Computational Methods: Problem Set 2

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Problem One

A competitive equilibrium is a set of prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$, an allocation $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ for the firm, and an allocation $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s, \}_{t=0}^{\infty}$ for the household such that:

1. The firm's allocation solves:

$$\max_{k_t, n_t} \pi = \sum_{t=0}^{\infty} p_t[y_t - r_t k_t - w_t n_t]$$
$$\text{s.t.} \quad y_t \leq F(k_t, n_t)$$

2. The household's allocation solves:

$$\max_{c_t, n_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$$
$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t[c_t + i_t] \leq \sum_{t=0}^{\infty} p_t[r_t k_t + w_t n_t] + \pi_t$$
$$x_{t+1} = (1-\delta)x_t + i_t, \quad 0 \leq n_t \leq 1, \quad 0 \leq k_t \leq x_t$$
$$k_0 \quad \text{given}, \quad c_t \geq 0, \quad x_{t+1} \geq 0$$
$$\text{TVC:} \quad \lim_{t\to\infty} p_t k_{t+1} = 0$$

3. All markets clear for all t: $k_t^d = k_t^s$, $n_t^d = n_t^s$, $c_t + i_t = y_t$

Problem 2

We have that $F(k_t, l_t) = z k_t^{\alpha} l_t^{1-\alpha}$.

The first order conditions are:

$$\begin{aligned}
\beta^t c_t^{-\sigma} &= \lambda p_t \\
\beta^t \chi l_t^{\eta} &= \lambda p_t w_t \\
\lambda p_t &= \lambda p_{t+1}(1 - \delta + r_{t+1}) \\
z\alpha k_t^{\alpha-1} l_t^{1-\alpha} - r_t &= 0 \\
z(1-\alpha) k_t^{\alpha} l_t^{-\alpha} - w_t &= 0 \\
\text{Market Clearing:} \quad c_t + [k_{t+1} - (1-\delta)k_t] &= z k_t^{\alpha} l_t^{1-\alpha}
\end{aligned}$$

The first and third FOCs, as well as the assumption of steady state gives:

$$\bar{r} = \frac{1}{\beta} - 1 + \delta \tag{1}$$

The fourth FOC gives:

$$l_t^{1-\alpha} = \frac{r_t}{z\alpha k_t^{\alpha-1}} \tag{2}$$

Subbing equation 2 into the market clearing condition, as well as the assumption of steady state, gives:

$$\bar{c} = (\frac{\bar{r}}{\alpha} - \delta)\bar{k} \tag{3}$$

The first, second and fifth FOCs give:

$$c_t^{-\sigma} = \frac{\chi l_t^{\eta+1}}{z(1-\alpha)k_t^{\alpha} l_t^{1-\alpha}} \tag{4}$$

Subbing the second equation into the fourth gives:

$$\bar{c} = \chi^{\frac{-1}{\sigma}} [r^{\frac{\eta+\alpha}{1-\alpha}} (\frac{1}{z\alpha})^{\frac{1+\eta}{1-\alpha}} (\frac{\alpha}{1-\alpha})]^{\frac{-1}{\sigma}} k^{\frac{-\eta}{\sigma}} \tag{5}$$

Subbing equation 5 into the market clearing condition gives:

$$\bar{k} = [\chi^{\frac{-1}{\sigma}} D^{\frac{-1}{\sigma}} (\frac{\bar{r}}{\alpha} - \delta)^{-1}]^{\frac{\sigma}{\sigma+\eta}} \tag{6}$$

Where $D = r^{\frac{\eta+\alpha}{1-\alpha}} (\frac{1}{z\alpha})^{\frac{1+\eta}{1-\alpha}} (\frac{\alpha}{1-\alpha})$

Problem 3

The planners problem is:

$$\begin{gathered}
\max_{c_t, k_t, l_t} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\
\text{s.t.} \quad F(k_t, l_t) = c_t + k_{t+1} - (1-\delta)k_t \\
c_t \geq 0, \quad k_t \geq 0, \quad 0 \leq l_t \leq 1 \quad \forall t \\
k_0 \quad \text{given} \\
\text{TVC:} \quad \lim_{t\to\infty} \beta^t u'(F(k_t, l_t) - k_{t+1}) F_k(k_t, l_t) k_t = 0
\end{gathered}$$

The bellman equation is:

$$\begin{gathered}
v(k_t) = \max_{k_{t+1}, l_t} \frac{\left(z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta)k_t\right)^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} + \beta[v(k_{t+1})] \\
\text{s.t.} \quad 0 \leq l_t \leq 1
\end{gathered}$$

Problem 4

I use $\chi = 34$ which gives $\bar{l} = 0.4004$. (I did this by trial and error)

Problem 5

Part A: 8.18s, 215 iterations