# Wasserstein stability for the $p$-Dowker nerve

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## 1 Setting

We fix a real number $p \geq 1$ and endow the interval $[0, \infty]$ with the binary operation $\otimes\colon [0, \infty]^2 \to [0, \infty]$ defined by $t \otimes s := (t^p + s^p)^{\frac{1}{p}}$.

**Definition 1.1.** Given a Dowker dissimilarity $\Lambda\colon L \times W \to [0, \infty]$, define the *$p$-Dowker nerve* $D^p\Lambda$ of $\Lambda$ by letting

$$D^p\Lambda_t := \left\{ \sigma \subseteq L \mid \exists w \in W \text{ such that } \bigotimes_{l \in \sigma} \Lambda(l, w) \leq t \right\} \text{ for } t \in [0, \infty).$$

We sometimes allow ourselves to suppress the $p$ and just write $D\Lambda$.

Let $W = \mathbb{R}^d$ be equipped with the Euclidean distance $d(x, y) = \|x - y\|_2$, and let $L$ and $L'$ be finite subsets of $W$ with $|L| = |L'|$[1]. Define the Dowker dissimilarity $\Gamma\colon W \times W \to [0, \infty]$ by letting $\Gamma(w, w') = d(w, w')$, and let $\Lambda = \Gamma\mid_{L \times W}$ and $\Lambda' = \Gamma\mid_{L' \times W}$. Setting $p = \infty$ we have that $D\Lambda$ and $D\Lambda'$ are the ambient Čech filtrations of the point clouds $L$ and $L'$, respectively. In this case, we have classical stability, i.e., the bottleneck distance between the persistence diagrams are upper-bounded by the Hausdorff distance between the two point clouds. In this note, we apply the Cellular Wasserstein Stability Theorem of [2] to establish an analogous stability result for the case $1 \leq p < \infty$.

## 2 Definitions

Let $D$ and $D'$ be two persistence diagrams. A *matching* $M$ between $D$ and $D'$ is a set of pairs $(x, y)$ with $x \in X \cup \Delta$, $y \in Y \cup \Delta$ and every $x \in X$ and $y \in Y$ is

[1] It would be nice if we could drop this assumption.

used exactly once. Here $\Delta$ denotes the diagonal.

**Definition 2.1** ($p$-Wasserstein distance)**.** Given $p \geq 1$, the *$p$-Wasserstein distance* between $D$ and $D'$ is defined as

$$\mathbf{W}_p(D, D') := \inf_M \left( \sum_{(x,y) \in M} \|x - y\|_p^p \right)^{\frac{1}{p}}$$

where $M$ ranges over all matchings between the two diagrams. If $D = (D_k)_k$ and $D' = (D'_k)_k$, which is the case when we consider different homological dimensions, we define the *total $p$-Wasserstein distance* by

$$\mathbf{W}_p(D, D') := \left( \sum_k W_p(D_k, D'_k)^p \right)^{\frac{1}{p}}.$$

**Definition 2.2** ($L^p$ norm)**.** Let $K$ be a finite CW-complex. The *$L^p$ norm* of a function $f \colon K \to \mathbb{R}$ is given by $\|f\|_p = (\sum_{\sigma \in K} |f(\sigma)|^p)^{\frac{1}{p}}$. Given two monotone functions $f, g \colon K \to \mathbb{R}$, the *$L^p$ distance* between $f$ and $g$ is the distance induced by the $L^p$ norm. That is,

$$\|f - g\|_p = \left( \sum_{\sigma \in K} |f(\sigma) - g(\sigma)|^p \right)^{\frac{1}{p}}.$$

The following definition of the point set Wasserstein distance appears in [2].

**Definition 2.3** ($p$-Wasserstein point set distance)**.** Given $p \geq 1$ and two point sets $L$ and $L'$ with $|L| = |L'|$, we define the *$p$-Wasserstein distance* between $L$ and $L'$ by

$$\mathbf{WP}_p(L, L') := \inf_\phi \left( \sum_{l \in L} \|l - \phi(l)\|_2^p \right)^{\frac{1}{p}}$$

where $\phi \colon L \to L'$ ranges over all bijections.

# 3 Stability

**Theorem 3.1** (Cellular Wasserstein Stability Theorem [2])**.** Let $K$ be a finite CW complex and let $f, g \colon K \to \mathbb{R}$ be monotone functions. Then

$$\mathbf{W}_p(\mathrm{Dgm}(f), \mathrm{Dgm}(g)) \leq \|f - g\|_p.$$

For a fixed homological dimension $k$, we have

$$\mathbf{W}_p(\mathrm{Dgm}_k(f), \mathrm{Dgm}_k(g))^p \leq \sum_{k \leq \dim(\sigma) \leq k+1} |f(\sigma) - g(\sigma)|^p.$$

For the rest of this section, let $L$ and $L'$ be finite subsets of $W = \mathbb{R}^d$ with $n = |L| = |L'|$. Define the Dowker dissimilarity $\Gamma\colon W \times W \to [0, \infty]$ by letting $\Gamma(w, w') = d(w, w')$, and let $\Lambda = \Gamma\mid_{L \times W}$ and $\Lambda' = \Gamma\mid_{L' \times W}$. Let $\mathrm{Dgm}(\Lambda)$ denote the persistence diagram corresponding to the sublevel set homology of the filtration function $D\Lambda \to \mathbb{R}$ defined by

$$\sigma \mapsto \inf\{t \in [0, \infty] \mid \sigma \in D\Lambda_t\} = \inf_{w \in W} \bigotimes_{l \in \sigma} \Lambda(l, w) = \inf_{w \in W} \left( \sum_{l \in \sigma} \|l - w\|_2^p \right)^{\frac{1}{p}}. \quad (1)$$

**Corollary 3.1.1** (Stability of the $p$-Dowker nerve)**.** Let $\Lambda$ and $\Lambda'$ be defined as above. Then

$$\mathbf{W}_p(\mathrm{Dgm}(\Lambda), \mathrm{Dgm}(\Lambda')) \leq C \cdot \mathbf{WP}_p(L, L').$$

where $C = (2^n - 1)^{\frac{1}{p}}$.

*Proof.* Let $\phi\colon L \to L'$ be a bijection minimizing the point set $p$-Wasserstein distance $\mathbf{WP}_p(L, L')$. Let $f\colon D\Lambda \to \mathbb{R}$ be the filtration function defined in Equation (1). Similarly, let $f'\colon D\Lambda' \to \mathbb{R}$ be the filtration function for $D\Lambda'$ and define $g := f' \circ \varphi\colon D\Lambda \to \mathbb{R}$. For $\sigma \in D\Lambda$, we have

$$\begin{aligned}
|f(\sigma) - g(\sigma)| &= \left| \inf_{w \in W} \bigotimes_{l \in \sigma} \Lambda(l, w) - \inf_{w \in W} \bigotimes_{l \in \sigma} \Lambda(\varphi(l), w) \right| \\
&\overset{(1)}{\leq} \sup_{w \in W} \left| \bigotimes_{l \in \sigma} \Lambda(l, w) - \bigotimes_{l \in \sigma} \Lambda(\varphi(l), w) \right| \\
&= \sup_{w \in W} \left| \left( \sum_{l \in \sigma} \|l - w\|_2^p \right)^{\frac{1}{p}} - \left( \sum_{l \in \sigma} \|\varphi(l) - w\|_2^p \right)^{\frac{1}{p}} \right| \\
&\overset{(2)}{\leq} \sup_{w \in W} \left| \left( \sum_{l \in \sigma} |\|l - w\|_2 - \|\varphi(l) - w\|_2|^p \right)^{\frac{1}{p}} \right| \\
&\overset{(3)}{\leq} \sup_{w \in W} \left( \sum_{l \in \sigma} \|l - \varphi(l)\|_2^p \right)^{\frac{1}{p}} = \left( \sum_{l \in \sigma} \|l - \varphi(l)\|_2^p \right)^{\frac{1}{p}} \leq \mathbf{WP}_p(L, L')
\end{aligned}$$

In (1) we used the inequality $|\inf f - \inf g| \leq \sup |f - g|$. In (2) and (3) we used the reverse triangle inequality $|\|x\| - \|y\|| \leq \|x - y\|$ for the $L^p$ norm and the $L^2$ norm, respectively. Applying Theorem 3.1 we then get

$$\begin{aligned}
\mathbf{W}_p(\mathrm{Dgm}(\Lambda), \mathrm{Dgm}(\Lambda')) \leq \|f - g\|_p &= \left( \sum_{\sigma \in D\Lambda} |f(\sigma) - g(\sigma)|^p \right)^{\frac{1}{p}} \\
&\leq \left( \sum_{\sigma \in D\Lambda} \mathbf{WP}_p(L, L')^p \right)^{\frac{1}{p}} \\
&= \left( \sum_{k=1}^{n} \binom{n}{k} \right)^{\frac{1}{p}} \mathbf{WP}_p(L, L') \\
&= (2^n - 1)^{\frac{1}{p}} \mathbf{WP}_p(L, L')
\end{aligned}$$

Similarly, if we fix the homological dimension $k \geq 0$, we get that

$$\mathbf{W}_p(\mathrm{Dgm}_k(\Lambda), \mathrm{Dgm}_k(\Lambda')) \leq \left( \binom{n}{k} + \binom{n}{k+1} \right)^{\frac{1}{p}} \mathbf{WP}_p(L, L') = \binom{n+1}{k+1}^{\frac{1}{p}} \mathbf{WP}_p(L, L').$$

□

# 4 Duplicating points

The Čech nerve $N^{\infty}\Lambda$ does not change, up to homotopy, if we duplicate a point that is already in the point set. This is however not the case for the $p$-nerve in general as the following example shows.

**Example 4.0.1** (Duplicating a point can change the Wasserstein distance between persistence diagrams)**.** Let $p = 2$ and consider the point set $L = \{x, y\} \subset \mathbb{R}^2$ with $x \neq y$. Clearly, the Dowker $p$-nerve $D\Lambda$ has trivial 1-homology. Now, suppose we duplicate the point $x$. That is, let $L' = \{x, x', y\}$ with $x = x'$. The 2-simplex $\sigma = \{x, x', y\}$ is born at time $d = \sqrt{\frac{2}{3}}\|x - y\|_2$. However, the faces of $\sigma$ are born at time 0 or $b = \sqrt{\frac{1}{2}}\|x - y\|_2$, meaning that we have a non-trivial persistent 1-cycle in $D\Lambda'$. Note that in the $p = \infty$ case, the 2-simplex will have the same birth time as its faces, killing any new cycles. The 2-Wasserstein distance between the persistence diagrams (in homological dimension 1) corresponding to $\Lambda$ and $\Lambda'$ can then be computed to be $\frac{d-b}{\sqrt{2}}$ which is proportional to the distance between $x$ and $y$.

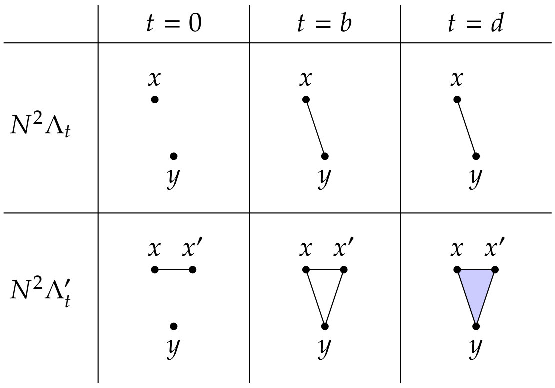


Figure 1: Duplicating a point can lead to the birth of a new non-trivial cycle. In the $p = \infty$ case, the triangle would already be filled in at $t = b$.

# 5 Network distance

A correspondence between two sets $X$ and $X'$ is a subset $C \subseteq X \times X'$ such that the projection maps to $X$ and $X'$ are onto. A non-negatively weighted network $\omega_X$ consists of a set $X$ together with a function $\omega_X : X \times X \to [0, \infty)$. We will typically be interested in weighted networks on the form $\omega_L = \Lambda = d : L \times L \to [0, \infty)$.

**Definition 5.1.** Let $\omega : X \times X \to [0, \infty)$ and $\omega' : X' \times X' \to [0, \infty)$ be weighted networks and let $C \subseteq X \times X'$. We define the *p-distortion of* $C$ as

$$\operatorname{dis}^p(C) = \left( \sum_{(x,x'),(y,y') \in C} |\omega(x, y) - \omega'(x', y')|^p \right)^{\frac{1}{p}}.$$

**Definition 5.2.** Let $\mathcal{R} = \mathcal{R}(X, X')$ denote the set of all correspondences $C \subseteq X \times X'$ and let $\omega$ and $\omega'$ be weighted networks as in the above definition. Define the *p-network distance* between $X$ and $X'$ as

$$d_N^p(X, X') = \frac{1}{2} \inf_{C \in \mathcal{R}} \operatorname{dis}^p(C).$$

Note that if $(M, d_M)$ and $(N, d_N)$ are metric spaces, and $\omega(x, y) = d_M(x, y)$, $\omega'(x', y') = d_N(x', y')$, then the $p$-network distance agrees with the Gromov-Hausdorff distance when $p = \infty$.

**Example 5.2.1** (Duplicating points does not change the network distance)**.** Let $X = \{x_1, x_2, \ldots, x_n\}$ be finite point set in $\mathbb{R}^d$, and let $X'$ be $X$ with one of the points $x' = x_i$ duplicated. We consider the weighted networks with the weight function $\omega$ being the $L_2$ distance on $\mathbb{R}^d$. Let $C \subseteq X \times X'$ be the correspondence consisting of all pairs on the form $(x_j, x_j)$ for $1 \leq j \leq n$, and the pair $(x_i, x')$. Then, clearly $\mathrm{dis}^p(C) = 0$ and hence $d_N^p(X, X') = 0$.

# 6 Computing witnesses

In this section, suppose $L \subseteq W = \mathbb{R}^d$ and $\Lambda\colon L \times W \to [0, \infty]$ is given by $\Lambda(l, w) = \|l - w\|_q$ for some fixed $q$. Fix some $p$ and consider the $p$-nerve $D\Lambda = D^p\Lambda$ of $\Lambda$. Given a simplex $\sigma \in L$, we want to know at what time $t$ the simplex $\sigma$ appears in the $p$-nerve. In other words, we want to compute the infimum

$$\inf_{w \in W} \left( \sum_{l \in \sigma} \Lambda(l, w)^p \right)^{\frac{1}{p}}.$$

In this section, we give a description of $w^* \in W$ minimizing the above sum for different values of $p$ and $q$. Let $g_\sigma\colon W \to [0, \infty]$ be given by $g_\sigma(w) = \sum_{l \in \sigma} \Lambda(l, w)^p$. The partial derivatives of $g_\sigma$ are

$$\frac{\partial g_\sigma}{\partial w_i}(w) = p \sum_{l \in \sigma} \|l - w\|_q^{p-q} (w_i - l_i)|l_i - w_i|^{q-2}$$

so any extremal point $w \in W$ must satisfy

$$w_i = \frac{\sum_{l \in \sigma} l_i \|l - w\|_q^{p-q} |l_i - w_i|^{q-2}}{\sum_{l \in \sigma} \|l - w\|_q^{p-q} |l_i - w_i|^{q-2}}$$

for all $i = 1, 2, \ldots, d$. For simplicity, we set $q = 2$ to get rid of the factors $|l_i - w_i|^{q-2}$. In the case $p = 2$, $w$ is the centroid of the points in $\sigma$. In the case $p = 1$, $w$ is the geometric median of $\sigma$.[2] We are looking for roots of the functions $G_\sigma\colon W \to W$ defined by

$$G_\sigma(w) = \frac{\boldsymbol{\sigma} D(w)}{\mathbf{1}^\top D(w)} - w.$$

where $\boldsymbol{\sigma}$ is the $d \times (k+1)$ matrix with $\boldsymbol{\sigma}_{ij} = l_i^j$ representing the $k$-simplex $\sigma = \{l^0, \ldots, l^k\}$, and $D(w)$ is the column vector with $D(w)_i = \|l^i - w\|_2^{p-2}$.

[2]The geometric median has many names, including the Fermat-Weber point, $L_1$-median, spatial median, Euclidean minisum point and Torricelli point.

Show that the minimizer $w$ is global. We need to show that the Hessian of $g_\sigma$ is positive semi-definite. This is equivalent with convexity (as $g_\sigma$ is twice differentiable). Then, for a differentiable convex function $g$, $w^*$ is a global minimizer iff $\nabla f(w^*) = 0$. Or maybe we can just show convexity directly from definition.

## 7 Comments

1. We use $q = p$ in the Wasserstein distance. Can we generalize for other choices of $q$? (See definition in [2] for example.)

2. In the definition of the point set $p$-Wasserstein distance and the dissimilarities we use $q = 2$. Other choices of $q$ are probably interleaved with ours.

3. What about the case when $|L| \neq |L'|$?

4. Could be interesting to look at the part about stability in [1][p. 14-16].

## References


[1] Morten Brun and Nello Blaser. "Sparse Dowker nerves". In: *Journal of Applied and Computational Topology* 3.1-2 (2019), pp. 1–28. DOI: 10.1007/s41468-019-00028-9. URL: https://doi.org/10.1007%2Fs41468-019-00028-9.

[2] Primoz Skraba and Katharine Turner. *Wasserstein Stability for Persistence Diagrams*. 2022. arXiv: 2006.16824 [math.AT].