Dowker dissimilarities

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1 Introduction

In this note, we summarize our progress attempting to generalize the results in [1] replacing $[0, \infty)$ by a poset P.

Definition 1.1 (Poset stuff)**.** For the following definitions, let P and Q be posets, and let $[1]$ denote the poset $0 \leq 1$.

- The *product poset* $P \times Q$ with $(p, q) \leq (p', q') \iff p \leq p'$ and $q \leq q'$.
- The *extended poset* $P^* := \hom(P, [1])$ with $\infty \equiv 1$.
- The *opposite poset* P^{op} with the same underlying set as P but with the reverse order. That is, $p \leq p'$ in P^{op} if and only if $p' \leq p$ in P.

Definition 1.2 (Dowker stuff)**.** Some fundamental definitions related to Dowker dissimilarities.

- A *Dowker dissimilarity* Λ consists of two sets L and W, and a function $\Lambda \colon L \times W \to P^*$ where P is some poset.
- Given $l \in L$ and $p \in P$, define the *ball of radius p centred at l* to be the set $B_\Lambda(l, p) := \{w \in W \mid \Lambda(l, w)(p) = 1\}$.
- Given some p in P, define $\Lambda_p := \{(l, w) \in L \times W \mid w \in B_\Lambda(l, p)\}$.
- The *Dowker nerve* $N\Lambda$ of Λ is the collection of simplicial complexes indexed over P defined in degree p as

  $$(N\Lambda)_p = \{\sigma \subseteq L \mid \sigma \text{ is finite and } \exists w \in W \text{ with } w \in B_\Lambda(l, p) \text{ for all } l \in \sigma\}.$$

In other words, the Dowker nerve $(N\Lambda)_p$ is the nerve of the covering $\{B_\Lambda(l,p)\}_{l\in L}$.

Note that if $p \leq p'$ for $p, p' \in P$, then we have an inclusion $N\Lambda_{p\leq p'} \colon N\Lambda_p \hookrightarrow N\Lambda_{p'}$.

Example 1.2.1 (Čech filtration)**.** Consider the poset $P = [0,\infty)$ and let $L \subset W = \mathbb{R}^d$. If we set $\Lambda \colon L \times W \to P^*$ to be the dissimilarity defined for $t \in P$ by

$$\Lambda(l,w)(t) := \begin{cases} 1 & \text{if } t \geq d(l,w) \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

then we have $\sigma \in N\Lambda_t$ if and only if there exists some $w \in \mathbb{R}^d$ with $d(l,w) \leq t$ for all $l \in \sigma$. The existence of such a w is equivalent to the intersection $\bigcap_{l\in L} B_t(l)$ being non-empty or in other words, the Dowker nerve of Λ is the usual ambient Čech complex of L.

Example 1.2.2 (Multicover filtration)**.** Consider the poset $P = \mathbb{N}^{op}\times[0,\infty)$ and let $L = \mathcal{P}(X)$ where $X \subset W = \mathbb{R}^d$. Define the Dowker dissimilarity $\Lambda \colon L \times W \to P^*$ be the Dowker dissimilarity defined by

$$\Lambda(A,y)(k,t) := \begin{cases} 1 & \text{if } (k,t) \geq (|A|,\, d_H(\{y\},A)) \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since $d_H(\{y\},A) = \max_{a\in A} d(y,a)$, we see that

$$B_\Lambda(A,(k,t)) := \begin{cases} \bigcap_{a\in A} B_t(a) & \text{if } k \leq |A| \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Following the notation in [2], let $\binom{X}{k}$ denote the set of all subsets $A \subseteq X$ with $|A| = k$, and define the lens $L_t(A)$ corresponding to $A \in \binom{X}{k}$ by $L_t(A) = \bigcap_{a\in A} B_t(a)$. The *k-fold cover* $L(t,X)$ of X at scale t is defined as $L(t,X) = \bigcup_{A\in\binom{X}{k}} L_t(A)$. The *kth order Čech complex with radius t* over X, denoted by $\check{\mathrm{C}}\mathrm{ech}_t(X,k)$, is defined as the nerve of the covering $\{L_t(A)\}_{A\in\binom{X}{k}}$ of $L(t,X)$.
The Dowker nerve $N\Lambda_{(k,t)}$ is homotopy equivalent to the higher order Čech complex $\check{\mathrm{C}}\mathrm{ech}_t(X,k)$. By the nerve lemma, the Dowker nerve is homotopy

equivalent to the union

$$\bigcup_{A\in\mathcal{P}(X)} B_\Lambda(A,(k,t)) = \bigcup_{\substack{A\in\mathcal{P}(X)\\ |A|\geq k}} \bigcap_{a\in A} B_t(a) = \bigcup_{A\in\binom{X}{k}} L_t(A) = L(t,X)$$

and $L(t,X) \simeq \check{\mathrm{C}}\mathrm{ech}_t(X,k)$, again by application of the nerve lemma. (The second equality above follows from the fact that if $|A| > k$ then we can remove points from A to obtain A' with $|A'| = k$ and $B_\Lambda(A,(k,t)) \subset B_\Lambda(A',(k,t))$.)
Note that $\check{\mathrm{C}}\mathrm{ech}_t(X,k) \subsetneq N\Lambda_{(k,t)}$ since the two complexes have different vertex sets $\binom{X}{k}$ and $\mathcal{P}(X)$, respectively.

2 Truncated nerves

Definition 2.1 (Translation function)**.** A *translation function* $\alpha\colon P \to P$ is an order preserving function which satisfies $p \leq \alpha(p)$ for all $p \in P$.

Example 2.1.1. Let $P = [0,\infty)$ and $a,b \in \mathbb{R}$ with $a > 1$ and $b \geq 0$. Then $\alpha\colon t \mapsto at + b$ is a translation function.

Definition 2.2 (Interleaving)**.** Given a translation function $\alpha\colon P \to P$ and a morphism $G\colon C \to C'$ of filtered objects in some category $\mathscr{C}$, then G is said to be an *α-interleaving* if for every $p \in P$ there exists a morphism $F_p\colon C'_p \to C_{\alpha(p)}$ such that both triangles in the following diagram commute:

$$\begin{tikzcd} C_p \arrow[r, "G_p"] \arrow[dr] & C'_p \arrow[d, "F_p"] \arrow[dr] & \\ & C_{\alpha(p)} \arrow[r, "G_{\alpha(p)}"] & C'_{\alpha(p)} \end{tikzcd}$$

Definition 2.3 (Truncation)**.** Given a dissimilarity $\Lambda\colon L \times W \to P^*$ and a translation function $\alpha\colon P \to P$, a function $T\colon L \to P$ is said to be an *α-truncation function for* Λ if for all $p \in P$ and all $l \in L$, there exists $l' \in L$ such that for all $w \in W$

$$w \in B_\Lambda(l,p) \implies w \in B_\Lambda(l',\alpha(p)) \cap B_\Lambda(l',T(l')). \tag{1}$$

The *T-truncation* $\Gamma\colon L \times W \to P^*$ of a dissimilarity Λ is the dissimilarity defined by letting

$$\Gamma(l,w) := \begin{cases} \Lambda(l,w) & \text{if } w \in B_\Lambda(l,T(l)), \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 2.3.1. Given a dissimilarity $\Lambda\colon L \times W \to P^*$, a translation function $\alpha\colon P \to P$, and an α-truncation function $T\colon L \to P$, let Γ be the T-truncation of Λ. The inclusion of $N\Gamma$ in $N\Lambda$ is an α-interleaving in the homotopy category of topological spaces.

Proof. Define the function $f_p\colon L \to L$ by picking for each $l \in L$ an element $l' \in L$ satisfying Equation (1) and set $f_p(l) = l'$. This induces a simplicial map $f_p\colon N\Lambda_p \to N\Gamma_{\alpha(p)}$. To see this, let $\sigma \in N\Lambda_p$ and pick some $w \in W$ such that $w \in B_\Lambda(l, p)$ for all $l \in \sigma$. By construction of f_p we have $w \in B_\Lambda(l', \alpha(p))$ and $w \in B_\Lambda(l', T(l'))$ and hence $w \in B_\Gamma(l', \alpha(p))$. In other words, $f_p(\sigma) \in N\Gamma_{\alpha(p)}$ by the definition of the T-truncation. What is left to show, is that the triangles in the following diagram commute up to homotopy:

$$
\begin{tikzcd}
N\Gamma_p \arrow[r, hook] \arrow[dr] & N\Lambda_p \arrow[d, "f_p"] \arrow[dr] & \\
& N\Gamma_{\alpha(p)} \arrow[r, hook] & N\Lambda_{\alpha(p)}
\end{tikzcd}
$$

Recall that two simplicial maps $f, g\colon K \to K'$ are homotopic if for all simplices $\sigma \in K$, the union $f(\sigma) \cup g(\sigma)$ is a simplex in K', and that this induces a homotopy on the level of geometric realizations. It is therefore enough to show that

1. for all $\sigma \in N\Lambda_p$ we have $f_p(\sigma) \cup \sigma \in N\Lambda_{\alpha(p)}$, and
2. for all $\sigma \in N\Gamma_p$ we have $f_p(\sigma) \cup \sigma \in N\Gamma_{\alpha(p)}$.

1. This follows from the fact that for all $l' \in f_p(\sigma)$ we have $w \in B_\Lambda(l', \alpha(p))$ and the fact that for all $l \in \sigma$ we have $B_\Lambda(l, p) \subseteq B_\Lambda(l, \alpha(p))$ since $p \leq \alpha(p)$.
2. Similar argument. $\square$

- check definition 4.5 in [1].
- what does the *farthest point sampling* look like in the multicover case?

References

[1] Nello Blaser and Morten Brun. *Sparse Filtered Nerves*. 2019. arXiv: `1810.02149 [math.AT]`.

[2] Mickaël Buchet, Bianca B. Dornelas, and Michael Kerber. *Sparse Higher Order Čech Filtrations*. 2023. arXiv: 2303.06666 [cs.CG].