# Dowker dissimilarities

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## 1 Introduction

In this note, we summarize our progress attempting to generalize the results in [1] replacing $[0, \infty)$ by a poset $P$.

**Definition 1.1** (Poset stuff)**.** For the following definitions, let $P$ and $Q$ be posets, and let $[1]$ denote the poset $0 \leq 1$.

- The *hom-poset* $\hom(P, Q)$ consisting of all order preserving maps $P \to Q$ endowed with the partial order $f \leq g \iff \forall p \in P : f(p) \leq g(p)$.
- The *product poset* $P \times Q$ with $(p, q) \leq (p', q') \iff p \leq p'$ and $q \leq q'$.
- The *extended poset* $P^* := \hom(P, [1])$ with $\infty \equiv 1$.
- The *opposite poset* $P^{op}$ with the same underlying set as $P$ but with the reverse order. That is, $p \leq p'$ in $P^{op}$ if and only if $p' \leq p$ in $P$.

**Definition 1.2** (Dowker stuff)**.** Some fundamental definitions related to Dowker dissimilarities.

- A *Dowker dissimilarity* $\Lambda$ consists of two sets $L$ and $W$, and a function $\Lambda\colon L \times W \to P^*$ where $P$ is some poset.
- Given $l \in L$ and $p \in P$, define the *ball of radius $p$ centred at $l$* to be the set $B_\Lambda(l, p) := \{w \in W \mid \Lambda(l, w)(p) = 1\}$.
- Given some $p$ in $P$, define $\Lambda_p := \{(l, w) \in L \times W \mid w \in B_\Lambda(l, p)\}$.

- The *Dowker nerve* $N\Lambda$ of $\Lambda$ is the collection of simplicial complexes indexed over $P$ defined in degree $p$ as

  $$(N\Lambda)_p = \{\sigma \subseteq L \mid \sigma \text{ is finite and } \exists w \in W \text{ with } w \in B_\Lambda(l,p) \text{ for all } l \in \sigma\}\,.$$

  In other words, the Dowker nerve $(N\Lambda)_p$ is the nerve of the covering $\{B_\Lambda(l,p)\}_{l\in L}$.

Note that if $p \leq p'$ for $p, p' \in P$, then we have an inclusion $N\Lambda_{p\leq p'} \colon N\Lambda_p \hookrightarrow N\Lambda_{p'}$.

**Example 1.2.1** (Čech filtration)**.** Consider the poset $P = [0,\infty)$ and let $L \subset W = \mathbb{R}^d$. If we set $\Lambda\colon L \times W \to P^*$ to be the dissimilarity defined for $t \in P$ by

$$\Lambda(l,w)(t) := \begin{cases} 1 & \text{if } t \geq d(l,w) \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

then we have $\sigma \in N\Lambda_t$ if and only if there exists some $w \in \mathbb{R}^d$ with $d(l,w) \leq t$ for all $l \in \sigma$. The existence of such a $w$ is equivalent to the intersection $\bigcap_{l\in L} B_t(l)$ being non-empty or in other words, the Dowker nerve of $\Lambda$ is the usual ambient Čech complex of $L$.

**Example 1.2.2** (Multicover filtration)**.** Consider the poset $P = \mathbb{N}^{op}\times[0,\infty)$ and let $L = \mathcal{P}(X)$ where $X \subset W = \mathbb{R}^d$. Define the Dowker dissimilarity $\Lambda\colon L\times W \to P^*$ be the Dowker dissimilarity defined by

$$\Lambda(A,y)(k,t) := \begin{cases} 1 & \text{if } (k,t) \geq (|A|,\, d_H(\{y\},A)) \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since $d_H(\{y\},A) = \max_{a\in A} d(y,a)$, we see that

$$B_\Lambda(A,(k,t)) := \begin{cases} \bigcap_{a\in A} B_t(a) & \text{if } k \leq |A| \text{ and} \\ \emptyset & \text{otherwise.} \end{cases}$$

Following the notation in [2], let $\binom{X}{k}$ denote the set of all subsets $A \subseteq X$ with $|A| = k$, and define the lens $L_t(A)$ corresponding to $A \in \binom{X}{k}$ by $L_t(A) = \bigcap_{a\in A} B_t(a)$. The *k-fold cover* $L(t,X)$ of $X$ at scale $t$ is defined as $L(t,X) =$

$\bigcup_{A \in \binom{X}{k}} L_t(A)$. The *kth order Čech complex with radius $t$* over $X$, denoted by $\check{\mathrm{C}}\mathrm{ech}_t(X, k)$, is defined as the nerve of the covering $\{L_t(A)\}_{A \in \binom{X}{k}}$ of $L(t, X)$. The Dowker nerve $N\Lambda_{(k,t)}$ is homotopy equivalent to the higher order Čech complex $\check{\mathrm{C}}\mathrm{ech}_t(X, k)$. By the nerve lemma, the Dowker nerve is homotopy equivalent to the union

$$\bigcup_{A \in \mathcal{P}(X)} B_\Lambda(A, (k, t)) = \bigcup_{\substack{A \in \mathcal{P}(X) \\ |A| \geq k}} \bigcap_{a \in A} B_t(a) = \bigcup_{A \in \binom{X}{k}} L_t(A) = L(t, X)$$

and $L(t, X) \simeq \check{\mathrm{C}}\mathrm{ech}_t(X, k)$, again by application of the nerve lemma. (The second equality above follows from the fact that if $|A| > k$ then we can remove points from $A$ to obtain $A'$ with $|A'| = k$ and $B_\Lambda(A, (k, t)) \subset B_\Lambda(A', (k, t))$.) Note that $\check{\mathrm{C}}\mathrm{ech}_t(X, k) \subsetneq N\Lambda_{(k,t)}$ since the two complexes have different vertex sets $\binom{X}{k}$ and $\mathcal{P}(X)$, respectively.

## 2 Truncated nerves

**Definition 2.1.** A *translation function* $\alpha \colon P \to P$ is an order preserving function which satisfies $p \leq \alpha(p)$ for all $p \in P$.

**Example 2.1.1.** Let $P = [0, \infty)$ and $a, b \in \mathbb{R}$ with $a > 1$ and $b \geq 0$. Then $\alpha \colon t \mapsto at + b$ is a translation function.

Given a dissimilarity $\Lambda \colon L \times W \to P^*$ and a translation function $\alpha \colon P \to P$, a function $T \colon L \to P$ is said to be an *$\alpha$-truncation function for* $\Lambda$ if for all $p \in P$ and all $l \in L$, there exists $l' \in L$ such that for all $w \in W$

$$w \in B_\Lambda(l, p) \implies w \in B_\Lambda(l', \alpha(p)) \cap B_\Lambda(l', T(l')). \tag{1}$$

The *$T$-truncation* $\Gamma \colon L \times W \to P^*$ of a dissimilarity $\Lambda$ is the dissimilarity defined by letting

$$\Gamma(l, w) := \begin{cases} \Lambda(l, w) & \text{if } w \in B_\Lambda(l, T(l)), \\ \infty & \text{otherwise.} \end{cases}$$

Given a dissimilarity $\Lambda \colon L \times W \to P^*$, a translation function $\alpha \colon P \to P$ and a $\alpha$-truncation function $T \colon L \to P$, define the function $f_p \colon L \to L$ by picking for each $l \in L$ an element $l' \in L$ satisfying Equation (1) and set $f_p(l) = l'$.

The map $f_p$ induces a simplicial map $N\Lambda_p \to N\Gamma_{\alpha(p)}$. To see this, let $\sigma \in N\Lambda_p$ and pick some $w \in W$ such that $w \in B_\Lambda(l, p)$ for all $l \in \sigma$. By construction of $f_p$ we have $w \in B_\Lambda(l', \alpha(p))$ and $w \in B_\Lambda(l', T(l'))$.

⋮

- define interleavings.
- complete the proof of proposition 4.3 in [1].
- check definition 4.5 in [1].

## References


[1] Nello Blaser and Morten Brun. *Sparse Filtered Nerves*. 2019. arXiv: `1810.02149 [math.AT]`.

[2] Mickaël Buchet, Bianca B. Dornelas, and Michael Kerber. *Sparse Higher Order Čech Filtrations*. 2023. arXiv: `2303.06666 [cs.CG]`.