Let P be the X-ray transform in n dimensions,

$$Pf(\omega, \mathbf{x}) = \int_{\mathbb{R}} f(\mathbf{x} + t\omega)\, dt \tag{1}$$

where $\omega \in S^{n-1}$, the unit sphere in $\mathbb{R}^n$, and $x \in \omega^\perp$, the hyperpane orthogonal to ω. Let Y_n denote the set of pairs $(\omega, \mathbf{x})$ with $\omega \in S^{n-1}$ and $\mathbf{x} \in \omega^\perp$.

For any measure μ on S^{n-1}, let P^*_μ be the operator defined by

$$P^*_\mu g(\mathbf{x}) = \int_{S^{n-1}} g(\omega, E_\omega \mathbf{x})\, d\mu(\omega), \tag{2}$$

Here E_ω denotes orthogonal projection onto $\omega^\perp$.

From now on we will specialize to the case when μ is concentrated to a curve $S_0 \subset S^{n-1}$. Suppose S_0 is parametrized by $\omega(\tau)$ where $\tau \in I$, an interval on the real line. Let us also assume that μ can be written $w(\tau)d\tau$ where $w(\tau)$ is an integrable function on I. Then P^*_μ can be written

$$P^*_\mu g(\mathbf{x}) = \int_I g(\omega(\tau), E_{\omega(\tau)}\mathbf{x}) w(\tau)\, d\tau. \tag{3}$$

An important special case is when $w(\tau) = |\omega'(\tau)|$; in this case μ is arc length measure on S_0.

For functions (or distributions) g and h defined on Y_n, define

$$[g *_{\omega^\perp} h](\omega, \mathbf{x}) = \int_{\omega^\perp} g(\omega, \mathbf{y}) h(\omega, \mathbf{x} - \mathbf{y})\, d\mathbf{y}, \tag{4}$$

the convolution of g and h in planes perpendicular to ω. For a function f defined in $\mathbb{R}^n$, let

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(\mathbf{x}) \exp(-i\mathbf{x} \cdot \xi)\, d\mathbf{x} \tag{5}$$

denote the Fourier transform of f, and for g defined on Y_n let

$$\hat{g}(\omega, \xi) = \int_{\omega^\perp} g(\omega, \mathbf{x}) \exp(-i\mathbf{x} \cdot \xi)\, d\mathbf{x} \tag{6}$$

for all $(\omega, \xi) \in Y_n$.

Inversion formulas for the X-ray transform are based on the identity

$$P^*_\mu(h * Pf) = P^*_\mu(h) * f. \tag{7}$$

If an h can be found such that $P^*_\mu h = \delta_0$, this identity shows that f can be recovered from Pf. The problem of finding such h has been extensively studied when μ is Lebesgue measure on an open subset of S^{n-1}, but here we deal with the case when the support of μ is a curve.

Let A_{S_0} be the set of all $\xi \neq 0$ in $\mathbb{R}^n$ such that $\xi^\perp$ is not tangent to S_0 and intersects S_0 in a finite number of points. Let N_ξ denote the number of these intersection points.

Lemma 1. *Suppose $\hat{h}(\omega, \xi)$ is a locally integrable function of ξ for each $\omega \in S_0$. If $\xi \in A_\gamma$, then*

$$\widehat{P^*_\mu h}(\xi) = 2\pi \sum_{i=0}^{N_\xi} \frac{w(\tau_i)}{|\xi \cdot \omega'(\tau_i)|} \hat{h}(\omega(\tau_i), \xi) \tag{8}$$

where $\tau_1, \ldots, \tau_{N_\xi}$ are all points $\tau \in I$ such that $\xi \cdot \omega(\tau) = 0$.

Lemma 2. *A necessary condition for $P_\mu^* h = \delta_0$ is that*

$$2\pi \sum_{i=0}^{N_\xi} \frac{w(\tau_i)}{|\xi \cdot \omega'(\tau_i)|} \hat{h}(\omega(\tau_i), \xi) = 1 \tag{9}$$

for all $\xi \in A_{S_0}$.

Important question: Can we find a necessary and sufficient condition? What can go wrong in the complement of A_{S_0}?

Lemma 2 suggests a natural choice of h: Choose h as the inverse Fourier transform of

$$\hat{h}(\omega(\tau), \xi) = \frac{|\xi \cdot \omega'(\tau)|}{2\pi N_\xi w(\tau)}. \tag{10}$$

Note, however, that when $N_\xi > 1$ for some ξ there are other choices also.

Proof of Lemma 1: Here is a not completely rigorous argument following the proof of Theorem 2.17 in Natterer and Wübbeling.

$$\begin{aligned}
\widehat{P_\mu^* h}(\xi) &= \int_{\mathbb{R}^n} P_\mu^* h(\mathbf{x}) \exp(-i\xi \cdot \mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbb{R}^n} \int_I h(\omega(\tau), E_{\omega(\tau)}\mathbf{x}) w(\tau) \exp(-i\xi \cdot \mathbf{x})\, d\tau\, d\mathbf{x} \\
&= \int_I w(\tau) \int_{\mathbb{R}^n} h(\omega(\tau), E_{\omega(\tau)}\mathbf{x}) \exp(-i\xi \cdot \mathbf{x})\, d\mathbf{x}\, d\tau \\
&= \int_I w(\tau) \int_{\omega(\tau)^\perp} h(\omega(\tau), \mathbf{y}) \int_{\mathbb{R}} \exp(-i\xi \cdot (\mathbf{y} + t\omega(\tau)))\, dt\, d\mathbf{y}\, d\tau \\
&= 2\pi \int_I w(\tau) \int_{\omega(\tau)^\perp} h(\omega(\tau), \mathbf{y}) \exp(-i\xi \cdot \mathbf{y}) \delta_0(\xi \cdot \omega(\tau))\, d\mathbf{y}\, d\tau \\
&= 2\pi \int_I w(\tau) \hat{h}(\omega(\tau), E_{\omega(\tau)}\xi) \delta_0(\xi \cdot \omega(\tau))\, d\tau \\
&= 2\pi \sum_{i=1}^{N_\xi} \frac{w(\tau_i)}{|\xi \cdot \omega(\tau_i)|} \hat{h}(\omega(\tau), \xi).
\end{aligned}$$