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Actuator Placement for Optimizing Network Performance under Controllability Constraints

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***Abstract*— With the rising importance of large-scale network control, the problem of actuator placement has received increasing attention. Our goal in this paper is to find a set of actuators minimizing the metric that measures the average energy consumption of the control inputs while ensuring structural controllability of the network. As this problem is intractable, the greedy algorithm can be used to obtain an approximate solution. To provide a performance guarantee for this approach, we first define a new notion of submodularity ratio and show that the metric under consideration enjoys the notion of weak submodularity corresponding to this ratio. We then reformulate the structural controllability constraint as a matroid constraint. This shows that the problem under study can be characterized by the optimization of a weakly submodular function under a matroid constraint. For the greedy algorithm applied to this class of optimization problems, we derive a novel performance guarantee. Finally, we show that the matroid feasibility check for the greedy algorithm can be cast as a maximum matching problem in a certain auxiliary bipartite graph related to the network graph.**

I. Introduction

Actuator placement is the problem of finding a subset from a finite set of possible placements for actuators to optimize a desired network performance metric. With the increased importance of large-scale network control problems, such as those arising in power grids and transportation systems, there has been a surge of interest to study the problem of actuator placement. Past works have discussed several controllability-based performance metrics and derived properties of the resulting optimization problems [1]–[3].

The problem of actuator placement is in general NP-hard [4]. Hence, earlier studies have adopted the greedy algorithm to derive an approximate solution [5]. Under a submodular metric and a cardinality constraint on the number of actuators, the greedy algorithm is shown to enjoy a provable suboptimality guarantee [6]. However, some metrics do not exhibit submodularity including the metric in this work, that is, the average energy required to reach any arbitrary direction of the state space [7], [8]. To alleviate this issue, the notion of submodularity has been extended to weak submodularity using the notion of submodularity ratio quantifying how close a function is to being submodular [9], [10]. Given this ratio, it is possible to derive a performance guarantee for the greedy algorithm applied to a larger class of performance metrics [7].

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Nonetheless, the guarantees above are restricted to optimization problems subject to simple cardinality constraints. Given a cardinality constraint, the resulting actuator set might not render the system controllable. To address this issue, we need to include controllability as a constraint in the optimization problem. However, to the best of our knowledge, there is no approach to ensure feasibility of the iterates of the greedy algorithm applied to this problem, nor to quantify its performance guarantee. On the other hand, structural controllability constraints have been well-studied. This controllability concept exploits only the graphical interconnection structure of the dynamical system [11], [12]. Structurally controllable systems are those controllable after a slight perturbation of the system parameters corresponding to the fixed set of edges in the underlying network graph. The authors in [13] have studied a leader selection problem to obtain a structurally controllable system while minimizing a submodular objective function. The structural controllability constraint arising in the leader selection problem is proven to be equivalent to a so-called matroid constraint [13]. However, the leader selection problem is different from the actuator placement problem. The former selects a set of leader nodes whose states can arbitrarily be dictated to steer the remaining states to desired positions, while the latter does not permit the states to be dictated arbitrarily; instead, it selects a set of actuators which can influence all the states through the dynamics. To this end, our first goal is to show that the actuator placement problem under structural controllability constraint can also be cast as a matroid optimization.

To obtain a performance guarantee for the greedy algorithm applied to a matroid optimization problem, [14] has considered submodular objective functions. As an extension, [15] has considered weakly submodular objective functions. This setting captures the actuator placement problems under structural controllability constraints. However, the performance guarantees in [15] are restricted to the residual random greedy algorithm. To the best of our knowledge, there is no guarantee obtained for the greedy algorithm applied to a matroid optimization if the objective is weakly submodular. Therefore, our second goal is to obtain a performance guarantee for the greedy algorithm applied to this problem.

Our contributions are as follows. First, we show that the actuator placement problem optimizing a nonsubmodular controllability metric under structural controllability constraints can be cast as a matroid optimization, see Theorem 1. Second, by introducing and utilizing a new notion of submodularity ratio, we bound the worst-case performance of the greedy algorithm applied to the class of optimization problems with weakly submodular objective functions and matroid constraints, see Theorem 2. This enables us to bound the greedy algorithm's performance on the actuator placement problem under structural controllability constraints. Finally, we show that the matroid feasibility check for the greedy algorithm is equivalent to a maximum matching problem in a bipartite graph related to the network graph, see Theorem 3. This result extends work on the feasibility check for leader selection problems, where the existing algorithm proposed in [13] could only ensure a correct result for the case where the minimum required cardinality for structural controllability was considered.

The remainder of this paper is organized as follows. In Section II, we introduce the problem. In Section III, we study the properties of the network objective function and reformulate the structural controllability constraints as a matroid. Section IV obtains a performance guarantee for the greedy algorithm. In Section V, we discuss the implementation of the algorithm and include case studies.

II. Problem Formulation

A. System Model

Consider a linear system with state vector $x \in \mathbb{R}^n$. To each state variable $x_i \in \mathbb{R}$, we associate a node v_i. A control input $u_i \in \mathbb{R}$ can be exerted at each node $v_i \in V := \{v_1, \dots, v_n\}$. Given a set $S \subset V$ chosen as the actuator set, dynamics can be written as

$$\dot{x} = Ax + B(S)u. \tag{1}$$

Above, $B(S) := \mathrm{diag}(\mathbf{1}(S))$, where $\mathbf{1}(S)$ denotes a vector of size n whose i^{th} element is 1 if v_i belongs to S and 0 otherwise. We let $G = (V, E)$ denote a directed graph with nodes V and edges E, where the edge $(v_j, v_i) \in E$ if $(A)_{ij} \neq 0$. Similar to the previous studies on structural controllability [3], [13], we assume that the graph G is strongly connected.

B. Problem Statement

The pair $(A, B(S))$ is called controllable if the states x can be steered arbitrarily in $\mathbb{R}^n$ in any given finite time. Controllability can be verified by the rank of the controllability matrix $P = \begin{bmatrix} B(S) & AB(S) & \dots & A^{n-1}B(S) \end{bmatrix} \in \mathbb{R}^{n \times n^2}$. If $(A, B(S))$ is not controllable, it might still be possible to slightly perturb the entries in A and $B(S)$ to ensure controllability [11]. Considering that the entries in A are generally not exactly known but only approximately determined with small errors, we need a robust controllability notion. To this end, we bring in structural controllability.

Definition 1: (A, B) and $(\hat{A}, \hat{B})$ with $A, B, \hat{A}, \hat{B} \in \mathbb{R}^{n \times n}$ are said to have the same structure if matrices $[A\ B]$ and $[\hat{A}\ \hat{B}]$ have fixed zeros at the same entries. Given $S \subset V$, $(A, B(S))$ is structurally controllable if there exists a controllable pair $(\hat{A}, \hat{B})$ having the same structure as $(A, B(S))$. The set S is a capable actuator set if $(A, B(S))$ is structurally controllable.

Even if a system is controllable, an unacceptably large amount of energy might be needed to reach a desired state. Hence, it is crucial to minimize this energy consumption. The minimum energy required to steer the system from zero at $t = 0$ to $x_0 \in \mathbb{R}^n$ at $t = T$ is given by $x_0^\top W_T^{-1}(S) x_0$, where $W_T(S) = \int_0^T e^{A\tau} B(S) B^\top(S) e^{A^\top \tau} d\tau$ is the controllability Gramian. To obtain an expression independent of the initial state x_0, we can calculate the average energy required over the unit sphere, $||x_0||_2 = 1$, as $\mathrm{tr}(W_T^{-1}(S))$. This expression is well-defined only when the set S renders the system controllable. Inspired by [16], we introduce a small positive number $\epsilon \in \mathbb{R}_+$ and propose the following metric

$$F(S) = \mathrm{tr}((W_T(S) + \epsilon I)^{-1}), \ \forall S \subset V. \tag{2}$$

To make a system easier to control, we seek a set $S \subset V$ minimizing the metric above. Since in a large-scale network, the number of actuators allowed is in general limited, we consider a cardinality bound of $K \in \mathbb{N}$ on the number of actuators allowed. Additionally, we need a controllability constraint to ensure that the actuator set is capable. Therefore, our main problem is formulated as

$$\begin{aligned} \min_{S} \quad & F(S) \\ \text{s.t.} \quad & |S| \leq K \text{ and } S \text{ is a capable actuator set.} \end{aligned} \tag{3}$$

For the remainder, assume that K is large enough to ensure feasibility. To the best of our knowledge, no computationally feasible method of finding the optimal solution to Problem (3) has ever been proposed. A heuristic method, called the greedy algorithm, has been broadly adopted to derive an approximate solution. This algorithm starts from the empty set and iteratively adds the element with the largest marginal gain. In the following, our goal is to derive a performance guarantee for the greedy algorithm applied to this problem.

III. Characterization of Problem Structure

For the maximization of submodular functions under a matroid constraint, the greedy algorithm achieves a performance guarantee of $1/2$ [14]. However, the work in [7] shows that the set function $-F$ is not submodular. Moreover, there is no work characterizing the constraints found in Problem (3). In the following, we show that a) one can analyze the submodularity ratio of the function $-F$ and b) the constraints in Problem (3) form a matroid via a reformulation.

A. Properties of the Objective

A set function $f : 2^V \to \mathbb{R}$ is (strictly) increasing if $f(S_1) \leq (<) f(S_2)$ for any $S_1 \subsetneq S_2 \subset V$. Similarly, we say that f is (strictly) decreasing if $-f$ is (strictly) increasing. Intuitively, with more input nodes, system (1) would be easier to control, and thus the metric F in (2) would be smaller. This intuition can be readily verified as follows.

Lemma 1: The metric F in (2) is strictly decreasing.

Proof: For any $S \subset V$ and any $\omega \in V \setminus S$, let $H(z) = (W_T(S) + zW_T(\{\omega\}) + \epsilon I)^{-1}$. Notice that $\mathrm{tr}(H(1)) = \mathrm{tr}((W_T(S \cup \{\omega\}) + \epsilon I)^{-1}) = F(S \cup \{\omega\})$, since $W_T(S) + W_T(\{\omega\}) = W_T(S \cup \{\omega\})$. Via the matrix inverse formula [17], if $H(z)$ is invertible $\forall z \in (0, 1)$, then $\mathrm{tr}(H(z))$

is continuous and differentiable, and we have

$$\frac{d(\mathrm{tr}(H(z)))}{dz} = -\mathrm{tr}(H(z)W_T(\{\omega\})H(z)) < 0.$$

This inequality holds since $H(z)$ is invertible and symmetric, and $W_T(\{\omega\})$ is positive semidefinite. Invoking the mean-value theorem, we have $\mathrm{tr}(H(1)) - \mathrm{tr}(H(0)) < 0$. ∎

To introduce the notion of submodularity ratio, define the marginal gains as

$$\rho_U(S) := f(S \cup U) - f(S), \quad \forall S, U \subset V.$$

Definition 2: For an increasing function $f\colon 2^V \to \mathbb{R}$, submodularity ratio is the largest $\gamma \in \mathbb{R}_+$ such that

$$\gamma \rho_\omega(S \cup U) \leq \rho_\omega(S), \ \forall S, U, \{\omega\} \subset V. \tag{4}$$

A set function f with submodularity ratio γ is called γ-submodular. A γ-submodular set function is said to be submodular if $\gamma = 1$ and weakly submodular if $0 < \gamma < 1$.

For any increasing set function, $\gamma \in [0, 1]$. Since the metric F is decreasing, we instead consider the submodularity ratio γ of $-F$. Due to the strict monotonicity of the metric F, we have $\gamma > 0$. Thus, $-F$ is weakly submodular. In the appendix, we connect Definition 2 with other existing notions of submodularity ratio, and discuss the need to introduce this notion as per Definition 2.

B. Reformulation of the Constraint Set

Since F is strictly decreasing, the optimal solution to Problem (3), denoted as S^*, satisfies $|S^*| = K$. As a result, we can define $\mathcal{C}_K = \{S \subset V \mid |S| = K$ and S is a capable actuator set$\}$ and rewrite Problem (3) as the minimization of F over the set $\mathcal{C}_K$. The greedy algorithm starts from the empty set and iteratively expands this set by adding an element, which maximizes the marginal gain. Let S^t denote the actuator set obtained at the t^{th} iteration. This set S^t has to be a subset of some set in $\mathcal{C}_K$. Otherwise, the greedy solution S^K would not belong to $\mathcal{C}_K$. Thus, define $\tilde{\mathcal{C}}_K = \{\Omega \mid \exists\ S \in \mathcal{C}_K$ such that $\Omega \subset S\}$ and reformulate Problem (3) as

$$\min_S F(S) \ \text{ s.t. } S \in \tilde{\mathcal{C}}_K. \tag{5}$$

The strict monotonicity of F again ensures that the optimal solution of Problem (5) coincides with that of (3). As such, for the rest of the paper, we consider solving Problem (5) as an equivalent characterization of Problem (3).

Next, we show that the feasible region of Problem (5) has a matroid structure and this helps us bound the worst-case performance of the greedy algorithm. To this end, we first bring in the definition of a matroid.

Definition 3: A matroid $\mathcal{M}$ is a pair $(V, \mathcal{F})$ consisting of a ground set V and a collection $\mathcal{F}$ of subsets of V which satisfies (i) $\emptyset \in \mathcal{F}$, (ii) if $S \in \mathcal{F}$ and $S' \subset S$, then $S' \in \mathcal{F}$, (iii) if $S_1, S_2 \in \mathcal{F}$ and $|S_1| < |S_2|$, there exists $\omega \in S_2 \setminus S_1$ such that $\{\omega\} \cup S_1 \in \mathcal{F}$. Every set in $\mathcal{F}$ is called independent.

Theorem 1: $\mathcal{M} = (V, \tilde{\mathcal{C}}_K)$ is a matroid.

Proof: To prove this theorem, we show that given an actuator set S, structural controllability of $(A, B(S))$ can equivalently be formulated as structural controllability of the system with the set S chosen as a leader set. Then, we use a result from [13] showing the matroid structure of the structural controllability constraints in leader selection problems. This result builds on [12], which shows the equivalence between structural controllability and existence of a perfect matching in an auxiliary bipartite graph whenever the graph G is strongly connected.

Define $N = V \setminus S$ and partition the state vector x into x_S and x_N. The dynamics can equivalently be written as

$$\begin{bmatrix} \dot{x}_N \\ \dot{x}_S \end{bmatrix} = \begin{bmatrix} A_{NN} & A_{NS} \\ A_{SN} & A_{SS} \end{bmatrix} \begin{bmatrix} x_N \\ x_S \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_{|S|} \end{bmatrix} u, \tag{6}$$

where $I_{|S|} \in \mathbb{R}^{|S| \times |S|}$ is the identity matrix.

In the leader selecting problem studied in [18], if the set S is chosen as a leader set, it is assumed that the values of x_S are directly dictated and are not influenced by the dynamics of x_N. Under this assumption, by treating x_S as the input, the dynamics of x_N are given by $\dot{x}_N = A_{NN}x_N + A_{NS}x_S$. Then, the leader set S achieves structural controllability if (A_{NN}, A_{NS}) is structurally controllable, which would allow the values of x_N to be steered to desired positions. Note that it is not clear whether we would achieve structural controllability when this set is chosen as the set of actuators in our original actuator placement problem.

From Definition 1, the set S is a capable actuator set if and only if there exists a pair $(\hat{A}, \hat{B})$ with the same structure as $(A, B(S))$ such that the controllability matrix $P \in \mathbb{R}^{n \times n^2}$,

$$P = \begin{bmatrix} 0 & 0 & 0 & \hat{A}_{NS} & 0 & \hat{A}_{NN}\hat{A}_{NS} + \hat{A}_{NS}\hat{A}_{SS} & \cdots \\ 0 & I_{|S|} & 0 & \hat{A}_{SS} & 0 & \hat{A}_{SN}\hat{A}_{NS} + \hat{A}_{SS}^2 & \cdots \end{bmatrix},$$

has full rank. Next, we claim that P has full rank if and only if the following matrix $\tilde{P}_1 \in \mathbb{R}^{|N| \times n^2}$ has full rank,

$$\tilde{P}_1 = \begin{bmatrix} 0 & 0 & 0 & \hat{A}_{NS} & \cdots & 0 & \hat{A}_{NN}^{j-1}\hat{A}_{NS} & \cdots \end{bmatrix}.$$

To see this, notice that P has full rank if and only if the submatrix $P_1 \in \mathbb{R}^{|N| \times n^2}$ containing the first $|N|$ rows of P has full rank. One can then show that there exists an upper triangular matrix $U \in \mathbb{R}^{n^2 \times n^2}$ with unit diagonal entries such that $\tilde{P}_1 = P_1 U$. Since U is invertible, $\tilde{P}_1$ and P_1 have the same rank.

Then, we further claim that $\tilde{P}_1$ has full rank if and only if the following matrix $\bar{P}_1$ has full rank

$$\bar{P}_1 = \begin{bmatrix} \hat{A}_{NS} & \hat{A}_{NN}\hat{A}_{NS} & \cdots & \hat{A}_{NN}^{|N|-1}\hat{A}_{NS} \end{bmatrix}.$$

Considering $|S| > 0$ and thus $|N| - 1 \leq n - 2$, for any $i > |N| - 1$, $\hat{A}_{NN}^i \hat{A}_{NS}$ is in the span of the matrices $\hat{A}_{NN}^j \hat{A}_{NS}$, $j = \{0, 1, \ldots, |N|-1\}$ by Cayley-Hamilton theorem. Hence, $\bar{P}_1$ has the same rank as $\tilde{P}_1$. This proves the claim.

In summary, P has full rank if and only if $\bar{P}_1$ has full rank. By the definition of $\bar{P}_1$, $\bar{P}_1$ being full rank is equivalent to controllability of $(\hat{A}_{NN}, \hat{A}_{NS})$. Hence, structural controllability of $(A, B(S))$ is equivalent to structural controllability of (A_{NN}, A_{NS}).

Now, define $\mathcal{L}_K = \{S \mid |S| = K$ and (A_{NN}, A_{NS}) is structurally controllable$\}$ and conclude that $\mathcal{L}_K = \mathcal{C}_K$. The

Algorithm 1 Greedy Algorithm on Matroid Optimization

Input: set function f, ground set V and matroid $(V, \mathcal{F})$
Output: actuator set S^{G}

function GREEDYONMATROID($f, V, \mathcal{F}$)
  $S^0 = \emptyset$, $U^0 = \emptyset$, $t = 1$
  while $U^{t-1} \neq V$ **do**
    $i^*(t) = \arg\max_{i \in V \setminus U^{t-1}} \rho_i(S^{t-1})$
    if $S^{t-1} \cup \{i^*(t)\} \notin \mathcal{F}$ **then**
      $U^{t-1} \leftarrow U^{t-1} \cup \{i^*(t)\}$
    else
      $\rho_{t-1} \leftarrow \rho_{i^*(t)}(S^{t-1})$ and $v_t^{\mathrm{G}} = i^*(t)$
      $S^t \leftarrow S^{t-1} \cup \{v_t^{\mathrm{G}}\}$ and $U^t \leftarrow U^{t-1} \cup \{v_t^{\mathrm{G}}\}$
      $t \leftarrow t + 1$
    end if
  end while
  $S^{\mathrm{G}} \leftarrow S^{t-1}$
end function

set collection $\mathcal{L}_K$ consists of all the K cardinality leader sets achieving structural controllability. From [13, Thm. 4], we have that the pair $(V, \tilde{\mathcal{L}}_K)$, where $\tilde{\mathcal{L}}_K := \{\Omega \mid \exists\, S \in \mathcal{L}_K$ such that $\Omega \subset S\}$, is a matroid if the graph G is strongly connected. Therefore, the pair $(V, \tilde{\mathcal{C}}_K)$ is also a matroid. ■

The proof above establishes the equivalence between finding a capable actuator set in our original problem and finding a leader set achieving structural controllability in a corresponding leader selection problem.

IV. Performance Guarantee

By considering $f = -F$ as the objective, Problem (5) falls into the following class of optimization problems:

$$\begin{aligned} \max_{S \subset V} \quad & f(S), \text{ increasing and } \gamma\text{-submodular} \\ \text{s.t.} \quad & S \in \mathcal{F}, \text{ where } \mathcal{M} = (V, \mathcal{F}) \text{ is a matroid,} \end{aligned} \tag{7}$$

where the cardinality of the largest set in $\mathcal{F}$ is K. Our goal is to derive a performance guarantee for the greedy algorithm applied to Problem (7).

The greedy algorithm is presented in Algorithm 1. Let S^t denote the actuator set returned by the t^{th} iteration. At the t^{th} iteration, we check the feasibility of the node with the largest marginal gain in $V \setminus S^{t-1}$. If the actuator set obtained by adding this node to S^{t-1} does not belong to $\tilde{\mathcal{C}}_K$, we exclude the node from consideration. Among the remaining ones, we check the feasibility of the node with the largest marginal gain until a feasible node v_t^{G} is found. Then $S^t = \{v_t^{\mathrm{G}}\} \cup S^{t-1}$ is the actuator set returned by the t^{th} iteration. The final actuator set is $S^{\mathrm{G}} := S^K$. The feasibility check ensures that $S^t \in \mathcal{F}$. Hence, S^{G} belongs to $\mathcal{F}$.

We define $U^{-1} = \emptyset$, $U^K = V$, and also use $U^t \subset V$ for $0 \leq t \leq K-1$ to denote all the nodes having been considered by the feasibility check before v_{t+1}^{G}. We define the marginal gains of the greedy algorithm as $\rho_t = f(S^{t+1}) - f(S^t)$.

Our main result is as follows.

Theorem 2: If Algorithm 1 is applied to Problem (7), then

$$\frac{f(S^{\mathrm{G}}) - f(\emptyset)}{f(S^*) - f(\emptyset)} \geq \frac{\gamma^3}{\gamma^3 + 1}. \tag{8}$$

The idea of the proof extends the work in [14], which derives a performance guarantee for matroid optimization featuring a submodular objective.

To assess the suboptimality of the actuator set S^{G}, we need to find an upper bound for $f(S^*) - f(S^{\mathrm{G}})$. We denote $S^* = \{v_1^*, \ldots, v_K^*\}$ and notice

$$\begin{aligned} f(S^*) - f(S^{\mathrm{G}}) &\leq f(S^* \cup S^{\mathrm{G}}) - f(S^{\mathrm{G}}) \\ &= \sum_{k=1}^{K} \rho_{v_k^*}(\{v_1^*, \ldots, v_{k-1}^*\} \cup S^{\mathrm{G}}) \\ &\leq \gamma^{-1} \sum_{j \in S^* \setminus S^{\mathrm{G}}} \rho_j(S^{\mathrm{G}}), \end{aligned} \tag{9}$$

where the first inequality is due to the monotonicity of f and the equality follows from a telescoping sum. The last inequality is from Definition 2. To further bound $\sum_{j \in S^* \setminus S^{\mathrm{G}}} \rho_j(S^{\mathrm{G}})$, we have the following lemmas. For these lemmas, define

$$s_t = |S^* \cap (U^{t+1} \setminus U^t)|.$$

Lemma 2: It holds that

$$\sum_{j \in S^* \setminus S^{\mathrm{G}}} \rho_j(S^{\mathrm{G}}) \leq \gamma^{-1} \sum_{t=1}^{K} \rho_{t-1} s_{t-1}. \tag{10}$$

Proof: From Definition 2, we have

$$\rho_j(S^{\mathrm{G}}) \leq \gamma^{-1} \rho_j(S^{t-1}),\ \forall t \leq K,\ \forall j \in V. \tag{11}$$

Since $U^{t_1} \subset U^{t_2}$ for any $t_1 < t_2$, notice that

$$V = U^K = \bigcup_{t=0}^{K} (U^t \setminus U^{t-1}).$$

Considering $U^{t_1} \setminus U^{t_1 - 1}$ and $U^{t_2} \setminus U^{t_2 - 1}$ are disjoint, we know that these sets constitute a partition of V. Since there is no subset of U^0 belonging to $\mathcal{F}$, we have $S^* \cap U^0 = \emptyset$. Using the partition of V, we can partition S^* as: $S^* = \bigcup_{t=1}^{K} (S^* \cap (U^t \setminus U^{t-1}))$. Combining this with (11), we have

$$\begin{aligned} \sum_{j \in S^* \setminus S^{\mathrm{G}}} \rho_j(S^{\mathrm{G}}) &\leq \sum_{j \in S^*} \rho_j(S^{\mathrm{G}}) \\ &= \sum_{t=1}^{K} \sum_{j \in S^* \cap (U^t \setminus U^{t-1})} \frac{1}{\gamma} \rho_j(S^{t-1}). \end{aligned} \tag{12}$$

Notice that all the nodes in U^{t-1} have been considered by the feasibility check before v_t^{G}. Since the greedy algorithm first checks the elements in $V \setminus U^{t-1}$ with larger marginal gains when added to U^{t-1}, we have that $\rho_{t-1} = \max_{j \in V \setminus U^{t-1}} \rho_j(S^{t-1})$. Considering $V \setminus U^{t-1} = \cup_{i=t}^{K} (U^i \setminus U^{i-1})$, for any $t' \geq t$,

$$\rho_{t-1} \geq \rho_j(S^{t-1}), \forall j \in U^{t'} \setminus U^{t'-1}. \tag{13}$$

Thus, for any $j \in S^* \cap (U^t \setminus U^{t-1})$, we have $\rho_j(S^{t-1}) \leq \rho_{t-1}$ and

$$\sum_{j \in S^* \cap (U^t \setminus U^{t-1})} \rho_j(S^{t-1}) \leq \rho_{t-1} s_{t-1}. \tag{14}$$

Now combining (12) and (14), it is straightforward that

$\sum_{j\in S^*\setminus S^{\mathrm{G}}} \rho_j(S^{\mathrm{G}}) \le \sum_{t=1}^{K} \gamma^{-1}\rho_{t-1}s_{t-1}$. ■

Lemma 3: For any $t \in \{1,\dots,K\}$, we have

$$\sum_{i=1}^{t} s_{i-1} \le t. \tag{15}$$

This has been proven by [14] for $\gamma = 1$. Since the proof exploits the matroid structure, the above lemma holds also when $\gamma \neq 1$. The proof is included for the sake of completeness.

Proof: We claim that any independent subset of U^t has a cardinality at most t. Otherwise, due to $\mathcal{F}$ being a matroid, there exists $j \in U^t \setminus S^t$ such that $S^t \cup \{j\}$ is independent. Since $j \in U^t$ and $U^t = \cup_{i=0}^{t}(U^i \setminus U^{i-1})$ is a partition, there exists $t' \le t$ such that $j \in U^{t'} \setminus U^{t'-1}$. Since $S^{t'} \cup \{j\} \subset S^t \cup \{j\}$, $S^{t'} \cup \{j\}$ is independent. By the mechanism of the greedy algorithm, we know j passes the feasibility check ahead of $v^{\mathrm{G}}_{t'+1}$, which contradicts the fact that j is discarded. Then, notice that $S^* \cap U^t$ is an independent subset of U^t. Hence, its cardinality is no more than t according to the above claim. The partition $U^t = \cup_{i=0}^{t}(U^i \setminus U^{i-1})$ gives us that $\sum_{i=1}^{t} s_{i-1} = |S^* \cap U^t| \le t$. ■

We use (15) to obtain an upper bound to the right-hand side of (10) and consequently to derive an upper bound of $f(S^*) - f(S^{\mathrm{G}})$. The following explains these steps in detail.

Proof: (Proof of Theorem 2) First, we consider the case in which ρ_i, $i = 0,\dots,K-1$, are distinct. We define t_1 such that ρ_{t_1-1} is the largest among $\rho_0, \rho_1, \dots, \rho_{K-1}$ and t_2 such that ρ_{t_2-1} is the largest among $\rho_{t_1}, \rho_{t_1+1}, \dots, \rho_{K-1}$. Following the same pattern we have $t_1, t_2, \dots, t_p$, where $t_p = K$. Since $s_i \ge 0$ is bounded by (15), to give an upper bound to the right-hand side of (10), we construct a linear program as follows,

$$\begin{aligned} \max_{s_0,s_1,\dots,s_{K-1}} \quad & \sum_{i=1}^{K} \rho_{i-1}s_{i-1} \\ \text{s.t.} \quad & \sum_{i=1}^{t} s_{i-1} \le t,\ t = 1,2,\dots,K, \\ & s_{t-1} \ge 0,\ t = 1,2,\dots,K. \end{aligned} \tag{16}$$

Let s^*_{i-1}, $i = 1,2,\dots,K$, denote the optimal solution. We claim $s^*_{t_1-1} = t_1$. Otherwise, $s^*_{t_1-1} < t_1$ and due to (15) two situations might happen, a) $\sum_{i=1}^{t_1} s^*_{i-1} = t_1$ or b) $\sum_{i=1}^{t_1} s^*_{i-1} < t_1$.

For case a), we obtain $\sum_{i=1}^{t_1-1} s^*_{i-1} > 0$. It follows that there exists $l < t_1$ such that $s^*_{l-1} > 0$. Then, we decrease s^*_{l-1} by $\delta > 0$ and increase $s^*_{t_1-1}$ also by δ. The value of δ is small enough so that $s^*_{l-1} > 0$. This operation decreases $\sum_{i=1}^{t} s^*_{i-1}$ for $l \le t \le t_1 - 1$ and keeps the sum unchanged for any other t, so the constraints of (16) are not violated. Also considering that $\rho^*_{t_1-1} > \rho^*_{l-1}$, after these changes, the objective function is strictly greater than the value obtained at the original optimum. Thus, case a) is impossible.

For case b), we collect all the integers $l > t_1$ satisfying $s^*_{l-1} > 0$. Assume they are $l_q > \dots > l_1 > t_1$. We have $q \ge 1$. Otherwise, $s^*_{l-1} = 0$ for any $l > t_1$ and we can increase $s^*_{t_1-1}$ by a small amount to obtain a greater value of the objective function without violating the constraints. Knowing that $s^*_{l_1-1} > 0$ and following the same reasoning provided in the discussion for the case a), we increase $s^*_{t_1-1}$ and decrease $s^*_{l_1-1}$ with the same amount. This way, an objective value is obtained larger than that evaluated at the original optimum. Thus, case b) is impossible.

In conclusion, $s^*_{t_1-1} = t_1$ and (16) is equivalent to

$$\begin{aligned} \max_{s_{t_1},\dots,s_{K-1}} \quad & \sum_{i=t_1+1}^{K} \rho_{i-1}s_{i-1} \\ \text{s.t.} \quad & \sum_{i=t_1+1}^{t} s_{i-1} \le t - t_1,\ t = t_1+1,\dots,K, \\ & s_{t-1} \ge 0,\ t = t_1+1,\dots,K. \end{aligned} \tag{17}$$

We determine $s^*_{t_2-1}$ in the same way as we determine $s^*_{t_1-1}$ in (16). By repeating the above procedure we obtain the solution to (16) as

$$s^*_{i-1} = \begin{cases} t_1, & \text{if } i = t_1, \\ t_j - t_{j-1}, & \text{if } i = t_j \text{ and } j \neq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Now, if ρ_i, $i = 0,\dots,K-1$ are not distinct, that is, there exist $i_1 < i_2 < \dots < i_q$ such that $\rho_{i_1} = \rho_{i_2} = \dots = \rho_{i_q}$, we can let $s^*_{i_1} = s^*_{i_2} = \dots = s^*_{i_{q-1}} = 0$ and obtain the same solution as (18).

Next, notice that

$$\begin{aligned} \rho_{i_2} &= f(S^{i_2+1}) - f(S^{i_2}) \\ &\le \gamma^{-1}(f(S^{i_1} \cup \{v^{\mathrm{G}}_{i_2+1}\}) - f(S^{i_1})) \\ &\le \gamma^{-1}\rho_{i_1}, \end{aligned} \tag{19}$$

where the first inequality comes from the definition of submodularity ratio, while the second is due to (13). Substituting the optimal solution into the objective function and considering (19), we have

$$\begin{aligned} \sum_{i=1}^{K} \rho_{i-1}s^*_{i-1} &= t_1\rho_{t_1-1} + \dots + (t_p - t_{p-1})\rho_{t_p-1} \\ &\le \gamma^{-1} \sum_{k=1}^{p} \sum_{i=t_{k-1}+1}^{t_k} \rho_{i-1} \\ &= \gamma^{-1} \sum_{i=1}^{K} \rho_{i-1} \\ &= \gamma^{-1}(f(S^{\mathrm{G}}) - F(\emptyset)). \end{aligned} \tag{20}$$

Combining (9), (10) and (20), we have

$$\begin{aligned} f(S^*) - f(S^{\mathrm{G}}) &\le \gamma^{-1} \sum_{j\in S^*\setminus S^{\mathrm{G}}} \rho_j(S^{\mathrm{G}}) \\ &\le \gamma^{-2} \sum_{i=1}^{K} \rho_{i-1}s^*_{i-1} \\ &\le \gamma^{-3}\Big(f(S^{\mathrm{G}}) - f(\emptyset)\Big). \end{aligned}$$

By rewriting the above inequality, we have

$$\frac{f(S^{\mathrm{G}})-f(\emptyset)}{f(S^*)-f(\emptyset)} \geq \frac{\gamma^3}{\gamma^3+1}.$$

■

When $\gamma = 1$, the guarantee in (8) coincides with that of [14], derived for a submodular f. We refer to the appendix for a comparison with the guarantee given by [15] for the residual random greedy algorithm. Since in the proof, only (9), (11) and (19) utilize γ, we denote the maximum γ satisfying these three inequalities as γ_G. We call γ_G the greedy submodularity ratio since it can be derived only after the greedy algorithm is completed. If γ is replaced by γ_G, the performance guarantee in (8) would still hold. Clearly, this gives a better performance guarantee because $\gamma_G \geq \gamma$. One can calculate γ_G after obtaining S^{G} by analyzing the three inequalities, which involves analyzing $\mathcal{O}(\binom{n}{K})$ inequalities. On the other hand, computing the submodularity ratio involves $\mathcal{O}(2^n)$ inequalities. Notice that γ_G changes with the constraint set of the problem since the inequalities defining γ_G would be different. In contrast, submodularity ratio γ depends only on the objective function.

By substituting $f = -F$ and γ_G, the greedy submodularity ratio of $-F$, into the performance guarantee (8), we obtain

$$\frac{F(\emptyset)-F(S^{\mathrm{G}})}{F(\emptyset)-F(S^*)} \geq \frac{\gamma_G^3}{\gamma_G^3+1}. \tag{21}$$

Since we have $F(\emptyset) = n\epsilon^{-1}$, and ϵ is generally small, the guarantee above can be loose. We show this in the numerics.

V. Implementation and Numerical Results

A. Feasibility check over $\tilde{\mathcal{C}}_K$

When applied to Problem (5), the greedy algorithm has to ensure that the actuator set returned by each iteration lies in $\tilde{\mathcal{C}}_K$. The work of [12] proposes a method to determine whether a given set S with $|S| = K$ belongs to $\mathcal{C}_K$. As will be explained later in theory and examples, this result is not directly applicable to answer whether an actuator set S with $|S| < K$ returned by a greedy iteration belongs to $\tilde{\mathcal{C}}_K$. In the following, we extend the work of [12] for a feasibility check over $\tilde{\mathcal{C}}_K$ by constructing auxiliary bipartite graphs associating structural controllability with the existence of a perfect matching.

We introduce the concept of matchings and bipartite graphs. An undirected graph is called bipartite and denoted as (V^1, V^2, E) if its vertices are partitioned into V^1 and V^2 while any edge in E connects a vertex in V^1 to another in V^2. A matching m is a subset of E if no two edges in m share a vertex in common. Given a subset L of $V^1 \cup V^2$, we say L is covered by m if any $v \in L$ is connected to an edge in m. Matching m is maximum if it has the largest cardinality among all the matchings and is perfect if V^2 is covered.

Given the graph $G = (V, E)$ describing system (1), we first build an auxiliary bipartite graph to determine whether a given actuator set S is capable. $V' = \{v'_1, \dots, v'_n\}$ is built as a copy of $V = \{v_1, \dots, v_n\}$ and $S' \subset V'$ denotes the copy of S. For the auxiliary edges, the edge set E includes an undirected edge connecting v_i to v'_j if $(v_i, v_j) \in E$. The edge set $\mathsf{E}_S \subset \mathsf{E}$ consists of edges incident to v'_k for any k such that $v_k \in S$. Using these sets, we build the bipartite graph defined by a function that maps from S to subgraphs of (V, V', E), specifically, $\mathcal{H}_b(S) = (V, V' \setminus S', \mathsf{E} \setminus \mathsf{E}_S)$.

If the graph G is strongly connected, the set S achieves structural controllability if and only if there exists a perfect matching in $\mathcal{H}_b(S)$, see [19, Thm. 2]. Using this result, [13] develops a recursive feasibility check algorithm for leader selection problems with structural controllability constraints. This method states that the set S lies in $\tilde{\mathcal{C}}_K$ if and only if there is a maximum matching for the bipartite graph $\mathcal{H}_b(\emptyset)$ with all the nodes in the set $S' \subset V'$ unmatched. However, this statement is true only if we consider the minimum required cardinality for structural controllability of the system, see the proof of [13, Lem. 3]. In Section V-B, we provide a counterexample where the feasibility check in [13] may not work.

We are now ready to provide our feasibility check.

Theorem 3: Given the strongly connected graph G, the cardinality limit K and an actuator set S with $|S| = k \leq K$, $S \in \tilde{\mathcal{C}}_K$ if and only if $|\bar{m}(S)| \geq n - K$, where $\bar{m}(S)$ is a maximum matching in $\mathcal{H}_b(S)$.

Proof: "$\Rightarrow$": If $S \in \tilde{\mathcal{C}}_K$, there exist $Q \in \mathcal{C}_K$ such that $S \subset Q$. This implies that there exists a perfect matching m in $\mathcal{H}_b(Q)$ [19]. By the definition of perfect matching, $|m| = n - K$. Since $\mathcal{H}_b(Q)$ is a subgraph of $\mathcal{H}_b(S)$, m is also a matching in $\mathcal{H}_b(S)$. Thus $|\bar{m}(S)| \geq |m| = n - K$.

"$\Leftarrow$": We pick a maximum matching in $\mathcal{H}_b(S)$ and denote it as $\bar{m}$. Suppose in $\mathcal{H}_b(S)$, $P' = \{v'_{i_1}, \dots, v'_{i_d}\}$ is the largest subset of $V' \setminus S'$ missed by $\bar{m}$. Since in $V' \setminus S'$ there are at least $n - K$ vertices covered by $\bar{m}$, we have $d \leq K - k$. Let $Q = P \cup S$, where $P = \{v_{i_1}, \dots, v_{i_d}\}$. Matching $\bar{m}$ is perfect in $\mathcal{H}_b(Q)$ because no vertex in $V' \setminus (P' \cup S')$ is missed by $\bar{m}$. Hence, Q makes the system structurally controllable. Also considering $|Q| \leq |J| + |S| \leq K$, we have $Q \in \tilde{\mathcal{C}}_K$. Since $S \subset Q$, we also have $S \in \tilde{\mathcal{C}}_K$. ■

For our feasibility check, we still need a method to obtain a maximum matching in $\mathcal{H}_b(S)$. It is well-established that this can equivalently be done by solving a maximum flow problem [20]. There are several algorithms for solving maximum flow problems. For instance, the Edmonds-Karp algorithm that we adopt in the numerical studies requires $O(pq^2)$ steps, where p and q respectively denote node cardinality and edge cardinality in the flow graph generated based on $\mathcal{H}_b(S)$ [21]. For example, in $\mathcal{H}_b(\emptyset)$, $p = 2n + 2$ and $q = 2n + |E|$. Thus, at each greedy iteration, we can examine in polynomial time whether $\{\omega\} \cup S^t$ belongs to $\tilde{\mathcal{C}}_K$ by finding the cardinality of the maximum matching in $\mathcal{H}_b(\{\omega\} \cup S^k)$.

B. Example on a 4-Node Network

Consider a system described by 4 nodes and the dynamic equations (1) where

$$A = \begin{bmatrix} 0 & -0.5 & -0.8 & -0.6 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

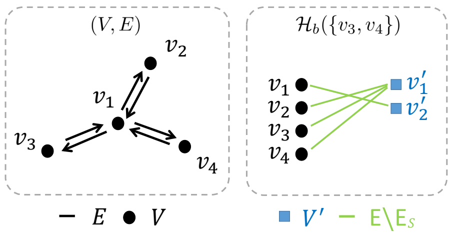

Fig. 1. Original graph (V, E) and the auxiliary graph $\mathcal{H}_b(\{v_3, v_4\})$

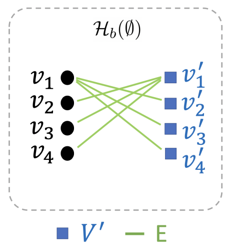

Fig. 2. The auxiliary graph $\mathcal{H}_b(\emptyset)$

The graph $G = (V, E)$ corresponding to this system is provided in Figure 1. To calculate the metric $F(S)$ in (2), we let $T = 2$ and $\epsilon = 10^{-9}$.

Suppose at most two input nodes are allowed, that is, $K = 2$. The greedy algorithm first examines v_3, because $F(\{v_3\})$ is the smallest among $F(\{v_i\})$, $i = 1, 2, 3, 4$. Notice that in $\mathcal{H}_b(\{v_3\})$ there exists a matching consisting of two edges. Since $n - K = 2$, we know from Theorem 3 that $\{v_3\}$ belongs to $\tilde{\mathcal{C}}_K$. Hence, the first node selected is v_3. Even if $F(\{v_1\})$ were the smallest, v_1 would not pass the feasibility check, because any maximum matching in $\mathcal{H}_b(\{v_1\})$ only contains one edge. Thus, $\bar{m}(\{v_1\}) < n - K$. This implies that v_1 does not belong to any capable actuator set with 2 elements. Then, the second node selected is v_4. Thus, $S^{\mathrm{G}} = \{v_3, v_4\}$. We illustrate the bipartite graph $\mathcal{H}_b(\{v_3, v_4\})$ in Fig. 1. It can be seen that there exists a perfect matching. Consequently, S^{G} is indeed a capable actuator set.

We now provide a counterexample based on the example above to show that the feasibility check method in [13] excludes feasible nodes from the consideration of the greedy algorithm. In this example, the minimum required cardinality for structural controllability is $K = 2$ since any maximum matching in $\mathcal{H}_b(\emptyset)$ misses 2 nodes in V', see Fig. 2. Now, assume $K = 3$. The feasibility check method in [13] indicates that $\{v_1\} \notin \tilde{\mathcal{C}}_K$, because $\{v_1'\}$ is not missed by any maximum matching in $\mathcal{H}_b(\emptyset)$. However, since $\{v_3, v_4\}$ is structurally controllable, so is $\{v_1, v_3, v_4\}$. Then, $\{v_1\} \subset \{v_1, v_3, v_4\}$ implies that $\{v_1\} \in \tilde{\mathcal{C}}_K$.

C. Experiment on a 23-Node Network

We study a system model based on an undirected unweighted graph given in Fig. 3 generated via Octave [22]. To gain insight into how the sets S^{G} and S^* depend on the node connectivity, we assign different degrees to each vertex. Specifically, vertex i has a degree of i if $i < 13$ and a degree of $24 - i$ if $i \geq 12$. To calculate the objective function $F(S)$ in (2), we let $\epsilon = 1.9 \times 10^{-4}$ and $T = 1$.

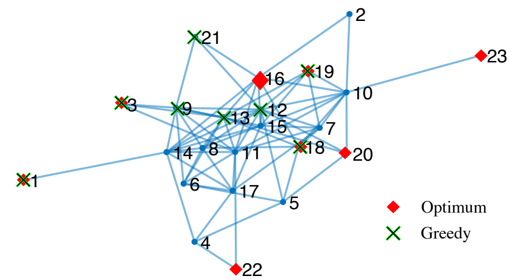

Fig. 3. Greedy selection versus the optimal

Illustrated in Fig. 3, the greedy selection includes some nodes with high degrees that are discarded by the optimal selection. The objective values are $F(S^{\mathrm{G}}) = 6193.5$ versus $F(S^*) = 4914.9$. The greedy algorithm picks the nodes in the order of 16, 13, 4, 8, 6, 20, 10, 21 while $S^* = \{1, 3, 16, 18, 19, 20, 22, 23\}$. At earlier stages the greedy algorithm tends to pick high-degree nodes that are not in S^*. This is because the high-degree nodes generally result in larger marginal gains at earlier stages of the greedy algorithm when compared to the low-degree nodes since, intuitively, they help influencing more nodes. The optimal solution S^* suggests that we can potentially improve the performance if we avoid these high-degree nodes. This shows a disadvantage of the greedy algorithm.

In this example, we numerically verified that $\gamma_G = 1$ satisfies (9), (11) and (19). Thus, $\gamma_G = 1$ is the greedy submodularity ratio. By rewriting (21), the performance guarantee is equivalent to $F(S^{\mathrm{G}}) \leq \frac{1}{2}F(\emptyset) + \frac{1}{2}F(S^*) = 6.1 \times 10^4 + \frac{1}{2}F(S^*)$. In this example, the appearance of $F(\emptyset)$ in the performance guarantee undermines its tightness.

VI. Conclusions

In this paper, we aimed to pick an actuator set to minimize a controllability metric based on average energy consumption while ensuring that the system is structurally controllable. We showed that this problem can be reformulated as a weakly submodular optimization problem over a matroid constraint. Given the submodularity ratio of the objective function, we bounded the worst-case performance of the greedy algorithm applied to this class of problems. To implement the greedy algorithm, we proved that the feasibility check over the structural controllability matroid can be done via calculating a maximum matching on a certain auxiliary bipartite graph resulting from the network graph.

Our future work is focused on exploring how to obtain a performance guarantee that can avoid $F(\emptyset)$. Inspired by the numerics, we aim to investigate the network structures under which our controllability metric is submodular.

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Appendix

Definitions of submodularity ratio

Let γ_1 denote the submodularity ratio of f from Definition 2. It is straightforward to see that $\gamma = \gamma_1$ satisfies

$$\gamma \rho_U(S) \leq \sum_{\omega \in U \setminus S} \rho_\omega(S), \forall S, U \subset V. \tag{22}$$

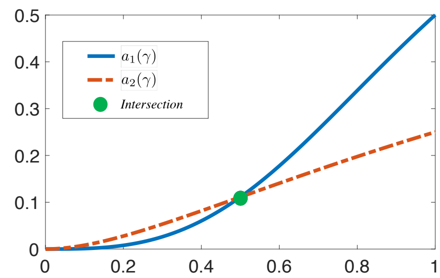

Fig. 4. A comparison between two guarantees

However, the largest γ satisfying the above set of inequalities, denoted as γ_2, does not necessarily satisfy (4) given in Definition 2. Hence, we have $\gamma_2 \geq \gamma_1$.

There are previous studies in the literature defining the submodularity ratio as γ_2 instead of γ_1 [7], [10], [15]. In the proof of Theorem 2, as we are deriving (11), we use the inequalities (4) from Definition 2. One can verify that the inequalities in (22) would not allow us to derive (11). Hence, the performance guarantee (8) does not extend to the submodularity ratio γ_2.

The work in [10] provides a performance guarantee for the greedy algorithm applied to weakly submodular optimization involving cardinality constraints. This guarantee improves with increasing γ_2. Since we have $\gamma_2 \geq \gamma_1$, the guarantee also holds if γ_2 is replaced by γ_1. In addition, the work of [7] obtains a lower bound for γ_2 for the metric $-F$ in (2) based on eigenvalue inequalities for sum and product of matrices. One can easily verify that this lower bound is also applicable to γ_1 from Definition 2.

To the best of our knowledge, the guarantee in (8) is the first performance guarantee for the greedy algorithm applied to matroid optimization problems featuring weakly submodular objective functions. The work of [15] exploited the submodularity ratio defined by (22) and obtained a guarantee for the residual random greedy algorithm on the same problem. We denote the final set returned by this algorithm as S^{RRG}. The guarantee provided in [15] for this class of randomized algorithms is

$$\frac{f(S^{\text{RRG}}) - f(\emptyset)}{f(S^*) - f(\emptyset)} \geq \frac{\gamma_2^2}{(1+\gamma_2)^2}. \tag{23}$$

Let γ denote the theoretical lower bound derived in [7] for the metric $-F$ in (2). This lower bound satisfies $\gamma_2 \geq \gamma_1 \geq \gamma$. Since γ is applicable to both (8) and (23), we let $a_1(\gamma) = \gamma^3/(1+\gamma^3)$ and $a_2(\gamma) = \gamma^2/(1+\gamma)^2$ denote the theoretical guarantees associated with (8) and (23), respectively. These two functions are plotted in Figure 4. We see that the guarantee we derived in (8) is tighter than the one from [15], if the lower bound satisfies $\gamma > 0.5$.

Besides these two definitions (4) and (22), there are other notions that can characterize the incremental changes of set functions, for instance, curvature for approximating supermodularity. These notions are beyond the scope of the present paper. We refer the interested readers to [23], [24].