrarily well by a kernel of this form and a constant term. Since the constant term is arbitrarily well approximated by very small values of  $\beta_{e',e,i}$ , we can approximate any other continuous kernel function arbitrarily well by a linear combination of decreasing exponentials. **This proof needs more detail. Use triangle inequality**

According to <https://ieeexplore.ieee.org/document/7416001> <https://arxiv.org/pdf/> hawkes processes are fully determined by the first two moments of the intensity function. Proof in the paper.

#### 2.2.4 State Dependence

Just as the arrivals of events in the limit order book governs the evolution of its state over time, we might reasonably expect that the arrival rate of certain kinds of events is dependent on the present state of the order book, as captured by a variety of features. Vinkovskaya (2014) [?] proposed a regime-switching variant of the Hawkes process model for which the kernel function additionally depends on whether the market was one tick wide at the time of the parent event, finding a greater self-excitation effect for the various event types in the one-tick-wide regime.

Moriaru-Patrichi and Pakkanen (2022) [?] find similarly that a regime-switching variant learns different parameters for the one-tick-wide regime, as well as in a model using five different states based on queue imbalance. Their work is distinguished from prior regime-switching approaches in the sense that they also model the evolution of the order book state over time as a finite state-switching process, with transition probabilities at each event time also dependent on the event type.

While this coupling of a point process model with state-switching behaviour admits broader application than simple modeling of arrival times, this is not the primary focus of the work, and the simple nature of their state model leaves room for further extension.

The model of [?] takes the form

$$\lambda_e(\omega, t) = \nu_{\mathcal{X}(\omega, t), e} + \sum_{e' \in E} \int_{t_{\min}}^t k_{\mathcal{X}(\omega, s), e', e}(t - s) d\Lambda_{\omega}^{e'}(s),$$

where  $\mathcal{X}(\omega, t)$  is a discrete-valued stochastic process representing the order book state, with jumps only at the event times of the point process.

Notably, the focus of prior work has been primarily on the case with a finite number of discrete states, possibly obtained by discretisation of a continuous quantity such as queue imbalance. Consideration of multiple state variables quickly becomes difficult, as the number of states grows exponentially in the number of independent state variables. No prior information is taken into account regarding the monotonicity of excitation with respect to continuous state variables, and no extrapolation can take place on this basis.

It also seems important for the model to update based on the very latest information about the state of the order book. Presently, such information is only taken into account at the times of the parent events, and may well be out of date at the time when the intensity function is evaluated. To address this, I will first propose a more general model of the form

$$\lambda_e(\omega, t) = \nu_{\mathcal{X}(\omega, t), e} + \sum_{e' \in E} \int_{t_{\min}}^t k_{\mathcal{X}(\omega, s), \mathcal{X}(\omega, t), e', e}(t - s) d\Lambda_{\omega}^{e'}(s).$$

Assuming that for each  $(e', e)$  the kernel function  $k$  is separable in  $\mathcal{X}(s)$ ,  $\mathcal{X}(t)$ , and  $t - s$ , **explain why this assumption still lets us approximate any true situation with enough component kernels**, we obtain

$$\lambda_e(\omega, t) = \nu_{\mathcal{X}(\omega, t), e} + \sum_{e' \in E} \int_{t_{\min}}^t A(\mathcal{X}(\omega, s)) B(\mathcal{X}(\omega, t)) k_{e', e}(t - s) d\Lambda_{\omega}^{e'}(s).$$

$$\lambda_e(\omega, t) = \nu_{\mathcal{X}(\omega, t), e} + \sum_{e' \in E} \int_{t_{\min}}^t k_{\mathcal{X}(\omega, s), \mathcal{X}(\omega, t), e', e}(t - s) d\Lambda_{\omega}^{e'}(s).$$

### 2.2.5 Regression on marks

Multiply the kernel by  $\exp(X \cdot \beta)$  where  $X$  is a vector of mark variables and  $\beta$  are coefficients. **transfer functions should be separable, see [6.25 Definition] in <https://www.research-collection.ethz.ch/bitstream/handle/20.500.11850/151886/eth-1112-02.pdf>**

### *2.2.6 Raising the Intensity to a Power*

### *2.2.7 Hidden States*

“Cohen and Elliott (2013) introduce a one-dimensional Markov-modulated Hawkes process following (2.4), where  $X$  is an exogenous Markov process in a finite state space (exogenous in the sense that the counting process does not influence  $X$ ). The key feature of their work is that they assume  $X$  to be unobservable, leading them to derive a filtering procedure for the estimation of the current state of  $X$ .” (hidden states)

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## CHAPTER 3

### Simulation

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Simulation of the limit order book allows for

Will play a key role in (1) use of the model for ‘experimental’ studies (2) parametric bootstrap (3) metaorder monte carlo em (if i end up doing this)

#### 3.1 Simulation Methods for Point Process Models

Pseudocode

Ogata thinning

Pseudocode

Do these have different time complexity? Memory complexity?

#### 3.2 Simulation Study of Estimation Methods

Convergence analysis for simple models (eg univariate)

#### 3.3 Impulse Response Function

Causal analysis, price impact of orders.

What does inserting a single exogenous event do to the order book, price, etc? Are there analytical formulas for this?

[19] [20] sources that use hawkes process models to simulate an order book for analysis purposes

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## CHAPTER 4

### Estimation

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To enable analysis of empirical order book data, I will now explain the principles involved in parametric estimation of point process models.

We will work with a dataset of the form  $\mathcal{D} = (\mathcal{T}_{\text{obs}}, \mathcal{X}_{\text{obs}})$ , consisting of a sequence of event times and event marks observed in the time interval

$$[t_{\min}, t_{\max}].$$

We can define a counting measure  $\Lambda_{\mathcal{D}}$  on the space of events times  $(T, \Sigma_T)$  by

$$\Lambda_{\mathcal{D}} : \Sigma_T \rightarrow \mathbb{R}_{\geq 0}$$

$$\Lambda_{\mathcal{D}}(A) := |A \cap \mathcal{T}_{\text{obs}}|.$$

Similarly, we can define a càdlàg counting function

$$N_{\mathcal{D}} : T \rightarrow \mathbb{N}$$

$$N_{\mathcal{D}}(t) := |\mathcal{T}_{\text{obs}} \cap [t_{\min}, t]| = \int_{[t_{\min}, t]} d\Lambda_{\mathcal{D}}.$$

In the multivariate case we will further have  $\Lambda_{\mathcal{D}}^E$  and  $N_{\mathcal{D}}^E$  for each event type  $E$ .

Given a subset of the data  $\mathcal{D}_t$  containing only the event times and marks up until some time  $t$ , it is reasonable to ask whether we can predict the remainder of the dataset using some model of the point process that generated the data.

If we consider a point process  $\mathcal{T}$  defined with respect to  $\mathbb{P}$  and having natural filtration  $\mathcal{F}$ , we can construct an *empirical intensity function*

$$\lambda_{\mathcal{D}}(t) := \lambda_{\mathbb{P}|\mathcal{F}_t, \mathcal{D}_t}(t)$$

as the intensity conditional on the  $\sigma$ -algebra  $\mathcal{F}_t$  and the contents of  $\mathcal{D}_t$ . **How is this well-defined? The point process realising  $\mathcal{D}_t$  exactly has probability zero, and probability-zero events can't be conditioned on. I suspect the solution here is to construct it using the janossy density. In particular, instead of conditioning on  $\mathcal{D}_t$ , it is proper to condition on  $\mathcal{D}_t$  being in a certain event set and having a certain number of events, then use this to define a measure which  $\lambda_{\mathcal{D}}$  is constructed as a density of**

**For the purposes of defining the likelihood function, the janossy density also needs to be derived or cited**

$$p(\mathcal{D}) = \exp \left( \int_{t_{\min}}^{t_{\max}} \log(\lambda_{\mathcal{D}}(t)) d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}(t) dt \right).$$

#### 4.1 Branching Probabilities

I originally wrote this for hawkes processes, but can be made more general now since I'm gonna describe what it means to condition on  $\mathcal{D}, \mathcal{F}$  above, and these are just probabilities wrt this conditional measure. Endo/exo can still be given as an example. If I don't make it more general I need to make it clear that it's only relevant to hawkes processes. It is common to regard events in the point process as having either been generated by a "background" poisson process with arrival rate  $\nu$ , or else "triggered" by some previous event. The first kind are known as *exogenous events*, while the second kind are *endogenous events*. From the form of the intensity, it follows **why?** that the probability of an event at time  $t$  being from the background process, conditioning on the natural filtration  $\mathcal{F}_t$ , **Is this defined properly?** will be

$$B_{\text{exo},t} := \frac{\nu}{\lambda(\omega, t)},$$

while the probability of it being endogenous will be

$$B_{\text{endo},t} := \int_{t_{\min}}^t B_{s,t} d\Lambda_{\omega}(s)$$

where

$$B_{s,t} := \frac{k(t-s)}{\lambda(\omega, t)}$$

is the probability of an event at  $t$  having been triggered by an event at  $s$ .

Notice the dependence on  $\omega$ , implying that these probabilities (often known as *branching probabilities*) are specific to a particular realisation of the process.

**Observe that these are truly probabilities for composite point processes, but if some components of intensity are negative this is no longer the correct interpretation; nonetheless we will still use the  $B$  notation**

#### 4.2 Maximum Likelihood Estimation

Given a parametric family of multivariate point processes,

$$\{(\mathcal{T}_{\theta}, \mathcal{X}) : \theta \in \Theta\},$$

we may wish to select one model from the family that best describes the empirical data.

Letting  $p_{\theta}(\mathcal{D})$  be the probability density of an observed dataset  $\mathcal{D}$  under  $\mathcal{T}_{\theta}$ , we can write the *likelihood function* for the dataset

$$L(\theta; \mathcal{D}) := p_{\theta}(\mathcal{D}) = \exp \left( \int_{t_{\min}}^{t_{\max}} \log(\lambda_{\mathcal{D}}^{\theta}(t)) d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}^{\theta}(t) dt \right),$$

where  $\lambda_{\mathcal{D}}^{\theta}$  represents the empirical intensity function of  $\mathcal{T}_{\theta}$ .

#### 4.2.1 Maximum Likelihood Estimator

Because the likelihood function tells us the probability density of the observed data under the corresponding model, a higher likelihood function corresponds to a better fit of model to data. Assuming that the likelihood has a unique global maxima

$$\hat{\theta}_{\text{MLE}} := \underset{\theta}{\operatorname{argmax}} L(\theta; \mathcal{D}),$$

this is known as the *maximum likelihood estimate*, and is a popular method of selecting a single “best” model from a parametric family. **Discuss why and what properties it has. Ogata proves consistency and asymptotic normality for stationary processes estimate = on a particular dataset. estimator = considered as a random variable** Ogata paper [https://www.ism.ac.jp/editsec/aism/pdf/030\\_20243.pdf](https://www.ism.ac.jp/editsec/aism/pdf/030_20243.pdf)

**Is it biased? preferably downward, this is nice for hidden events to stop the event count from increasing too much**

Because finding global maxima is in general a hard problem, it is common to first weaken our search to finding stationary points of  $L$  using its first derivative with respect to  $\theta$ . We can then use second-order information to eliminate local minima and saddle points, and select the best local maxima among those found in hopes that it is either the global maxima, or at least has a similar enough likelihood to be a good model choice.

#### 4.2.2 Log-Likelihood Maximisation

Observe that we can write  $L$  in the form

$$L(\theta; \mathcal{D}) = \exp(\ell(\theta; \mathcal{D}))$$

where  $\ell(\theta; \mathcal{D})$  is the *log-likelihood function*,

$$\begin{aligned} \ell(\theta; \mathcal{D}) &:= \log(p_{\theta}(\mathcal{D})) \\ &= \int_{t_{\min}}^{t_{\max}} \log(\lambda_{\mathcal{D}}^{\theta}(t)) d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}^{\theta}(t) dt \\ &= \int_{t_{\min}}^{t_{\max}} \log(\lambda_{\mathcal{D}}^{\theta}(t)) d\Lambda_{\mathcal{D}} - \bar{\Lambda}(t_{\max}). \end{aligned}$$

Then, by the chain rule, it follows that

$$\frac{\partial}{\partial \theta} L(\theta; \mathcal{D}) = \left( \frac{\partial}{\partial \theta} \ell(\theta; \mathcal{D}) \right) \exp(\ell(\theta; \mathcal{D})).$$

Because  $\exp(\ell(\theta; \mathcal{D})) > 0$  for all  $\theta$ , the left-hand side is zero exactly when

$$\frac{\partial}{\partial \theta} \ell(\theta; \mathcal{D}) = 0,$$

meaning that the stationary points of  $L$  and  $\ell$  coincide. This is also clear if we consider that  $\log$  is a monotonic function, meaning that if  $L$  has a unique maxima in a region, it will also be the unique maxima of  $\ell$  in that region.

Furthermore, we have

$$\begin{aligned}
\frac{\partial}{\partial \theta} \ell(\theta; \mathcal{D}) &= \int_{t_{\min}}^{t_{\max}} \frac{\partial}{\partial \theta} \log(\lambda_{\mathcal{D}}^{\theta}(t)) d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \frac{\partial}{\partial \theta} \lambda_{\mathcal{D}}^{\theta}(t) dt \\
&= \int_{t_{\min}}^{t_{\max}} \frac{\frac{\partial}{\partial \theta} \lambda_{\mathcal{D}}^{\theta}(t)}{\lambda_{\mathcal{D}}^{\theta}(t)} d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \frac{\partial}{\partial \theta} \lambda_{\mathcal{D}}^{\theta}(t) dt, \\
\frac{\partial^2}{\partial \theta^2} \ell(\theta; \mathcal{D}) &= \int_{t_{\min}}^{t_{\max}} \frac{\frac{\partial^2}{\partial \theta^2} \lambda_{\mathcal{D}}^{\theta}(t) - \left( \frac{\partial}{\partial \theta} \lambda_{\mathcal{D}}^{\theta}(t) \right) \left( \frac{\partial}{\partial \theta} \lambda_{\mathcal{D}}^{\theta}(t) \right)^T}{\lambda_{\mathcal{D}}^{\theta}(t)} d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \frac{\partial^2}{\partial \theta^2} \lambda_{\mathcal{D}}^{\theta}(t) dt.
\end{aligned}$$

why can we bring the derivative inside the integral?

#### 4.2.3 Poisson Processes

Suppose we have a family of Poisson process models parameterised by their constant intensity  $\nu \in \mathbb{R}_{>0}$ .

The log-likelihood of each model is then given by

$$\begin{aligned}
\ell(\nu; \mathcal{D}) &= \int_{t_{\min}}^{t_{\max}} \log(\nu) d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \nu dt \\
&= \log(\nu) \int_{t_{\min}}^{t_{\max}} d\Lambda_{\mathcal{D}} - \nu \int_{t_{\min}}^{t_{\max}} dt \\
&= \log(\nu) N_{\mathcal{D}}(t_{\max}) - \nu (t_{\max} - t_{\min}).
\end{aligned} \tag{4.2.1}$$

At stationary points, we will then have

$$\frac{\partial}{\partial \nu} \ell(\nu; \mathcal{D}) = \frac{1}{\nu} N_{\mathcal{D}}(t_{\max}) - (t_{\max} - t_{\min}) = 0,$$

implying that

$$\hat{\nu}_{\text{MLE}} = \frac{N_{\mathcal{D}}(t_{\max})}{t_{\max} - t_{\min}}. \tag{4.2.2}$$

Intuitively, the estimate for the arrival rate  $\nu$  is the average number of events per unit time in the observed data. Note also that for the maximum likelihood model, the sample average of the residuals  $r$  will be exactly equal to one, since we have

$$\sum_{i=1}^{N_{\mathcal{D}}(t_{\max})} r_i = \bar{\Lambda}(t_{\max}) = \hat{\nu}_{\text{MLE}}(t_{\max} - t_{\min}) = N_{\mathcal{D}}(t_{\max}).$$

In the multivariate case, where we have a vector  $\nu \in \mathbb{R}_{>0}^n$  containing the arrival rates for each event type  $E$ , we similarly obtain

$$(\hat{\nu}_{\text{MLE}})_E = \frac{N_{\mathcal{D}}^E(t_{\max})}{t_{\max} - t_{\min}}.$$

---

<sup>1</sup>Subject to the Hessian of  $\ell$  being negative definite, which is true here.



#### 4.2.4 Composite Processes

Suppose that the intensity of each point process  $\mathcal{T}_{\theta_1, \theta_2, \dots}$  in a parametric family has the form

$$\lambda^{\theta_1, \theta_2, \dots}(\omega, t) := \sum_i \lambda_i^{\theta_i}(\omega, t),$$

where each component  $\lambda_i(\omega, t)$  is parameterised by some vector  $\theta_i$  and adapted with respect to the natural filtration  $\mathcal{F}$  of  $\mathcal{T}_{\theta_1, \theta_2, \dots}$ . Then the loglikelihood will be

$$\ell(\theta_1, \theta_2, \dots; \mathcal{D}) = \int_{t_{\min}}^{t_{\max}} \log \left( \sum_i (\lambda_i^{\theta_i})_{\mathcal{D}}(t) \right) d\Lambda_{\mathcal{D}}(t) - \sum_i \bar{\Lambda}_i(t_{\max}),$$

where

$$(\lambda_i^{\theta_i})_{\mathcal{D}} := \lim_{s \rightarrow t^+} \mathbb{E}_{\mathbb{P}|\mathcal{F}_t, \mathcal{D}_t} [\lambda_i^{\theta_i}(s)]$$

Is this the right definition to use? it should coincide with component intensities in the case of a composite point process, as noted below. It seems to depend on continuity of the first moment, and so maybe more restrictions need to be placed on the component processes

and

$$\bar{\Lambda}_i(t) := \int_{t_{\min}}^t (\lambda_i^{\theta_i})_{\mathcal{D}}(s) ds.$$

Define

$$B_{i,t}^{\theta_1, \theta_2, \dots} := \frac{(\lambda_i^{\theta_i})_{\mathcal{D}}(t)}{\lambda_{\mathcal{D}}^{\theta_1, \theta_2, \dots}(t)}$$

to be the ratios of each component to the overall intensity.

By the chain rule, we then have [Why can I exchange derivative and integral here](#)

$$\begin{aligned} \frac{\partial}{\partial \theta_i} \ell(\theta_1, \theta_2, \dots; \mathcal{D}) &= \int_{t_{\min}}^{t_{\max}} \frac{\partial}{\partial \theta_i} \log \left( \sum_i (\lambda_i^{\theta_i})_{\mathcal{D}}(t) \right) d\Lambda_{\mathcal{D}} - \frac{\partial}{\partial \theta_i} \bar{\Lambda}_i(t_{\max}) \\ &= \int_{t_{\min}}^{t_{\max}} \frac{\frac{\partial}{\partial \theta_i} (\lambda_i^{\theta_i})_{\mathcal{D}}(t)}{\lambda_{\mathcal{D}}^{\theta_1, \theta_2, \dots}(t)} d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \frac{\partial}{\partial \theta_i} (\lambda_i^{\theta_i})_{\mathcal{D}}(t) dt \\ &= \int_{t_{\min}}^{t_{\max}} \frac{\frac{\partial}{\partial \theta_i} (\lambda_i^{\theta_i})_{\mathcal{D}}(t)}{(\lambda_i^{\theta_i})_{\mathcal{D}}(t)} B_{i,t} d\Lambda_{\mathcal{D}} - \int_{t_{\min}}^{t_{\max}} \frac{\partial}{\partial \theta_i} (\lambda_i^{\theta_i})_{\mathcal{D}}(t) dt \\ &= \int_{t_{\min}}^{t_{\max}} \left( \frac{\partial}{\partial \theta_i} \log (|(\lambda_i^{\theta_i})_{\mathcal{D}}(t)|) \right) (|B_{i,t}| d\Lambda_{\mathcal{D}}) - \int_{t_{\min}}^{t_{\max}} \frac{\partial}{\partial \theta_i} (\lambda_i^{\theta_i})_{\mathcal{D}}(t) dt, \end{aligned}$$

Can the last equation be written in terms of log?

Fixing  $B$  and iteratively maximising (or newtons method, or gradient steps) can only stop at a stationary point; these are the fixed points of such a procedure. If the  $B$ s are positive then convergence is guaranteed, covered in next section. Also notice that expected gradient is equal to  $\ell$  gradient, meaning that small improvements to  $E[\ell]$  should also improve  $\ell$ . So it makes sense to iteratively maximise, and hope for convergence.

### 4.3 Expectation Maximisation

If  $(\lambda_i^{\theta_i})_{\mathcal{D}}$  are positive,<sup>2</sup> we have

$$\log \left( \sum_i (\lambda_i^{\theta_i})_{\mathcal{D}}(t) \right) = \sum_i \log \left( (\lambda_i^{\theta_i})_{\mathcal{D}}(t) B_{i,t}^{\theta_1, \theta_2, \dots} \right) - \sum_i \log \left( B_{i,t}^{\theta_1, \theta_2, \dots} \right) B_{i,t}^{\theta_1, \theta_2, \dots},$$

By Gibb's inequality, it follows that

$$- \sum_i \log \left( B_{i,t}^{\theta_1, \theta_2, \dots} \right) B_{i,t}^{\theta_1, \theta_2, \dots} \leq - \sum_i \log \left( B_{i,t}^{\theta'_1, \theta'_2, \dots} \right) B_{i,t}^{\theta_1, \theta_2, \dots}$$

EM algorithm and proof of convergence.

Can cite original dempster paper? As well as the one that corrects the errors in that paper

#### 4.3.1 Hawkes Processes

$$\begin{aligned} \ell(\alpha, \beta; \mathcal{D}) &= \sum_j \left( \int_{t_{\min}}^{t_{\max}} \log \left( \sum_i \int_{t_{\min}}^t \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^i(s) \right) d\Lambda_{\mathcal{D}}^j(t) \right) \\ &\quad - \sum_j \sum_i \left( \int_{t_{\min}}^{t_{\max}} \int_{t_{\min}}^t \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^i(s) dt \right) \\ \frac{\partial \ell}{\partial \alpha_{i,j}} &= \int_{t_{\min}}^{t_{\max}} \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) - \int_{t_{\min}}^{t_{\max}} \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}} dt \\ \frac{\partial^2 \ell}{\partial \alpha_{i,j} \partial \alpha_{i,j}} &= \int_{t_{\min}}^{t_{\max}} \left( \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}^2 \lambda_{\mathcal{D}}^j(t)} - \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}^2 \lambda_{\mathcal{D}}^j(t)} - \left( \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) \\ &\quad - \int_{t_{\min}}^{t_{\max}} \left( \frac{\lambda_{\mathcal{D}}^j(t)}{\alpha_{i,j}^2} - \frac{\lambda_{\mathcal{D}}^j(t)}{\alpha_{i,j}^2} \right) dt \\ &= - \int_{t_{\min}}^{t_{\max}} \left( \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \right)^2 d\Lambda_{\mathcal{D}}^j(t) \\ \frac{\partial^2 \ell}{\partial \alpha_{i,j} \partial \alpha_{k,j}} &= \int_{t_{\min}}^{t_{\max}} \frac{\lambda_{\mathcal{D}}^{i,j}(t) \lambda_{\mathcal{D}}^{k,j}(t)}{\alpha_{i,j} \alpha_{k,j} \lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) \\ \frac{\partial^2 \ell}{\partial \alpha_{i,j} \partial \alpha_{k,l}} &= 0 \end{aligned}$$

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<sup>2</sup>In the case where  $\lambda^{\theta_1, \theta_2, \dots}$  is the intensity of a composite point process whose components are adapted to the natural filtration of  $\mathcal{T}_{\theta_1, \theta_2, \dots}$ , and have intensities  $\lambda_i^{\theta_i}$ , the functions  $(\lambda_i^{\theta_i})_{\mathcal{D}}$  will just be the empirical intensity of the components. Is this true? is there a simple proof?, and therefore will each be positive definite functions. The values of  $B_{i,t}$  will then represent the probability that an event at  $t$  is from the  $i$ th component point process. Is this true? is there a simple proof?

$$\begin{aligned}
\gamma_{\mathcal{D}}^{i,j}(t) &= -\frac{\partial}{\partial \beta_{i,j}} \lambda_{\mathcal{D}}^{i,j}(t) = \int_{t_{\min}}^t (t-s) \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \\
\gamma_{\mathcal{D}}^{i,j}(t+\delta) &= \int_{t_{\min}}^{t+\delta} (t+\delta-s) \alpha_{i,j} \exp(-\beta_{i,j}(t+\delta-s)) d\Lambda_{\mathcal{D}}^j(s) \\
&= \delta \exp(-\beta_{i,j}\delta) \left( \lambda_{\mathcal{D}}^{i,j}(t) + \int_t^{t+\delta} \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \right) \\
&\quad + \exp(-\beta_{i,j}\delta) \left( \gamma_{\mathcal{D}}^{i,j}(t) + \int_t^{t+\delta} (t-s) \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \right) \\
\Gamma_{\mathcal{D}}^{i,j}(t) &= \int_{t_{\min}}^t \gamma_{\mathcal{D}}^{i,j}(s) ds \\
\Gamma_{\mathcal{D}}^{i,j}(t+\delta) &= \Gamma_{\mathcal{D}}^{i,j}(t) + \int_0^{\delta} \gamma_{\mathcal{D}}^{i,j}(t+\delta') d\delta' \\
&= \Gamma_{\mathcal{D}}^{i,j}(t) + \lambda_{\mathcal{D}}^{i,j}(t) \int_0^{\delta} \delta' \exp(-\beta_{i,j}\delta') d\delta' + \gamma_{\mathcal{D}}^{i,j}(t) \int_0^{\delta} \exp(-\beta_{i,j}\delta') d\delta' \\
&\quad + \int_{t_{\min}}^t \int_t^{t+\delta} \dots d\Lambda_{\mathcal{D}}^j(s) du \\
&= \Gamma_{\mathcal{D}}^{i,j}(t) + \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\beta_{i,j}^2} (1 - (\beta_{i,j}\delta + 1) \exp(-\beta_{i,j}\delta)) + \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\beta_{i,j}} (1 - \exp(-\beta_{i,j}\delta)) \\
&\quad + \int_{t_{\min}}^t \int_t^{t+\delta} \dots d\Lambda_{\mathcal{D}}^j(s) du
\end{aligned}$$

$$\begin{aligned}
\phi_{\mathcal{D}}^{i,j}(t) &= -\frac{\partial}{\partial \beta_{i,j}} \gamma_{\mathcal{D}}^{i,j}(t) = \int_{t_{\min}}^t (t-s)^2 \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \\
\phi_{\mathcal{D}}^{i,j}(t+\delta) &= \int_{t_{\min}}^{t+\delta} (t+\delta-s)^2 \alpha_{i,j} \exp(-\beta_{i,j}(t+\delta-s)) d\Lambda_{\mathcal{D}}^j(s) \\
&= \exp(-\beta_{i,j}\delta) \int_{t_{\min}}^{t+\delta} (\delta^2 + 2\delta(t-s) + (t-s)^2) \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \\
&= \delta^2 \exp(-\beta_{i,j}\delta) \left( \lambda_{\mathcal{D}}^{i,j}(t) + \int_t^{t+\delta} \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \right) \\
&\quad + 2\delta \exp(-\beta_{i,j}\delta) \left( \gamma_{\mathcal{D}}^{i,j}(t) + \int_t^{t+\delta} (t-s) \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \right) \\
&\quad + \exp(-\beta_{i,j}\delta) \left( \phi_{\mathcal{D}}^{i,j}(t) + \int_t^{t+\delta} (t-s)^2 \alpha_{i,j} \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^j(s) \right) \\
\Phi_{\mathcal{D}}^{i,j}(t) &= \int_{t_{\min}}^t \phi_{\mathcal{D}}^{i,j}(s) ds \\
\Phi_{\mathcal{D}}^{i,j}(t+\delta) &= \Phi_{\mathcal{D}}^{i,j}(t) + \int_0^\delta \phi_{\mathcal{D}}^{i,j}(t+\delta') d\delta' \\
&= \Phi_{\mathcal{D}}^{i,j}(t) + \lambda_{\mathcal{D}}^{i,j}(t) \int_0^\delta \delta'^2 \exp(-\beta_{i,j}\delta') d\delta' \\
&\quad + 2\gamma_{\mathcal{D}}^{i,j}(t) \int_0^\delta \delta' \exp(-\beta_{i,j}\delta') d\delta' + \phi_{\mathcal{D}}^{i,j}(t) \int_0^\delta \exp(-\beta_{i,j}\delta') d\delta' \\
&\quad + \int_t^{t+\delta} \int_t^{t+\delta'} \dots d\Lambda_{\mathcal{D}}^j(s) \\
&= \Phi_{\mathcal{D}}^{i,j}(t) + \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\beta_{i,j}^3} (2 - \exp(-\beta_{i,j}\delta) ((\beta_{i,j}\delta)^2 - 2\beta_{i,j}\delta + 2)) \\
&\quad + \frac{2\gamma_{\mathcal{D}}^{i,j}(t)}{\beta_{i,j}^2} (1 - \exp(-\beta_{i,j}\delta) (\beta_{i,j}\delta + 1)) \\
&\quad + \frac{\phi_{\mathcal{D}}^{i,j}(t)}{\beta_{i,j}} (1 - \exp(-\beta_{i,j}\delta)) \\
&\quad + \int_t^{t+\delta} \int_t^{t+\delta'} \dots d\Lambda_{\mathcal{D}}^j(s)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \beta_{i,j}} &= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) + \Gamma_{\mathcal{D}}^{i,j}(t_{\max}) \\
&= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) + \Gamma_{\mathcal{D}}^{i,j}(t_{\max}) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \beta_{i,j}} &= \int_{t_{\min}}^{t_{\max}} \left( \frac{\phi_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} - \left( \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) - \Phi(t_{\max}) \\
&= \int_{t_{\min}}^{t_{\max}} \left( \frac{\phi_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} - \left( \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) - \Phi(t_{\max}) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \beta_{k,j}} &= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t) \gamma_{\mathcal{D}}^{k,j}(t)}{\lambda_{\mathcal{D}}^j(t)^2} d\Lambda_{\mathcal{D}}^j(t) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \beta_{k,l}} &= 0 \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \alpha_{i,j}} &= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \left( 1 - \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right) d\Lambda_{\mathcal{D}}^j(t) + \frac{\Gamma_{\mathcal{D}}^{i,j}(t_{\max})}{\alpha_{i,j}} \\
&= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \left( 1 - \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right) d\Lambda_{\mathcal{D}}^j(t) + \frac{\Gamma_{\mathcal{D}}^{i,j}(t_{\max})}{\alpha_{i,j}} \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \alpha_{k,j}} &= \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t) \lambda_{\mathcal{D}}^{k,j}(t)}{\alpha_{k,j} \lambda_{\mathcal{D}}^j(t)^2} d\Lambda_{\mathcal{D}}^j(t) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \alpha_{k,l}} &= 0
\end{aligned}$$

#### 4.3.2 Continuous-State Hawkes Processes

Piecewise constant multivariate left-continuous state process  $X(t)$

$$\begin{aligned}
\lambda_{i,j}(t) &= \alpha_{i,j} \exp(X^-(t) \cdot b_{i,j}) \int_{t_{\min}}^t \exp(X^+(s) \cdot a_{i,j}) \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^i(s) \\
\lambda_j(t) &= \sum_i \lambda_{i,j}(t) \\
\gamma_{i,j}(t) &= - \frac{\partial \lambda_{i,j}(t)}{\partial \beta_{i,j}} = \alpha_{i,j} \exp(X^-(t) \cdot b_{i,j}) \int_{t_{\min}}^t \exp(X^+(s) \cdot a_{i,j}) (t-s) \exp(-\beta_{i,j}(t-s)) d\Lambda_{\mathcal{D}}^i(s)
\end{aligned}$$

$$\begin{aligned}
\ell(\alpha, \beta, a, b; \mathcal{D}) &= \sum_j \left( \int_{t_{\min}}^{t_{\max}} \log(\lambda_j(t)) d\Lambda_{\mathcal{D}}^j(t) - \int_{t_{\min}}^{t_{\max}} \lambda_j(t) dt \right) \\
\frac{\partial \ell}{\partial \alpha_{i,j}} &= \int_{t_{\min}}^{t_{\max}} \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) - \int_{t_{\min}}^{t_{\max}} \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}} dt \\
\frac{\partial^2 \ell}{\partial \alpha_{i,j} \partial \alpha_{i,j}} &= \int_{t_{\min}}^{t_{\max}} \left( \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}^2 \lambda_{\mathcal{D}}^j(t)} - \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}^2 \lambda_{\mathcal{D}}^j(t)} - \left( \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) \\
&\quad - \int_{t_{\min}}^{t_{\max}} \left( \frac{\lambda_{\mathcal{D}}^j(t)}{\alpha_{i,j}^2} - \frac{\lambda_{\mathcal{D}}^j(t)}{\alpha_{i,j}^2} \right) dt \\
&= - \int_{t_{\min}}^{t_{\max}} \left( \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \right)^2 d\Lambda_{\mathcal{D}}^j(t) \\
\frac{\partial^2 \ell}{\partial \alpha_{i,j} \partial \alpha_{k,j}} &= \int_{t_{\min}}^{t_{\max}} \frac{\lambda_{\mathcal{D}}^{i,j}(t) \lambda_{\mathcal{D}}^{k,j}(t)}{\alpha_{i,j} \alpha_{k,j} \lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) \\
\frac{\partial^2 \ell}{\partial \alpha_{i,j} \partial \alpha_{k,l}} &= 0 \\
\frac{\partial \ell}{\partial \beta_{i,j}} &= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) + \Gamma_{\mathcal{D}}^{i,j}(t_{\max}) \\
&= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) + \Gamma_{\mathcal{D}}^{i,j}(t_{\max}) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \beta_{i,j}} &= \int_{t_{\min}}^{t_{\max}} \left( \frac{\phi_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} - \left( \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) - \Phi(t_{\max}) \\
&= \int_{t_{\min}}^{t_{\max}} \left( \frac{\phi_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} - \left( \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) - \Phi(t_{\max}) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \beta_{k,j}} &= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t) \gamma_{\mathcal{D}}^{k,j}(t)}{\lambda_{\mathcal{D}}^j(t)^2} d\Lambda_{\mathcal{D}}^j(t) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \beta_{k,l}} &= 0 \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \alpha_{i,j}} &= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \left( 1 - \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right) d\Lambda_{\mathcal{D}}^j(t) + \frac{\Gamma_{\mathcal{D}}^{i,j}(t_{\max})}{\alpha_{i,j}} \\
&= - \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j} \lambda_{\mathcal{D}}^j(t)} \left( 1 - \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right) d\Lambda_{\mathcal{D}}^j(t) + \frac{\Gamma_{\mathcal{D}}^{i,j}(t_{\max})}{\alpha_{i,j}} \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \alpha_{k,j}} &= \int_{t_{\min}}^{t_{\max}} \frac{\gamma_{\mathcal{D}}^{i,j}(t) \lambda_{\mathcal{D}}^{k,j}(t)}{\alpha_{k,j} \lambda_{\mathcal{D}}^j(t)^2} d\Lambda_{\mathcal{D}}^j(t) \\
\frac{\partial^2 \ell}{\partial \beta_{i,j} \partial \alpha_{k,l}} &= 0 \frac{\partial \ell}{\partial b_{i,j,k}} \\
\frac{\partial^2 \ell}{\partial b_{i,j,k} \partial b_{i,j,l}} &= \int_{t_{\min}}^{t_{\max}} \frac{X^-(t)_k X^-(t)_l \lambda_{i,j}(t)}{\lambda_j(t)} \frac{1}{27} \left( 1 - \frac{\lambda_{i,j}(t)}{\lambda_j(t)} \right) d\Lambda_{\mathcal{D}}^j(t) \\
&\quad - \int_{t_{\min}}^{t_{\max}} X^-(t)_k X^-(t)_l \lambda_{i,j}(t) dt
\end{aligned}$$

This nests ordinary state-dependence if you have one positive and one negative kernel

#### *4.3.3 Continuous-State Power Hawkes Processes*

“We introduce and establish the main properties of QHawkes (“Quadratic” Hawkes) models. QHawkes models generalize the Hawkes price models introduced in E. Bacry et al. (2014), by allowing all feedback effects in the jump intensity that are linear and quadratic in past returns. A non-parametric fit on NYSE stock data shows that the off-diagonal component of the quadratic kernel indeed has a structure that standard Hawkes models fail to reproduce. Our model exhibits two main properties, that we believe are crucial in the modelling and the understanding of the volatility process: first, the model is time-reversal asymmetric, similar to financial markets whose time evolution has a preferred direction. Second, it generates a multiplicative, fat-tailed volatility process, that we characterize in detail in the case of exponentially decaying kernels, and which is linked to Pearson diffusions in the continuous limit. Several other interesting properties of QHawkes processes are discussed, in particular the fact that they can generate long memory without necessarily be at the critical point. Finally, we provide numerical simulations of our calibrated QHawkes model, which is indeed seen to reproduce, with only a small amount of quadratic non-linearity, the correct magnitude of fat-tails and time

reversal asymmetry seen in empirical time series.”

$$\begin{aligned}
\lambda_{\mathcal{D}}^{i,j}(t) &= \alpha_{i,j} \exp(X^-(t) \cdot b_{i,j}) \int_{t_{\min}}^t \exp(X^+(s) \cdot a_{i,j}) \exp(-\beta(t-s)) d\Lambda_{\mathcal{D}}^i(s) \\
\lambda_{\mathcal{D}}^j(t) &= \sum_i \lambda_{\mathcal{D}}^{i,j}(t) \\
\ell(\alpha, \beta, a, b, k; \mathcal{D}) &= \sum_j \left( \int_{t_{\min}}^{t_{\max}} k \log(\lambda_{\mathcal{D}}^j(t)) d\Lambda_{\mathcal{D}}^j(t) - \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}^j(t)^k dt \right) \\
\frac{\partial \ell}{\partial k} &= \sum_j \left( \int_{t_{\min}}^{t_{\max}} \log(\lambda_{\mathcal{D}}^j(t)) d\Lambda_{\mathcal{D}}^j(t) - \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}^j(t)^k \log(\lambda_{\mathcal{D}}^j(t)) dt \right) \\
\Lambda(t+\epsilon) - \Lambda(t) &= 0 \Rightarrow \int_t^{t+\epsilon} \lambda_{\mathcal{D}}^j(s)^k \log(\lambda_{\mathcal{D}}^j(s)) ds \\
&= \lambda_{\mathcal{D}}^j(t)^k \log(\lambda_{\mathcal{D}}^j(t)) \int_0^\epsilon \exp(-\beta ks) ds - \lambda_{\mathcal{D}}^j(t)^k \beta \int_0^\epsilon s \exp(-\beta ks) ds \\
\frac{\partial^2 \ell}{\partial k \partial k} &= - \sum_j \left( \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}^j(t)^k \log(\lambda_{\mathcal{D}}^j(t))^2 dt \right) \\
\Lambda(t+\epsilon) - \Lambda(t) &= 0 \Rightarrow \int_t^{t+\epsilon} \lambda_{\mathcal{D}}^j(s)^k \log(\lambda_{\mathcal{D}}^j(s))^2 ds \\
&= \int_0^\epsilon \lambda_{\mathcal{D}}^j(t)^k \exp(-\beta ks) (\log(\lambda_{\mathcal{D}}^j(t)) - \beta s)^2 ds \\
&= \int_0^\epsilon \lambda_{\mathcal{D}}^j(t)^k \exp(-\beta ks) (\log(\lambda_{\mathcal{D}}^j(t))^2 - 2 \log(\lambda_{\mathcal{D}}^j(t)) \beta s + \beta^2 s^2) ds \\
\frac{\partial \lambda_{\mathcal{D}}^j(t)}{\partial \beta} &= - \sum_i \alpha_{i,j} \exp(X^-(t) \cdot b_{i,j}) \int_{t_{\min}}^t (t-s) \exp(X^+(s) \cdot a_{i,j}) \exp(-\beta(t-s)) d\Lambda_{\mathcal{D}}^i(s) \\
\Lambda(t+\epsilon) - \Lambda(t) &= 0 \Rightarrow \frac{\partial \lambda_{\mathcal{D}}^j}{\partial \beta}(t+\epsilon) = \exp(-\beta\epsilon) \left( \frac{\partial \lambda_{\mathcal{D}}^j}{\partial \beta}(t) - \epsilon \lambda_{\mathcal{D}}^j(t) \right) \\
\frac{\partial^2 \lambda_{\mathcal{D}}^j(t)}{\partial \beta \partial \beta} &= \sum_i \alpha_{i,j} \exp(X^-(t) \cdot b_{i,j}) \int_{t_{\min}}^t (t-s)^2 \exp(X^+(s) \cdot a_{i,j}) \exp(-\beta(t-s)) d\Lambda_{\mathcal{D}}^i(s) \\
\Lambda(t+\epsilon) - \Lambda(t) &= 0 \Rightarrow \frac{\partial^2 \lambda_{\mathcal{D}}^j}{\partial \beta \partial \beta}(t+\epsilon) = \exp(-2\beta\epsilon) \left( \frac{\partial^2 \lambda_{\mathcal{D}}^j}{\partial \beta \partial \beta}(t) - 2\epsilon \frac{\partial \lambda_{\mathcal{D}}^j}{\partial \beta}(t) + \epsilon^2 \lambda_{\mathcal{D}}^j(t) \right) \\
\frac{\partial \ell}{\partial \beta} &= k \sum_j \left( \int_{t_{\min}}^{t_{\max}} \frac{\frac{\partial}{\partial \beta} \lambda_{\mathcal{D}}^j(t)}{\lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) - \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}^j(t)^{k-1} \frac{\partial \lambda_{\mathcal{D}}^j(t)}{\partial \beta} dt \right) \\
\frac{\partial^2 \ell}{\partial \beta \partial \beta} &= k \sum_j \int_{t_{\min}}^{t_{\max}} \left( \frac{\frac{\partial^2}{\partial \beta \partial \beta} \lambda_{\mathcal{D}}^j(t)}{\lambda_{\mathcal{D}}^j(t)} - \left( \frac{\frac{\partial}{\partial \beta} \lambda_{\mathcal{D}}^j(t)}{\lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) \\
&\quad - k \sum_j \int_{t_{\min}}^{t_{\max}} \left( (k-1) \lambda_{\mathcal{D}}^j(t)^{k-2} \left( \frac{\partial \lambda_{\mathcal{D}}^j(t)}{\partial \beta} \right)^2 + \lambda_{\mathcal{D}}^j(t)^{k-1} \frac{\partial^2 \lambda_{\mathcal{D}}^j(t)}{\partial \beta \partial \beta} \right) dt
\end{aligned}$$



$$\begin{aligned}
\frac{\partial \ell}{\partial \theta_{i,j}} &= k \int_{t_{\min}}^{t_{\max}} \frac{\frac{\partial}{\partial \theta_{i,j}} \lambda_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} d\Lambda_{\mathcal{D}}^j(t) - k \int_{t_{\min}}^{t_{\max}} \lambda_{\mathcal{D}}^j(t)^{k-1} \frac{\partial \lambda_{\mathcal{D}}^{i,j}(t)}{\partial \theta_{i,j}} dt \\
\frac{\partial^2 \ell}{\partial \theta_{i,j} \partial \theta_{i,j}} &= k \int_{t_{\min}}^{t_{\max}} \left( \frac{\frac{\partial^2}{\partial \theta_{i,j} \partial \theta_{i,j}} \lambda_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} - \left( \frac{\frac{\partial}{\partial \theta_{i,j}} \lambda_{\mathcal{D}}^{i,j}(t)}{\lambda_{\mathcal{D}}^j(t)} \right)^2 \right) d\Lambda_{\mathcal{D}}^j(t) \\
&\quad - k \int_{t_{\min}}^{t_{\max}} \left( (k-1) \lambda_{\mathcal{D}}^j(t)^{k-2} \left( \frac{\partial \lambda_{\mathcal{D}}^{i,j}(t)}{\partial \theta_{i,j}} \right)^2 + \lambda_{\mathcal{D}}^j(t)^{k-1} \frac{\partial^2 \lambda_{\mathcal{D}}^{i,j}(t)}{\partial \theta_{i,j} \partial \theta_{i,j}} \right) dt \\
\frac{\partial \lambda_{\mathcal{D}}^{i,j}(t)}{\partial \alpha_{i,j}} &= \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}} \\
\frac{\partial^2 \lambda_{\mathcal{D}}^{i,j}(t)}{\partial \alpha_{i,j} \partial \alpha_{i,j}} &= \frac{\lambda_{\mathcal{D}}^{i,j}(t)}{\alpha_{i,j}^2} \\
\frac{\partial \lambda_{\mathcal{D}}^{i,j}(t)}{\partial b_{i,j}} &= X(t) \lambda_{\mathcal{D}}^{i,j}(t) \\
\frac{\partial^2 \lambda_{\mathcal{D}}^{i,j}(t)}{\partial b_{i,j} \partial b_{i,j}} &= X(t) X(t)^T \lambda_{\mathcal{D}}^{i,j}(t) \\
\frac{\partial \lambda_{\mathcal{D}}^{i,j}(t)}{\partial a_{i,j}} &= \alpha_{i,j} \exp(X(t) \cdot b_{i,j}) \int_{t_{\min}}^t X(s) \exp(X(s) \cdot a_{i,j}) \exp(-\beta(t-s)) d\Lambda_{\mathcal{D}}^i(s) \\
\frac{\partial^2 \lambda_{\mathcal{D}}^{i,j}(t)}{\partial a_{i,j} \partial a_{i,j}} &= \alpha_{i,j} \exp(X(t) \cdot b_{i,j}) \int_{t_{\min}}^t X(s) X(s)^T \exp(X(s) \cdot a_{i,j}) \exp(-\beta(t-s)) d\Lambda_{\mathcal{D}}^i(s)
\end{aligned}$$

This nests the previous with  $n^2$  kernels and a possible sign flip. Explain why  $\beta$  needs to be a single number rather than a matrix as before.

#### 4.3.4 EM for State-Dependent Hawkes Processes

- Equations
- Visualisation of branching matrix. How does it evolve throughout the fitting procedure?
- Quasi EM approximation
- Closed form constant time M step for multivariate state dependent hawkes processes
- Method of scoring (Newton's method)
- Negative probabilities. Conditions for kernel nonnegativity? (Probably not tractable)

#### 4.4 Inference for the Mark Process

Kalman filter? Ways to couple this with the point process? (Monte carlo EM?)

## 4.5 Hidden Events

$$\begin{aligned}
& \mathbb{P}^\theta(t \in \mathcal{T} | \mathcal{D}, \mathcal{H}_t) \\
&= \mathbb{P}^\theta(t \in \mathcal{T} | \mathcal{D}_t, \mathcal{H}_t) \frac{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \in \mathcal{T}, \mathcal{D}_t, \mathcal{H}_t)}{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \in \mathcal{T}, \mathcal{D}_t, \mathcal{H}_t) \lambda_{\mathcal{D}, \mathcal{H}} dt + \mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \notin \mathcal{T}, \mathcal{D}_t, \mathcal{H}_t) (1 - \lambda_{\mathcal{D}, \mathcal{H}} dt)} \\
&= \mathbb{P}^\theta(t \in \mathcal{T} | \mathcal{D}_t, \mathcal{H}_t) \frac{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \in \mathcal{T}, \mathcal{D}_t, \mathcal{H}_t)}{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \notin \mathcal{T}, \mathcal{D}_t, \mathcal{H}_t)} \\
& \lambda_{\mathbb{P} | \mathcal{D}, \mathcal{H}_t}^\theta = \lambda_{\mathcal{D}, \mathcal{H}}^\theta \frac{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \in \mathcal{T}, \mathcal{D}_t, \mathcal{H}_t)}{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \notin \mathcal{T}, \mathcal{D}_t, \mathcal{H}_t)}
\end{aligned}$$

$$\begin{aligned}
& \lambda_{\mathbb{P} | \mathcal{D}}^\theta \\
&= \lambda_{\mathcal{D}}^\theta \frac{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \in \mathcal{T}, \mathcal{D}_t)}{\mathbb{P}^\theta(\mathcal{D}_{t \rightarrow T} | t \notin \mathcal{T}, \mathcal{D}_t)} \\
&= \lambda_{\mathcal{D}}^\theta \exp \left( \int_t^T \log \left( 1 + \frac{\lambda_{\text{from } t \text{ and hidden descendants}}}{\lambda_{\text{exogenous to } t \text{ plus observed and descendants}}} \right) d\Lambda - \int_t^T \lambda_{\text{from } t \text{ and hidden descendants}} \right)
\end{aligned}$$

Can bound this above by bounding denominator below and numerator above using monotonicity of exponential kernel :)

For metaorders with fixed duration and no descendants this can be evaluated quickly. Unfortunately this makes things nondifferentiable: ( But can approximated with a ‘bump’ function with bounded support More generally, can use bump times exp(polynomial), possibly with a dynamic ‘duration’ component that varies as a function of history, and possibly with descendants

## 4.6 Monte Carlo EM

I think this makes it possible to do hidden marks

$$\begin{aligned}
& \mathbb{E} \left[ \int \log \lambda d\Lambda - \int \lambda dt | \mathcal{D} \right] \\
&= \int \left[ \int \log \lambda d\Lambda - \int \lambda dt \right] p(\lambda | \mathcal{D}) d\mu(\lambda) \\
&= \int \left[ \int \log \lambda d\Lambda - \int \lambda dt \right] \frac{p(\mathcal{D}_{\text{after}} | \lambda, \mathcal{D}_{\text{before}}) p(\lambda | \mathcal{D}_{\text{before}})}{p(\mathcal{D}_{\text{after}} | \mathcal{D}_{\text{before}})} d\mu(\lambda) \\
&= \frac{\mathbb{E} \left[ \left( \int \log \lambda d\Lambda - \int \lambda dt \right) p(\mathcal{D} | \lambda) \right]}{p(\mathcal{D})} \\
&= \frac{\mathbb{E} \left[ \left( \int \log \lambda d\Lambda - \int \lambda dt \right) p(\mathcal{D} | \lambda) \right]}{\mathbb{E} [p(\mathcal{D} | \lambda)]}
\end{aligned}$$

This is basically particle filtering, I think?

$$H(s) = UH(s) + \epsilon_u(s)$$

$$X(s) = OH(s) + \epsilon_i(s)$$

$$\sum_s \mathbb{E} \left[ \int (\log(\alpha) + X(s)c + H(s)c_H - \beta(t-s)) B_{s,t} d\Lambda(t) \right. \\ \left. - \int \alpha \exp(X(s)c + H(s)c_H - \beta(t-s)) dt \middle| \mathcal{D} \right]$$

$$\mathbb{E} \left( \sum_s \int H(s) B_{s,t} d\Lambda(t) - \int \alpha H(s) \exp(X(s)c + H(s)c_H - \beta(t-s)) dt \middle| \mathcal{D} \right)$$

## 4.7 Hidden Marks

### 4.7.1 Discrete

Markov

## 4.8 Quadratic Hawkes Processes

$$\int k \log(\alpha \sum \exp(-\beta(t-s))) d\Lambda - \alpha^k \int \left( \sum \exp(-\beta(t-s)) \right)^k dt \\ = \int k \log(\alpha) d\Lambda + \int k \log(\sum \exp(-\beta(t-s))) d\Lambda - \alpha^k \int \left( \sum \exp(-\beta(t-s)) \right)^k dt$$

$$\frac{\partial}{\partial \alpha} \\ = \int \frac{k}{\alpha} d\Lambda - k \alpha^{k-1} \int \left( \sum \exp(-\beta(t-s)) \right)^k dt$$

$$\frac{\partial}{\partial \beta} \\ = \int k \frac{\sum \exp(-\beta(t-s))(s-t)}{\sum \exp(-\beta(t-s))} d\Lambda \\ - \alpha^k \int k \left( \sum \exp(-\beta(t-s)) \right)^{k-1} \left( \sum \exp(-\beta(t-s))(s-t) \right) dt$$

$$\frac{\partial}{\partial k} \\ = \int \log(\alpha \sum \exp(-\beta(t-s))) d\Lambda - \int \log \left( \alpha \sum \exp(-\beta(t-s)) \right) \left( \alpha \sum \exp(-\beta(t-s)) \right)^k dt$$

$$\int_a^b \log(c e^{sx}) c e^x dx \\ = \log(c) \int_a^b e^x dx + cs \int_a^b x e^x dx \\ = \log(c) (e^b - e^a) + cs \left( [x e^x]_a^b - \int_a^b e^x dx \right) \\ = \log(c) (e^b - e^a) + cs (b e^b - a e^a - e^b + e^a)$$

## 4.9 Uncertainty Quantification

Fisher information calculations (different to observed information)

Asymptotic normality

Parametric Bootstrap

## 4.10 Computational Concerns

Sensor fusion for parallelisation

Momentum (analyse autocorrelation of parameter changes throughout the learning process)

Exploiting sparsity [14]

## 4.11 Model Selection

Performance of information criteria for selection of Hawkes process models of financial data <https://www.tandfonline.com/doi/full/10.1080/14697688.2017.1403140>

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## CHAPTER 5

### Application to KOSPI/SPY Data

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#### Introduce

#### 5.1 Dataset

20240920 and before - ESU4 september expiry 20240922 and after - ESZ4 december expiry

#### 5.2 Point Process Modelling

##### 5.2.1 Santa Fe Model

Poisson Processes Add a mixture distribution to model the arrivals to the book  
Make base rate depend on volume in front and distance from mid

##### 5.2.2 Inhomogeneity

Piecewise Linear Spline

Components have the form  $\lambda^e(t) = a \max(t - k, 0)$

$$\begin{aligned}\ell &= \sum_e \int (\log(a_e) + \log(\max(t - k, 0))) d\Lambda_e - \sum_e \int a_e \max(t - k, 0) dt \\ &= \sum_e \int (\log(a_e) + \log(\max(t - k, 0))) d\Lambda_e - \frac{1}{2}(t_{\max} - k)^2 \sum_e a_e \\ \frac{\partial \ell}{\partial a_e} &= \frac{1}{a_e} \Lambda_e(t_{\max}) - \frac{1}{2}(t_{\max} - k)^2 \\ \frac{\partial^2 \ell}{\partial a_e^2} &= -\frac{\Lambda_e(t_{\max})}{a_e^2} \\ \frac{\partial \ell}{\partial k} &= -\sum_e \int \frac{1_{t > k}}{\max(t - k, 0)} d\Lambda_e - (k - t_{\max}) \sum_e a_e \\ \frac{\partial^2 \ell}{\partial k^2} &= \sum_e \int \frac{1_{t > k}}{\max(t - k, 0)^2} d\Lambda_e - \sum_e a_e \\ \frac{\partial^2 \ell}{\partial k \partial a_e} &= t_{\max} - k\end{aligned}$$

### 5.2.3 *Hawkes Kernel*

### 5.2.4 *State Dependence*

discrete and regressive, as well as ‘reverse’ state dependence - dependence on state just before triggered event rather than just state at triggering event. Quantify which of these is the best. Feature selection using hessian info

More event types with ‘factor’ model. Marks should play as little a role as possible

State variables in original MPP paper - spread, queue imbalance.

### 5.2.5 *Regression*

### 5.2.6 *Power Hawkes*

quadratic and beyond

### 5.2.7 *Idiosyncratic Daily Behaviour*

## 5.3 notes

Baseline time series model - Zero-inflated gaussian or tdist mixture with GARCH vol and autoregressive midprice changes

Move from complex mark models, simple point process models → simple mark models, complex point process models. Reduces behaviour to emergence.

Models for the point process

1. Poisson Process
2. Inhomogenous Poisson Process on multiple days of data, check for residual autocorrelation (Polynomial, spline, mixture of truncated gaussians - almost KDE but not quite. Information criteria model selection.)
3. Hawkes Process with Inhomogenous Background (+ sinusoid kernel)
4. IHP with state dependence. Which variables?
5. IHP with regression - generalisation of state dependence. Again, which variables?
6. IHP with ‘reverse’ state dependence - depends on state just before triggered event rather than just state at triggering event. Is this worth it?
7. More event types. With ‘factor’ model. Marks should play as little a role as possible.
8. Quadratic and higher order
9. Day-specific metaorders with some distribution of schedules (eg volume, time, price-sensitive, liquidity-sensitive). These can insert in response to various market phenomena. This then ‘explains away’ some of the activity, depressing the estimates for the regular kernels. They’re basically ordinary kernels of their own. But the parameters are drawn from a distribution. And there can be multiple metaorders, drawn from some mixture distribution.

Models for the marks/state

1. State dependence & markov switching
2. Gaussian mixture. Examine residuals. May need to try tdist mixture.
3. Autoregression
4. Autoregression w/ GARCH

5. Kalman Filter (on marks only. Does this work, though? Does it mess with the point process somehow?); can I have KF with heteroskedasticity somehow?
6. Options factor model. How to weight strikes?

#### Diagnostics

- Likelihood, parameter count, information criteria, CV likelihood
- Residual autocorrelation (Ljung-Box pval), residual distribution (KS pval)
- Microprice quality: signature plot of realised vol against time horizon
- Predicting volatility. Baseline = GARCH w/ EMA features, Baseline = implied vol.
- Predicting volume traded
- Predicting liquidity
- Price impact of trades (function of size) and passive quotes
- Check whether things have a relationship with: spread, recent volume, predicted volatility, realised volatility, etc.; include plot if needed
- Filtered vs smoothed metaorders

#### Model Characteristics

- Shape of background rate
- Kernel function learned for each event type against each other
- How kernel function changes throughout the day based on time-varying coefficients
- How kernel function responds to various book features
- Metaorder duration/size/impact
- TWAP and VWAP (what DMA calls percent of volume) cost for various sizes, urgencies; can also test with foreknowledge of closing price. TWAP with altered schedule over the course of a day (DMA book calls this 'tilting'). What are the recommendations here?

Strategies - The ideal situation is that these look good wrt midprice, ok wrt bid/ask spread costs, bad when market impact is considered. Don't know fees

- Market making into a spread arb
- Microprice crossing spread arb - shouldn't happen often
- Vol factors stat arb - does it just sell vol or wings?

how to quantify compute time? number of EM steps, etc

Hawkes processes and their applications to finance: a review <https://www.tandfonline.com/doi/https://www.tandfonline.com/journals/rej20/collections/Hawkes-Processes-in-Finance>

- Replicate findings from [13]
- Hidden events (& events on different exchanges) - either Poisson distributed or more complex
- Modeling changes in the entire order book
- Market impact (are there any datasets on this? square-root law, other common findings. power law impact for hawkes processes is explicitly studied here <https://arxiv.org/pdf/1805.07134>)
- Optimal execution - VWAP, TWAP, Almgren-Chriss

- Midprice change prediction/explanation - explicit formula or simulation?
- Realised volatility prediction
- Correlated products (with low beta, preferably - or see what is done in literature studies of correlated products)
- Options (if I can get data) - would give lots of (nonlinearly) correlated products. Can estimate the correlation between products at any point in time using factor loadings & historical factor correlations. Here is one source: [https://www.nber.org/system/files/working\\_papers/w29369/w29369.pdf](https://www.nber.org/system/files/working_papers/w29369/w29369.pdf). <https://arxiv.org/abs/1602.03043> (*Closed form for optimal execution of sig impact*)
- This paper [arxiv.org/pdf/2401.11495](https://arxiv.org/pdf/2401.11495) shows a functional limit theorem for Hawkes processes behaving as integrated CIR processes. These are a popular volatility model so this makes sense!

<https://quant.stackexchange.com/questions/59593/what-are-some-currently-open-p>

[https://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=4844711](https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4844711) *Option Pricing Using H*

A slightly depressing jump model: intraday volatility pattern simulation

<https://www.tandfonline.com/doi/full/10.1080/14697688.2017.1403139> Implementat.

### 5.3.1 Clustering Ratio

Variance of interevent time divided by expectation. Bouchaud p 165. Should this section be elsewhere? Does this mean something for residuals too?

Point process control [https://pages.stern.nyu.edu/~rcaldent/courses/B60.4308\\_files/Process.pdf](https://pages.stern.nyu.edu/~rcaldent/courses/B60.4308_files/Process.pdf)

Recent Paper - Limit Order Book Dynamics and Order Size Modelling Using

Compound Hawkes Process <https://arxiv.org/abs/2312.08927> <https://www.sciencedirect.com/science/article/pii/S0950068724000111>

Databento info: CME: ES MES OPRA: SPX - Index options SPY - Index ETF

options <https://databento.medium.com/getting-futures-tick-sizes-and-notional-t>

<https://databento.com/portal/datasets/XNAS.ITCH/ETF/SPY> <https://databento.com/>

<https://databento.com/portal/datasets/GLBX.MDP3/Futures/ES> <https://databento.com/portal/datasets/GLBX.MDP3/Futures/ES>

## 5.4 Summary of datasets

- data structure and granularity - which assets are involved

## 5.5 Inhomogeneity

Two choices for handling inhomogeneity: - Reparameterise to 'business time' in preprocessing, as recommended by [3] in 9.3.1 - Splines/polynomials etc - Could also combine both - Can also focus on a small part of the data. counting process should look roughly linear

etc - Could also combine both - Can also focus on a small part of the data. counting process should look roughly linear

## 5.6 Multidimensionality

```
poisson model
```

correlations in trade sign (buys vs sells arrival rate nonstationary)

trades vs inserts vs cancels



different price levels have different arrival rates -> sante fe model is wrong

how to not need a separate event type for each price level

## 5.7 Self-Excitation and Mutual Excitation

i.e. poisson vs hawkes

## 5.8 Queue Size Dependence

queue size (esp. cancellations) and also book imbalance

state-dependent process can be used here

queue-reactive process is a little bit more fit for purpose

## 5.9 Regression on marks

not sure what this will look like yet. how to justify? which variables are reasonable to use? Show that discrete state dependence is nested inside composite model with regression on appropriately engineered features

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## CHAPTER 6

### Conclusion

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This is the conclusion

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## CHAPTER 7

### Appendix: Foundations of Probabilistic Models

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In order to describe precisely the various models explored in this thesis, it is necessary to introduce some key mathematical concepts that form the basis for model specifications and related derivations. In this appendix, I cover the basics of measure, probability, and stochastic processes.

#### 7.1 Measure Theory Fundamentals

In order to formalise the concept of a stochastic process, it is necessary to introduce the concept of a measure space.

For any set  $X$ , we can construct the power set

$$\mathcal{P}(X) = \{S : S \subseteq X\}$$

which is a new set that contains as its elements every subset  $S$  of  $X$ . We say that a subset  $\Sigma$  of  $\mathcal{P}(X)$  is a  $\sigma$ -*algebra* on  $X$  if and only if it satisfies the following three properties:

1. Containment of the full space, i.e.

$$X \in \Sigma.$$

2. Closure under complements, i.e.

$$\forall S \in \Sigma, X \setminus S \in \Sigma.$$

3. Closure under countable union, i.e.

$$\forall \{A_n\}_{n=0}^{\infty} \in \Sigma^{\mathbb{N}}, \quad \bigcup_{n=0}^{\infty} A_n \in \Sigma.$$

Elements of  $\Sigma$  are known as *measurable sets*. A common example of a  $\sigma$ -algebra is the Borel  $\sigma$ -algebra  $B(X)$  of a topological space  $X$  (e.g.  $\mathbb{R}$ ), defined as the smallest  $\sigma$ -algebra such that every open set is measurable.

We then say that a *measure* on  $(X, \Sigma)$  is any function  $\mu : \Sigma \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$  that is *countably additive*, meaning that for any finite or countable sequence of disjoint sets  $A_n \in \Sigma$ , we have

$$\mu \left( \bigcup_n A_n \right) = \sum_n \mu(A_n).$$

Assuming that at least one set  $S \in \Sigma$  has finite measure, we then have

$$\mu(S) = \mu(S \cap \emptyset) = \mu(S) + \mu(\emptyset) \Rightarrow \mu(\emptyset) = 0.$$

We refer to the combined triple  $(X, \Sigma, \mu)$  as a *measure space*.

Informally, a measure formalises intuitions about the size, mass, or significance of a set of points. For instance, a set  $S \subseteq X$  with measure zero is known as a *null set*, and a property that holds only for points in a null set is said to be true *almost nowhere* in  $X$ . Conversely, a property that holds for every point except those in a null set is said to be true *almost everywhere* (a.e.) in  $X$ . In this way, a measure quantifies how significant or negligible the exceptions to a heuristic principle may be.

Similarly, familiar concepts of *length*, *area* and *volume* are all formalised by a family of translation-invariant measures on  $\mathbb{R}^n$ , known as the *n-dimensional Lebesgue measures*.

The concept of a measure is foundational to the definition of the Lebesgue integral. Integration of a function  $f$  over a measurable set  $S$  with respect to a measure  $\mu$  is written as

$$\int_{t \in S} f(t) d\mu(t),$$

while integration with respect to the Lebesgue measure will often be written simply as

$$\int_S f(t) dt.$$

I will not cover the definition of the Lebesgue measure or Lebesgue integral here, but they can be found in most textbooks on measure theory.

## 7.2 Probability Spaces

In the special case where  $\mu(X) = 1$ , we refer to  $\mu$  as a *probability measure*, and to  $X$  as a *sample space*. Correspondingly, the term *almost everywhere* is replaced with *almost surely* (a.s.), indicating a property that holds on a set of points with probability measure one.

A function from one  $\sigma$ -algebra to another is called *measurable* if and only if the preimage of any measurable set in the codomain is a measurable set in the domain. A measurable function whose domain is a sample space  $\Omega$  equipped with a probability measure  $\mathbb{P}$  is known as a *random variable*.

Measurable sets in a sample space are often referred to as *events*, and the measure of such a set is called the probability of the event. For instance, the set  $S$  of points  $\omega \in \Omega$  for which a random variable  $X : \Omega \rightarrow \mathbb{R}$  satisfies a particular property  $P : \mathbb{R} \rightarrow \{\text{True}, \text{False}\}$  will have measure equal to the probability of that property being true.<sup>1</sup>

---

<sup>1</sup>The predicate  $P$  must be measurable.

A more general concept is *expectation*. For a real-valued random variable  $X : \Omega \rightarrow \mathbb{R}$ , we define the expectation of  $X$  with respect to  $\mathbb{P}$  to be the linear functional

$$\mathbb{E}_{\mathbb{P}}[X(\omega)] := \int_{\omega \in \Omega} X(\omega) d\mathbb{P}(\omega).$$

For binary-valued random variables  $X : \Omega \rightarrow \{0, 1\}$ , we have the identity

$$\mathbb{E}_{\mathbb{P}}[X(\omega)] = \mathbb{P}(\{\omega \in \Omega : X(\omega) = 1\}).$$

### 7.2.1 Conditionalisation

By defining ‘events’ as  $\Sigma$ -measurable subsets of  $\Omega$ , we allow knowledge of an event to inform us about which possible values of the hidden  $\omega$  could have produced the result we actually observe. In this way, the knowledge that  $\omega$  has produced one observed event allows us to draw probabilistic conclusions about the occurrence of a different, related event. For instance, learning that a die has rolled an even number tells us that it cannot possibly have rolled a 3, and makes the proposition that a 2 has been rolled a more reasonable guess than otherwise.

In order to capture the relationships between events, and to make systematic inferences about hidden events from observed ones, it is necessary to describe mathematical rules for *conditionalisation*.

Given two  $\sigma$ -algebras  $\Sigma_1, \Sigma_2$  on  $\omega$ , we say that  $\Sigma_1$  is a *sub- $\sigma$ -algebra* of  $\Sigma_2$  if and only if  $\Sigma_1 \subseteq \Sigma_2$ . In the case where  $\Sigma_1 \neq \Sigma_2$ ,  $\Sigma_1$  is said to be *coarser* than  $\Sigma_2$ , in the sense that  $\Sigma_2$  contains events that cannot be expressed as  $\Sigma_1$ -measurable sets. Knowing which  $\Sigma_2$ -events the hidden  $\omega$  falls under can therefore give us more information about the exact value of  $\omega$ .

One common example of a sub- $\sigma$ -algebra arises by considering the set

$$\{X^{-1}(A) : A \in \mathbb{R}\} \subseteq \mathcal{P}(\Omega),$$

consisting of all the preimages of Borel-measurable sets under a random variable  $X : \Omega \rightarrow \mathbb{R}$ . This is known as the  $\sigma$ -algebra generated by  $X$ .

expectation conditional on sigma algebra <https://math.stackexchange.com/questions/expectation-conditional-on-an-event>

probability is expectation of indicator function (notation for indicator function is  $1_A$ )

random measure

What is independence independence of RVs if and only if independence of generated sigma algebras

Conditional probability

### 7.3 Density of a Measure

Absolute continuity:  $\mu_1 \ll \mu_2$  ( $\mu_1$  is dominated by  $\mu_2$ ) if and only if  $\mu_2(A) = 0 \Rightarrow \mu_1(A) = 0$  for every measurable set  $A$ . Can also say that one measure is absolutely continuous wrt another on a subset of the measure space Radon-Nikodym Theorem

### 7.4 Stochastic Process Fundamentals

What is a filtration

What is a stochastic process

What is the natural filtration

What is a realisation of a stochastic process

What is a cadlag The order book is a cadlag over time. Also the counting process is a cadlag.

What is a martingale - do I need this?

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## References

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