Reaction Wheel Stabilized Inverted Pendulum

optimal control, state space, and quaternions, oh my

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Abstract

In an attempt to develop some new skills, I started development of an inverted pendulum. This paper describes the system and the optimal controller development for the reaction wheel stabilized inverted pendulum. Not included in this paper is the printed circuit board design, part selection, CAD models for the wheels, and machine paths for wheel manufacture. I drew from papers concerning optimal control of a cart stabilized inverted pendulum, PID control of a reaction wheel stabilized inverted pendulum, and finally satellite control systems papers that utilize quaternions.

System Description

This inverted pendulum is stabilized by the spinning of reaction wheels mounted to the end of the pendulum. Only the case of a pendulum which is initiated and is controlled to its inverted position is being considered. In addition to the wheels at the end of the pendulum rod, there is mass associated with motors, a printed circuit board, batteries, and structure. Theses are not shown in the image below but their masses are configurable variables for the control system.

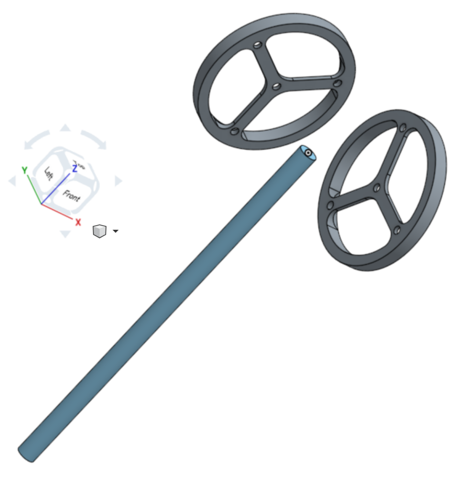


Figure 1: System rotated.

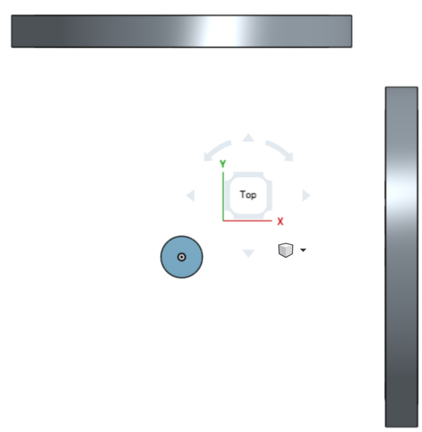


Figure 2: Top of system.

The equations of motion were determined and are described below. These equations are used to build a state space representation of the system. The state space representation takes the form $\dot{\boldsymbol{x}}\boldsymbol{(t)} = \boldsymbol{Ax(t)} + \boldsymbol{Bu(t)}$ where $x(t)$ is the system state at time t, $u(t)$ is the input to the system at time t, and A and B are their respective coefficients. Below are the expanded matrices with our variables ω, rotational speed for the respective axis and quaternions represented by $\boldsymbol{q} = q_0 + q_1\,i + q_2\,j + q_3\,k$. $\boldsymbol{q}$ and $\boldsymbol{w}$ are measured by a 3-axis accelerometer and a 3-axis gyroscope, respectively.

$$\begin{bmatrix} \underset{3\times1}{\dot{\boldsymbol{q}}} \\ \underset{3\times1}{\dot{\boldsymbol{\omega}}_{\boldsymbol{p}}} \\ \underset{3\times1}{\dot{\boldsymbol{\omega}}_{\boldsymbol{w}}} \end{bmatrix} = \begin{bmatrix} \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\boldsymbol{A_{12}}} & \underset{3\times3}{\mathbf{0}} \\ \underset{3\times3}{\boldsymbol{A_{21}}} & \underset{3\times3}{\boldsymbol{A_{22}}} & \underset{3\times3}{\boldsymbol{A_{23}}} \\ \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\boldsymbol{A_{33}}} \end{bmatrix} \begin{bmatrix} \underset{3\times1}{\boldsymbol{q}} \\ \underset{3\times1}{\boldsymbol{\omega_p}} \\ \underset{3\times1}{\boldsymbol{\omega_w}} \end{bmatrix} + \begin{bmatrix} \underset{3\times3}{\mathbf{0}} \\ \underset{3\times3}{\boldsymbol{B_{21}}} \\ \underset{3\times3}{\boldsymbol{B_{31}}} \end{bmatrix} \begin{bmatrix} \underset{3\times1}{\boldsymbol{\mu}} \end{bmatrix}$$

See [1] for more information about this representation.

Quaternion Equations

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

The above is apparently from [3] but I was not able to get access to it. It makes sense though: The rate of change in attitude should have the same units as angular speed times an attitude. In the equation below we simply zero-out the $\dot{q}_0$ term and substitute in for q_0 using $q_0 = \sqrt{1 - q_1^2 - q_2^2 - q_3^2}$. This reduces the number of terms in the equation and per [2] is still controllable.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{1 - q_1^2 - q_2^2 - q_3^2} & -q_3 & q_2 \\ q_3 & \sqrt{1 - q_1^2 - q_2^2 - q_3^2} & -q_1 \\ -q_2 & q_1 & \sqrt{1 - q_1^2 - q_2^2 - q_3^2} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Finally we linearlize this using small angle approximation and the Taylor series expansion with the following equation.

$$\boldsymbol{q} = [q_0, q_1, q_2, q_3]^T = \left[\cos\frac{\theta}{2}, \hat{\boldsymbol{e}}^T \sin\frac{\theta}{2}\right]^T$$

We can do this if we assume the pendulum system is started in appoximately the upright position and maintains that attitude. The Taylor series expansion is then centered around $\theta = 0$, reducing the $\sin \frac{\theta}{2}$ terms to 0.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Angular Rotation Equations

Again using the small angle approximation we can write the equation of motion for the change in rotational speed in terms of quaterions.

$$\dot{\omega}_{px} = \frac{-b_{wx}}{I_{px}} \omega_{wx} + \frac{b_{px}}{I_{px}} \omega_{px} + \frac{2glm}{I_{px}} q_1 + \frac{k_s}{I_{px}} \mu$$

$$\dot{\omega}_{py} = \frac{-b_{wy}}{I_{py}} \omega_{wy} + \frac{b_{py}}{I_{py}} \omega_{py} + \frac{2glm}{I_{py}} q_2 + \frac{k_s}{I_{py}} \mu$$

The third equation does not have to be solved for basic operation because we're not controlling around that axis. Below is what I think may work if a third wheel was added. F_{spin} is a term that covers force of rotation around the Z-axis while the pendulum is not vertical. We can simply set it equal to 1 for this simulation.

$$\dot{\omega}_{pz} = \frac{-b_{wy}}{I_{pz}} \omega_{wz} + \frac{b_{pz}}{I_{pz}} \omega_{pz} + \frac{2gF_{spin}}{I_{pz}} q_3 + \frac{k_s}{I_{pz}} \mu$$

Working our way through the terms on the right side of the equal sign we have force from friction in the wheel bearings, force from friction at the pendulum point of rotation, force of gravity pulling the pendulum down, and then finally the force of the wheels transferring momentum. The terms used are defined here:

- $-b_w$ is the friction in the wheel bearings.
- ω_w is rotational speed of the wheel.
- I_p is the moment of inertia of the pendulum system.
- b_p is the friction in the rotation of the pendulum rod.

- ω_p is the rotational speed of the pendulum rod.
- g is gravitational acceleration (-9.81 m/s^2).
- l is the length of the pendulum rod.
- m is the combined mass at the end of the pendulum rod.
- k_s is the voltage applied to the motor.

Finally we have the equation for the rate of change of the reaction wheel rotational speed. There are two terms to account for here: friction in the wheel bearings and torque into the wheels from the motor that we are controlling.

$$\dot{\omega}_{wx} \;=\; \frac{-b_{wx}}{I_{wx}}\,\omega_{wx} \;+\; \frac{k_s}{I_{wx}}\,\mu$$

$$\dot{\omega}_{wy} \;=\; \frac{-b_{wy}}{I_{wy}}\,\omega_{wy} \;+\; \frac{k_s}{I_{wy}}\,\mu$$

Where I_w is the moment of inertia of the wheel.

Final State and Input Matrices

Taking the equations from the previous two sections we can build out the original state matrices.

$$\boldsymbol{A_{12}} = \frac{1}{2}\boldsymbol{I_3}$$

$$\boldsymbol{A_{21}} = \begin{bmatrix} \frac{2glm}{I_{px}} & 0 & 0 \\ 0 & \frac{2glm}{I_{py}} & 0 \\ 0 & 0 & \frac{2gF_{spin}}{I_{pz}} \end{bmatrix} \qquad \boldsymbol{A_{22}} = \begin{bmatrix} \frac{B_{px}}{I_{px}} & 0 & 0 \\ 0 & \frac{B_{py}}{I_{py}} & 0 \\ 0 & 0 & \frac{B_{pz}}{I_{pz}} \end{bmatrix}$$

$$\boldsymbol{A_{23}} = \begin{bmatrix} -\frac{B_{wx}}{I_{px}} & 0 & 0 \\ 0 & -\frac{B_{wy}}{I_{py}} & 0 \\ 0 & 0 & -\frac{B_{wz}}{I_{pz}} \end{bmatrix} \qquad \boldsymbol{A_{33}} = \begin{bmatrix} -\frac{B_{wx}}{I_{wx}} & 0 & 0 \\ 0 & -\frac{B_{wy}}{I_{wy}} & 0 \\ 0 & 0 & -\frac{B_{wz}}{I_{wz}} \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} \frac{k_s}{I_{px}} & 0 & 0 \\ 0 & \frac{k_s}{I_{py}} & 0 \\ 0 & 0 & \frac{k_s}{I_{pz}} \end{bmatrix} \qquad B_{31} = \begin{bmatrix} \frac{k_s}{I_{wx}} & 0 & 0 \\ 0 & \frac{k_s}{I_{wy}} & 0 \\ 0 & 0 & \frac{k_s}{I_{wz}} \end{bmatrix}$$

Controller Development

Here the controller design work will be described.

Software Operation

The simulation software was developed on Mac OSX and Python 3.5.

Setup and Execution

The user of this script should first set up a profile in the config files that represents the motors, pendulum, and wheels that they wish to simulate. They then need to update the three lines in pendulum.py where the config is chosen.

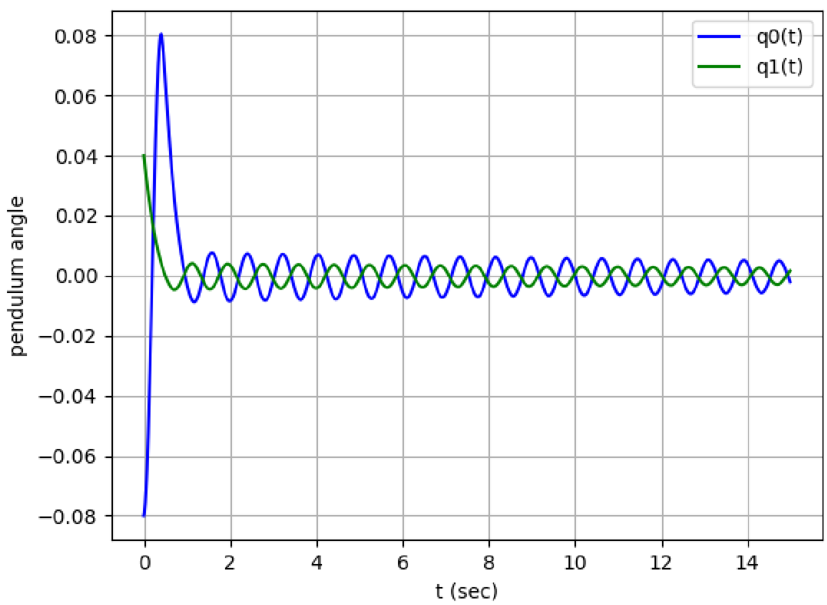
```
# Set configuration
motor = config_motor.maxon_70w
wheel = config_wheel.estimate
pend = config_pendulum.estimate
```

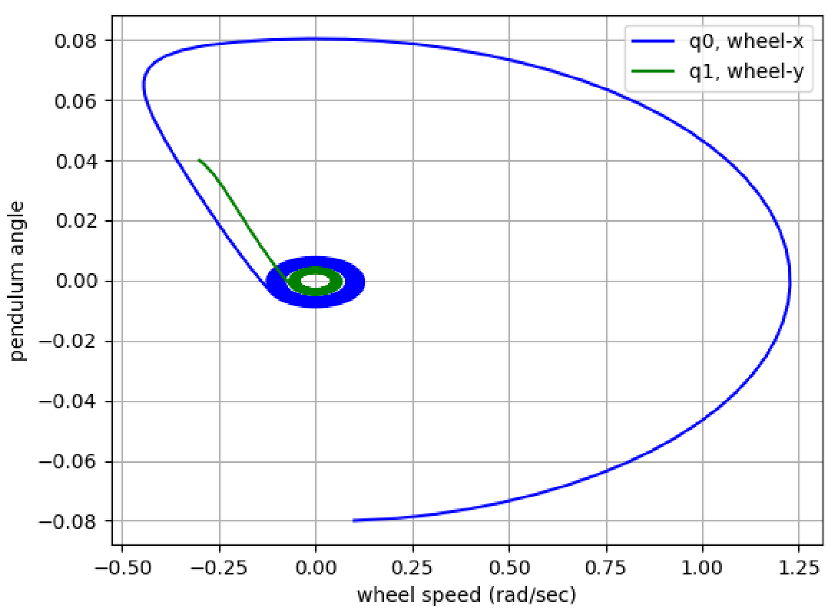
The script can then be invoked with:

```
$ python pendulum.py
```

Output

The output will provide the user with the K values and a plot of the simulated control given the initial gains and starting position of the pendulum. Below is the output from a simulation using the default initialization.


q0(t)
q1(t)
pendulum angle
t (sec)
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
0
2
4
6
8
10
12
14


q0, wheel-x
q1, wheel-y
pendulum angle
wheel speed (rad/sec)
0.08
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
1.25

References

[1] State-Space Representation of LTI Systems, Derek Rowell October 2002
`http://web.mit.edu/2.14/www/Handouts/StateSpace.pdf`

[2] Yaguang Yang, *Quaternion based model for momentum biased nadir pointing spacecraft*, Aerospace Science and Technology 14 (2010) 199–202

[3] B. Wie, *Vehicle Dynamics and Control*, AIAA Education Series, AIAA, Reston, VA, 1998.