

Wasserstein stability for the p -Dowker nerve

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1 Setting

We fix a real number $p \geq 1$ and endow the interval $[0, \infty]$ with the binary operation $\otimes: [0, \infty]^2 \rightarrow [0, \infty]$ defined by $t \otimes s := (t^p + s^p)^{\frac{1}{p}}$.

Definition 1.1. Given a Dowker dissimilarity $\Lambda: L \times W \rightarrow [0, \infty]$, define the p -Dowker nerve $D^p \Lambda$ of Λ by letting

$$D^p \Lambda_t := \left\{ \sigma \subseteq L \mid \exists w \in W \text{ such that } \bigotimes_{l \in \sigma} \Lambda(l, w) \leq t \right\} \text{ for } t \in [0, \infty).$$

We sometimes allow ourselves to suppress the p and just write $D\Lambda$.

Let $W = \mathbb{R}^d$ be equipped with the Euclidean distance $d(x, y) = \|x - y\|_2$, and let L and L' be finite subsets of W with $|L| = |L'|^1$. Define the Dowker dissimilarity $\Gamma: W \times W \rightarrow [0, \infty]$ by letting $\Gamma(w, w') = d(w, w')$, and let $\Lambda = \Gamma|_{L \times W}$ and $\Lambda' = \Gamma|_{L' \times W}$. Setting $p = \infty$ we have that $D\Lambda$ and $D\Lambda'$ are the ambient Čech filtrations of the point clouds L and L' , respectively. In this case, we have classical stability, i.e., the bottleneck distance between the persistence diagrams are upper-bounded by the Hausdorff distance between the two point clouds. In this note, we apply the Cellular Wasserstein Stability Theorem of [2] to establish an analogous stability result for the case $1 \leq p < \infty$.

2 Definitions

Let D and D' be two persistence diagrams. A *matching* M between D and D' is a set of pairs (x, y) with $x \in X \cup \Delta$, $y \in Y \cup \Delta$ and every $x \in X$ and $y \in Y$ is

¹It would be nice if we could drop this assumption.

used exactly once. Here Δ denotes the diagonal.

Definition 2.1 (p -Wasserstein distance). Given $p \geq 1$, the p -Wasserstein distance between D and D' is defined as

$$\mathbf{W}_p(D, D') := \inf_M \left(\sum_{(x,y) \in M} \|x - y\|_p^p \right)^{\frac{1}{p}}$$

where M ranges over all matchings between the two diagrams. If $D = (D_k)_k$ and $D' = (D'_k)_k$, which is the case when we consider different homological dimensions, we define the *total p -Wasserstein distance* by

$$\mathbf{W}_p(D, D') := \left(\sum_k \mathbf{W}_p(D_k, D'_k)^p \right)^{\frac{1}{p}}.$$

Definition 2.2 (L^p norm). Let K be a finite CW-complex. The L^p norm of a function $f: K \rightarrow \mathbb{R}$ is given by $\|f\|_p = (\sum_{\sigma \in K} |f(\sigma)|^p)^{\frac{1}{p}}$. Given two monotone functions $f, g: K \rightarrow \mathbb{R}$, the L^p distance between f and g is the distance induced by the L^p norm. That is,

$$\|f - g\|_p = \left(\sum_{\sigma \in K} |f(\sigma) - g(\sigma)|^p \right)^{\frac{1}{p}}.$$

The following definition of the point set Wasserstein distance appears in [2].

Definition 2.3 (p -Wasserstein point set distance). Given $p \geq 1$ and two point sets L and L' with $|L| = |L'|$, we define the p -Wasserstein distance between L and L' by

$$\mathbf{WP}_p(L, L') := \inf_{\phi} \left(\sum_{l \in L} \|l - \phi(l)\|_2^p \right)^{\frac{1}{p}}$$

where $\phi: L \rightarrow L'$ ranges over all bijections.

3 Stability

Theorem 3.1 (Cellular Wasserstein Stability Theorem [2]). Let K be a finite CW complex and let $f, g: K \rightarrow \mathbb{R}$ be monotone functions. Then

$$\mathbf{W}_p(\text{Dgm}(f), \text{Dgm}(g)) \leq \|f - g\|_p.$$

For a fixed homological dimension k , we have

$$\mathbf{W}_p(\mathrm{Dgm}_k(f), \mathrm{Dgm}_k(g))^p \leq \sum_{k \leq \dim(\sigma) \leq k+1} |f(\sigma) - g(\sigma)|^p.$$

For the rest of this section, let L and L' be finite subsets of $W = \mathbb{R}^d$ with $n = |L| = |L'|$. Define the Dowker dissimilarity $\Gamma: W \times W \rightarrow [0, \infty]$ by letting $\Gamma(w, w') = d(w, w')$, and let $\Lambda = \Gamma|_{L \times W}$ and $\Lambda' = \Gamma|_{L' \times W}$. Let $\mathrm{Dgm}(\Lambda)$ denote the persistence diagram corresponding to the sublevel set homology of the filtration function $D\Lambda \rightarrow \mathbb{R}$ defined by

$$\sigma \mapsto \inf\{t \in [0, \infty] \mid \sigma \in D\Lambda_t\} = \inf_{w \in W} \bigotimes_{l \in \sigma} \Lambda(l, w) = \inf_{w \in W} \left(\sum_{l \in \sigma} \|l - w\|_2^p \right)^{\frac{1}{p}}. \quad (1)$$

Corollary 3.1.1 (Stability of the p -Dowker nerve). Let Λ and Λ' be defined as above. Then

$$\mathbf{W}_p(\mathrm{Dgm}(\Lambda), \mathrm{Dgm}(\Lambda')) \leq C \cdot \mathbf{W}_p(L, L').$$

where $C = (2^n - 1)^{\frac{1}{p}}$.

Proof. Let $\phi: L \rightarrow L'$ be a bijection minimizing the point set p -Wasserstein distance $\mathbf{W}_p(L, L')$. Let $f: D\Lambda \rightarrow \mathbb{R}$ be the filtration function defined in Equation (1). Similarly, let $f': D\Lambda' \rightarrow \mathbb{R}$ be the filtration function for $D\Lambda'$ and define $g := f' \circ \phi: D\Lambda \rightarrow \mathbb{R}$. For $\sigma \in D\Lambda$, we have

$$\begin{aligned} |f(\sigma) - g(\sigma)| &= \left| \inf_{w \in W} \bigotimes_{l \in \sigma} \Lambda(l, w) - \inf_{w \in W} \bigotimes_{l \in \sigma} \Lambda(\phi(l), w) \right| \\ &\stackrel{(1)}{\leq} \sup_{w \in W} \left| \bigotimes_{l \in \sigma} \Lambda(l, w) - \bigotimes_{l \in \sigma} \Lambda(\phi(l), w) \right| \\ &= \sup_{w \in W} \left| \left(\sum_{l \in \sigma} \|l - w\|_2^p \right)^{\frac{1}{p}} - \left(\sum_{l \in \sigma} \|\phi(l) - w\|_2^p \right)^{\frac{1}{p}} \right| \\ &\stackrel{(2)}{\leq} \sup_{w \in W} \left| \left(\sum_{l \in \sigma} \left| \|l - w\|_2 - \|\phi(l) - w\|_2 \right|^p \right)^{\frac{1}{p}} \right| \\ &\stackrel{(3)}{\leq} \sup_{w \in W} \left(\sum_{l \in \sigma} \|l - \phi(l)\|_2^p \right)^{\frac{1}{p}} = \left(\sum_{l \in \sigma} \|l - \phi(l)\|_2^p \right)^{\frac{1}{p}} \leq \mathbf{W}_p(L, L') \end{aligned}$$

In (1) we used the inequality $|\inf f - \inf g| \leq \sup |f - g|$. In (2) and (3) we used the reverse triangle inequality $|||x|| - ||y||| \leq \|x - y\|$ for the L^p norm and the L^2 norm, respectively. Applying Theorem 3.1 we then get

$$\begin{aligned}
\mathbf{W}_p(\text{Dgm}(\Lambda), \text{Dgm}(\Lambda')) &\leq \|f - g\|_p = \left(\sum_{\sigma \in D\Lambda} |f(\sigma) - g(\sigma)|^p \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{\sigma \in D\Lambda} \mathbf{WP}_p(L, L')^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{k=1}^n \binom{n}{k} \right)^{\frac{1}{p}} \mathbf{WP}_p(L, L') \\
&= (2^n - 1)^{\frac{1}{p}} \mathbf{WP}_p(L, L')
\end{aligned}$$

Similarly, if we fix the homological dimension $k \geq 0$, we get that

$$\mathbf{W}_p(\text{Dgm}_k(\Lambda), \text{Dgm}_k(\Lambda')) \leq \left(\binom{n}{k} + \binom{n}{k+1} \right)^{\frac{1}{p}} \mathbf{WP}_p(L, L') = \binom{n+1}{k+1}^{\frac{1}{p}} \mathbf{WP}_p(L, L').$$

□

4 Duplicating points

The Čech nerve $N^\infty \Lambda$ does not change, up to homotopy, if we duplicate a point that is already in the point set. This is however not the case for the p -nerve in general as the following example shows.

Example 4.0.1 (Duplicating a point can change the Wasserstein distance between persistence diagrams). Let $p = 2$ and consider the point set $L = \{x, y\} \subset \mathbb{R}^2$ with $x \neq y$. Clearly, the Dowker p -nerve $D\Lambda$ has trivial 1-homology. Now, suppose we duplicate the point x . That is, let $L' = \{x, x', y\}$ with $x = x'$. The 2-simplex $\sigma = \{x, x', y\}$ is born at time $d = \sqrt{\frac{2}{3}}\|x - y\|_2$. However, the faces of σ are born at time 0 or $b = \sqrt{\frac{1}{2}}\|x - y\|_2$, meaning that we have a non-trivial persistent 1-cycle in $D\Lambda'$. Note that in the $p = \infty$ case, the 2-simplex will have the same birth time as its faces, killing any new cycles. The 2-Wasserstein distance between the persistence diagrams (in homological dimension 1) corresponding to Λ and Λ' can then be computed to be $\frac{d-b}{\sqrt{2}}$ which is proportional to the distance between x and y .

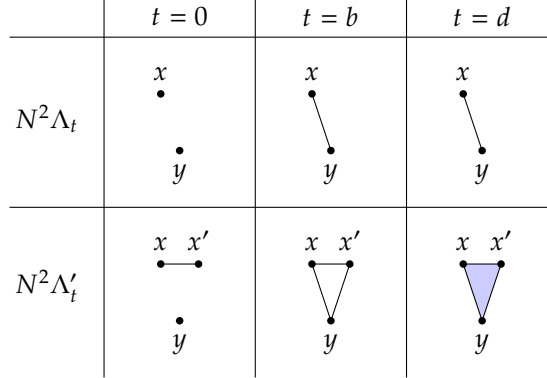


Figure 1: Duplicating a point can lead to the birth of a new non-trivial cycle. In the $p = \infty$ case, the triangle would already be filled in at $t = b$.

5 Network distance

A correspondence between two sets X and X' is a subset $C \subseteq X \times X'$ such that the projection maps to X and X' are onto. A non-negatively weighted network ω_X consists of a set X together with a function $\omega_X: X \times X \rightarrow [0, \infty)$. We will typically be interested in weighted networks on the form $\omega_L = \Lambda = d: L \times L \rightarrow [0, \infty)$.

Definition 5.1. Let $\omega: X \times X \rightarrow [0, \infty)$ and $\omega': X' \times X' \rightarrow [0, \infty)$ be weighted networks and let $C \subseteq X \times X'$. We define the p -distortion of C as

$$\text{dis}^p(C) = \left(\sum_{(x, x'), (y, y') \in C} |\omega(x, y) - \omega'(x', y')|^p \right)^{\frac{1}{p}}.$$

Definition 5.2. Let $\mathcal{R} = \mathcal{R}(X, X')$ denote the set of all correspondences $C \subseteq X \times X'$ and let ω and ω' be weighted networks as in the above definition. Define the p -network distance between X and X' as

$$d_N^p(X, X') = \frac{1}{2} \inf_{C \in \mathcal{R}} \text{dis}^p(C).$$

Note that if (M, d_M) and (N, d_N) are metric spaces, and $\omega(x, y) = d_M(x, y)$, $\omega'(x', y') = d_N(x', y')$, then the p -network distance agrees with the Gromov-Hausdorff distance when $p = \infty$.

Example 5.2.1 (Duplicating points does not change the network distance). Let $X = \{x_1, x_2, \dots, x_n\}$ be finite point set in \mathbb{R}^d , and let X' be X with one of the

points $x' = x_i$ duplicated. We consider the weighted networks with the weight function ω being the L_2 distance on \mathbb{R}^d . Let $C \subseteq X \times X'$ be the correspondence consisting of all pairs on the form (x_j, x_j) for $1 \leq j \leq n$, and the pair (x_i, x') . Then, clearly $\text{dis}^p(C) = 0$ and hence $d_N^p(X, X') = 0$.

6 Comments

1. We use $q = p$ in the Wasserstein distance. Can we generalize for other choices of q ? (See definition in [2] for example.)
2. In the definition of the point set p -Wasserstein distance and the dissimilarities we use $q = 2$. Other choices of q are probably interleaved with ours.
3. What about the case when $|L| \neq |L'|$?
4. Could be interesting to look at the part about stability in [1][p. 14-16].

References

- [1] Morten Brun and Nello Blaser. “Sparse Dowker nerves”. In: *Journal of Applied and Computational Topology* 3.1-2 (2019), pp. 1–28. DOI: [10.1007/s41468-019-00028-9](https://doi.org/10.1007/s41468-019-00028-9). URL: <https://doi.org/10.1007%2Fs41468-019-00028-9>.
- [2] Primož Skraba and Katharine Turner. *Wasserstein Stability for Persistence Diagrams*. 2022. arXiv: 2006.16824 [math.AT].