# CSE 213 – Foundations of Computer Science II
## Sample Solutions for Selected Exercises on Binary Relations and Composition and Closures of Relations

1. **Exercise 4.1.1**

   (b) The congruence relation on the set of triangles is reflexive, transitive, and symmetric.

   (d) The subset relation on sets is reflexive, transitive, and antisymmetric.

   (f) The relation on people that relates people with bachelor's degrees in computer science is reflexive, transitive, and symmetric.

   (h) The "has a common national language" relation on countries is reflexive and symmetric.

   (j) The "is father of" relation on the set of people is irreflexive and antisymmetric.

2. **Exercise 4.1.2**

   (b) The relation $\{(a, b) \in \mathbf{R}^2 \mid a^2 + b^2 = 1\}$ is symmetric.

   (d) The relation $\{(a, b) \in \mathbf{R}^2 \mid a^2 = b^2\}$ is reflexive, symmetric, and transitive.

3. **Exercise 4.1.3**

   (b) The universal relation $A \times A$ is reflexive because $(x, x) \in A \times A$, for all $x \in A$. It is transitive and symmetric because $(x, y) \in A \times A$, for all $x \in A$ and $y \in A$. If $|A| = 1$, i.e., $A = \{x\}$ for some $x$, then $A \times A = \{(x, x)\}$ and the relation is also antisymmetric as it contains only one pair in which the first and second component are identical.

4. **Exercise 4.1.4**

   (b) The relation $\{(a, b), (b, a)\}$ is one possible example. Another example is $\{(a, c), (c, a)\}$. Note that the relation must be nonempty, for otherwise it would be transitive.

   (d) The relation $\{(a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b)\}$ is reflexive and symmetric but not transitive.

   (f) The empty relation is symmetric and transitive but not reflexive.

5. **Exercise 4.1.7**

   (b) Let $R$ be the binary relation $\{(a,a),(a,b),(b,c),(c,a)\}$, which is antisymmetric. The relation $R^2$ contains pairs $(a,b)$ and $(b,a)$ and hence is not antisymmetric.

6. **Exercise 4.1.8**

   (b) Let $R$ be the less-than relation on the natural numbers. Then $R^3 = \{(x,y) : (x+2,y) \in R\}$.

7. **Exercise 4.1.9**

   (b) The composition of the greater-than relation with the less-than relation on the natural numbers results in the relation $R = \{(m,n) : m \neq 0 \text{ and } n \neq 0\}$.

   (d) The composition of the greater-than relation with the not-equal relation on the natural numbers results in the relation $R = \{(m,n) : m \neq 0\} - \{(1,0)\}$.

8. **Exercise 4.1.10**

   Let $R$ be the relation $\{(x,y) \in \mathbf{Z} \times \mathbf{Z} : x+y \text{ is even}\}$. Note that a pair $(x,y)$ is in $R$ if, and only if, the integers $x$ and $y$ are either both even or else both odd. Consequently, if $(x,y) \in R$ and $(y,z) \in R$, then either $x$ and $z$ are both even (if $y$ is even) or else they are both odd (if $y$ is odd). Therefore $R^2 \subseteq R$. It can also be easily seen that $R \subseteq R^2$ and, hence, $R^2 = R$.

9. **Exercise 4.1.12**

   (b) The symmetric closure of the relation $R = \{(a,b),(b,a)\}$ is $R$ itself (as the relation is symmetric).

   (d) The symmetric closure of the relation $R = \{(a,a),(a,b),(c,b),(c,a)\}$ is $R \cup \{((b,a),(b,c),(a,c)\}$.

10. **Exercise 4.1.13**

    (b) The transitive closure of the relation $R = \{(a,b),(a,c),(b,c)\}$ is $R$ itself (as the relation is transitive).

    (d) The transitive closure of the relation $R = \{(a,b),(b,c),(c,d),(d,a)\}$ is $A \times A$, where $A = \{a,b,c,d\}$.

11. **Exercise 4.1.20**

    (b) We prove that if $R$ is symmetric, so is $R^2$.

Suppose $R$ is symmetric and $(x, y) \in R^2$. Then there exists an element $z$ such that $(x, z) \in R$ and $(z, y) \in R$. By the symmetry of $R$, we must have $(z, x) \in R$ and $(y, z) \in R$, which implies that $(y, x) \in R^2$. We thus have shown that whenever $(x, y) \in R^2$ then $(y, x) \in R^2$. In short, $R^2$ is symmetric.

12. **Exercise 4.1.24**

    (b) Let $R$ be a symmetric relation. We show that $r(R)$ and $t(R)$ are also symmetric.

    We will use the observations that (i) a binary relation $S$ is symmetric if and only if $X = X^c$, (ii) $(X \circ Y)^c = Y^c \circ X^c$, and (iii) $\bigcup_i X_i^c = (\bigcup_i X_i)^c$ (where $X^c$ denotes the converse of $X$).

    Since $R$ is symmetric and $E^c = E$, we have

    $$r(R) = R \cup E = R^c \cup E^c = (R \cup E)^c = r(R)^c$$

    and hence $r(R)$ is symmetric.

    We can use mathematical induction to show that $(R^k)^c = (R^c)^k$, for all $k \geq 2$. Since $R^c = R$ we then obtain

    $$t(R) = \bigcup_{k \geq 1} R^k = \bigcup_{k \geq 1} (R^c)^k = \bigcup_{k \geq 1} (R^k)^c = (\bigcup_{k \geq 1} R^k)^c = t(R)^c$$

    which establishes that $t(R)$ is transitive.

13. **Exercise 4.1.25**

    (b) Let $R$ be a binary relation over a set $A$. We prove that $rs(R) = sr(R)$. By the definition of reflexive and symmetric closures we have

    $$rs(R) = r(R \cup R^c) = (R \cup R^c) \cup E$$

    and

    $$sr(R) = s(R \cup E) = (R \cup E) \cup (R \cup E)^c$$

    where $E$ denotes the identity relation on $A$ and $S^c$ denotes the converse of a binary relation $S$. Using basic properties of sets (cf., list 1.4 on p. 19) we obtain

    $$\begin{aligned} & (R \cup E) \cup (R \cup E)^c \\ & \quad = \quad (R \cup E) \cup (R^c \cup E^c) \\ & \quad = \quad (R \cup E) \cup (R^c \cup E) \end{aligned}$$

$$
\begin{aligned}
&= R \cup (E \cup (R^c \cup E)) \\
&= R \cup ((E \cup R^c) \cup E) \\
&= R \cup ((R^c \cup E) \cup E) \\
&= R \cup (R^c \cup (E \cup E)) \\
&= (R \cup R^c) \cup (E \cup E) \\
&= (R \cup R^c) \cup E
\end{aligned}
$$

which completes the proof.