Exercise sheet 4 – NLAA– Numerical Linear Algebra with Applications

Instructions: Please submit your scripts by 1pm Monday 29 April using the white box labeled D. Loghin on the first floor of the Watson Building. Please upload to Canvas by the same deadline the file `eulertest.m` corresponding to question **2(f)**. Please **name your file as suggested** in the question.
Plagiarism check: Your handwritten and electronic submissions should be your own work and should not be identical or similar to other submissions. A check for plagiarism will be performed on all submissions.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with boundary Γ and consider the following reaction-diffusion problem

$$\text{(BVP)} : \begin{cases} -\varepsilon\Delta u(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) & = f(\mathbf{x}) \quad \mathbf{x} \in \Omega; \\ u(\mathbf{x}) & = g(\mathbf{x}) \quad \mathbf{x} \in \Gamma; \end{cases}$$

where $\varepsilon > 0$ and $0 < c_1 \leq c(\mathbf{x}) \leq c_2$ for all $\mathbf{x} \in \Omega$. The variational formulation reads

$$\text{(VF)} : \begin{cases} \text{Find } u \in V \text{ such that for all } v \in V, \\ k(u,v) := \varepsilon a(u,v) + m_c(u,v) = \ell(v) \end{cases},$$

where V is a suitable function space and

$$a(u,v) = \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad m_c(u,v) = \int_\Omega c(\mathbf{x})u(\mathbf{x})v(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad \ell(v) = \int_\Omega f(\mathbf{x})v(\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

Let

$$V_h = \operatorname{span}\{\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \cdots, \phi_N(\mathbf{x})\} \subset V$$

be a finite dimensional space of functions defined on a subdivision of Ω into triangles of maximum size h. The finite element method applied to (VF) with V replaced by V_h yields a linear system of the form

$$K\mathbf{u} = (\varepsilon A + M_c)\mathbf{u} = \mathbf{f},$$

where

$$A := [a(\phi_j, \phi_i)], \quad M_c := [m_c(\phi_j, \phi_i)], \quad \mathbf{f}_i = \ell(\phi_i).$$

The time-dependent problem corresponding to (BVP) is included below

$$\text{(IBVP)} : \begin{cases} u_t(\mathbf{x},t) - \varepsilon\Delta u(\mathbf{x},t) + c(\mathbf{x})u(\mathbf{x},t) & = f(\mathbf{x}) \quad \mathbf{x} \in \Omega, t > 0; \\ u(\mathbf{x},t) & = g(\mathbf{x}) \quad \mathbf{x} \in \Gamma, t > 0; \\ u(\mathbf{x},0) & = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{cases}$$

A finite element discretisation of (IBVP) yields the system of ODE

$$M\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + K\mathbf{u} = \mathbf{f}, \quad \mathbf{u}(0) = \mathbf{u}^0, \tag{1}$$

where

$$M := [m(\phi_j, \phi_i)], \quad m(u,v) := \int_\Omega u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}.$$

Notation: in the following questions, the notation K_N, M_N, A_N, M_N^c is sometimes used to indicate that the matrices $K =: K_N, M =: M_N, M_c =: M_{cN}, A =: A_N$ are matrices of size $N \times N$.

1. Let $P_N, Q_N \in \mathbb{R}^{N\times N}$ be symmetric and positive-definite matrices. We say P_N is spectrally equivalent to Q_N uniformly with respect to N (or uniformly spectrally equivalent) if there exist positive constants γ, Γ independent of N such that for all $\mathbf{u} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$

$$\gamma \leq \frac{\mathbf{u}^T P_N \mathbf{u}}{\mathbf{u}^T Q_N \mathbf{u}} \leq \Gamma.$$

We write $P_N \approx Q_N$. The constants γ, Γ are called constants of equivalence.

(a) Consider the set of linear systems

$$K_N \mathbf{u}_N = \mathbf{f}_N,$$

where $K_N \in \mathbb{R}^{N\times N}$ with $N \in \mathbb{N}$, $N \leq N_{\text{max}}$. Let P_N denote a set of preconditioners for K_N such that $P_N \approx K_N$. What are the practical implications of uniform spectral equivalence of P_N to K_N, assuming an iterative solver such as the preconditioned Steepest Descent method is used?

(b) Show that if $P_N \approx Q_N$, then $Q_N \approx P_N$. Show that if we further have $Q_N \approx R_N$ then $P_N \approx R_N$.

(c) Show that $M \approx M_c$ (i.e., $M_N \approx M_{cN}$), indicating the constants of equivalence.

(d) It is given that for any function $u_h \in V_h$ there holds

$$\eta_1 \int_\Omega u_h^2(\mathbf{x}) \mathrm{d}\mathbf{x} \leq \int_\Omega \nabla u_h(\mathbf{x}) \cdot \nabla u_h(\mathbf{x}) \mathrm{d}\mathbf{x} \leq \frac{\eta_2}{h^2} \int_\Omega u_h^2(\mathbf{x}) \mathrm{d}\mathbf{x}.$$

Use these inequalities to show that if $h > \sqrt{\varepsilon}$, then $M_N \approx \varepsilon A_N$.

(e) Use parts **(c–d)** to show that $K_N \approx M_N$ provided $h > \sqrt{\varepsilon}$, indicating the constants of equivalence.

(f) Problem (BVP) is solved on the L-shaped domain $\Omega = (0,2)^2 \setminus [1,2)^2$ using the following data:

$$f(\mathbf{x}) = (1+x+y)(x+y), \quad g(\mathbf{x}) = x+y, \quad \varepsilon = 10^{-5}, \quad c(\mathbf{x}) = 1+x+y.$$

Use `femsol` with quasi-uniform subdivisions (set `fem.Hmax=[]`) and values of `fem.level` ranging between 1 and 5 to investigate the performance of the preconditioned Steepest Descent method. As preconditioner you should use the (sparse) diagonal matrix D containing the main diagonal of M. For each linear system write down the value of h^2 and the least number of iterations k for which $\|\mathbf{f} - K\mathbf{u}_k\|_2 \leq 10^{-6}\|\mathbf{f}\|_2$. Comment on your results in view of your statement from part **(a)**. Using parts **(b)** and **(e)**, discuss what your experiments imply about the spectral equivalence of D and M.

2. **(a)** Let $\mathbf{v}^0, \mathbf{g} \in \mathbb{R}^n, G \in \mathbb{R}^{n\times n}$ be given and assume $\rho(G) < 1$. Consider the recurrence

$$\mathbf{v}^{k+1} = G\mathbf{v}^k + \mathbf{g}. \tag{2}$$

Show by induction that for $k > 0$

$$\mathbf{v}^k = G^k\mathbf{v}^0 + \left(\sum_{j=0}^{k-1} G^j\right)\mathbf{g},$$

and therefore that

$$\mathbf{v}^k = G^k\mathbf{v}^0 + (I - G)^{-1}(I - G^k)\mathbf{g}.$$

Hence, show that

$$\lim_{k\to\infty} \mathbf{v}^k = (I - G)^{-1}\mathbf{g}.$$

(b) Write down explicit Euler's method for the system (1) in the form

$$\mathbf{U}^{k+1} = \mathbf{U}^k + h_t\mathbf{F}(\mathbf{U}^k), \tag{3}$$

where h_t is some constant time-step and $\mathbf{F}$ is a function which you should specify in terms of K, M and $\mathbf{f}$.

(c) Write recurrence (3) in the form (2), for a matrix G and vector $\mathbf{g}$ which you should specify. Hence find the limit

$$\mathbf{U}^\infty := \lim_{k\to\infty} \mathbf{U}^k.$$

Comment on this result, in view of the problems (IBVP), (BVP) described above.

(d) Show that the eigenvalues of G are real and positive.

(e) Show that if $0 < h_t < 2/\lambda_1(M^{-1}K)$ then $\rho(G) < 1$.

(f) Write a script file `eulertest.m` to verify the result in part **(c)**. In particular, your file should

- generate the matrices K, M and $\mathbf{f}$ using `femsol` given the data

$$\Omega = (0,1)^2, \quad f(\mathbf{x}) = 1, \quad g(\mathbf{x}) = 0, \quad \varepsilon = 10^{-2}, \quad c(\mathbf{x}) = 1 + x + y;$$

- use $\mathbf{u}^0 = \mathbf{1}$;
- estimate $\lambda_1(M^{-1}K)$ using the power method and find a suitable value of h_t;
- use Euler's method (3) to compute an approximation $\tilde{\mathbf{U}}$ to $\mathbf{u}(\mathbf{x}, T)$, where $T = 5$ on three subdivisions of Ω corresponding to `fem.level` equal to 4, 5 and 6;
- evaluate and display the norm $\|\tilde{\mathbf{U}} - \mathbf{U}^\infty\|_2$ for each of the three subdivisions above.