Exercise 2)

Prove that the following spaces are normed linear spaces $(X, \|\cdot\|)$

ii) $X = C[a,b]$ and, for $p = 1$ and $p = 2$

$$\|f\| = \|f\|_p := \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

Exercise 7)

Let $f : [-1,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

Prove that

$$\lim_{k\to\infty} \left\|f^{(k)}\right\|_\infty := \lim_{k\to\infty} \sup_{x\in[-1,1]} |f^{(k)}(x)| = \infty$$

We have seen that if every deriviative of a function is bounded by some constant in a region then the (full) Taylor expansion of that function is equivalent to the original expression.

Suppose that such a constant exists, so that $\sup_{x\in[-1,1]}|f^{(k)}(x)| = C$. If so, we would be able to do a Taylor expansion of the function. However, we have seen in class that the (complete) Taylor expansion T_f of f is just $T_f(x) = 0$.

Clearly $T_f \neq f$. For instance $f\left(\frac{1}{2}\right) = e^{-4} \neq 0 = T_f\left(\frac{1}{2}\right)$. Therefore, a bound C for all the derivatives of f cannot exist. Hence, the supremum of $f^{(k)}$ will increase arbitrarily so the limit as $k \to \infty$ is ∞.

Exercise 8)

Let X be the space of all polynomials with real coefficients, and consider the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,\mathrm{d}x$$

Show that the Hermite polynomials, defined as, $H_n(x) = (-1)^n e^{x^2}\frac{\mathrm{d}^n}{(\mathrm{d}x)^n}e^{-x^2}$ form an orthogonal set with respect to this inner product.

Show that $\|H_n\|^2 = \langle H_n, H_n\rangle = 2^n n!\sqrt{\pi}$.

Derivatives of e^{-x^2}:

0. e^{-x^2}
1. $-2xe^{-x^2}$
2. $(-2+4x^2)e^{-x^2}$
3. $(8x + (-2+4x^2)\cdot(-2x))e^{-x^2}$

The recursive relation seems to be that you always have an expression of $p_n(x)e^{-x^2}$ where $p_n(x)$ is a polynomial. The relation is $\left(p_n(x)e^{-x^2}\right)' = (p_n'(x) - 2xp_n(x))e^{-x^2}$ so the recursive relation between polynomials is $p_{n+1}(x) = p_n'(x) - 2xp_n(x)$. This makes it easier to enumerate:

0. 1
1. $-2x$
2. $-2+4x^2$
3. $4x + 4x - 8x^3 = 8x - 8x^3$
4. $4 + 4 - 24x^2 - 8x^2 - 8x^2 + 16x^4 = 8 - 40x^2 + 16x^4$
5. $-48x - 16x - 16x + 64x^3 - 8x - 8x + 48x^3 + 16x^3 + 16x^3 - 32x^5 = -96x + 144x^3 - 32x^5$

For each p_n, we get one side with terms reduced via p_n' and another side with terms augmented via $2xp_n$.

What is important here is to notice that the structure of the numbers is

$$p_n(x) = \sum_{k\in S_n} c_k x^k$$

Where $S_n = \{k \mid k < n, k = 2 \pmod 2\}$. That is, it has only powers of the parity of n.

Or, to be more explicit:

$$p_n(x) = \begin{cases} \sum_{k=0}^{\frac{n}{2}} c_k x^{2k} & \text{if } n \text{ is even} \\ \sum_{k=0}^{\frac{n-1}{2}} c_k x^{2k+1} & \text{if } n \text{ is odd} \end{cases}$$

So

$$\begin{aligned} p_{n+1}(x) = p_n'(x) - 2xp_{n(x)} &= \left(\sum_{k\in S_n} c_k x^k\right)' - 2x\sum_{k\in S_n} c_k x^k \\ &= \sum_{k\in S_n} c_k k x^{k-1} - 2\sum_{k\in S_n} c_k x^{k+1} \end{aligned}$$

To prove the original statements, we will show that $\langle H_n, H_m\rangle = 0$ if $m > n$, which by symmetry implies that $\langle H_n, H_m\rangle = 0$ if $n \neq m$.

We start with the base case $\langle H_n, H_{n+1}\rangle$. This is:

$$\begin{aligned} \langle H_n, H_{n+1}\rangle &= \int_{-\infty}^{\infty} p_n(x)p_{n+1}(x)e^{-x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} p_n(x)(p_n'(x) - 2xp_n(x))e^{-x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} p_n(x)p_n'(x)e^{-x^2}\,\mathrm{d}x + \int_{-\infty}^{\infty} -2x(p_n(x))^2 e^{-x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \sum_{k,l\in S_n} c_k c_l x^{k+l-1} e^{-x^2}\,\mathrm{d}x + \int_{-\infty}^{\infty} -2x(p_n(x))^2 e^{-x^2}\,\mathrm{d}x \end{aligned}$$

Exercise 9)

Let $f(x) = e^x$. Give explicitly the polynomial of degree 2 that best approximates f with respect to the norm

$$\|g\| = \left(\int_{-\infty}^{\infty} |g(x)|^2\, e^{-x^2}\,\mathrm{d}x\right)^{\frac{1}{2}}$$

In other words: for which polynomial p of degree 2 is $\|f - p\|$ minimal? What is the minimal error?

To solve this exercise we:

1. Prove that the norm comes from an inner product
2. Make an orthonormal basis for polynomials of degree 2 given that inner product
3. Find the coefficients for the basis that makes $\|f - p\|$ minimal.
4. Compute $\|f - p\|$

Step 1: norm as inner product

The norm can be constructed from an inner product $\langle g, g\rangle^{\frac{1}{2}}$, where

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,\mathrm{d}x$$

We prove that this satisfies all properties of inner products:

i) $\langle f, f\rangle > 0$ if $f \neq 0$ and $\langle f, f\rangle = 0$ if $f = 0$.

Let $f(x) = a + bx + cx^2$. If $f = 0$ then $\langle 0, 0\rangle = \int_{-\infty}^{\infty} 0\,\mathrm{d}x = 0$. If $f \neq 0$ then one of the coefficients of $f(x) = a + bx + cx^2$ will be non-zero. e^{-x^2} is strictly positive and $(f(x))^2$ is also for any non-zero coefficient of either a, b or c **⟨ Does this need more explanation? ⟩**

ii) $\langle f, g\rangle = \langle g, f\rangle$

$$\langle f, g\rangle = \int_{-\infty}^{\infty} |f(x)g(x)|\, e^{-x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} |g(x)f(x)|\, e^{-x^2}\,\mathrm{d}x = \langle g, f\rangle$$

iii) $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$

$$\begin{aligned} \langle \alpha f + \beta g, h\rangle &= \int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x))h(x)e^{-x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \alpha f(x)h(x)e^{-x^2}\,\mathrm{d}x + \int_{-\infty}^{\infty} \beta g(x)h(x)e^{-x^2}\,\mathrm{d}x \\ &= \alpha\int_{-\infty}^{\infty} f(x)h(x)e^{-x^2}\,\mathrm{d}x + \beta\int_{-\infty}^{\infty} g(x)h(x)e^{-x^2}\,\mathrm{d}x \\ &= \alpha\langle f, h\rangle + \beta\langle g, h\rangle \end{aligned}$$

Since the norm comes from an inner product, we can find the best approximation given an orthonormal basis $G = \{g_0, g_1, g_2\}$ as $\sum_{g_k\in G}\langle g_k, f\rangle g_k$.

Step 2: orthonormal basis

First, we need to find an orthonormal basis. We start with a linearly independent basis for polynomials of degree 2 which is $\{1, x, x^2\}$. Then, we orthogonalize it. 1 and x and x and x^2 are already orthogonal, since:

$$\langle 1, x\rangle = \int_{-\infty}^{\infty} 1\cdot xe^{-x^2}\,\mathrm{d}x = -\frac{1}{2}e^{-x^2}\Big|_{-\infty}^{\infty} = 0$$

$$\langle x, x^2\rangle = \int_{-\infty}^{\infty} x^3 e^{-x^2}\,\mathrm{d}x \overset{\text{by parts}}{=} -\frac{1}{2}x^2 e^{-x^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x = 0$$

However, 1 and x^2 are not orthogonal. That is,

$$\langle 1, x^2\rangle = \int_{-\infty}^{\infty} x^2 e^{-x^2}\,\mathrm{d}x \overset{\text{by parts}}{=} -\frac{1}{2}e^{-x^2}\Big|_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \overset{\text{well known}}{=} \frac{\sqrt{\pi}}{2}$$

Therefore, we have to apply the Gram Schmidt procedure. Before doing it we are going to normilize the other polynomials first to reduce computations. That is:

$$\|1\| = \sqrt{\langle 1, 1\rangle} = \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)^{\frac{1}{2}} = \pi^{\frac{1}{4}}$$

$$\|x\| = \sqrt{\langle x, x\rangle} = \left(\int_{-\infty}^{\infty} x^2 e^{-x^2}\,\mathrm{d}x\right)^{\frac{1}{2}} = \frac{\pi^{\frac{1}{4}}}{\sqrt{2}}$$

we know the integral of $x^2 e^{-x^2}$ from computing $\langle 1, x^2\rangle$ before

Therefore our two first elements of the basis are $g_0 = \frac{1}{\pi^{\frac{1}{4}}}$ and $g_1 = \frac{x\sqrt{2}}{\pi^{\frac{1}{4}}}$.

We can now apply Gram Schmidt to x^2. from linearity we have that $\left\langle \frac{1}{\pi^{\frac{1}{4}}}, x^2\right\rangle = \frac{\sqrt{\pi}}{2} \div \pi^{\frac{1}{4}} = \frac{\pi^{\frac{1}{4}}}{2}$ and that, still, $\left\langle \frac{x\sqrt{2}}{\pi^{\frac{1}{4}}}, x^2\right\rangle = 0$, so a third orthogonal element is

$$x^2 - \frac{\pi^{\frac{1}{4}}}{2}g_0 = x^2 - \frac{1}{2}$$

We still have to normalize this element to get the final basis element, so we have:

$$\begin{aligned} \left\langle x^2 - \frac{1}{2}, x^2 - \frac{1}{2}\right\rangle &= \int_{-\infty}^{\infty} \left(x^2 - \frac{1}{2}\right)^2 e^{-x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left(x^4 - x^2 + \frac{1}{4}\right)e^{-x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} x^4 e^{-x^2}\,\mathrm{d}x - \int_{-\infty}^{\infty} x^2 e^{-x^2}\,\mathrm{d}x + \frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \\ &= \frac{3}{2}\int_{-\infty}^{\infty} x^2 e^{-x^2}\,\mathrm{d}x - \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{4} \\ &= \frac{3\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{4} \\ &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

So $\left\|x^2 - \frac{1}{2}\right\| = \sqrt{\langle x^2 - \frac{1}{2}, x^2 - \frac{1}{2}\rangle} = \frac{\pi^{\frac{1}{4}}}{\sqrt{2}}$, with which we can get our final basis element:

$$g_2 = \left(x^2 - \frac{1}{2}\right)\frac{\sqrt{2}}{\pi^{\frac{1}{4}}}$$

The basis is, then:

$$G = \{g_0, g_1, g_2\} = \left\{\frac{1}{\pi^{\frac{1}{4}}}, \frac{x\sqrt{2}}{\pi^{\frac{1}{4}}}, \left(x^2 - \frac{1}{2}\right)\frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\right\}$$

Step 3: Coefficients

Now we compute the coefficients. First, let's calculate a common integral by completing the square. Namely, the integral of e^{-x^2+x}, which is:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{x-x^2}\,\mathrm{d}x &= \int_{-\infty}^{\infty} e^{-\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}}\,\mathrm{d}x \\ &= e^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-\left(x-\frac{1}{2}\right)^2}\,\mathrm{d}x \\ &= e^{\frac{1}{4}}\pi^{\frac{1}{2}} \end{aligned}$$

Now let's compute the coefficients. Remember that $c_k = \langle f, g_k\rangle$, so

$$\begin{aligned} c_0 = \langle f, g_0\rangle = \left\langle e^x, \frac{1}{\pi^{\frac{1}{4}}}\right\rangle &= \int_{-\infty}^{\infty} e^x \frac{1}{\pi^{\frac{1}{4}}}e^{-x^2}\,\mathrm{d}x \\ &= \frac{1}{\pi^{\frac{1}{4}}}\int_{-\infty}^{\infty} e^{x-x^2}\,\mathrm{d}x \\ &= \frac{1}{\pi^{\frac{1}{4}}}e^{\frac{1}{4}}\pi^{\frac{1}{2}} \\ &= (\pi e)^{\frac{1}{4}} \end{aligned}$$

$$\begin{aligned} c_1 = \langle f, g_1\rangle &= \left\langle e^x, \frac{x\sqrt{2}}{\pi^{\frac{1}{4}}}\right\rangle \\ &= \int_{-\infty}^{\infty} \frac{x\sqrt{2}}{\pi^{\frac{1}{4}}}e^{x-x^2}\,\mathrm{d}x \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\int_{-\infty}^{\infty} xe^{x-x^2}\,\mathrm{d}x \\ &\overset{\text{by parts}}{=} \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\cancel{-\frac{1}{2}e^{x-x^2}\Big|_{-\infty}^{\infty}} + \frac{1}{2}\int_{-\infty}^{\infty} e^{x-x^2}\,\mathrm{d}x\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\frac{1}{2}e^{\frac{1}{4}}\pi^{\frac{1}{2}} \\ &= \frac{\sqrt{2}}{2}(\pi e)^{\frac{1}{4}} \end{aligned}$$

And

$$\begin{aligned} c_2 = \langle f, g_2\rangle &= \left\langle e^x, \left(x^2 - \frac{1}{2}\right)\frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\right\rangle \\ &= \int_{-\infty}^{\infty} \left(x^2 - \frac{1}{2}\right)\frac{\sqrt{2}}{\pi^{\frac{1}{4}}}e^{x-x^2}\,\mathrm{d}x \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\int_{-\infty}^{\infty} x^2 e^{x-x^2}\,\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{\infty} e^{x-x^2}\,\mathrm{d}x\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\cancel{-\frac{1}{2}xe^{x-x^2}\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} (e^x + xe^x)\left(-\frac{1}{2}e^{-x^2}\right)\mathrm{d}x - \frac{e^{\frac{1}{4}}\pi^{\frac{1}{2}}}{2}\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\frac{1}{2}\int_{-\infty}^{\infty} (e^x + xe^x)e^{-x^2}\,\mathrm{d}x - \frac{e^{\frac{1}{4}}\pi^{\frac{1}{2}}}{2}\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\frac{1}{2}\left(\int_{-\infty}^{\infty} e^{x-x^2}\,\mathrm{d}x + \int_{-\infty}^{\infty} xe^{x-x^2}\,\mathrm{d}x\right) - \frac{e^{\frac{1}{4}}\pi^{\frac{1}{2}}}{2}\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\frac{1}{2}\left(e^{\frac{1}{4}}\pi^{\frac{1}{2}} + \frac{1}{2}e^{\frac{1}{4}}\pi^{\frac{1}{2}}\right) - \frac{e^{\frac{1}{4}}\pi^{\frac{1}{2}}}{2}\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\frac{1}{2}\left(\frac{3}{2}e^{\frac{1}{4}}\pi^{\frac{1}{2}}\right) - \frac{e^{\frac{1}{4}}\pi^{\frac{1}{2}}}{2}\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\left(\frac{3}{4}e^{\frac{1}{4}}\pi^{\frac{1}{2}} - \frac{e^{\frac{1}{4}}\pi^{\frac{1}{2}}}{2}\right) \\ &= \frac{\sqrt{2}}{\pi^{\frac{1}{4}}}\frac{1}{4}e^{\frac{1}{4}}\pi^{\frac{1}{2}} \\ &= \frac{\sqrt{2}}{4}(\pi e)^{\frac{1}{4}} \end{aligned}$$

So, the best polynomial approximation to f with the given norm is

$$\begin{aligned} p(x) &= (\pi e)^{\frac{1}{4}}\left(1 + \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{4}\left(x^2 - \frac{1}{2}\right)\right) \\ &= (\pi e)^{\frac{1}{4}}\left(1 - \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{4}x^2\right) \\ &= \sqrt{2}(\pi e)^{\frac{1}{4}}\left(\frac{7}{8} + \frac{x}{2} + \frac{x^2}{4}\right) \end{aligned}$$

Step 4: Computing distance

The distance $\|e^x - p(x)\|$ is:

$$\left\|e^x - \sqrt{2}(\pi e)^{\frac{1}{4}}\left(\frac{7}{8} + \frac{x}{2} + \frac{x^2}{4}\right)\right\|$$

This would be a monstrosity to compute, but we can instead use the Pythagorean theorem for inner product spaces. We can manipulate a bit the expression $\|e^x - p\|^2 = \langle e^x - p, e^x - p\rangle$:

$$\begin{aligned} \langle e^x - p, e^x - p\rangle &= \langle e^x, e^x - p\rangle - \langle p, e^x - p\rangle \\ &= \langle e^x, e^x\rangle - \langle e^x, p\rangle - \langle p, e^x\rangle + \langle p, p\rangle \\ &= \langle e^x, e^x\rangle - 2\langle e^x, p\rangle + \langle p, p\rangle \end{aligned}$$

Here we have three distinct terms. The first is simple:

$$\langle e^x, e^x\rangle = 1$$

For the second, we can break it down into the basis vectors and use orthogonality:

$$\begin{aligned} \langle e^x, p\rangle &= \langle e^x, c_0g_0 + c_1g_1 + c_2g_2\rangle \\ &= c_0\langle e^x, g_0\rangle + c_1\langle e^x, g_1\rangle + c_2\langle e^x, g_2\rangle \end{aligned}$$

However, the inner products that multiply the coefficients are just the definition of the coefficients themselves! So:

$$\langle e^x, p\rangle = c_0^2 + c_1^2 + c_2^2$$

Finally, we need the term $\langle p, p\rangle$ which is also the sum of the coefficients, but this time just by simple Pythagorean theorem.

Another way to put it is that $e^x - p$ and p are perpendicular since that is what makes a best approximation, so $\|p + e^x - p\|^2 = \|p\|^2 + \|e^x - p\|^2 \Rightarrow \|e^x - p\|^2 = \|e^x\|^2 - \|p\|^2$. This is simpler but I didn't invoke it because we haven't explicitly stated in the course that a best approximation is the orthogonal projection, and the derivation for that is basically what I have written above.

Therefore, the inner product is:

$$\begin{aligned} \langle e^x - p, e^x - p\rangle &= 1 - c_0^2 + c_1^2 + c_2^2 \\ &= 1 - (\pi e)^{\frac{1}{4}}\left(1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}\right) \\ &= 1 - (\pi e)^{\frac{1}{4}}\left(1 + 3\frac{\sqrt{2}}{4}\right) \end{aligned}$$

This value is the norm squared, so the distance is:

$$\|e^x - p\| = \sqrt{\langle e^x - p, e^x - p\rangle} = \sqrt{1 - (\pi e)^{\frac{1}{4}}\left(1 + 3\frac{\sqrt{2}}{4}\right)}$$

Exercise 10)

Which polynomial p of degree 2 best approximates the function $|x|^{-\frac{1}{3}}$ on $[-1,1]$ with respect to the norm

$$\|f\| = \left(\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

What is the minimal error? Can $|x|^{-\frac{1}{3}}$ be approximated by polynomials with respect to the supremum norm to arbitrary precision?

Exercise 11)

Consider the 2π-periodic function

$$f : \mathbb{R} \to \mathbb{R}$$

defined by

$$f(x) = \frac{\pi^2}{3} - \pi|x| + \frac{1}{2}x^2, \quad x \in (-\pi, \pi]$$

Find a number n and a trigonometric polynomial

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx))$$

with real coefficients such that

$$\|f - S_n\|_\infty := \sup_{x\in(-\pi,\pi]} |f(x) - S_n(x)| < 10^{-1}$$

Plot f and S_n to visualize the error.