Pencil and Paper Exercises Week 2

1. Prove that a binary relation R is transitive iff $R \circ R \subseteq R$.
2. Give an example of a transitive binary relation R with the property that $R \circ R \neq R$.
3. Let R be a binary relation on a set A. Prove that $R \cup \{(x,x) \mid x \in A\}$ is the reflexive closure of R.
4. Prove that $R \cup R\check{}$ is the symmetric closure of R.
5. A partition P of a set A is a set of subsets of A with the following properties:
   (a) every member of P is non-empty.
   (b) every element of A belongs to some member of P.
   (c) different members of P are disjoint.

   If R is an equivalence relation on A and $a \in A$ then $|a|_R$, the R-class of a, is the set $\{b \in A \mid bRa\}$.

   Show that the set of R-classes

   $$\{|a|_R \mid a \in A\}$$

   of an equivalence relation R on A forms a partition of A.
6. Give the euclidean closure of the relation $\{(1,2),(2,3)\}$.
7. Show that for any relation R, the relation $R\check{} \circ (R \cup R\check{})^* \circ R$ is symmetric, transitive and euclidean.
8. Prove by induction that if R is an euclidean relation, then $R\check{} \circ (R \cup R\check{})^* \circ R \subseteq R$.
9. Prove that

   $$R \cup (R\check{} \circ (R \cup R\check{})^* \circ R)$$

   is the euclidean closure of R.

   Hint: look at the example proof in the course slides, and use the results of the previous exercises.
10. Prove that **lfp** $(\lambda S.S \cup (S \circ S))$ R is the transitive closure of R. (**lfp** f c is the least fixpoint of the operation f, starting from c.)
11. Prove that **lfp** $(\lambda S.S \cup (S\check{} \circ S))$ R is the euclidean closure of R.
12. Give an example of a formula ϕ and a Kripke model M, with the following properties: (i) $M \models \phi$, and (ii) $M \mid \phi \not\models \phi$. In other words, public announcement of ϕ has the effect that ϕ becomes false.