

# Dowker dissimilarities

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## 1 Introduction

In this note, we summarize our progress attempting to generalize the results in [1] replacing  $[0, \infty)$  by a poset  $P$ .

**Definition 1.1** (Poset stuff). For the following definitions, let  $P$  and  $Q$  be posets, and let  $[1]$  denote the poset  $0 \leq 1$ .

- The *product poset*  $P \times Q$  with  $(p, q) \leq (p', q') \iff p \leq p' \text{ and } q \leq q'$ .
- The *extended poset*  $P^* := \text{hom}(P, [1])$  with  $\infty \equiv 1$ .
- The *opposite poset*  $P^{op}$  with the same underlying set as  $P$  but with the reverse order. That is,  $p \leq p'$  in  $P^{op}$  if and only if  $p' \leq p$  in  $P$ .

**Definition 1.2** (Dowker stuff). Some fundamental definitions related to Dowker dissimilarities.

- A *Dowker dissimilarity*  $\Lambda$  consists of two sets  $L$  and  $W$ , and a function  $\Lambda: L \times W \rightarrow P^*$  where  $P$  is some poset.
- Given  $l \in L$  and  $p \in P$ , define the *ball of radius  $p$  centred at  $l$*  to be the set  $B_\Lambda(l, p) := \{w \in W \mid \Lambda(l, w)(p) = 1\}$ .
- Given some  $p$  in  $P$ , define  $\Lambda_p := \{(l, w) \in L \times W \mid w \in B_\Lambda(l, p)\}$ .
- The *Dowker nerve*  $N\Lambda$  of  $\Lambda$  is the collection of simplicial complexes indexed over  $P$  defined in degree  $p$  as

$$(N\Lambda)_p = \{\sigma \subseteq L \mid \sigma \text{ is finite and } \exists w \in W \text{ with } w \in B_\Lambda(l, p) \text{ for all } l \in \sigma\}.$$

In other words, the Dowker nerve  $(N\Lambda)_p$  is the nerve of the covering  $\{B_\Lambda(l, p)\}_{l \in L}$ .

Note that if  $p \leq p'$  for  $p, p' \in P$ , then we have an inclusion  $N\Lambda_{p \leq p'}: N\Lambda_p \hookrightarrow N\Lambda_{p'}$ .

**Example 1.2.1** (Čech filtration). Consider the poset  $P = [0, \infty)$  and let  $L \subset W = \mathbb{R}^d$ . If we set  $\Lambda: L \times W \rightarrow P^*$  to be the dissimilarity defined for  $t \in P$  by

$$\Lambda(l, w)(t) := \begin{cases} 1 & \text{if } t \geq d(l, w) \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

then we have  $\sigma \in N\Lambda_t$  if and only if there exists some  $w \in \mathbb{R}^d$  with  $d(l, w) \leq t$  for all  $l \in \sigma$ . The existence of such a  $w$  is equivalent to the intersection  $\bigcap_{l \in L} B_t(l)$  being non-empty or in other words, the Dowker nerve of  $\Lambda$  is the usual ambient Čech complex of  $L$ .

**Example 1.2.2** (Multicover filtration). Consider the poset  $P = \mathbb{N}^{op} \times [0, \infty)$  and let  $L = \mathcal{P}(X)$  where  $X \subset W = \mathbb{R}^d$ . Define the Dowker dissimilarity  $\Lambda: L \times W \rightarrow P^*$  be the Dowker dissimilarity defined by

$$\Lambda(A, y)(k, t) := \begin{cases} 1 & \text{if } (k, t) \geq (|A|, d_H(\{y\}, A)) \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since  $d_H(\{y\}, A) = \max_{a \in A} d(y, a)$ , we see that

$$B_\Lambda(A, (k, t)) := \begin{cases} \bigcap_{a \in A} B_t(a) & \text{if } k \leq |A| \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Following the notation in [2], let  $\binom{X}{k}$  denote the set of all subsets  $A \subseteq X$  with  $|A| = k$ , and define the lens  $L_t(A)$  corresponding to  $A \in \binom{X}{k}$  by  $L_t(A) = \bigcap_{a \in A} B_t(a)$ . The  $k$ -fold cover  $L(t, X)$  of  $X$  at scale  $t$  is defined as  $L(t, X) = \bigcup_{A \in \binom{X}{k}} L_t(A)$ . The  $k$ th order Čech complex with radius  $t$  over  $X$ , denoted by  $\check{\text{Cech}}_t(X, k)$ , is defined as the nerve of the covering  $\{L_t(A)\}_{A \in \binom{X}{k}}$  of  $L(t, X)$ .

The Dowker nerve  $N\Lambda_{(k, t)}$  is homotopy equivalent to the higher order Čech complex  $\check{\text{Cech}}_t(X, k)$ . By the nerve lemma, the Dowker nerve is homotopy

equivalent to the union

$$\bigcup_{A \in \mathcal{P}(X)} B_\Lambda(A, (k, t)) = \bigcup_{\substack{A \in \mathcal{P}(X) \\ |A| \geq k}} \bigcap_{a \in A} B_t(a) = \bigcup_{A \in \binom{X}{k}} L_t(A) = L(t, X)$$

and  $L(t, X) \simeq \check{\text{Cech}}_t(X, k)$ , again by application of the nerve lemma. (The second equality above follows from the fact that if  $|A| > k$  then we can remove points from  $A$  to obtain  $A'$  with  $|A'| = k$  and  $B_\Lambda(A, (k, t)) \subset B_\Lambda(A', (k, t))$ .)

Note that  $\check{\text{Cech}}_t(X, k) \subsetneq N\Lambda_{(k, t)}$  since the two complexes have different vertex sets  $\binom{X}{k}$  and  $\mathcal{P}(X)$ , respectively.

## 2 Truncated nerves

**Definition 2.1** (Translation function). A *translation function*  $\alpha: P \rightarrow P$  is an order preserving function which satisfies  $p \leq \alpha(p)$  for all  $p \in P$ .

**Example 2.1.1.** Let  $P = [0, \infty)$  and  $a, b \in \mathbb{R}$  with  $a > 1$  and  $b \geq 0$ . Then  $\alpha: t \mapsto at + b$  is a translation function.

**Definition 2.2** (Interleaving). Given a translation function  $\alpha: P \rightarrow P$  and a morphism  $G: C \rightarrow C'$  of filtered objects in some category  $\mathcal{C}$ , then  $G$  is said to be an  $\alpha$ -*interleaving* if for every  $p \in P$  there exists a morphism  $F_p: C'_p \rightarrow C_{\alpha(p)}$  such that both triangles in the following diagram commute:

$$\begin{array}{ccc} C_p & \xrightarrow{G_p} & C'_p \\ & \searrow & \downarrow F_p \\ & & C_{\alpha(p)} \xrightarrow{G_{\alpha(p)}} C'_{\alpha(p)} \end{array}$$

**Definition 2.3** (Truncation). Given a dissimilarity  $\Lambda: L \times W \rightarrow P^*$  and a translation function  $\alpha: P \rightarrow P$ , a function  $T: L \rightarrow P$  is said to be an  $\alpha$ -*truncation function* for  $\Lambda$  if for all  $p \in P$  and all  $l \in L$ , there exists  $l' \in L$  such that for all  $w \in W$

$$w \in B_\Lambda(l, p) \implies w \in B_\Lambda(l', \alpha(p)) \cap B_\Lambda(l', T(l')). \quad (1)$$

The  $T$ -*truncation*  $\Gamma: L \times W \rightarrow P^*$  of a dissimilarity  $\Lambda$  is the dissimilarity defined by letting

$$\Gamma(l, w) := \begin{cases} \Lambda(l, w) & \text{if } w \in B_\Lambda(l, T(l)), \\ \infty & \text{otherwise.} \end{cases}$$

**Proposition 2.3.1.** Given a dissimilarity  $\Lambda: L \times W \rightarrow P^*$ , a translation function  $\alpha: P \rightarrow P$ , and an  $\alpha$ -truncation function  $T: L \rightarrow P$ , let  $\Gamma$  be the  $T$ -truncation of  $\Lambda$ . The inclusion of  $N\Gamma$  in  $N\Lambda$  is an  $\alpha$ -interleaving in the homotopy category of topological spaces.

*Proof.* Define the function  $f_p: L \rightarrow L$  by picking for each  $l \in L$  an element  $l' \in L$  satisfying Equation (1) and set  $f_p(l) = l'$ . This induces a simplicial map  $f_p: N\Lambda_p \rightarrow N\Gamma_{\alpha(p)}$ . To see this, let  $\sigma \in N\Lambda_p$  and pick some  $w \in W$  such that  $w \in B_\Lambda(l, p)$  for all  $l \in \sigma$ . By construction of  $f_p$  we have  $w \in B_\Lambda(l', \alpha(p))$  and  $w \in B_\Lambda(l', T(l'))$  and hence  $w \in B_\Gamma(l', \alpha(p))$ . In other words,  $f_p(\sigma) \in N\Gamma_{\alpha(p)}$  by the definition of the  $T$ -truncation. What is left to show, is that the triangles in the following diagram commute up to homotopy:

$$\begin{array}{ccccc} N\Gamma_p & \hookrightarrow & N\Lambda_p & & \\ & \searrow & \downarrow f_p & \searrow & \\ & & N\Gamma_{\alpha(p)} & \hookrightarrow & N\Lambda_{\alpha(p)} \end{array}$$

Recall that two simplicial maps  $f, g: K \rightarrow K'$  are homotopic if for all simplices  $\sigma \in K$ , the union  $f(\sigma) \cup g(\sigma)$  is a simplex in  $K'$ , and that this induces a homotopy on the level of geometric realizations. It is therefore enough to show that

1. for all  $\sigma \in N\Lambda_p$  we have  $f_p(\sigma) \cup \sigma \in N\Lambda_{\alpha(p)}$ , and
  2. for all  $\sigma \in N\Gamma_p$  we have  $f_p(\sigma) \cup \sigma \in N\Gamma_{\alpha(p)}$ .
1. This follows from the fact that for all  $l' \in f_p(\sigma)$  we have  $w \in B_\Lambda(l', \alpha(p))$  and the fact that for all  $l \in \sigma$  we have  $B_\Lambda(l, p) \subseteq B_\Lambda(l, \alpha(p))$  since  $p \leq \alpha(p)$ .
  2. Similar argument. □

- check definition 4.5 in [1].
- what does the *farthest point sampling* look like in the multicover case?

## References

- [1] Nello Blaser and Morten Brun. *Sparse Filtered Nerves*. 2019. arXiv: 1810.02149 [math.AT].

- [2] Mickaël Buchet, Bianca B. Dornelas, and Michael Kerber. *Sparse Higher Order Čech Filtrations*. 2023. arXiv: 2303.06666 [cs.CG].