TTK4130 MODELING AND SIMULATION | ASSIGNMENT 7

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1) We have $e_k = x_{k+1} - x(t_{k+1}|k)$. Taylor expanding $x(t_{k+1}|k)$:

$$x(t_{k+1}|k) = x_k + \Delta t f(x_k, u_k) + \underbrace{\frac{\Delta t^2}{2} \cdot \frac{\partial f}{\partial x} \cdot f(x_k, u_k)}_{= \dot{f}} + \mathcal{O}(\Delta t^3) \quad (1)$$

($\ddot{x}(t_{k+1})$)

ERK2 for x_{k+1} gives

$$x_{k+1} = x_k + \Delta t \sum_{i=1}^{2} b_i k_i$$

$$= x_k + \Delta t b_1 \cdot k_1 + \Delta t b_2 k_2$$

$$= x_k + \Delta t b_1 \cdot f(x_k, u_k) + \Delta t b_2 f(x_k + a\Delta t \cdot f(x_k, u_k), u_k) \quad (2)$$

$u(t_k + c\Delta t) = u_k$, $0 < c < 1$

Taylor expanding the last term gives:

$$\Delta t b_2 f(x_k + a\Delta t \cdot f(x_k, u_k), u_k) = \Delta t \cdot b_2 \left(f(x_k, u_k) + a\Delta t \cdot f(x_k, u_k) \frac{\partial f}{\partial x} + \mathcal{O}(\Delta t^2) \right)$$

$$= \Delta t \cdot b_2 \cdot f(x_k, u_k) + a b_2 \Delta t^2 f(x_k, u_k) \frac{\partial f}{\partial x} + \mathcal{O}(\Delta t^3) \quad (3)$$

Inserting (3) into (2) and taking the difference with (1) gives

$$e_k = x_k + \Delta t f(x_k,u_k) + \frac{\Delta t^2}{2} \cdot \frac{\partial f}{\partial x} f(x_k,u_k) + \mathcal{O}(\Delta t^3)$$
$$- \left[x_k + \Delta t b_1 f(x_k,u_k) + \Delta t b_2 f(x_k,u_k) + a b_2 \Delta t^2 f(x_k,u_k) \cdot \frac{\partial f}{\partial x} + \mathcal{O}(\Delta t^3) \right]$$
$$= \Delta t f(x_k,u_k) \cdot (1 - b_1 - b_2) + \Delta t^2 \frac{\partial f}{\partial x} \cdot f(x_k,u_k) \left(\frac{1}{2} - a b_2 \right) + \mathcal{O}(\Delta t^3)$$

For the error to be $\mathcal{O}(\Delta t^3)$, the two first terms must be 0.

$\Rightarrow$ $1 - b_1 - b_2 = 0$ and $\frac{1}{2} - a b_2 = 0$

$\Rightarrow$ $b_1 + b_2 = 1$ and $a b_2 = \frac{1}{2}$

As long as $0 < c < 1$, it can be whatever as u_k is constant.

b) The global or total error is the sum of errors over all steps, so

$$e = \| x_N - x(T) \| = N \cdot \| x_{k+1} - x(t_{k+1}) \|. \quad N = \frac{T}{\Delta t},$$

so $$e = \frac{T}{\Delta t} \| x_{k+1} - x(t_{k+1}) \| \leq \frac{T}{\Delta t} \cdot \mathcal{O}(\Delta t^3) = \underline{\underline{\mathcal{O}(\Delta t^2)}}$$

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a) ERK1 and ERK2 ~~has~~ have approximately the same accuracy, but are on the lower or upper side of the true trajectory. ERK4 is the most accurate.

b) See added plots. RK1 : order 1 : global error $\mathcal{O}(\Delta t)$

RK2 : order 2 : global error $\mathcal{O}(\Delta t^2)$

(RK3 : order 3 : global error $\mathcal{O}(\Delta t^3)$)

RK4 : order 4 : global error $\mathcal{O}(\Delta t^4)$

c) RK1 and RK2 becomes unstable at ~~about~~ $\approx \Delta t = 1{,}2$ with $\lambda = -2$ $\Rightarrow$

RK4 $\approx \Delta t = 2{,}5$. (more like $\approx \Delta t = 2{,}12$)