Ulm University
Faculty of Mathematics and Economics

An Euler-Poisson Scheme for Lévy driven Stochastic Differential Equations

Master's Thesis

in Finance

by
Odunayo O. Rotimi
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Advisors
Prof. Dr. Robert Stelzer
Prof. Dr. Mitja Stadje

Contents

1 Introduction

1.1 Preview

Stochastic differential equations play important roles in modeling quantitative phenomena. Their field of application includes but is not limited to: Physics [Feller, 13], Biometrics [Barndorff-Nielsen, Mikosch, and Resnick, 3], modeling contingencies in Actuarial Sciences as well as valuation of financial instruments and entities in the field of Finance [Black and Scholes, 5]. Particularly, the ever evolving nature of financial markets around the world is synchronized with an increase in its risk characteristics, which requires the use of sophisticated models that are robust enough to capture such dynamics; hence the prominence of Lévy driven stochastic differential equations in the discipline of Financial Mathematics; see for example, [Tankov, 38].

In this thesis, we are interested in the discrete approximation of the stochastic process $Y = (Y_t)_{t\in[0,T]}$; defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$; which is the strong solution to

$$Y_t = y_0 + \int_0^t a(Y_{s-})dX_s \qquad t \in [0,\, T], \tag{1.1}$$

with $T < \infty$, $y_0 \in \mathbb{R}^d$, $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $X = (X_t)_{t\in[0,T]}$ is a d-dimensional Lévy process (see Section 2.2 for more details on (1.1)). Since only a class of (1.1) admits closed form solutions, it is important to construct discrete time approximations. In our case, we consider a discrete time approximation of Y, the solution of (1.1), constructed on the time discretization with maximum step size $\delta \in (0, \delta_0)$, where $\delta_0 \in (0,\, 1)$. We adopt the following in accordance with [Tankov, 38] and [Jum, 22]: if the jump times of the driving process X are not included in the time discretization, then such a discretization is called *regular*; if on the other hand, the jump times are included in the time discretization, such discretization is termed *jump-adapted*. For this reason, the method which constructs a discrete-time approximation on a regular time discretization is called a *regular scheme*; whereas the one constructed on jump-adapted time discretization is *jump-adapted scheme*. Furthermore, a widely used measure of efficiency of a discretization scheme is its order of convergence, which is a measure of the rate at which a discrete approximation converges to a true one in a certain sense. The two commonly used modes of convergence in literature are the *strong* order and the *weak* order of convergence (for example, see, [Glasserman, 20]).

Definition 1.1. *A discrete time approximation Y^*_T constructed on a grid $0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq T$ with a maximum step size $\delta > 0$, converges with strong order p at time T to the solution Y of a given stochastic differential equation, if there exists a positive* *constant* $\mathbf{C}$*, independent of* δ*, and a finite number* $\delta_0 \in (0, 1)$*, such that*

$$\mathbb{E}[|Y_T - Y_T^*|^2] \leq \mathbf{C}\delta^{2p}, \tag{1.2}$$

for all $\delta \in (0, \delta_0)$*.*

Definition 1.2. *A discrete time approximation* Y_T^* *constructed on a grid* $0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq T$ *with maximum step size* $\delta > 0$*, converges with weak order* β *at time* T *to the solution* Y *of a given stochastic differential equation, if for a sufficiently smooth function* g*, there exists a positive constant* $\mathbf{C}$*, independent of* δ*, and a finite number* $\delta_0 \in (0, 1)$*, such that*

$$|\mathbb{E}[g(Y_T) - g(Y_T^*)]| \leq \mathbf{C}\delta^{\beta}, \tag{1.3}$$

for all $\delta \in (0, \delta_0)$*.*

The most frequently encountered numerical scheme for the approximation of (1.1), if $X = (X_t)_{t\in[0,T]}$ is a Wiener process, is the Euler scheme on an equally-spaced time grid $0 = t_0 < t_1 < \cdots < t_n = T$, of the interval $[0, T]$ which is given by

$$Y_{t_i}^* := Y_{t_{i-1}}^* + a(Y_{t_{i-1}}^*)(X_{t_i} - X_{t_{i-1}}) \qquad \hat{Y}_0^* = y_0 \qquad i \in [0, n], \tag{1.4}$$

where $n \in \mathbb{N}$ and $t_i := iT/n$. The increments $X_{t_i} - X_{t_{i-1}}$ of the Wiener process X are independent and identically distributed random variables that follow a normal distribution and, thus, these increments can be simulated by a closed form formula. By contrast, there are no closed formula for the simulation of Lévy processes in general, which makes it somewhat difficult to simulate the paths of Y using (1.4). This in turn has inspired novel ideas from various scholars in different fields where (1.1) constitutes a vital modeling tool.

1.2 Literature Review

The case where X is a Brownian motion has enjoyed rich scholarly work. We refer the reader to [Kloeden and Platen, 23] for a comprehensive treatment of numerical approximations of (1.1) of the mentioned case. The literature on weak numerical approximation of (1.1) is scarce, and even less extensive when it comes to strong numerical approximations.

The standard work on discrete-time approximation of (1.1) goes back to [Protter and Talay, 32], in which various conditions under which the weak convergence rates of (1.4) realistic were considered. There, it is required that the function g (cf. Definition 1.2) satisfies the regularity condition $g \in \mathcal{C}^4(\mathbb{R})$ with additional impositions on the first moment of X in order to show the order of convergence of (1.4) is $O(n^{-\frac{1}{2}})$, provided that the increments of the driving Lévy process are available. Similarly, in [Jacod et al., 21] the increments of X were approximated by independent and identically distributed random variables and the associated weak order of convergence was shown. The strong error (1.2); on the other hand, according to [Ferreiro-Castilla, Kyprianou, and Scheichl, 14] can be inferred from [Dereich and Heidenreich, 11] to be of the order $O(n^{-1})$ under the assumptions of finite second moments of X.

Since the simulation of Lévy processes is not generally straightforward and an extra source of error has to be incorporated into the convergence rates due to the approximation, the aforementioned convergence rates are therefore theoretical.

A more frequently encountered approach which relies on the Lévy-Itô decomposition is to approximate X by a jump-diffusion process, that is, a Lévy process that can be expressed as the sum of a linear Brownian motion plus an independent compound Poisson process. This approximation entails the truncation of the Lévy measure, removing all small jumps below a certain threshold in magnitude and compensating for their removal by making appropriate adjustment to the linear and/or Gaussian component. The truncation of small jumps ensures that the remaining jumps conform to a compound Poisson structure. Hence one is left with the task of simulating the paths of a linear Brownian motion interlaced with jumps that are distributed according to a normalized truncated Lévy measure arriving at an appropriate Poissonian rate (cf. Section 2, [Ferreiro-Castilla et al., 17]).

For instance, [Rubenthaler, 34] approximated X by a suitable compound Poisson process using the jump time of the compound Poisson process as discretization points by thus obtaining a weak numerical approximation of (1.1). This scheme however performs poorly when the driving Lévy measure has strong singularity at the origin, i.e., when the jump component has path of infinite p-variation, with p close to 2. Furthermore, [Dereich and Heidenreich, 11] take the approach of truncating small jumps in their design of Multilevel Monte Carlo simulation for the Lévy driven stochastic differential equation (1.1). There, it is observed that when the jump components of the driving process X is of finite variation, one may reasonably replace the jumps by a linear trend. On the other hand, if the jump component of X is of infinite variation, an appropriate approximation is to replace small jumps by a Gaussian process. The shortcomings of this method are discussed in [Asmussen and Rosiński, 2].

Finally, the most recent method of simulation which is attracting increasing attention is the concept of Multilevel Monte Carlo simulation introduced by [Giles, 19] and applied to a jump-diffusion model in [Xia and Giles, 39]. The Multilevel Monte Carlo simulation has witnessed a suitable application to the Wiener-Hopf factorization for Lévy processes, see, for example, [Kuznetsov et al., 27]. The Wiener-Hopf factorization for Lévy processes entails the decomposition of the paths of a Lévy process in terms of the running infimum and running supremum. In [Ferreiro-Castilla et al., 17] this factorization is used to sample from the bivariate distribution of $(X_t, \sup_{s<t} X_s)$ by constructing a random walk approximation, with the choice of the time steps according to an exponential distribution, i.e., the arrival time of a Poisson process.

1.3 Aims and Objectives

The earlier mentioned Wiener-Hopf Multilevel Monte Carlo simulation performed by [Ferreiro-Castilla et al., 17] effectively constructs a numerical path of X; based on this exposition, this thesis aims to investigate the performance of this numerical solution when applied in order to obtain numerical approximation of (1.1). Albeit the scheme constructs a random walk approximation of paths that captures both the the end point and supremum over each exponentially distributed time step; here, we shall only consider an Euler scheme for the solution Y_T of (1.1) at the end point T only. In this case, our proposed scheme can be thought of as a random modification of the Euler scheme, where we assume that we can sample exactly from the distribution of $X_{\xi(n/T)}$. $\xi(n/T)$ are the exponentially distributed time steps, with mean n/T, independent of X. More precisely, the grid points in our Euler-Poisson scheme are dictated by a Poisson point process with rate n/T denoted by $\mathcal{N}(n/T)$, where the mean T/n plays the role of the grid size. The analysis in this thesis does not assume any specific way of obtaining the distribution of $X_{\xi(n/T)}$ and there is no reason why it should demand a lesser degree of technicality than would X_1 for general Lévy processes.

That notwithstanding, the meromorphic class is a large class of of Lévy processes that have enjoyed contributions from the likes of [Kuznetsov, Kyprianou, and Pardo, 26]. This class provides one with processes whose Weiner-Hopf factors are explicit, hence there is the possibility of sampling from the distribution of $X_{\xi(n/T)}$. Additionally, several popularly used Lévy processes in finance can be approximated by the Meromorphic class of Lévy processes (cf. [Corcuera et al., 10]) (see Section 2.4 for examples). Hence, the proposed scheme can be taken as an alternative to dealing with stochastic differential equations driven by such financial models, whilst preserving the stylized properties for the driving process.

In contrast to the more classical methods mentioned thus far, it is worth mentioning that the advantage of our numerical approximation is that approximation given by our scheme does not depend on the jump structure of X. The main result of this thesis derives the rate of convergence for the mean squared error for the approximation $\tilde{Y}_n$ of Y_T obtained through the Euler-Poisson scheme, showing that $\mathbb{E}[|Y_T - \tilde{Y}_n|^2] = O(n^{-\frac{1}{2}})$.

It shall also be shown that our algorithm tracks closely the classical discretization scheme for the partial integro-differential equation associated with computing $\mathbb{E}[g(Y_T)]$ for a given function g.

The rest of the thesis is structured as follow. In Chapter two, we give a general overview of Lévy processes, introducing the tools needed to perform the numerical analysis of (1.1) and set up notation and terminologies. Furthermore, we introduce the notion of meromorphic Lévy processes with applicable examples, and finally outlay the our Euler-Poisson discretization scheme. Since we are interested in numerical approximation of a given Lévy driven stochastic differential equation, Chapter three is devoted to the numerical analysis of our scheme where the convergence rate in the mean square error is derived. Chapter four is intended to motivate the investigation of the feasibility of our scheme, we thus collect several remarks and observations regarding feasibility, and extensions and its relation with partial integro-differential equations; the simulation result is also presented in the same chapter. Finally, the Appendix collects some standard inequalities as well as the simulation code.

2 The Euler-Poisson scheme

2.1 Preliminaries and Notations

Throughout this thesis, we will always assume given a *complete* probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$, i.e., the σ-algebra $\mathcal{F}$ contains additionally all subsets of nullsets. We also assume a filtration $(\mathcal{F}_t)_{t\geq 0}$, which is a family of σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$. Our filtration is assumed to satisfy the usual hypothesis, i.e., $\mathcal{F}_0$ contains all $\mathcal{P}$-nullsets of $\mathcal{F}$, and the filtration is right continuous. By $\mathcal{G}_t = \mathcal{F}_t \bigvee \mathcal{H}$ we mean $\mathcal{G}_t$ is a filtration generated by the union of $\mathcal{F}_t$ and $\mathcal{H}$.
Furthermore, we will denote by $\mathbb{R}$ the set of all real numbers, and $\mathbb{R}_+$ the set of all positive real numbers. Equality in distribution is denoted by $\stackrel{d}{=}$. We will employ the notation $|\cdot|$, as used by [Ferreiro-Castilla, Kyprianou, and Scheichl, 14], to indistinctly denote the Euclidean norm for vectors or the Frobenius norm of matrices. For $x, y \in \mathbb{R}$, $(x \wedge y) := \min\{x,\, y\}$ and $(x \vee y) := \max\{x,\, y\}$. $\mathcal{C}^n$ is the Space of n times continuously differentiable functions and $\mathcal{C}^{1,n}$ is the Space of functions which are continuously differentiable with respect to the first variable, and n times continuously differentiable with respect to the second variable. L^p denotes the Space of measurable functions with a finite p-th norm. Finally, $\langle \cdot, \cdot \rangle$ denotes inner product.

2.2 Overview of Lévy Processes

In this section we give the definition of a Lévy process and closely related distributions useful for the study of its characteristics as well as performing numerical analysis. We refer the reader to [Sato and Ken-Iti, 35] for detailed treatment of general Lévy Processes and to [Tankov, 38] for their applications in financial modelling.

Definition 2.1 (Lévy process)**.** *A d-dimensional adapted stochastic process $X = (X_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$, is a called a Lévy process if the following conditions are satisfied.*

- $X_0 = 0$ *a.s.,*
- *X is a.s. càdlàg (i.e. right-continuous with left limits),*
- *X has independent increments, i.e., for all $n \in \mathbb{N}$ and all sequences $0 = t_0 \leq t_1 \leq \cdots \leq t_n < \infty$, $X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent,*
- *X has stationary increments, i.e., for all $s, t \in [0,\, T]$, $s < t$, $X_t - X_s \stackrel{d}{=} X_{t-s}$.*

Given the properties listed above, we consider an $\mathbb{R}^d$-valued adapted stochastic process $Y = (Y_t)_{t\in[0,T]}$ defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{P})$, which is a strong solution to

$$Y_t = y_0 + \int_0^t a(Y_{s-})dX_s \qquad t \in [0,\, T], \tag{2.1}$$

where $T < \infty$, $y_0 \in \mathbb{R}^d$ is the deterministic initial value; $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a coefficient function, on which we impose the standard Lipschitz assumption to ensure the existence of a unique strong solution. The smoothness of a is specified in the sequel. $X = (X_t)_{t\in[0,T]}$ is a d-dimensional Lévy process. We write Y_{t-} instead of Y_t in order that the integrand be predictable, and that it is well-defined as an Itô integral with

$$Y_{t-} = \lim_{s\uparrow t} Y_s.$$

A very crucial tool to the analysis of distributions of Lévy Process is the so called characteristic function, i.e. the Fourier transform of a probability distribution.

Definition 2.2 (Characteristic Function)**.** *The characteristic function $\tilde{\Phi}(\vartheta)$ of a probability measure on $\mathbb{R}^d$ is*

$$\tilde{\Phi}(\vartheta) = \int_{\mathbb{R}^d} e^{i\langle\vartheta,x\rangle}\Phi(dx), \qquad \vartheta \in \mathbb{R}^d. \tag{2.2}$$

Similarly, the characteristic function of the distribution $\mathcal{P}_X$ of a random variable X on $\mathbb{R}^d$ denoted by $\tilde{\mathcal{P}}_X(\vartheta) : \mathbb{R}^d \to \mathbb{R}$ is given as

$$\tilde{\mathcal{P}}_X(\vartheta) = \int_{\mathbb{R}^d} e^{i\langle\vartheta,x\rangle}\mathcal{P}_X(dx) = \mathbb{E}[e^{i\langle\vartheta,X\rangle}]. \tag{2.3}$$

Furthermore, another very important notion for the study of Lévy processes is that of infinite divisibility of a distribution.

Definition 2.3. *Let $\tilde{\mathcal{P}}^n$ denote the n-fold convolution of a probability measure $\tilde{\mathcal{P}}$ with itself. A probability measure $\tilde{\mathcal{P}}$ on $\mathbb{R}^d$ is infinitely divisible if for all $n \in \mathbb{N}$, there is a probability measure $\tilde{\mathcal{P}}_n$ on $\mathbb{R}^d$ such that,*

$$\tilde{\mathcal{P}}_n^n := \underbrace{\tilde{\mathcal{P}}_n * \ldots * \tilde{\mathcal{P}}_n}_{n\ times} = \tilde{\mathcal{P}}.$$

The following is the famous *Lévy-Khintchine formula* which gives a representation of the characteristic function of all infinitely divisible distributions.

Theorem 2.4 (Theorem 8.2 p. 38, [Sato and Ken-Iti, 35])**.**

(i) if $\mathcal{P}$ is an infinitely divisible distribution on $\mathbb{R}^d$, then

$$\tilde{\mathcal{P}}(\vartheta) = \exp\left[-\frac{1}{2}\langle\vartheta, \Sigma\vartheta\rangle + i\langle a, \vartheta\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle\vartheta,x\rangle} - 1 - i\langle\vartheta, x\rangle\mathbb{1}_{\{||x||\leq 1\}}\right)v(dx)\right], \tag{2.4}$$

where $a \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ is a symmetric and non-negative definite matrix and v is a measure on $\mathbb{R}^d$ satisfying

$$v(\{0\}) = 0 \qquad and \qquad \int_{\mathbb{R}^d}(1 \wedge |x|^2)v(dx) < \infty. \tag{2.5}$$

(ii) The representation of $\tilde{\mathcal{P}}$ in (2.4) by a, Σ and v is unique.

(iii) Conversely, if $a \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ is symmetric and non-negative definite matrix, v is a measure on $\mathbb{R}^d$ satisfying (2.5), then there is an infinitely divisible distribution $\mathcal{P}$ whose characteristic function is given by (2.4).

Definition 2.5. *The triplet (a, Σ, v) in Theorem 2.4 is called generating triplet. Moreover, if $\Sigma = 0$, $\mathcal{P}$ is called a pure jump Lévy process.*

The following gives an explicit representation of the characteristic function of the law of a Lévy process.

Proposition 2.6 (Characteristic function of a Lévy process)**.** *For any Lévy process $(Y_t)_{t\in\mathbb{R}_+} in \mathbb{R}^d$, there exist a unique Lévy triplet (a, Σ, v) such that $\forall t > 0$,*

$$\mathbb{E}[e^{i\langle z, Y_t\rangle}] = e^{t\Psi(\vartheta)}, \qquad \forall\, \vartheta \in \mathbb{R}^d,$$

where

$$\Psi(\vartheta) = -\frac{1}{2}\langle \vartheta, \Sigma\vartheta\rangle + i\langle a, \vartheta\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle \vartheta, x\rangle} - 1 - i\langle \vartheta, x\rangle \mathbb{1}_{\{|x|\le 1\}}\Big) v(dx). \tag{2.6}$$

For the Euler-Poison Scheme, we assume that the marginals of the process are in $L^2(\Omega, \mathcal{F}, \mathcal{P})$ hence omit the truncation function, leading to the so called characteristic exponent of the Lévy process expressed as

$$\mathbb{E}[e^{i\langle \vartheta, X_t\rangle}] = e^{t\Psi(\vartheta)}, \qquad \forall\, \vartheta \in \mathbb{R}^d,$$

where

$$\Psi(\vartheta) = -\frac{1}{2}\langle \vartheta, \Sigma\Sigma^T\vartheta\rangle + i\langle b, \vartheta\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle \vartheta, x\rangle} - 1 - i\langle \vartheta, x\rangle\Big)\, v(dx), \tag{2.7}$$

and $b \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ and v is a measure concentrated on $\mathbb{R}^d \setminus \{0\}$ and $\int_{\mathbb{R}^d}(1 \wedge |x|^2)v(dx) < \infty$.

2.3 Lévy-Itô Decomposition

Here we state the classical Lévy-Itô decomposition, which describes the structure of the sample path of a Lévy process. It expresses the sample path of a Lévy process as a sum of four independent parts-the drift, the Guassian part, the small jump part and the large jump part. In order to state this, we need the notion of a Poisson random measure (PRM).

Definition 2.7. *Let $G \subset \mathbb{R}^d$. A Radon measure on the space $(G, \mathcal{G})$ is a measure μ such that for every compact measurable set $B \in \mathcal{G}$, $\mu(B) < \infty$.*

Definition 2.8. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, $A \subset \mathbb{R}^d$ and μ a given Radon measure on $(G, \mathcal{G})$. A Poisson random measure on G with intensity μ is an integer valued random measure $N : \Omega \times \mathcal{G} \to \mathbb{N}$, $(\omega, B) \mapsto N(\omega, B)$, such that* *(i) For any bounded measurable set $B \subset G$, $N(B) < \infty$ is an integer valued random variable.*

(ii) For each measurable set $B \subset G$, $N(\omega, B) = N(B)$ is a PRM with parameter $\mu(B)$,

$$\mathcal{P}(N(B) = \kappa) = e^{-\mu(B)} \frac{(\mu(B))^\kappa}{\kappa!}, \qquad \forall \kappa \in \mathbb{N}. \tag{2.8}$$

(iii) For disjoint measurable sets $B_1, \ldots, B_n \in \mathcal{G}$, the variables $N(B_1), \ldots, N(B_n)$ are independent.

Moreover, $\hat{N}(B) = N(B) - \mu(B)$ is called the compensated PRM.

Remark 2.9. *N can be represented through Dirac masses located at random points $(V_i)_{i \geq 1}$, $V_i : \Omega \to A$ given by*

$$N(B) = \sum_i \delta_{V_i}(B),$$

provided $B \cap (V_i)_{i \geq 1} < \infty$ for any compact set $B \subset G$

The following allows one to construct a PRM from a given Radon measure.

Proposition 2.10. *For any Radon measure μ on $G \subset \mathbb{R}^d$, there exists a PRM N on G with intensity μ.*

Proof. See (Section 2.6, [Tankov, 38]). □

The process stated below corresponds to the PRM and its compensated counterpart

Proposition 2.11. *Let N be an adapted PRM on $G = [0, T] \times \mathbb{R}^d \setminus \{0\}$ with intensity μ with compensated PRM $\hat{N} = N - \mu$, and $f : G \to \mathbb{R}^d$ such that $\mu(f) < \infty$. Then the process*

$$\begin{aligned} \tilde{X}_t &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y) \hat{N}(dsdy) \\ &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y) N(dsdy) - \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} f(s, y), \end{aligned} \tag{2.9}$$

is a martingale.

Definition 2.12 (Jump Measure)**.** *Let V be an $\mathbb{R}^d$−valued càdlàg process. The jump measure of V is the random measure on $\mathcal{B}((0, \infty) \times \mathbb{R}^d)$ defined by*

$$N_V(A) = \#\{t \,;\, \Delta V_t \neq 0 \, and \, (t, \Delta V_t) \in A\}. \tag{2.10}$$

Thus the jump measure counts the number of jumps of V of a particular size that falls into the set A. The Lévy measure builds on this and is given as follows.

Theorem 2.13. *Let V be $\mathbb{R}^d-$valued Lévy process. The measure v given by*

$$v(A) = \mathbb{E}[\#\{t \in [0,1];\, \Delta V_t \neq 0 \, and \, \Delta V_t \in A\}], \qquad A \in \mathcal{B}(\mathbb{R}^d), \tag{2.11}$$

is called a Lévy measure.

Interpretation 2.14. *The Lévy measure of a Borel set is equal to the expected number of jumps in a time interval* $[0,\, 1]$ *with jump sizes in the Borel set. The jumps are described by the PRM.*

The Lévy-Itô decomposition is stated in the following proposition.

Proposition 2.15 (Theorem 1, p.11, Lévy-Itô decomposition [Tankov, 38])**.** *Let $X = (X_t)_{t\geq 0}$ be an $\mathbb{R}^d-$valued Lévy process, with Lévy measure v. Then*

1. *The jump measure N_X (cf. (2.10)) of X is a Poison random measure on* $[0\,\infty) \times \mathbb{R}^d$ *with intensity $dt \times v$.*
2. *The Lévy measure v satisfies $\int_{\mathbb{R}^d}(1 \wedge |x|^2)v(dx) < \infty$.*
3. *There exist $\gamma \in \mathbb{R}^d$ a d-dimensional Brownian Motion W and covariance matrix Σ such that*

$$X_t = \gamma t + \Sigma W_t + X_t^{(1)} + X_t^{(2)} \tag{2.12}$$

where

$$X_t^{(1)} = \int_0^t \int_{|x|\geq 1} x N_X(ds \times dx) \tag{2.13}$$

$$X_t^{(2)} = \int_0^t \int_{|x|\in[\varepsilon\,,1)} x(N_X(ds \times dx) - v(dx)ds) \tag{2.14}$$

$$=: \int_0^t \int_{|x|\in[\varepsilon\,,1)} x\tilde{N}_X(ds \times dx), \tag{2.15}$$

for $\varepsilon \in \mathbb{R}_+$.

The three terms are independent and the convergence in the last term is almost sure and uniform in t on compacts. The triple $(\gamma,\, v,\, \Sigma)$ is called the characteristic triple of X.

Proof. See [Tankov, 38]. □

Remark 2.16. *The mean value of X_1 exist if and only if (cf. Sato, 1999 p.39)*

$$\int_{|x|>1} |x| v(dx) < \infty,$$

in which case we can rewrite setting $\mathbb{E}[X_t] =: bt$ the representation (2.12)-(2.15) as

$$X_t = bt + \Sigma W_t + \int_0^t \int_{\mathbb{R}^d} x\tilde{N}_X(ds \times dx), \tag{2.16}$$

where the integral on the right-hand side of (2.16) is the compensated jump of X

Interpretation 2.17. *The jumps of X are contained in the discontinuous processes $X_t^{(1)}$ and $X_t^{(2)}$. While the sum*

$$X_t^{(1)} = \sum_{s\in[0,t]} \Delta X_s \mathbb{1}_{\{|\Delta X_s|\geq 1\}}$$

contains almost surely finite number of terms and is a well-defined compound Poisson process, the compensated jump integral $X_t^{(2)}$ is centered with $\varepsilon \to 0$ to avoid divergence since the jump measure v can have singularity at 0. Moreover, $X_t^{(2)}$ is a martingale by definition. The implication of this is that, every Lévy process can be approximated with arbitrary precision by the sum of a Brownian motion with drift and a compound Poisson process. Additionally from (2.12), if we set

$$A_t = bt + X_t^{(1)}$$

and

$$M_t = \Sigma W_t + X_t^{(2)},$$

it follows that every Lévy process is a semimartingale. This is because M is by definition a martingale while the condition $\int_{|x|>1} v(dx) < \infty$ verifies that X has a finite number of jumps with absolute value larger than 1 thus A is of finite variation (see [Barndorff-Nielsen and Shephard, 4] and the references provided therein).

Denoting by $W = (W_t)_{t\in[0,T]}$ a d-dimensional Wiener process independent of L, where $L = (L_t)_{t\in[0,T]}$ is the compensated jump process expressed in (2.16) which is an L^2-martingale, see for example [Applebaum, 1]. From Proposition 2.15, the Lévy-Ito decomposition guarantees that every L^2-Lévy process has the representation

$$X_t = \Sigma W_t + L_t + bt. \tag{2.17}$$

Without loss of generality, we impose the following conditions on our variables for tractability of our numerical analysis.

Assumption 1. *There exist a constant $k \in \mathbb{R}_+$ such that*

(i) $\int_{\mathbb{R}^d} (1 \wedge |x|^2) v(dx) \leq k^2$,

(ii) $|\Sigma| \leq k$,

(iii) $|b| \leq k$,

(iv) $|y_0| \leq k$.

The following theorem, whose proof can be found in [Rong, 33] sets up the usual condition that guarantees the existence of a unique strong solution to (2.1).

Theorem 2.18 (Section 3.1, [Rong, 33])**.** *Consider the SDE driven by a square-integrable Lévy process given in (2.1). Let* $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ *be a measurable function such that*

$$|a(x) - a(x')| \leq k'|x - x'| \qquad \textit{and} \qquad |a(y_0)| \leq k',$$

for x, $x' \in \mathbb{R}^d$ *and* $k' \in \mathbb{R}_+$*. Then, equation (2.1) has a unique strong solution adapted to the filtration generated by* X *denoted by* $\mathcal{F}^X := \sigma(X_t;\, t \geq 0)$*, such that*

$$\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^2] \leq \mathbf{C}_1,$$

where $\mathbf{C}_1$ *is a positive constant depending on* k' *and* T *only.*

Remark 2.19. *Subsequently, for* $a, b \in \mathbb{R}_+$ *we denote by* $a \lesssim b$ *the fact that* a/b *is uniformly bounded. We shall also denote constants depending on* k *and* T *by* $(\mathfrak{c}_i)_{i\geq 0}$ *and* $(\mathbf{C}_i)_{i\geq 0}$*. Without loss of generality, we let* $k' = k$ *in Theorem 2.18.*

2.3.1 Examples of Lévy processes

Example 2.20 (Brownian motion)**.** *A stochastic process* $(W_t)_{t\geq 0}$ *in* $\mathbb{R}$ *is called a Brownian Motion with variance* Σ*, if* B_1 *is normally distributed with mean 0 and sample paths of* $(W_t)_{t\geq 0}$ *are almost surely continuous. This the only Lévy process with continuous sample paths.*

Example 2.21 (The Poisson process)**.** *The Poisson process with intensity* $\lambda > 0$*, is a Lévy process* $(N_t)_{t\geq 0}$ *taking values in* $\mathbb{N} \cup \{0\}$*, where each* N_t *follows a Poisson distribution with parameter* λt*.*

Example 2.22 (The compound Poisson process)**.** *Let* $Y = (Y_n)_{n\in\mathbb{N}}$ *be a sequence of i.i.d. random variables in* $\mathbb{R}^d$ *with common distribution* μ_Y*. Let* N_t *be the given Poisson process in Example 2.21 independent of* Y_t *for all* $t \geq 0$*. The process* Z *defined as*

$$Z_t = \sum_{k=1}^{N_t} Y_k, \qquad \textit{for all} \quad t \geq 0,$$

is said to be a compound Poisson process.

Example 2.23 (Jump-diffusion process)**.** *Let* W *be the Brownian Motion in Example 2.20 and* Z *the compound Poisson process in Example 2.22 then the process* $(J_t)_{t\geq 0}$ *given as*

$$J_t = W_t + Z_t, \tag{2.18}$$

is a Lévy process called a jump-diffusion process.

Example 2.24 (Tempered stable family)**.** *A Lévy Process* Z *with the characteristic triplet* $(b,\, 0,\, v)$ *is said to belong to the Tempered stable family if its jump density is given by*

$$v(x) = c_+ \frac{e^{-\alpha_+ x}}{x^{\lambda_+}} \mathbb{1}_{\{x>0\}} + c_- \frac{e^{\alpha_- x}}{|x|^{\lambda_-}} \mathbb{1}_{\{x<0\}}, \tag{2.19}$$

where $c_\pm > 0$, $\lambda_\pm > 0$, $\alpha_\pm \in (0,2)$. With appropriate choice of parameters, one obtains the Normal Inverse Gaussian (NIG) process, the CGMY process proposed by [Carr et al., 7] and the KoBoL process introduced by [Koponen, 24], see Example 2.29.

2.4 Meromorphic Lévy processes

The name Meromorphic derives from the fact that the characteristic exponent Ψ in (2.7) can be extended to a meromorphic function (which may be understood as the ratio of two holomorphic functions) in $\mathbb{C}$. This class of meromorphic processes is rich, since the paths of bounded and unbounded variation as well as finite and infinite activity jumps can be generated (cf.[Ferreiro-Castilla and Van Schaik, 16], Section 5.1).

Definition 2.25. *A function $h : \mathbb{R}_+ \to \mathbb{R}$ is said to be* completely monotone *if $h \in \mathcal{C}^\infty$ and $(-1)^n h^{(n)}(x) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, Bernstein's theorem states that h is completely monotone if and only if it can be represented as Laplace transform of a positive measure μ on $[0, \infty)$:*

$$h(x) = \int_0^\infty e^{-xz} \mu(dz), \qquad x > 0. \tag{2.20}$$

If $h : \mathbb{R}_+ \to \mathbb{R}$ in (2.20) is discrete, it said to be a discrete completely monotone *function and can be represented as*

$$h(x) = \sum_{n \geq 1} a_n e^{-b_n x}, \qquad x > 0.$$

where $a_n > 0$, $b_n \geq 0$, $\lim_{n \to \infty} b_n = \infty$ and the sequence $(b_n)_{n \geq 1}$, assumed to be strictly increasing.

Remark 2.26.

(i) If h is completely monotone, *whose class we denote $\mathcal{CM}$; and $h(0^+) = 1$, then $\mu(dz)$ is a probability measure and $1 - h(x)$ is the cumulative distribution function of a positive infinitely divisible random variable X, whose distribution is mixture of exponential distributions (cf. [Kuznetsov, Kyprianou, and Pardo, 26] Section 2).*

(ii) The class of discrete completely monotone functions is denoted by $\mathcal{DCM}$.

Definition 2.27. *A Lévy process X is said to belong to the meromorphic class if its Lévy measure v decomposes as*

$$v^+(x) = v((x, \infty)) \quad and \quad v^- = v((-x, -\infty)), \qquad for\, all \quad x > 0,$$

and $v^+, v^- \in \mathcal{DCM}$.

2.4.1 Examples of meromorphic Lévy processes

Example 2.28 (The compound Poisson process). *See Example 2.22.*

Example 2.29 (β-family of Lévy processes). *A Lévy Process $Z = (Z_t)_{t\geq 0}$ in $\mathbb{R}^d$ with the generating triplet (b, Σ, v) is said to belong to the β-family if its jump density is given by*

$$v(x) = c_1 \frac{e^{-\alpha_1 \beta_1 x}}{(1-e^{-\beta_1 x})^{\lambda_1}} \mathbb{1}_{\{x>0\}} + c_2 \frac{e^{\alpha_2 \beta_2 x}}{(1-e^{\beta_2 x})^{\lambda_2}} \mathbb{1}_{\{x<0\}}, \tag{2.21}$$

where $\alpha_i, \beta_i > 0$, $c_i \geq 0$, $\lambda_i \in (0, 3)$, $i = 1, 2$; with Lévy-Khintchine representation given as

$$\Phi(\vartheta) = \frac{\Sigma^2}{2}\vartheta + i\rho\vartheta - \frac{c_1}{\beta_1}\mathcal{K}\left(\alpha_1 - \frac{i\vartheta}{\beta_1}, 1-\lambda_1\right) - \frac{c_2}{\beta_2}\mathcal{K}\left(\alpha_2 - \frac{i\vartheta}{\beta_2}, 1-\lambda_2\right) + \varrho,$$

where

$$\begin{aligned}
\varrho &= \frac{c_1}{\beta_1}\mathcal{K}(\alpha_1, 1-\lambda_1) + \frac{c_2}{\beta_2}\mathcal{K}(\alpha_2, 1-\lambda_2), \\
\rho &= \frac{c_1}{\beta_2^2}\mathcal{K}(\alpha_1, 1-\lambda_1)(\mathcal{K}'(1+\alpha_1-\lambda_1) - \mathcal{K}'(\alpha_1)) \\
&\quad - \frac{c_2}{\beta_2^2}\mathcal{K}(\alpha_2, 1-\lambda_2)(\mathcal{K}'(1+\alpha_2-\lambda_2) - \mathcal{K}'(\alpha_2)) - b,
\end{aligned}$$

with $b, \Sigma \in \mathbb{R}$, $\mathcal{K}(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\mathcal{K}'(x) = \frac{d}{dx}\log\Gamma(x)$. $\Gamma(\cdot)$ is the conventional Gamma function.
If in (2.21) we let $\beta \to 0^+$ and let $c_1 = c_+\beta^{\lambda_+}$, $c_- = c_-\beta^{\lambda_-}$, $\alpha_1 = \alpha_+\beta^{-1}$, $\alpha_2 = \alpha_-\beta^{-1}$, $\beta_1 = \beta_2 = \beta$; one obtains the generalized Tempered stable family. Particularly,

(i) if $\lambda_1 = \lambda_2$, the resulting process is called KoBoL process.

(ii) If $c_1 = c_2$, $\lambda_1 = \lambda_2$ and $\beta_1 = \beta_2$, the jump density converges to that of CGMY processes.

(iii) If $c_1 = c_2 = 4$, $\beta_1 = \beta_2 = 1/2$, $\lambda_1 = \lambda_2 = 2$, $\alpha_1 = 1-\alpha$, $\alpha_2 = 1+\alpha$, one obtains an analogue of the NIG process.

Remark 2.30. *For a detailed treatment and simulations of the given examples, see [Ferreiro-Castilla and Van Schaik, 16], [Kuznetsov, Kyprianou, and Pardo, 26] and [Kuznetsov et al., 27].*

2.5 Arrival Time of the Poisson Process

The Poisson process is an example of a stochastic process with discontinuous paths. It has been used by a wide variety of authors such as, [Tankov, 38, Sato and Ken-Iti, 35, Protter, 31], as a basis for building more complex jump processes. Furthermore, the major task of the numerical analysis in the proposed scheme relies on the interplay between the arrival and interarrival time of the Poisson process.

Definition 2.31 (Exponential distribution)**.** *A continuous random variable ξ is said to have an exponential distribution with parameter $\lambda > 0$, written as $\xi \sim Exponential(\lambda)$, if its probability density function is given by*

$$f_\xi(x) = \lambda e^{\lambda x} \mathbb{1}_{x \geq 0}. \tag{2.22}$$

Moreover, the distribution function of X is given by

$$F_\xi(x) = \mathcal{P}(\xi \leq x) = 1 - e^{-\lambda x}, \qquad \forall x \in [0, \infty), \tag{2.23}$$

with the n-th moment of ξ for any $n \in \mathbb{N}$ being

$$\mathbb{E}[\xi^n] = \frac{n!}{\lambda^n}.$$

Definition 2.32 (Poisson Process)**.** *A counting process $(\mathcal{N}_t)_{t \geq 0}$ is said to be a Poisson process with rate $\lambda > 0$ if the following conditions are satisfied*

(i) $\mathcal{N}_0 = 0$ *a.s.*

(ii) $\mathcal{N}_t$ *has independent increments, that is, for any* $t_1 < \cdots < t_n < \infty$, $\mathcal{N}_{t_2} - \mathcal{N}_{t_1}, \ldots, \mathcal{N}_{t_n} - \mathcal{N}_{t_{n-1}}$ *are independent.*

(iii) The number of events in any time interval of length $t > 0$ is Poisson distributed with mean $\mathbb{E}[\mathcal{N}_t] = \lambda t$. That is,

$$\mathcal{P}(N_t = \kappa) = \frac{(\lambda t)^\kappa}{\kappa!} e^{-\lambda t} \mathbb{1}_{\kappa > 0}. \tag{2.24}$$

The next closely related distribution is the Gamma distribution which could be viewed as a generalization of an exponential distribution.

Definition 2.33 (Gamma distribution)**.** *A continuous random variable X is said to have a Gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\lambda > 0$, written as $X \sim \Gamma(\alpha, \lambda)$, if its probability density function is given by*

$$f_X(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \mathbb{1}_{x \geq 0}, \tag{2.25}$$

where $\Gamma(\cdot)$ is the Gamma function.

Remark 2.34.

(i) If $\alpha = 1$, we recover an exponentially distributed random variable, i.e., $\Gamma(1, \lambda) \stackrel{d}{=} Exponential(\lambda)$.

(ii) One could also show by induction that the sum $X = \sum_{i=1}^n \xi_i$ of independent random variables following an exponential distribution, that is, $\xi_i \sim Exponential(\lambda)$, is equal in distribution to a Gamma distribution, that is, $X \sim \Gamma(\alpha, \lambda)$, $\alpha, \lambda > 0$.

The relationship between the Exponential distribution and the Poisson process is captured in the following proposition.

Proposition 2.35 (Section 2.5, p.47 [Tankov, 38])**.** *If $(T_i)_{i\geq 1}$ are independent exponential random variables with rate $\lambda > 0$ then, for any $t > 0$, the random variable*

$$\mathcal{N}_t = \inf\{k \geq 1; \sum_{i=1}^{k} T_i > t\} \tag{2.26}$$

follows a Poisson distribution with rate λt, i.e.,

$$\mathcal{P}(\mathcal{N}_t = \kappa) = \frac{(\lambda t)^\kappa}{\kappa!} e^{-\lambda t} \mathbb{1}_{\kappa>0}. \tag{2.27}$$

Proof. Let $\mathcal{T}_n = \sum_{i=1}^{n} T_i$ for all n. The density of $(\mathcal{T}_1, \dots \mathcal{T}_n)$ is given by

$$\lambda^n \mathbb{1}_{0<t_1<\dots<t_n} e^{-\lambda t_n} dt_1 \dots dt_n.$$

Since $\mathcal{P}(\mathcal{N}_t = \kappa) = \mathcal{P}(t \in [\mathcal{T}_\kappa, \mathcal{T}_{\kappa+1}))$, it can be estimated as

$$\begin{aligned}
\mathcal{P}(\mathcal{N}_t = \kappa) &= \int_{0<t_1<\dots<t_\kappa<t<t_{\kappa+1}} \lambda^\kappa e^{-\lambda t_{\kappa+1}} dt_1 \dots dt_\kappa dt_{\kappa+1} \\
&= \lambda^\kappa e^{-\lambda t} \int_{0<t_1<\dots<t_\kappa<t} dt_1 \dots dt_\kappa \\
&= \frac{(\lambda t)^\kappa}{\kappa!} e^{-\lambda t}.
\end{aligned}$$

□

2.6 The discretization scheme

For $n \geq 1$, let $\xi(n/T) := (\xi_i(n/T))_{i\geq 1}$ be a sequence on independent identically distributed (i.i.d) random variables defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where each ξ_i is exponentially distributed and dependent on rate n/T. For this we write $\xi_i \sim \mathit{Exponential}(n/T)$, where $\mathbb{E}[\xi_i] = T/n$. We denote further by $\mathcal{F}^\xi := \sigma(\xi_i;\, i \geq 1)$ the σ-algebra generated by ξ, which is assumed to be independent of X, and set $\xi_0 = 0$.

We recall that the Euler approximation of a stochastic differential equations is given on a time grid $0 = t_0 < t_1 < \ldots < t_n$ by

$$Y^*_{t_i} := Y^*_{t_{i-1}} + a(Y^*_{t_{i-1}})\Delta W_{t_{i-1}}, \qquad Y^*_0 = y_0, \qquad i = 1, 2, \dots n-1,$$

where $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a measurable function, $\Delta W_{t_{i-1}} := W_{t_i} - W_{t_{i-1}}$ is the increments of the Wiener process W and $\Delta W_{t_i} \sim N(0,\, \Delta_{t_i})$, i.e., the increment follows an independent Gaussian distribution; whereas, the Euler-Poisson scheme is then given by the sequence $\tilde{Y} := (\tilde{Y}_{t_i})_{i\geq 0}$ defined as

$$\tilde{Y}_{t_i} := \tilde{Y}_{t_{i-1}} + a(\tilde{Y}_{t_{i-1}})\Delta X_{\xi_i(n/T)}, \qquad \tilde{Y}_0 = y_0, \qquad i = 1, 2, \dots. \tag{2.28}$$

with independent and stationary increment $\Delta X_{\xi_i(n/T)} := X_{\xi_i(n/T)} - X_{\xi_{i-1}(n/T)} \stackrel{d}{=} X_{\xi(n/T)}$. We define the random grid $(t_i)_{i\geq 0}$ by the partial sum of sequences of these ξ_i's, specified as

$$t_i := \sum_{j=0}^{i} \xi_j(n/T), \tag{2.29}$$

and

$$\mathcal{N}(n/T) = \mathcal{N} := (\mathcal{N}_t)_{t\geq 0}$$

is the Poisson process with the arrival times $(t_i)_{i\geq 0}$. It is noteworthy that t_i follows a Gamma distribution with shape parameter i and rate parameter n/T, i.e., $t_i \stackrel{d}{=} \Gamma(i,\, n/T)$. The mean $\mathbb{E}[\xi_i] = T/n$ corresponds to the time-steps of the deterministic equally-spaced Euler scheme. Based on the foregoing construction, the claim here is that $\tilde{Y}_{t_n}$ is an approximation of the solution Y at end point T and the aim of this work is to derive the asymptotic behavior of

$$\lim_{n\to\infty} \mathbb{E}[|Y_T - \tilde{Y}_{t_n}|^2]. \tag{2.30}$$

In order to carry out the numerical analysis, an interpolant, which stochastically interpolates the Euler-Poisson scheme is introduced: let $\iota(t)$ denote the largest grid point before t in this scheme, expressed as:

$$\iota(t) := \sup[0,\, t] \cap (t_i)_{i\geq 0}$$

and define

$$\hat{Y}_t := y_0 + \int_0^t a(\hat{Y}_{\iota(s-)})dX_s = \hat{Y}_{\iota(t)} + a(\hat{Y}_{\iota(t)})(X_t - X_{\iota(t)}), \qquad t \in [0, t_n \vee T]. \tag{2.31}$$

Observe that for $t \in [t_i,\, t_{i+1})$ we have $\hat{Y}_{t_i} = \tilde{Y}_{t_i} = Y_{\iota(t)}$, i.e., the processes coincide almost surely for all random times $(t_i)_{t\geq 0}$. Therefore, $\hat{Y} = (\hat{Y})_{t\in[0,t_n\vee T]}$ interpolates the process $\tilde{Y}$ in a random way. Furthermore, another important variable crucial to the derivations carried out in this work is the largest distance on the random grid $(t_i)_{t\geq 0}$ restricted to $[0, T]$. This $\mathcal{F}^{\xi}$-measurable random variable is denoted

$$\tau := \sup_{s\in[0,T]} (s - \iota(s)). \tag{2.32}$$

2.6.1 The Moments of τ

In the classical Euler scheme, the maximum time step is bounded by a deterministic finite constant, however, this cannot be the case for the Euler-Poisson scheme since our time steps are generated by a the arrival times Poisson process. Thus in the following, we give the moments of τ following the expositions of [Ferreiro-Castilla, Kyprianou, and Scheichl, 14].

For a Poisson process $\mathcal{N}$, conditioned on d arrivals occurring in the interval from 0 up to and including T, the timing of the arrivals have the same distribution as d

ordered independent uniformly distributed random variables on $[0, T]$. Since τ is defined in (2.32) to be the largest random gap between two neighbouring points, an approach would be to study the maximum gap on the unit interval $[0, 1]$ defined by these d ordered independent uniformly distributed random variables using the well known principle of order statistics.

Definition 2.36. *For $d > 0$, let $\{D_i\,; i = 1, \dots, d-1\}$ be a sequence of i.i.d. random variables where each random variable D_i follows a uniform distribution on $[0\,,1]$. We say that $\{D_{(i)}\,; i = 0, \dots, d\}$ is the order statistic corresponding to $\{D_i\,; i = 1, \dots, d-1\}$ if $D_{(k)}$ is the k-th smallest value among $\{D_i\,; i = 1, \dots, d-1\}$, $k = 0, \dots, d$ where $D_0 = 0$ and $D_d = 1$. The value*

$$\Lambda_d := \max_{i=1,\dots,d}\{D_{(i)} - D_{(i-1)}\}, \tag{2.33}$$

is said to be the largest gap.

A comparison of the definition of τ and (2.33) reveals that $\frac{1}{T}\tau$ conditioned on $\mathcal{N}_T$,is equal in distribution to $\Lambda_{\mathcal{N}_T+1}$, and hence

$$\frac{1}{T}\mathbb{E}[\tau] = \mathbb{E}[\Lambda_{\mathcal{N}_T+1}]. \tag{2.34}$$

For an overview of the behavior of Λ_d we refer the reader to [Fisher, 18]. The following equation was taken from [Mauldon, 29]

$$\mathbb{E}[(1-\Lambda_d s)^{-d}] = \frac{d!}{1-s}\prod_{j=2}^{d}\frac{1}{j-s}, \qquad |s| < 1/2, d \geq 1.$$

For $d \geq 1$ we have that

$$\mathbb{E}[\Lambda_d] = \frac{\sum_{j=1}^{d}\frac{1}{j}}{d} = \frac{\digamma(d+1)+\nu}{d},$$

where $\digamma$ is the digamma function (cf. [Stegun and Abramowitz, 37], Sections 6.3.2 and 6.4.10). We note that

$$(\digamma(d+1)+\nu)\begin{cases} = 0 & \text{for } d = 0 \\ > 0 & \text{for } d > 0 \\ = \log(d+1) & \text{for } d \to \infty. \end{cases}$$

It follows that $\lim_{d\to 0}\frac{\digamma(d+1)}{\log(d+1)} = 1$, and we thus conclude that there is a constant $\mathfrak{c}_0$ independent of d such that $\digamma(d+1)+\nu \leq \mathfrak{c}_0\log(d+1)$. Therefore

$$\mathbb{E}[\Lambda_d] = \mathfrak{c}_0\frac{\log(d+1)}{d} \qquad \textit{for}\; d = 1, 2, \dots$$

Proposition 2.37. *Using the facts above, it holds that*

$$\mathbb{E}[\tau] + \mathbb{E}[\tau^2] \lesssim \frac{\log(n)}{n}.$$

Proof. (2.34) combined with the given arrival rate n/T for the Poisson process $\mathcal{N}$ yields

$$\begin{aligned}
\frac{1}{T}\mathbb{E}[\tau] = \mathbb{E}[\Lambda_{\mathcal{N}_T+1}] &= \sum_{\kappa=0}^{\infty} \mathbb{E}[\Lambda_{\mathcal{N}_T+1}|\mathcal{N}_T]\mathcal{P}(\mathcal{N}_t = \kappa) \\
&\leq \mathfrak{c}_0 \sum_{\kappa=0}^{\infty} \frac{\log(\kappa+1)}{\kappa+1} \exp\Big(-\frac{n}{T}\Big)\frac{(n/T)^\kappa}{\kappa!} \\
&= \frac{\mathfrak{c}_0 T}{n} \sum_{\kappa=0}^{\infty} \log(\kappa+1) \exp\Big(-\frac{n}{T}\Big)\frac{(n/T)^\kappa}{\kappa!} \\
&= \frac{\mathfrak{c}_0 T}{n}\mathbb{E}[\log(\mathcal{N}_T+1)].
\end{aligned}$$

By the concavity of $x \mapsto \log(x+1)$ for $x \in [0,\infty)$ and Jensen's inequality, one has that

$$\begin{aligned}
\frac{1}{T}\mathbb{E}[\tau] = \mathbb{E}[\Lambda_{\mathcal{N}_T+1}] &\leq \frac{\mathfrak{c}_0 T}{n}\log(E[\mathcal{N}_T]+1) \\
&= \frac{\mathfrak{c}_0 T \log(\frac{n}{T}+1)}{n}.
\end{aligned}$$

The assertion follows from the fact that, for $\Lambda_d \in [0,1]$, $\Lambda_d^2 \leq \Lambda_d$, and, thus,

$$\frac{1}{T^2}\mathbb{E}[\tau^2] = \mathbb{E}[\Lambda^2_{\mathcal{N}_T+1}] \leq \mathbb{E}[\Lambda_{\mathcal{N}_T+1}]$$

□

2.6.2 Main Result and Feasibility of the Euler-Poisson scheme

Having presented the preliminaries and notations, we now proceed to a formal statement of the main result of this thesis.

Theorem 2.38. *Under the assumption of Theorem 2.18 we have that*

$$\mathbb{E}[|Y_T - \tilde{Y}_{t_n}|^2] \leq \mathbf{C}_2\sqrt{\frac{1}{n}},$$

where $\mathbf{C}_2$ depends on k and T only, $\mathbf{C}_2, k \in \mathbb{R}_+, T < \infty$.

The Euler-Poisson scheme would benefit from the possibility of sampling from the distribution of X_ξ, which in general would require the same level of technicality as sampling from that of X_1. Thanks to the contributions of [Kuznetsov et al., 27], [Kuznetsov, 25] and [Kuznetsov, Kyprianou, and Pardo, 26] to the recent developments in Wiener-Hopf factorization theory for 1-dimensional Lévy processes, the authors provide one with a handful of examples for which distributional sampling can be performed. Moreover, one could infer from their discussions that the Euler-Poisson scheme is a possible simpler numerical technique for (2.1). The class of processes for which the distributional sampling can be performed is named meromorphic Lévy processes. [Ferreiro-Castilla, Kyprianou, and Scheichl, 14] claims that the Wiener-Hopf factorization gives a lot more information than is needed to implement the Euler-Poisson scheme, as it involves the running infimum and running supremum of the stochastic process X.

Definition 2.39. *Let X be a Lévy process and $\xi \sim$ Exponential(n/T) such that $\mathbb{E}[\xi] = T/n > 0$. The following processes*

$$\mathcal{S}_t = \sup\{X_s\,;\, s \in [0,t]\}, \qquad \mathcal{I}_t = \inf\{X_s\,;\, s \in [0,t]\}$$

are called running supremum and running infimum of X, respectively.

We drop the dependence of ξ on (n/T) as usual. The Wiener-Hopf factorization states that the random variables $\mathcal{S}_\xi$ and $X_\xi - \mathcal{S}_\xi$ are independent. Further, due to the equality in distribution of $(X_s - X_{(t-s)-})_{s\in[0,t]}$ and $(X_s)_{s\in[0,t]}$, it follows that $\mathcal{I}_\xi \overset{d}{=} X_\xi - \mathcal{S}_\xi$. The result is the following characteristic exponent factorization known as the Wiener-Hopf factorization:

$$\mathbb{E}[e^{i\vartheta X_\xi}] = \mathbb{E}[e^{i\vartheta \mathcal{S}_\xi}] \times \mathbb{E}[e^{i\vartheta \mathcal{I}_\xi}], \qquad \forall\, \vartheta \in \mathbb{R}. \tag{2.35}$$

The Wiener-Hopf factors are explicit for the class of meromorphic Lévy processes. Authors such as [Ferreiro-Castilla and Van Schaik, 16], [Kuznetsov, 25] and [Kuznetsov et al., 27] have shown that one can effectively sample from the law of X_ξ through (2.35) for simulating of a variety of Lévy processes. For instance, [Ferreiro-Castilla et al., 17] applied the Multilevel Monte Carlo simulation technique to the β-family pf meromorphic Lévy processes and concludes that the numerical algorithms involving the computation of X_ξ for such processes are very easy to implement and robust with respect to the jump structure. The β-family of meromorphic Lévy processes (cf. Subsection 2.4.1) offers the desirable properties similar to those in used mathematical finance; more so, [Kuznetsov, 25] argues that a large class of Lévy processes can be approximated by a member of the β-class and a compound Poisson process. In this light, [Schoutens and Van Damme, 36] examined the numerical performance of the β-family of meromorphic Lévy processes with parameters chosen such that the Lévy density is approximately equal to that of the classical Variance Gamma (VG) model. The resulting model referred to as β-VG model was found to track the original model, even though the β-VG required more computation time. Likewise, [Ferreiro-Castilla and Schoutens, 15] employed the same methodology to obtain a β-family analog of the Meixner model and the result was congruent to that of [Schoutens and Van Damme, 36]. This brings the possibility to study new processes associated to the stochastic differential equation (2.1). For instance, the following model

$$Y_t = y_0 + \int_0^t a(Y_{s-}, \mathcal{S}_{s-})dX_s \quad \text{or} \quad Y_t = y_0 + \int_0^t a(Y_{s-}, \mathcal{I}_{s-})dX_s \qquad t \in [0,\,T]. \tag{2.36}$$

can be found in [Ferreiro-Castilla, Kyprianou, and Scheichl, 14]. The proposition is that, the stochastic dynamics of populations (see expositions of [Rong, 33], Chapter 11) or chemical reactions can be modelled using (2.36), where the knowledge of $\mathcal{S}$ can replace the artificial barrier restrictions that are usually imposed on the driving process due to physical constraints.

3 Numerical Analysis

The Euler-Poisson scheme is constructed on a random grid that is supported on the interval that can be smaller or bigger than $[0, T]$. The mean squared error described in (2.30) is decomposed into two components, namely, the discretization error and the hitting error. In order to establish the proposed ideas, let the mean squared error be written as

$$|Y_T - \tilde{Y}_{t_n}| = |Y_T - \hat{Y}_{t_n}| \leq |Y_T - \hat{Y}_T| + |\hat{Y}_T - \hat{Y}_{t_n}| \tag{3.1}$$

where the term $|Y_T - \hat{Y}_T|$ corresponds to the discretization error; a measure of variation between the exact solution and the approximation proposed by the Euler-Poisson scheme on $[0, T]$; $|\hat{Y}_T - \hat{Y}_{t_n}|$ corresponds to the hitting error which accounts for the deviation of the Euler-Poisson scheme on $[t_n, T]$. Both errors are described in the following sections.

3.1 The Discretization Error

Heuristically, the discretization error should be an analog of the traditional Euler scheme with equally-spaced grid points. In order to show this, we first present a technical lemma with result for $\hat{Y}$ similar to the one described in Theorem 2.18 for Y.

Lemma 3.1. *Under the assumptions of Theorem 2.18, the process $\hat{Y}$ defined in (2.31) is adapted to the enlarged filtration $\mathcal{F}^\xi \bigvee \mathcal{F}^X := \sigma(\mathcal{F}^\xi \bigcup \mathcal{F}^X)$ such that*

(i) $\mathbb{E}[\sup_{t\in[0,T]} |\hat{Y}_t|^2] \leq \mathbf{C}_3$,

(ii) $\mathbb{E}[\sup_{t\in[0,T]} |\hat{Y}_t|^2 | \mathcal{F}^\xi] \leq \mathbf{C}_3$,

where $\mathbf{C}_3$ is a positive constant depending on $k \in \mathbb{R}_+$ and T only.

The proof of the lemma is achieved along the standard argument for proving bounds for second moments. As we shall see, the standard combination of Gronwall's inequality (cf. Theorem .5) together with Doob's inequality (cf. (.15)) yields the desired results.

Proof. That $\hat{Y}$ is adapted is clear from the left-hand side of (2.31) since it is almost surely a measurable function in the initial value y_0, and the processes $(W_s)_{s\leq t}$ and $(L_s)_{s\leq t}$ driving the stochastic differential equation up to time t. The square integrability property follows from the proof of Lemma 3.1(i) which shall be shown in the following.

By the Lévy-Itô decomposition of the Lévy process X in (2.17) and the definition of $\hat{Y}_t$, we have that

$$\hat{Y}_t = y_0 + \int_0^t a(\hat{Y}_{\iota(s)}) b ds + \int_0^t a(\hat{Y}_{\iota(s-)}) d(\Sigma W_s) + \int_0^t a(\hat{Y}_{\iota(s-)}) d(L_s), \qquad t \in [0, t_n \vee T].$$

Further, for $t \in [0, T]$ define the following stopping time

$$\sigma_N := \inf\{t > 0\,; |\hat{Y}_t| > N\}.$$

From the foregoing we have that

$$\begin{aligned}
|\hat{Y}_{t\wedge\sigma_N}|^2 &= \bigg| y_0 + \int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s)})bds \\
&\qquad + \int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s-)})d(\Sigma W_s) + \int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s-)})d(L_s)\bigg|^2 \\
\frac{1}{4}|\hat{Y}_{t\wedge\sigma_N}|^2 &\leq |y_0|^2 + \bigg|\underbrace{\int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s)})bds}_{=:(\mathcal{J}_t)}\bigg|^2 + \bigg|\underbrace{\int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s-)})d(\Sigma W_s)}_{=:(\mathcal{J}_t^*)}\bigg|^2 \\
&\qquad + \bigg|\underbrace{\int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s-)})d(L_s)}_{=:(\mathcal{J}_t^{**})}\bigg|^2,
\end{aligned} \tag{3.2}$$

where the second inequality holds by the application of $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, $a, b, c, d \in \mathbb{R}$. Using the Cauchy-Schwartz inequality (cf. (.16)) combined with the definition of the stopping time σ_N, one obtains

$$\mathbb{E}[|(\mathcal{J}_t)|^2] \leq tk^2\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N} |a(\hat{Y}_{\iota(s)})|^2 ds\bigg] \leq N^2\mathbf{C}_0\int_0^t (1+|N|^2)ds < \infty, \tag{3.3}$$

for a constant $\mathbf{C}_0$ depending on k and T only. Moreover, the Itô isometry guarantees that

$$\mathbb{E}[|(\mathcal{J}_t^*)|^2] \leq k^2\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N} \Big|a(\hat{Y}_{\iota(s)})\Big|^2 ds\bigg] \leq N^2\mathbf{C}_0\int_0^t (1+|N|^2)ds < \infty. \tag{3.4}$$

Observe that a similar bound holds for $\mathcal{J}_t^{**}$ as in (3.4) above. Hence we conclude that the integrals denoted by $\mathcal{J}_t^*$ and $\mathcal{J}_t^{**}$ are square-integrable martingales. Using the assumptions of Lipschitz conditions from Theorem 2.18, the linear growth bounds of $a(x)$ is given as

$$\begin{aligned}
|a(x)|^2 &= |a(x) - a(y_0) + a(y_0)|^2 \\
&\leq |k|x-y_0| + k|^2 \leq 2(k^2|x-y_0|^2 + k^2) \\
&\leq 2k^2(2|x|^2 + 2k^2) + 2k^2 \leq 4k^2|x|^2 + 2k^2(2k^2+1) \\
&\leq \mathbf{C}_0(1+|x|^2),
\end{aligned} \tag{3.5}$$

for a constant $\mathbf{C}_0$ depending on k and T only. Hence

$$\begin{aligned}
\frac{1}{4}\mathbb{E}\Big[\sup_{r\leq t\wedge\sigma_N}|\hat{Y}_r|^2\Big] &\leq k^2 + tk^2\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}|a(\hat{Y}_{\iota(s)})|^2 ds\bigg] \\
&+ \mathbb{E}\bigg[\sup_{r\leq t\wedge\sigma_N}\bigg|\int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s-)})d(\Sigma W_s)\bigg|^2\bigg] + \mathbb{E}\bigg[\sup_{r\leq t\wedge\sigma_N}\bigg|\int_0^{t\wedge\sigma_N} a(\hat{Y}_{\iota(s-)})d(L_s)\bigg|^2\bigg],
\end{aligned} \tag{3.6}$$

and by the application of Doob's inequality and Itô isometry we obtain

$$\mathbb{E}\Big[\sup_{r\leq t\wedge\sigma_N}|\mathcal{J}^*_r|^2\Big]\leq 4\mathbb{E}\Big[\mathcal{J}^{*2}_{t\wedge\sigma_N}\Big]\leq 4k^2\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}\Big|a(\hat{Y}_{\iota(s)})\Big|^2ds\bigg], \tag{3.7}$$

and, similarly,

$$\mathbb{E}\Big[\sup_{r\leq t\wedge\sigma_N}|\mathcal{J}^{**}_r|^2\Big]\leq 4\mathbb{E}\Big[\mathcal{J}^{**2}_{t\wedge\sigma_N}\Big]\leq 4k^2\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}\Big|a(\hat{Y}_{\iota(s)})\Big|^2ds\bigg]. \tag{3.8}$$

The following holds by substitution of (3.7) and (3.8) into (3.6) together with the bound provided in (3.5):

$$\begin{aligned}\frac{1}{4}\mathbb{E}[\sup_{r\leq t\wedge\sigma_N}|\hat{Y}_r|^2]&\leq k^2+(tk^2+8k^2)\bigg(\mathbf{C}_0\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}\Big(1+|\hat{Y}_{\iota(s)}|^2\Big)ds\bigg]\bigg)\\&\leq k^2+(tk^2+8k^2)\bigg(\mathbf{C}_0t+\mathbf{C}_0\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}|\hat{Y}_{\iota(s)}|^2ds\bigg]\bigg)\\&\leq\mathfrak{c}_1\bigg(1+\int_0^t\mathbb{E}[\sup_{r\leq s\wedge\sigma_N}|\hat{Y}_r|^2]ds\bigg),\end{aligned} \tag{3.9}$$

where $\mathfrak{c}_1\in\mathbb{R}$ depends on k and T only. Finally, Gronwall's inequality gives

$$\begin{aligned}\mathbb{E}[\sup_{r\leq t\wedge\sigma_N}|\hat{Y}_r|^2]&\leq 4\mathfrak{c}_1\bigg(1+\int_0^t\mathbb{E}[\sup_{r\leq s\wedge\sigma_N}|\hat{Y}_r|^2]ds\bigg)\\&\leq 4\mathfrak{c}_1e^{4\mathfrak{c}_1t}\leq 4\mathfrak{c}_1e^{4\mathfrak{c}_1T}=\mathbf{C}_3.\end{aligned}$$

Finally, the assertion of Lemma 3.1(i) follows by letting $N\to\infty$, while for that of Lemma 3.1(ii), one observes that X is independent of $\mathcal{F}^\xi$. Hence, the stochastic integrals $\mathcal{J}^*$ and $\mathcal{J}^{**}$ are martingales with respect to $\mathcal{F}^X$. Furthermore,

$$\begin{aligned}\frac{1}{4}\mathbb{E}[\sup_{r\leq t\wedge\sigma_N}|\hat{Y}_r|^2|\mathcal{F}^\xi]\leq|y_0|^2&+\mathbb{E}\bigg[\bigg|\int_0^{t\wedge\sigma_N}a(\hat{Y}_{\iota(s)})bds\bigg|^2\mathcal{F}^\xi\bigg]\\&+\mathbb{E}\bigg[\bigg|\int_0^{t\wedge\sigma_N}a(\hat{Y}_{\iota(s-)})d(\Sigma W_s)\bigg|^2\mathcal{F}^\xi\bigg]+\bigg|\int_0^{t\wedge\sigma_N}a(\hat{Y}_{\iota(s-)})d(L_s)\bigg|^2\mathcal{F}^\xi\bigg]\\&\leq k^2+tk^2\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}|a(\hat{Y}_{\iota(s)})|^2ds\bigg]+8k^2\mathbb{E}\bigg[\int_0^{t\wedge\sigma_N}|a(\hat{Y}_{\iota(s)})|^2ds\bigg],\end{aligned}$$

and every other step carries on from Lemma 3.1(i) using the conditional version of Doob's inequality as well as the conditional version of Itô isometry (cf. [Ferreiro-Castilla, Kyprianou, and Scheichl, 14], Section 3.1). Therefore

$$\mathbb{E}[\sup_{r\leq t\wedge\sigma_N}|\hat{Y}_r|^2|\mathcal{F}^\xi]\leq 4\mathfrak{c}_1e^{3\mathfrak{c}_1t}\leq 4\mathfrak{c}_1e^{4\mathfrak{c}_1T}=\mathbf{C}_3.$$

□

The following theorem derives the asymptotic behavior for the discretization error which ultimately depends on the maximum random step seize τ defined in (2.32). We shall apply the results of Proposition 2.37 for τ:

Theorem 3.2. *Under the assumptions of Theorem 2.18, we have*

$$\mathbb{E}[\sup_{t\in[0,\,T]} |Y_t - \hat{Y}_t|^2] \leq \mathbf{C}_4[\tau^2 + 2\tau] \lesssim \frac{\log(n)}{n},$$

where $\mathbf{C}_4$ is a positive constant depending on k and T only.

Proof. Let $t \in [0,\, T]$ and define

$$Z_t := Y_t - \hat{Y}_t = \int_0^t (a(Y_s) - a(\hat{Y}_{\iota(s)}))b ds + \int_0^t (a(Y_{s-}) - a(\hat{Y}_{\iota(s-)}))d(\Sigma W_s) + \int_0^t (a(Y_{s-}) - a(\hat{Y}_{\iota(s-)}))d(L_s). \tag{3.10}$$

Using the assertions of Lemma 3.1(i) and Lemma (ii), one concludes that stochastic integrals on the right-hand side of (3.10) are square integrable martingales with respect to the filtration $\mathcal{F}^\xi \bigvee \mathcal{F}^X$. In the following, the first inequality follows from the application of the relation $(a+b+c)^2 \leq 3(a^2+b^2+c^2),\ a,b,c \in \mathbb{R}$ to (3.10); the second inequality is obtained by the successive applications of the Cauchy-Schwartz inequality, Doob's inequality and Itô Isometry as in (3.6). Furthermore, the third and fifth inequalities hold due to the assumptions of Theorem 2.18 and the definition of Z_t, respectively.

$$\begin{aligned}
\frac{1}{3}\mathbb{E}[\sup_{r<t} |Z_r|^2] &\leq \mathbb{E}\left[\sup_{r<t}\left|\int_0^r (a(Y_s) - a(\hat{Y}_{\iota(s)}))b ds\right|^2\right] \\
&\quad + \mathbb{E}\left[\sup_{r<t}\left|\int_0^r (a(Y_{s-}) - a(\hat{Y}_{\iota(s-)}))d(\Sigma W_s)\right|^2\right] \\
&\quad + \mathbb{E}\left[\sup_{r<t}\left|\int_0^r (a(Y_{s-}) - a(\hat{Y}_{\iota(s-)}))d(L_s)\right|^2\right] \\
&\leq tk^2\mathbb{E}\left[\int_0^t \left|(a(Y_s) - a(\hat{Y}_{\iota(s)}))\right|^2 ds\right] + 4\mathbb{E}\left[\int_0^t \left|(a(Y_s) - a(\hat{Y}_{\iota(s)}))\right|^2 ds\right] \\
&\quad + 4\mathbb{E}\left[\int_0^t \left|(a(Y_s) - a(\hat{Y}_{\iota(s)}))\right|^2 ds\right] \qquad (3.11) \\
&\leq tk^2\mathbb{E}\left[\int_0^t |Y_s - \hat{Y}_{\iota(s)}|^2 ds\right] + 8k^2\mathbb{E}\left[\int_0^t |Y_s - \hat{Y}_{\iota(s)}|^2 ds\right] \\
&\leq (tk^2 + 8k^2)\left(\int_0^t \mathbb{E}[|Y_s - \hat{Y}_s|^2] + \mathbb{E}[|\hat{Y}_s - \hat{Y}_{\iota(s)}|^2] ds\right) \\
&\leq \mathfrak{c}_2 \int_0^t \mathbb{E}[|Z_s|^2] + \mathbb{E}[|\hat{Y}_s - \hat{Y}_{\iota(s)}|^2] ds \\
&\leq \mathfrak{c}_2 \int_0^t \mathbb{E}[\sup_{r<t} |Z_s|^2] + \mathbb{E}[|\hat{Y}_s - \hat{Y}_{\iota(s)}|^2] ds.
\end{aligned}$$

Our next objective is to apply the Gronwall's inequality to the last integral. In order to do this, we first compute a bound for $\mathbb{E}[|\hat{Y}_s - \hat{Y}_{\iota(s)}|^2]$. Using the growth condition of $a(x)$ obtained in (3.5), we have that

$$\begin{aligned}
\mathbb{E}[|\hat{Y}_s - \hat{Y}_{\iota(s)}|^2] &= \mathbb{E}[|a(\hat{Y}_{\iota(s)})(X_s - X_{\iota(s)})|^2] \\
&\leq \mathbb{E}[|a(\hat{Y}_{\iota(s)})|^2]\mathbb{E}[|X_s - X_{\iota(s)}|^2] \\
&\leq \bigg(\mathbf{C}_0\Big(1 + \mathbb{E}[|\hat{Y}_{\iota(s)}|^2]\Big)\bigg)\mathbb{E}[|X_s - X_{\iota(s)}|^2] \\
&\leq \bigg(\mathbf{C}_0\Big(1 + \mathbb{E}[|Z_{\iota(s)} + Y_{\iota(s)}|^2]\Big)\bigg)\mathbb{E}[|X_s - X_{\iota(s)}|^2] \\
&\leq \bigg(\mathbf{C}_0\Big(1 + 2\mathbb{E}[|Z_{\iota(s)}|^2] + 2\mathbb{E}[|Y_{\iota(s)}|^2]\Big)\bigg)\mathbb{E}[|X_s - X_{\iota(s)}|^2],
\end{aligned} \tag{3.12}$$

where the third inequality holds by the relation $|\hat{Y}_{\iota(s)}| \leq |Z_{\iota(s)}| + |Y_{\iota(s)}|$, while the fourth inequality is due to the relation $(a+b)^2 \leq 2(a^2+b^2)$, $a, b \in \mathbb{R}$. It remains to control $\mathbb{E}[|X_s - X_{\iota(s)}|^2]$ whose estimate follows from the knowledge of the increments of the stochastic process X: $|X_s - X_{\iota(s)}| \sim N(0, \Sigma^2|s - \iota(s)|)$, i.e., $|X_s - X_{\iota(s)}|$ follows a Gaussian distribution with mean 0 and variance $\Sigma^2|s - \iota(s)|$. We thus have

$$\begin{aligned}
\mathbb{E}[|X_s - X_{\iota(s)}|^2] &\leq \mathbb{E}\Big[|\Sigma W_s - \Sigma W_{\iota(s)}|^2 + |L_s - L_{\iota(s)}| + |b|^2|s - \iota(s)|\Big] \\
&\leq k^2\mathbb{E}\Big[|s - \iota(s)|^2 + |s - \iota(s)| + |s - \iota(s)|\Big] \\
&\leq k^2\mathbb{E}[\tau^2 + 2\tau] \leq k^2\mathbb{E}[T^2 + 2T],
\end{aligned} \tag{3.13}$$

where we have used the independence of the Wiener process and the random grid points in $(t_i)_{i\geq 0}$, and the third inequality holds since two neighbouring time points in $(t_i)_{i\geq 0}$ are at most τ units apart. Therefore, (3.12) and (3.13), when substituted in (3.11), together with Theorem 2.18 yield

$$\mathbb{E}[\sup_{r<t}|Z_r|^2] \leq \mathfrak{c}_2\bigg(\int_0^t \mathbb{E}[\sup_{r<t}|Z_s|^2] + \mathbb{E}[\tau^2 + 2\tau]\bigg),$$

where the constant k^2 has been replaced by $\mathfrak{c}_2$ and thus

$$\mathbb{E}[\sup_{t\in[0,T]}|Y_t - \hat{Y}_t|^2] \leq \mathbb{E}[\tau^2 + 2\tau]\mathfrak{c}_2 e^{T\mathfrak{c}_2} = \mathbb{E}[\tau^2 + 2\tau]\mathbf{C}_4$$

follows from Gronwall's inequality. □

3.2 The Hitting Error

We next turn to deriving the results for the asymptotic behaviour of the hitting error. The hitting error as would be used in this section refers to the first time $\hat{Y}_T$ hits $\hat{Y}_{t_n}$ on $[0, T]$, i.e.,

$$|\hat{Y}_T - \hat{Y}_{t_n}| = \inf\{T, t_n \in [0, T]; |\hat{Y}_T| \geq |\hat{Y}_{t_n}|\}.$$

It shall be shown that the error, in a sense, measures the rate at which at which the grid points t_n converges to T, which is in turn controlled by the variance of the Gamma distribution. In order to see this, we present two technical lemmas that are similar in nature to Lemma 3.1.

Lemma 3.3. *Under the assumptions of Theorem 2.18, the process $\hat{Y}$ defined in (2.31) is adapted to $\mathcal{F}^\xi \bigvee \mathcal{F}^X$ and we have that*

$$\max_{i\in[0,n]} \mathbb{E}[|\hat{Y}_{t_i}|^2] \le \mathbf{C}_5,$$

where $\mathbf{C}_5$ depends on k and T only, $\mathbf{C}_5, k \in \mathbb{R}_+$, $T < \infty$.

Proof. Fixing $i > 0$ and by recalling the definition of $\hat{Y}$ given by (2.31), we write

$$\begin{aligned}
\mathbb{E}[|\hat{Y}_{t_i}|^2] &= \mathbb{E}[|\hat{Y}_{t_{i-1}} + a(\hat{Y}_{t_{i-1}})(X_{t_i} - X_{t_{i-1}})|^2] \\
&= \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2] + \mathbb{E}[|a(\hat{Y}_{t_{i-1}})|^2]\mathbb{E}[|X_{t_i} - X_{t_{i-1}}|^2] + 2\mathbb{E}[\hat{Y}_{t_{i-1}}^T a(\hat{Y}_{t_{i-1}})]\mathbb{E}[X_{t_i} - X_{t_{i-1}}] \\
&\le \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2] + \mathbf{C}_0\Big(1 + \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\Big)k^2\Big(\mathbb{E}|t_i - t_{i-1}|^2 + 2\mathbb{E}|t_i - t_{i-1}|\Big) \\
&\qquad + 2\sqrt{\mathbf{C}_0}\Big(1 + \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\Big)k\mathbb{E}[|t_i - t_{i-1}|] \\
&\le \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2] + \mathbf{C}_0\Big(1 + \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\Big)2k^2\Big(\mathbb{E}|\xi|^2 + \mathbb{E}|\xi|\Big) \\
&\qquad + 2\sqrt{\mathbf{C}_0}\Big(1 + \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\Big)k\mathbb{E}|\xi| \\
&\le \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2] + \mathbf{C}_0\Big(1 + \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\Big)2k^2\frac{T}{n}\left(1 + \frac{T}{n}\right) \\
&\qquad + \sqrt{\mathbf{C}_0}\Big(1 + \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\Big)2k\frac{T}{n} \\
&\le \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\left(1 + 2k^2\frac{T}{n}\left(1 + \frac{T}{n}\right) + \sqrt{\mathbf{C}_0}2k\frac{T}{n}\right) \\
&\qquad + \mathbf{C}_0 2k^2\frac{T}{n}\left(1 + \frac{T}{n}\right) + \sqrt{\mathbf{C}_0}2k\frac{T}{n},
\end{aligned} \tag{3.14}$$

where we have used the knowledge of the moments of $t_i - t_{i-1} \stackrel{d}{=} Exponential(n/T)$ together with the following bounds which is derived from the growth condition of $a(x)$:

$$|x^T a(x)| \le \sqrt{\mathbf{C}_0}\Big(1 + |x|^2\Big).$$

It is evident from (3.14) that there exists a constant $\mathfrak{c}_3$, depending on K and T only, such that

$$\mathbb{E}[|\hat{Y}_{t_i}|^2] \le \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2]\left(1 + \frac{\mathfrak{c}_3}{n}\right) \le |y_0|^2\left(1 + \frac{\mathfrak{c}_3}{n}\right)^i + \exp\left(i\frac{\mathfrak{c}_3}{n}\right)i\frac{\mathfrak{c}_3}{n},$$

which follows from the argument that

$$\begin{aligned}
&\text{if} \qquad z_{m+1} \le \varphi z_m + \psi, \quad \varphi \ge 1, \\
&\text{then} \qquad z_m \le \varphi^m z_0 + \psi m e^{(m\varphi - m)}.
\end{aligned} \tag{3.15}$$

Finally using the fact that for $m, t, r \in \mathbb{R}_+$, $\lim_{m\to\infty}(1+r/m)^{mt} = e^{rt}$, we have that

$$\max_{i\in[0,n]} \mathbb{E}[|\hat{Y}_{t_i}|^2] \leq |y_0|^2\left(1+\frac{\mathfrak{c}_3}{n}\right)^n + e^{\mathfrak{c}_3}\mathfrak{c}_3 \leq e^{\mathfrak{c}_3}(k^2+\mathfrak{c}_3)$$

which is the assertion of the lemma. □

Lemma 3.4. *Under the assumptions of Theorem 2.18, the process $\hat{Y}$ defined in (2.31) is adapted to $\mathcal{F}^\xi \bigvee \mathcal{F}^X$ such that*

(i) $\mathbb{E}[\max_{i\in[0,n]} |\hat{Y}_{t_i}|^2] \leq \mathbf{C}_6$,

(ii) $\mathbb{E}\left[\left(\mathbb{E}\left[\max_{i\in[0,n]} |\hat{Y}_{t_i}|^2 | \mathcal{F}^\xi\right]\right)^2\right] \leq \mathbf{C}_6$,

where $\mathbf{C}_6$ is a positive constant depending on k and T only.

Proof. For $i \in [0,\, n-1]$, define $\Delta\hat{Y}_i := \hat{Y}_{t_{i+1}} - \hat{Y}_{t_i}$. By similar arguments applied in (3.14) and Lemma 3.3, we have that

$$\begin{aligned}
\mathbb{E}[|\Delta\hat{Y}_{t_i}|^2] &= \mathbb{E}[|a(\hat{Y}_{t_i})|^2]\mathbb{E}[|X_{t_{i+1}} - X_{t_i}|^2] \\
&\leq \Big(\mathbf{C}_0(1+\mathbb{E}[|\hat{Y}_{t_i}|^2])\Big)2k^2\frac{T}{n}\left(1+\frac{T}{n}\right) \\
&\leq \Big(\mathbf{C}_0(1+\mathbf{C}_5)\Big)2k^2\frac{T}{n}\left(1+\frac{T}{n}\right).
\end{aligned}$$

Therefore

$$\max_{i\in[0,n-1]} \mathbb{E}[|\Delta\hat{Y}_{t_i}|^2] \leq \frac{\mathfrak{c}_4}{n}, \tag{3.16}$$

for some constants $\mathfrak{c}_4$ depending on k and T only. For $i \in [0, n-1]$, define the auxiliary random variables

$$Z_i := \Delta\hat{Y}_{t_i} - \mathbb{E}[\Delta\hat{Y}_{t_i}|\mathcal{G}_i] \tag{3.17}$$

using the filtration $\mathcal{G}_i := \sigma(\hat{Y}_{t_j};\, j \in [0,i])$. Due to the tower property of conditional expectation and the fact that Z_i is $\mathcal{G}_{i+1}$-measurable, one can easily show that $\sum_{j=0}^{i} Z_j$ is a martingale such that $\mathbb{E}[Z_i Z_j] = 0$ for $i \neq j$, $i, j \in [0, n-1]$. We can thus write

$$\begin{aligned}
\max_{i\in[0,n]} |\Delta\hat{Y}_{t_i}|^2 &\leq \left(|y_0|^2 + \max_{i\in[0,n-1]}\left|\sum_{j=0}^{i}\Delta\hat{Y}_{t_j}\right|^2\right) \\
&= \left(|y_0|^2 + \max_{i\in[0,n-1]}\left|\sum_{j=0}^{i} Z_j + \mathbb{E}[\Delta\hat{Y}_{t_j}|\mathcal{G}_j]\right|^2\right) \\
&\leq 2\Bigg(|y_0|^2 + 2\underbrace{\max_{i\in[0,n-1]}\left|\sum_{j=0}^{i} Z_j\right|^2}_{=:(\tilde{\mathcal{J}}_t)} + 2\underbrace{\max_{i\in[0,n-1]}\left|\sum_{j=0}^{i}\mathbb{E}[\Delta\hat{Y}_{t_j}|\mathcal{G}_j]\right|^2}_{=:(\tilde{\mathcal{J}}_t^*)}\Bigg).
\end{aligned} \tag{3.18}$$

Furthermore, we have that

$$\begin{aligned}\mathbb{E}[(\tilde{\mathcal{J}}_t)] &= \mathbb{E}\left[\max_{i\in[0,n-1]}\left|\sum_{j=0}^{i} Z_j\right|^2\right] \le \mathbb{E}\left[\sum_{j=0}^{n-1}\left|Z_j\right|^2\right] \\ &\le 2\mathbb{E}\left[\sum_{j=0}^{n-1}|\Delta\hat{Y}_{t_j}|^2 + |\mathbb{E}[\Delta\hat{Y}_{t_j}|\mathcal{G}_j]|^2\right] \\ &\le 4\sum_{j=0}^{n-1}\mathbb{E}\Big[|\Delta\hat{Y}_{t_j}|^2\Big] \le 4\mathfrak{c}_4,\end{aligned}$$

where the first inequality results from Doob's inequality and the fact that $(\tilde{\mathcal{J}}_t)$ is a sum of uncorrelated random variables; the second inequality holds by the combination of the definition of Z_i and the relation $(a+b)^2 \le 2(a^2+b^2)$, $a, b \in \mathbb{R}$; the third and fourth inequalities follow from Jensen's inequality and the bound given in (3.16), respectively. Likewise, using Lemma 3.3 one has that

$$\begin{aligned}\mathbb{E}[(\tilde{\mathcal{J}}^*_t)] &= \mathbb{E}\left[\max_{i\in[0,n-1]}\left|\sum_{j=0}^{i}\Delta\hat{Y}_{t_j}|\mathcal{G}_j]\right|^2\right] \le \mathbb{E}\left[\left(\sum_{j=0}^{n-1}|\mathbb{E}[\Delta\hat{Y}_{t_j}|\mathcal{G}_j]|\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{j=0}^{n-1}|a(\hat{Y}_{t_j})\mathbb{E}[t_{j+1}-t_j]|\right)^2\right] \\ &\le \mathbb{E}\left[\left(\sum_{j=0}^{n-1}|a(\hat{Y}_{t_j})|k\frac{T}{n}\right)^2\right] \le k^2T^2\Big(\mathbf{C}_0(1+\mathbf{C}_5)\Big).\end{aligned}$$

From (3.18) we have that

$$\begin{aligned}\max_{i\in[0,n]}|\Delta\hat{Y}_{t_i}|^2 &\le 2(|y_0|^2 + 2\mathbb{E}[(\tilde{\mathcal{J}}_t)] + 2\mathbb{E}[(\tilde{\mathcal{J}}^*_t)]) \\ &\le 2(k^2 + 8\mathfrak{c}_4 + 2k^2T^2(\mathbf{C}_0(1+\mathbf{C}_5))) = \mathbf{C}_6.\end{aligned}$$

We are now in the position to prove the second part of the Lemma, consider the filtration $\mathcal{G}_i := \mathcal{F}^\xi \bigvee \sigma(\hat{Y}_{t_j};\, j\in[0,i])$. The procedure that produced (3.18) similarly yields

$$\mathbb{E}\Big[\max_{i\in[0,n]}|\hat{Y}_{t_i}|^2|\mathcal{F}^\xi\Big] \le 2(|y_0|^2 + 2\mathbb{E}[(\tilde{\mathcal{J}}_t)|\mathcal{F}^\xi] + 2\mathbb{E}[(\tilde{\mathcal{J}}^*_t)|\mathcal{F}^\xi]).$$

By definition of $\Delta\hat{Y}_{t_i}$, one obtains

$$\begin{aligned}\mathbb{E}[(\tilde{\mathcal{J}}_t)|\mathcal{F}^\xi] = 4\sum_{j=0}^{n-1}\mathbb{E}\Big[|\Delta\hat{Y}_{t_j}|^2|\mathcal{F}^\xi\Big] &\le 4\max_{i\in[0,n-1]}\Big|\mathbb{E}[|a(\hat{Y}_{t_i})|^2|X_{t_{i+1}}-X_{t_i}|^2]|\mathcal{F}^\xi\Big| \\ &\le 4\max_{i\in[0,n-1]}\Big|\mathbb{E}[|a(\hat{Y}_{t_i})|^2|\mathcal{F}^\xi]2k^2(\xi^2_{i+1}+\xi_{i+1})\Big| \qquad (3.19) \\ &\le 8k^2\max_{i\in[0,n-1]}\mathbb{E}[|a(\hat{Y}_{t_i})|^2|\mathcal{F}^\xi]\sum_{j=0}^{n}\xi_j(1+\xi_j),\end{aligned}$$

and, similarly,

$$\mathbb{E}[(\tilde{\mathcal{J}}_t^*)|\mathcal{F}^\xi] \leq \mathbb{E}\left[\left(\sum_{j=0}^{n-1} |a(\hat{Y}_{t_i})|k\xi_j\right)^2 \middle| \mathcal{F}^\xi\right] \leq k^2 \max_{i\in[0,n-1]} \mathbb{E}[|a(\hat{Y}_{t_i})|^2|\mathcal{F}^\xi] n \sum_{j=0}^{n} \xi_j^2. \tag{3.20}$$

Hence it suffices to use (3.18) and then show that

$$\mathbb{E}[(\mathbb{E}[(\tilde{\mathcal{J}}_t)|\mathcal{F}^\xi])^2 + (\mathbb{E}[(\tilde{\mathcal{J}}_t^*)|\mathcal{F}^\xi])^2] \leq \mathfrak{c}_5 \tag{3.21}$$

for $\mathfrak{c}_5$ depending on k and T only, $\mathfrak{c}_5$, $k \in \mathbb{R}_+$ in order to complete the proof. Using the Cauchy-Schwartz inequality, a sufficient condition for (3.21) is given by

$$\mathbb{E}\left[\left(\max_{i\in[0,n-1]} \mathbb{E}[|a(\hat{Y}_{t_i})|^2|\mathcal{F}^\xi]\right)^4\right] + \mathbb{E}\left[\left(\sum_{j=0}^{n} \xi_j(1+\xi_j)\right)^4 + \left(n\sum_{j=0}^{n} \xi_j^2\right)^4\right] \leq \mathfrak{c}_5 \tag{3.22}$$

where we have renamed $\mathfrak{c}_5$. Note that for $m \geq 1$, for any $\xi \sim \mathit{Exponential}(n/T)$ we have $\mathbb{E}[\xi^m] = m!(\frac{T}{n})^m$. Using the inequality $(c+g)^{2n} \leq 2^{2n-1}(c^{2n}+g^{2n}), n = 2, c, g \in \mathbb{R}$ and the fact that ξ_i, $i \geq 1$ are independent and identically distributed, it is easily seen that

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{j=0}^{n} \xi_j(1+\xi_j)\right)^4 + \left(n\sum_{j=0}^{n} \xi_j^2\right)^4\right] &\leq \left((n(\mathbb{E}[\xi]+\mathbb{E}[\xi^2]))^4 + (n^2\mathbb{E}[\xi^2])^4\right) \\ &\leq 8\left(4!T^4 + 8!\frac{T^8}{n^4}\right) + 8!T^8, \end{aligned} \tag{3.23}$$

and (3.22) holds. Our next objective is to determine the bound of $\mathbb{E}[|\hat{Y}_{t_i}|^2|\mathcal{F}^\xi]$, which we do by incorporating the conditional expectation to the right-hand side of (3.14), and, thus

$$\begin{aligned} \mathbb{E}[|\hat{Y}_{t_i}|^2|\mathcal{F}^\xi] &\leq \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi] + \mathbf{C}_0\left(1+\mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi]\right)k^2\Big(\mathbb{E}[|t_i - t_{i-1}|^2|\mathcal{F}^\xi] \\ &\qquad + 2\mathbb{E}[|t_i - t_{i-1}||\mathcal{F}^\xi]\Big) + 2\sqrt{\mathbf{C}_0}\left(1+\mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi]\right)k\mathbb{E}[|t_i - t_{i-1}||\mathcal{F}^\xi] \\ &\leq \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi] + \mathbf{C}_0\left(1+\mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi]\right)2k^2(\xi_i^2+\xi_i) \\ &\qquad + 2\sqrt{\mathbf{C}_0}\left(1+\mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi]\right)k\xi_i \\ &\leq \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi]\left(1+2k^2\mathbf{C}_0(\xi_i^2+\xi_i)+2k\sqrt{\mathbf{C}_0}\xi_i\right) \\ &\qquad + 2k^2\mathbf{C}_0(\xi_i^2+\xi_i) + 2k\sqrt{\mathbf{C}_0}\xi_i. \end{aligned} \tag{3.24}$$

Clearly, there exists a constant $\mathfrak{c}_6$ depending on k only such that

$$\begin{aligned} \mathbb{E}[|\hat{Y}_{t_i}|^2|\mathcal{F}^\xi] &\leq \mathbb{E}[|\hat{Y}_{t_{i-1}}|^2|\mathcal{F}^\xi]\left(1+\mathfrak{c}_6(\xi_i^2+\xi_i)\right) + \mathfrak{c}_6(\xi_i^2+\xi_i) \\ &\leq |y_0|^2\left(1+\mathfrak{c}_6(\xi_i^2+\xi_i)\right)^i + i\,\exp\left(i\,\mathfrak{c}_6(\xi_i^2+\xi_i)\right)\mathfrak{c}_6(\xi_i^2+\xi_i). \end{aligned}$$

Using the an analogue of the recurrence argument in (3.15), we conclude that

$$\max_{i\in[0,n]} \mathbb{E}[|\hat{Y}_{t_i}|^2|\mathcal{F}^\xi] \leq |y_0|^2 \prod_{i=1}^{n} \left(1+\mathfrak{c}_6(\xi_i^2+\xi_i)\right)+\sum_{i=1}^{n} \mathfrak{c}_6(\xi_i^2+\xi_i) \prod_{j=i+1}^{n} \left(1+\mathfrak{c}_6(\xi_j^2+\xi_j)\right). \quad (3.25)$$

A computation similar to (3.23) for the first term on the left-hand side of (3.22) combined with (3.25) yields

$$\begin{aligned}\mathbb{E}\left[\left(\max_{i\in[0,n-1]} \mathbb{E}[|a(\hat{Y}_{t_i})|^2|\mathcal{F}^\xi]\right)^4\right] &\leq \mathbb{E}\Bigg[\Bigg(k^2 \prod_{i=1}^{n} \left(1+\mathfrak{c}_6(\xi_i^2+\xi_i)\right) \\ &\qquad + \sum_{i=1}^{n} \mathfrak{c}_6(\xi_i^2+\xi_i) \prod_{j=i+1}^{n} \left(1+\mathfrak{c}_6(\xi_j^2+\xi_j)\right)\Bigg)^4\Bigg] \qquad (3.26) \\ &\leq k^2\mathbb{E}\left[\left(\prod_{i=1}^{n} \left(1+\mathfrak{c}_6(\xi_i^2+\xi_i)\right)\left(1+\sum_{i=1}^{n} \mathfrak{c}_6(\xi_i^2+\xi_i)\right)\right)^4\right] \\ &\leq 8k^2\left(1+n!\,\mathfrak{c}_6\left(2\frac{T^2}{n^2}+\frac{T}{n}\right)\right)^4\left(1+8\,\mathfrak{c}_6\left(4!T^4+8!\frac{T^8}{n^4}\right)\right).\end{aligned}$$

Since (3.22) holds, we see that (3.19) and (3.20) likewise hold, and by thus completing the proof. □

Proposition 3.5. *Under the assumption of Theorem 2.18 we have that*

$$\mathbb{E}[|\hat{Y}_T - \hat{Y}_{t_n}|^2] \leq \mathbf{C}_7\sqrt{\frac{1}{n}},$$

where $\mathbf{C}_7$ is a positive constant depending on k and T only, $k \in \mathbb{R}_+, T < \infty$.

Proof. Using the identity, for $n = 1, c, g \in \mathbb{R}$, $(c+g)^{2n} \leq 2^{2n-1}(c^{2n}+g^{2n})$ one obtains

$$\frac{1}{2}|\hat{Y}_T - \hat{Y}_{t_n}|^2 \leq \left|\int_{t_n}^{T} a(\hat{Y}_{\iota(s)})b ds\right|^2 + \left|\int_{t_n}^{T} a(\hat{Y}_{\iota(s-)})d(\Sigma W_s + L_s)\right|^2.$$

By Lemma 3.1(i) and Lemma 3.4(i), it is clear that the stochastic integral above is a square integrable martingale with respect to $\mathcal{F}^\xi \bigvee \mathcal{F}^X$. Furthermore,

$$\begin{aligned}\frac{1}{2}\mathbb{E}[|\hat{Y}_T - \hat{Y}_{t_n}|^2] &\leq \mathbb{E}\left[\left|k^2|T-t_n| \int_{t_n}^{T} |a(\hat{Y}_{\iota(s)})|^2 ds + 2k^2 \int_{t_n}^{T} |a(\hat{Y}_{\iota(s)})|^2 ds\right|\right] \\ &\leq \mathbb{E}\left[\left(k^2|T-t_n| + 2k^2\right) \int_{t_n}^{T} |a(\hat{Y}_{\iota(s)})|^2 ds\right] \\ &\leq k^2\mathbb{E}\left[\left(|T-t_n|+2\right) \int_{t_n}^{T} \mathbb{E}[|a(\hat{Y}_{\iota(s)})|^2|\mathcal{F}^\xi] ds\right],\end{aligned}$$

where we have used the square integrability of the integral under consideration together with the Itô isometry and the Cauchy-Schwartz inequality. Moreover, given that the random grid $(t_i)_{i\geq 0}$ is $\mathcal{F}^\xi$-measurable, apply the following to the inequality above

$$\int_{t_n}^{T} \mathbb{E}[|a(\hat{Y}_{\iota(s)})|^2|\mathcal{F}^\xi]ds \leq |T-t_n| \sup_{t\in[T\vee t_n]} \mathbb{E}[|a(\hat{Y}_{\iota(s)})|^2|\mathcal{F}^\xi].$$

We therefore arrive at

$$\frac{1}{2}\mathbb{E}[|\hat{Y}_T - \hat{Y}_{t_n}|^2] \leq k^2 \Bigg(\mathbb{E}\Big[\Big(|T-t_n|^2 + 2|T-t_n| \Big)^2 \Big] \\ \mathbb{E}\Bigg[\Bigg(\sup_{t\in[T\vee t_n]} \mathbb{E}\Big[|a(\hat{Y}_{\iota(t)})|^2|\mathcal{F}^\xi\Big] \Bigg)^2 \Bigg] \Bigg)^{\frac{1}{2}}. \tag{3.27}$$

Using Lemma 3.1(ii) and Lemma 3.4(ii), for the second term on the right-hand side of equation (3.27) we can write

$$\mathbb{E}\Bigg[\Bigg(\sup_{t\in[T\vee t_n]} \mathbb{E}\Big[|a(\hat{Y}_{\iota(t)})|^2|\mathcal{F}^\xi\Big] \Bigg)^2 \Bigg] \Bigg) \leq \mathbb{E}\Bigg[\Bigg(\sup_{t\in[0,T]} \mathbb{E}\Big[|a(\hat{Y}_{\iota(t)})|^2|\mathcal{F}^\xi\Big] \\ + \max_{i\in[0,n]} \mathbb{E}\Big[|a(\hat{Y}_{t_i})|^2|\mathcal{F}^\xi\Big] \Bigg) \Bigg] \leq \mathfrak{c}_7, \tag{3.28}$$

where $\mathfrak{c}_7$ is a positive constant depending on k and T only, $k \in \mathbb{R}_+, T < \infty$. Furthermore,

$$\mathbb{E}\Big[\Big(|T-t_n|^2 + 2|T-t_n| \Big)^2 \Big] \leq 2\mathbb{E}[|T-t_n|^4] + 4\mathbb{E}[|T-t_n|^2] \\ = 2\Bigg(\frac{3T^4(2+n)}{n^3} + \frac{2T^2}{n} \Bigg), \tag{3.29}$$

where the above holds by the assumption $t_n \overset{d}{=} \Gamma(\alpha, \lambda)$ coupled with the inequality $(c+g)^{2n} \leq 2^{2n-1}(c^{2n} + g^{2n}), n = 1, c, g \in \mathbb{R}$. The proof is completed by applying (3.28) and (3.29) to (3.27). □

Proof of Theorem 2.38. We observe that the error in Theorem 2.38 can be split into a hitting and discretization error as in equation (3.1). Then the assertion of the theorem, namely, that the convergence rate of the approximation of the Euler-Poisson scheme towards the solution of stochastic differential equation (2.1) is of order $O(n^{\frac{-1}{2}})$, is but a corollary following from the combination of Theorem 3.2 and Proposition 3.5. □

4 Remarks on the the Euler-Poisson Scheme

4.1 Enhanced Euler-Poisson Scheme

The Euler-Poisson scheme has deterministic number of iterations, but since it is supported on the random grid, the time where the algorithm ends is random. It is straightforward to show that $t_n \to T$ a.s. as $n \uparrow \infty$, but it is important to consider if there are more efficient methods to stop the algorithm other than doing n iterations.

Given the definition of Poisson process $\mathcal{N}$ in Section 2.5, define $\mathbb{T} := t_{\mathcal{N}_T+1}$; i.e., $\mathbb{T}$ is the closest random grid point greater than or equal to T; we omit the dependence on (n/T) for convenience. Consider the Euler-Poisson scheme stopped at a random iteration dictated by $\mathcal{N}_T + 1$, which means that the enhanced scheme approximates Y_T by $\hat{Y}_{\mathbb{T}}$.

Proposition 4.1. *Under the assumption of Theorem 2.18, there exist a constant* $\mathbf{C}_8 > 0$ *such that*

$$\mathbb{E}[|Y_T - \tilde{Y}_{\mathbb{T}}|^2] \leq \frac{\mathbf{C}_8 \log(n)}{n}.$$

Proof. We begin by showing a result similar to Proposition 3.5 for the random iteration $\mathcal{N}_T + 1$. By definition of the stochastic interpolant $\hat{Y}$, and given that $\hat{Y}_{\mathbb{T}} = \tilde{Y}_{\mathbb{T}}$, one has that

$$\begin{aligned} \mathbb{E}[|Y_T - \hat{Y}_{\mathbb{T}}|^2] &= \mathbb{E}[|a(\hat{Y}_{\iota(T)})|^2]\mathbb{E}[|X_{\mathbb{T}} - X_T|^2] \\ &\leq \mathbf{C}_0 k^2 (1 + \mathbf{C}_3)(\mathbb{E}[|\mathbb{T} - T|^2] + 2\mathbb{E}[|\mathbb{T} - T|]) \\ &= \mathbf{C}_0 k^2 (1 + \mathbf{C}_3)\left(\frac{T^2}{n^2} + 2\frac{T}{n}\right), \end{aligned} \tag{4.1}$$

where the first inequality holds using a combination of (3.5), Lemma 3.3 and Lemma 3.4 to bound $a(\hat{Y}_{\iota(T)})$. The last equation follows from the memoryless property $\mathbb{T} - T \overset{d}{=} \xi$. The proof of the claim is immediate from decomposing the error $|Y_T - \tilde{Y}_{\mathbb{T}}|$ into a discretization error and hitting error, as shown in (3.1), and then using Theorem 3.2 together with (4.1). $\square$

This modification, in a sense, iterates the Euler-Poisson scheme optimal amount of times to get to the final point in the random grid as close as possible to T by overlapping it. Thus, the enhanced Euler-Poisson scheme is quasi-optimal.

Alternatively, one can use the final point $\bar{\mathbb{T}} := t_{\mathcal{N}_T}$, i.e., the closest point in the Poisson grid that is smaller than T. A key observation to be made here is that in order to construct both $\hat{Y}_{\mathbb{T}}$ and $\hat{Y}_{\tilde{\mathbb{T}}}$, one needs to be able to sample from the bivariate distribution of $\left(\Delta X_{\xi_i(n/T)}, \xi_i(n/T)\right)$. If the latter is available, the distribution of X_t is given by

$$\mathcal{P}(\Delta X_{\xi(n/T)} \in dx, \xi(n/T) \in dt) = \mathcal{P}(X_t \in dx)\left(\frac{n}{T}\right) \exp\left(\frac{-nt}{T}\right) dt, \tag{4.2}$$

therefore one might as well apply the classical Euler scheme for stochastic differential equations (also known as the Euler-Maruyama scheme) rather than sample from the resolvent of X, i.e. rather than sample just from the univariate $\Delta X_{\xi_i(n/T)}$. However, the Wiener-Hopf factorisation does not provide the pair $\left(\Delta X_{\xi_i(n/T)}, \xi_i(n/T)\right)$, and to the best of my knowledge, there is also no other approach. Therefore, the enhancement is of little practical relevance. The only edge the enhanced Euler-Poisson algorithm has over Euler-Maruyama would be to avoid the Laplace transformation in (4.2).

4.2 Heuristics behind the Euler-Poisson Scheme

The Feynman-Kac formula states that conditional expectation with respect to some stochastic differential equation can be obtained as a solution of an associated partial integro-differential equation. In this section, we formalize the relationship between the discretization procedure of given by the Euler-Poisson scheme in (2.28) and its counterpart in the PIDE representation. We claim that, in some sense, sampling the solution Y of (2.1) over a random grid generated by the arrival times of a Poisson process is equivalent to performing a discretization in time by the method of lines to the associated Feynman-Kac equation; hence the Euler-Poisson scheme arises as a natural discretization scheme. [Carr, 6] and [Matache, Nitsche, and Schwab, 28] lend evidence to our claim: an approximation for American options of finite maturity is obtained by randomizing the time horizon using an Erlang distribution by the former; while the latter identified an informal relationship between a deterministic discretization in time of a Feynman-Kac PIDE and its probabilistic counterpart.

Theorem 4.2 (Section 8.17, p.271 [Rong, 33]). *Consider the following integro-differential operator*

$$\begin{aligned}\mathcal{A}_Y h(x) := \langle a(x)b, \nabla\rangle h(x) + \frac{1}{2}\langle a(x)\Sigma\Sigma^T a^T(x)\nabla, \nabla\rangle h(x) \\ + \int_{\mathbb{R}^d} \Big(h(x+a(x)z) - h(x) - \langle a(x)z, \nabla\rangle h(x)\Big) v(dz),\end{aligned} \tag{4.3}$$

taking values in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d, \mathbb{R})$. Assume the assumptions of Theorem 2.18 hold and:

(i) $a : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is bounded;

(ii) there exist $\delta_1,\ \delta_2 > 0$ such that $\delta_1|\lambda|^2 \leq \langle a(x)\Sigma\Sigma^T a^T(x)\lambda, \lambda\rangle \leq \delta_2|\lambda|^2$ for all $x, \lambda \in \mathbb{R}^d$.

Let $u(t,x) \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ be a classical solution to the Partial Integro Differential Equation

$$\frac{\partial}{\partial t} u(t,x) = \mathcal{A}_Y u(t,x), \tag{4.4}$$

with initial condition $u(0,x) = g(x)$ for some bounded function $g : \mathbb{R}^d \to \mathbb{R}$, i.e. $g \in \mathcal{C}_0$. Then

$$u(T-t,x) = \mathbb{E}[g(Y_T)|Y_t = x] = \mathbb{E}[g(Y_{T-t})|Y_0 = x] := \mathbb{E}_x[g(Y_T)], \tag{4.5}$$

where Y is the unique strong solution of (2.1) and $t \in [0,T]$.

Given the conditional expectation in (4.5) one can compute it as a solution to the Partial Integro Differential Equation (4.3) with further assumptions. A classical scenario where the stated relation is applicable is when (4.5) represents the value of a contingent claim at time t on a risky asset Y, depending on the value of the payoff at time T and computed by numerically solving the associated partial integro-differential equation. The popular Black-Scholes [Black and Scholes, 5] model is a typical example under the assumptions of a complete market, ensured by the existence of a unique equivalent martingale measure with the dynamics of the non-dividend paying underlying asset following a geometric Brownian motion. The classical Merton [Merton, 30] model for firm valuation is set up also under similar assumptions. [Chan, 8] relaxed the complete market assumption to develop a model whose underlying is driven by a geometric Lévy process.

Recall the random times $(t_i)_{i\geq 0}$ defined in Section 2.5 as the arrival times of Poisson process $\mathcal{N}$ and consider the Laplace-Carlson transform $\mathfrak{L}$ of $u(t,x)$ in the form

$$\begin{aligned}
\mathfrak{L}[u](x) &:= \int_0^\infty \frac{n}{T} \exp\Big(\frac{-nt}{T}\Big) u(t,x) dt \\
&= \int_0^\infty \frac{n}{T} \exp\Big(\frac{-nt}{T}\Big) \mathbb{E}_x[g(Y_t)] dt \\
&= \mathbb{E}^x[g(Y_{\xi(n/T)})] \\
&= \mathbb{E}^x[g(Y_{t_1})],
\end{aligned} \tag{4.6}$$

where we have used the boundedness of $g \in \mathcal{C}_0$ to apply Fubini's theorem. We note here that $\mathbb{E}^x[g(Y_{t_1})]$ is the expectation of the solution at the first arrival time of the Poisson process $\mathcal{N}$ in (2.1). More so, since the function $g : \mathbb{R}^d \to \mathbb{R}$ is bounded, one can interchange the infinitesimal generator $\mathcal{A}_Y u(t,x)$ and the Laplace-Carlson transform $\mathfrak{L}$ to obtain an integro-differential equation satisfied by the Laplace-Carlson transform:

$$\frac{\mathbb{E}^x[g(Y_{t_1})] - g(x)}{T/n} = \mathcal{A}_Y \mathfrak{L}[u](x). \tag{4.7}$$

Indeed, the result of $\mathcal{A}_Y \mathfrak{L}[u](x)$ is a first order finite difference approximation in time of (4.4) with respect to $\mathfrak{L}[u]$ instead of u due to homogeneity of $\mathcal{A}_Y$. To see this, in the proposition that follows, we link the solution Y at the arrival times of the Poisson process $\mathcal{N}$ with a numerical method called the method of lines or Rothe's method. The method of lines involves the discretization in time of a given partial integro-differential equation to obtain finite differences akin to (4.7) for (4.4), solved iteratively going backwards from the endpoint (cf. Section 12.5, p. 421, [Tankov, 38]).

Proposition 4.3. *Under the assumptions of Theorem 2.18 and Theorem 4.2, consider Rothe's discretization of (4.4) given by the backward difference*

$$\frac{u_i(x) - u_{i-1}(x)}{T/n} = \mathcal{A}_Y u_i(x), \tag{4.8}$$

for $i = 1, \ldots, n$ *with initial condition* $u_0(x) = g(x)$*. Then for all* $i = 1, \ldots, n$*,*

$$u_i(x) = \mathbb{E}^x[g(Y_{t_1})].$$

Proof. It is easily seen that the solution to (2.1) obtained by Theorem 2.18 has the strong Markov property (cf. Theorem 32 p. 294 [31]). Thus, one can write

$$\mathbb{E}^x[g(Y_{t_1})] = \mathbb{E}^x[\mathbb{E}^{Y_{t_1}}[\mathbb{E}^{Y_{t_2}}[\mathbb{E}^{Y_{t_3}}[\cdots \mathbb{E}^{Y_{t_{i-1}}}[g(Y_{t_1})]\cdots]]]],$$

and apply recursively the arguments obtained from (4.6) and (4.7) in the above nested expectations to arrive at the recursive solutions that solve the systems of differential equations in (4.8) □

We have justified the Euler-Poisson scheme in Subsection 2.6.2 by considering the scenarios for which the method is feasible. The above result strengthens the discussion.

4.3 Pathwise Convergence

As mentioned earlier, the Euler-Poisson scheme is supported on a random grid, hence there is no direct methodology to measure local (i.e. pathwise) errors in our scheme. Yet, by taking into account the analogy with Rothe's discretization method one may consider the following quantity

$$\mathbb{E}[\max_{i\in[1,n]} |Y_{iT/n} - \tilde{Y}_{t_i}|^2] = \mathbb{E}[\max_{i\in[1,n]} |Y_{iT/n} - \hat{Y}_{t_i}|^2]. \tag{4.9}$$

Theorem 3.2 indeed states a pathwise result for the discretization error of the Euler-Poisson scheme. Thus, in order to study the quantity (4.9), it suffices to split it into a discretization error and hitting error, where (4.9) then arises as an analogue of the hitting error, i.e., pathwise generalization of Proposition 3.5. Contrary to this expectation, the latter is not true. One can prove a weaker statement incorporating the entire path for the Euler-Poisson scheme:

$$\max_{i\in[1,n]} \mathbb{E}[|Y_{iT/n} - \tilde{Y}_{t_i}|^2] \leq \mathbf{C}_9\sqrt{\frac{1}{n}}, \tag{4.10}$$

where $\mathbf{C}_9$ depends on k and T only, $k \in \mathbb{R}_+, T < \infty$. To show the above we introduce the following lemma.

Lemma 4.4. *It holds that*

$$\mathbb{E}\left[\max_{i\in[1,n]}\left|t_i - \frac{iT}{n}\right|^p\right] \leq 8[|t_n - T|^p], \qquad for \quad p \geq 1. \tag{4.11}$$

Proof. Let

$$Z_i := \begin{cases} \xi_i(n/T) - \mathbb{E}[\xi(n/T)] & \text{for } i \in [1,n] \\ 0 & \text{for } i > n, \end{cases}$$

we have that $\mathbb{E}[Z_i] = 0$ for all $i \in [1,n]$. Hence the sequence $(Z_i)_{i\geq 1}$ is centered and mutually independent since it holds that $\mathbb{E}[Z_i Z_j] = 0$ for all $i \neq j$, $i,j \in [1,n]$. The result in (Theorem 5.1 [Doob, 12]) applies such that (4.11) holds. □

We give an outline of the proof of (4.10). Decomposing (4.10) into a discretization error and hitting error, the result in Theorem 2.18 obviously applies, and for the hitting error we have that

$$\begin{aligned} \frac{1}{2}\max_{i\in[1,n]}\mathbb{E}[|\tilde{Y}_{iT/n} - \tilde{Y}_{t_i}|^2] &\leq \max_{i\in[1,n]}\mathbb{E}\left|\int_{t_i}^{iT/n} a(\tilde{Y}_{\iota(s)})bds\right|^2 \\ &\quad + \max_{i\in[1,n]}\mathbb{E}\left|\int_{t_i}^{iT/n} a(\tilde{Y}_{\iota(s-)})d(\Sigma W_s + L_s)\right|^2 \\ &\leq k^2 \max_{i\in[1,n]}\mathbb{E}\left|\Big(\Big|\frac{iT}{n} - t_i\Big| + 2\Big)\int_{t_n}^{T}\mathbb{E}[|a(\tilde{Y}_{\iota(s)})|^2|\mathcal{F}^{\xi}]ds\right|. \end{aligned} \tag{4.12}$$

The last inequality combined with Proposition 3.5 and Lemma 4.4 yields the desired result.

4.4 Simulation Results

The aim of this section is to provide the outcome of the simulations of the Euler-Poisson scheme.

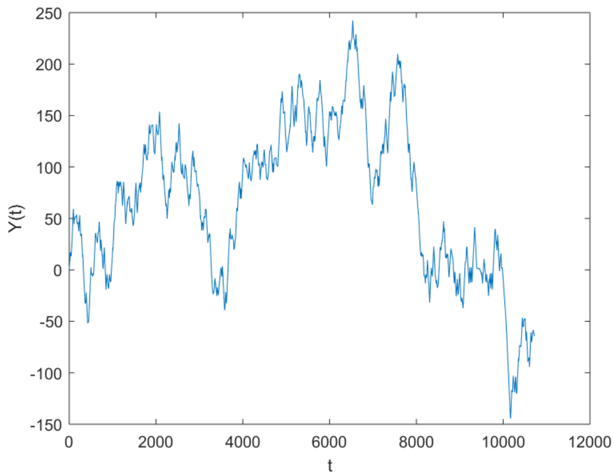
As discussed in Section 4.1, the Euler-Poisson scheme runs on a deterministic number of iterations with a random specification of the grid. Moreover, the enhanced Euler-Poisson scheme is subject to the availability of the pair $\big(\Delta X_{\xi_i(n/T)}, \xi_i(n/T)\big)$. Thus, our implementation is based on the distribution of $\Delta X_{\xi_i(n/T)}$, and is an adapted version of the traditional simulation algorithm of the Euler scheme. We simulate the discretization scheme

$$\tilde{Y}_{t_i} := \tilde{Y}_{t_{i-1}} + a(\tilde{Y}_{t_{i-1}})\Delta X_{\xi_i(n/T)}, \qquad \tilde{Y}_0 = y_0, \qquad i = 1, 2, \ldots$$

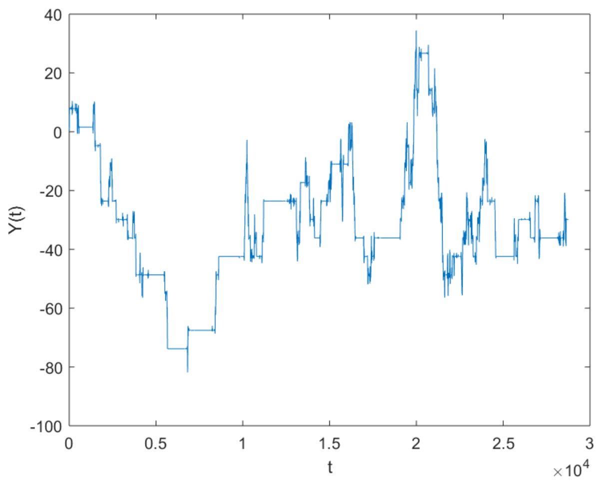
with terminal time point T, step size n, time step T/n and number of simulations N. ξ is generated as the interarrival time of a homogeneous Poisson process defined in (2.29) with the choice of the measurable function $a(x) := \cos(x)$, moreover X follows a Gamma distribution. Figure 4.1 displays the trajectories of the numerical solution obtained by the Euler-Poisson scheme for the following

$$\tilde{Y}_{t_i} := \tilde{Y}_{t_{i-1}} + \cos(\tilde{Y}_{t_{i-1}})\Delta X_{\xi_i(n/T)}, \qquad \tilde{Y}_0 = y_0, \qquad i = 1, 2, \ldots$$

Figure 4.1a displays the trajectories of the Euler-Poisson scheme for the choice of $N = 1,000$, $n = 10$ and $T = 1$; and in 4.1b, $N = 10,000$, $n = 20$ and $T = 7$, respectively.



(a) Figure 1



(b) Figure 2

Figure 4.1: Trajectories of the numerical solution obtained by the Euler-Poisson Scheme

Appendix

.1 Standard Inequalities

.1.1 Gronwall's Inequality

Theorem .5. *Let α and f be nonnegative, continuous functions defined for $t \in [0, T]$, and let $\mathbf{C}_0 \geq 0$ denote a constant. If*

$$\alpha(t) \leq \mathbf{C}_0 + \int_0^t f\alpha ds, \qquad \textit{for all} \quad t \in [0, T], \tag{.13}$$

then

$$\alpha(t) \leq \mathbf{C}_0 e^{\int_0^t f ds}, \qquad \textit{for all} \quad t \in [0, T]. \tag{.14}$$

Proof. Set

$$\beta(t) := \mathbf{C}_0 + \int_0^t f\alpha ds.$$

Then $\beta' = f\alpha \leq f\beta$, and so

$$\left(\beta\, e^{-\int_0^t f ds}\right)' = (\beta' - f\beta)e^{-\int_0^t f ds} \leq (f\alpha - f\alpha)e^{-\int_0^t f ds} = 0.$$

Therefore

$$\beta(t)e^{-\int_0^t f ds} \leq \beta(0)e^{-\int_0^t f ds} = \mathbf{C}_0,$$

and thus

$$\alpha(t) \leq \beta(t) \leq \mathbf{C}_0 e^{-\int_0^t f ds}.$$

□

.1.2 Doob's Inequality

Theorem .6. *Let $(M_t)_{t \geq 0}$ be a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$. If $p > 1$, $T > 0$ and $\mathbb{E}[|M_T|^p] \leq +\infty$, then we have that*

$$\mathbb{E}\Big[\sup_{t \in [0, T]} |M_t|^p\Big] \leq \left(\frac{p}{p-1}\right)\mathbb{E}[|M_T|^p]. \tag{.15}$$

Proof. See [Protter, 31], Theorem 74, p. 226. □

.1.3 Cauchy-Schwartz Inequality

The following which is an implication of the Cauchy-Schwartz Inequality taken from [Choe, 9].

Theorem .7. *Let $g : [0, t] \to \mathbb{R}^n$. For any $t > 0$ we have that*

$$\left| \int_0^t g\,ds \right|^2 \leq t \int_0^t |g|^2 ds. \tag{.16}$$

.2 Simulation Code

```
N = 10000;
% best N = 10000;
T=7; %Terminal Point
n=20; %number of partitions
lambda = T/n;
u = rand(1,N-1);
delta_X = -log(1-u)/lambda;%generates an exponential
    distribution
%disp(delta_X)
t = zeros(1,N);
t(1) = 0;
for i = 1:N-1
    t(i+1)= t(i) + delta_X(i); %generates the random time
end
%disp(t(i+1))

syms x; %defining 'x' as a symbol for symbolic manipulation
a = @(x) cos(x); %setting a(X(t)) = cos(X(t)) with an anonymous
    function

Y = zeros(1,N);
Y_0 = 1;
Y(1) = Y_0;
for i = 1:N-1
    Y(i+1)= Y(i) + a(Y(i))*delta_X(i);%
end
%disp(Y(i+1))

h = plot(t, Y, '-');
set(h, 'Markersize', 20)
xlabel('t');
ylabel('Y(t)');
```

```
%title("Trajectories of the Euler Poisson Scheme")
```

Bibliography

[1] Applebaum, David. *Lévy processes and stochastic calculus*. Cambridge University Press, 2009 (cit. on p. 11).

[2] Asmussen, Søren and Rosiński, Jan. "Approximations of small jumps of Lévy processes with a view towards simulation". In: *Journal of Applied Probability* 38.2 (2001), pp. 482–493 (cit. on p. 3).

[3] Barndorff-Nielsen, Ole E, Mikosch, Thomas, and Resnick, Sidney I. *Lévy processes: theory and applications*. Springer Science & Business Media, 2012 (cit. on p. 1).

[4] Barndorff-Nielsen, Ole E and Shephard, Neil. *Basics of Lévy processes*. University of Oxford, Department of Economics, 2012 (cit. on p. 11).

[5] Black, Fischer and Scholes, Myron. "The pricing of options and corporate liabilities". In: *Journal of political economy* 81.3 (1973), pp. 637–654 (cit. on pp. 1, 35).

[6] Carr, Peter. "Randomization and the American put". In: *The Review of Financial Studies* 11.3 (1998), pp. 597–626 (cit. on p. 34).

[7] Carr, Peter et al. "The fine structure of asset returns: An empirical investigation". In: *The Journal of Business* 75.2 (2002), pp. 305–332 (cit. on p. 13).

[8] Chan, Terence. "Pricing contingent claims on stocks driven by Lévy processes". In: *Annals of Applied Probability* (1999), pp. 504–528 (cit. on p. 35).

[9] Choe, Geon Ho. *Stochastic analysis for finance with simulations*. Springer, 2016 (cit. on p. 40).

[10] Corcuera, José Manuel et al. "Efficient pricing of contingent convertibles under smile conform models". In: *Available at SSRN 1954671* (2011) (cit. on p. 4).

[11] Dereich, Steffen and Heidenreich, Felix. "A multilevel Monte Carlo algorithm for Lévy-driven stochastic differential equations". In: *Stochastic Processes and their Applications* 121.7 (2011), pp. 1565–1587 (cit. on p. 3).

[12] Doob, Joseph Leo. *Stochastic processes*. Vol. 101. New York Wiley, 1953 (cit. on p. 37).

[13] Feller, William. "Diffusion processes in genetics". In: *Proceedings of the second Berkeley symposium on mathematical statistics and probability*. The Regents of the University of California. 1951 (cit. on p. 1).

[14] Ferreiro-Castilla, Albert, Kyprianou, Andreas E, and Scheichl, Robert. "An Euler–Poisson scheme for Lévy driven stochastic differential equations". In: *Journal of Applied Probability* 53.1 (2016), pp. 262–278 (cit. on pp. 2, 6, 17, 20, 21, 24).

[15] Ferreiro-Castilla, Albert and Schoutens, Wim. “The β-Meixner model”. In: *Journal of Computational and Applied Mathematics* 236.9 (2012), pp. 2466–2476 (cit. on p. 20).

[16] Ferreiro-Castilla, Albert and Van Schaik, Kees. “Applying the Wiener-Hopf Monte Carlo simulation technique for Lévy processes to path functionals”. In: *Journal of Applied Probability* 52.1 (2015), pp. 129–148 (cit. on pp. 13, 14, 20).

[17] Ferreiro-Castilla, Albert et al. “Multilevel Monte Carlo simulation for Lévy processes based on the Wiener–Hopf factorisation”. In: *Stochastic Processes and their Applications* 124.2 (2014), pp. 985–1010 (cit. on pp. 3, 4, 20).

[18] Fisher, Ronald Aylmer. “Tests of significance in harmonic analysis”. In: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 125.796 (1929), pp. 54–59 (cit. on p. 18).

[19] Giles, Michael B. “Multilevel monte carlo path simulation”. In: *Operations Research* 56.3 (2008), pp. 607–617 (cit. on p. 3).

[20] Glasserman, Paul. *Monte Carlo methods in financial engineering*. Vol. 53. Springer Science & Business Media, 2013 (cit. on p. 1).

[21] Jacod, Jean et al. “The approximate Euler method for Lévy driven stochastic differential equations”. In: *Annales de l’IHP Probabilités et statistiques*. Vol. 41. 3. 2005, pp. 523–558 (cit. on p. 2).

[22] Jum, Ernest. “Numerical Approximation of Stochastic Differential Equations Driven by Levy Motion with Infinitely Many Jumps”. In: (2015) (cit. on p. 1).

[23] Kloeden, Peter E and Platen, Eckhard. *Numerical solution of stochastic differential equations*. Vol. 23. Springer Science & Business Media, 2013 (cit. on p. 2).

[24] Koponen, Ismo. “Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process”. In: *Physical Review E* 52.1 (1995), p. 1197 (cit. on p. 13).

[25] Kuznetsov, Alexey. “Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes”. In: *The Annals of Applied Probability* 20.5 (2010), pp. 1801–1830 (cit. on pp. 19, 20).

[26] Kuznetsov, Alexey, Kyprianou, Andreas E, and Pardo, Juan C. “Meromorphic Lévy processes and their fluctuation identities”. In: *The Annals of Applied Probability* 22.3 (2012), pp. 1101–1135 (cit. on pp. 4, 13, 14, 19).

[27] Kuznetsov, Alexey et al. “A Wiener–Hopf Monte Carlo simulation technique for Lévy processes”. In: *The Annals of Applied Probability* 21.6 (2011), pp. 2171–2190 (cit. on pp. 3, 14, 19, 20).

[28] Matache, A-M, Nitsche, P-A, and Schwab, Christoph. “Wavelet Galerkin pricing of American options on Lévy driven assets”. In: *Quantitative Finance* 5.4 (2005), pp. 403–424 (cit. on p. 34).

[29] Mauldon, JG. “Random division of an interval”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 47. 2. Cambridge University Press. 1951, pp. 331–336 (cit. on p. 18).

[30] Merton, Robert C. “On the pricing of corporate debt: The risk structure of interest rates”. In: *The Journal of finance* 29.2 (1974), pp. 449–470 (cit. on p. 35).

[31] Protter, Philip E. “Stochastic differential equations”. In: *Stochastic integration and differential equations*. Springer, 2005 (cit. on pp. 14, 36, 39).

[32] Protter, Philip and Talay, Denis. “The Euler scheme for Lévy driven stochastic differential equations”. In: *The Annals of Probability* 25.1 (1997), pp. 393–423 (cit. on p. 2).

[33] Rong, Situ. *Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering*. Springer New York, 2005 (cit. on pp. 11, 12, 21, 34).

[34] Rubenthaler, Sylvain. “Numerical simulation of the solution of a stochastic differential equation driven by a Lévy process”. In: *Stochastic processes and their applications* 103.2 (2003), pp. 311–349 (cit. on p. 3).

[35] Sato, Ken-iti and Ken-Iti, Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, 1999 (cit. on pp. 6, 7, 14).

[36] Schoutens, Wim and Van Damme, Geert. “The β-variance gamma model”. In: *Review of Derivatives Research* 14.3 (2011), pp. 263–282 (cit. on p. 20).

[37] Stegun, Irene A and Abramowitz, Milton. “Handbook of Mathematical Functions with Formulas”. In: *Graphs and Mathematical Tables* (1970) (cit. on p. 18).

[38] Tankov, Peter. *Financial modelling with jump processes*. Chapman and Hall/CRC, 2003 (cit. on pp. 1, 6, 9, 10, 14, 16, 36).

[39] Xia, Yuan and Giles, Michael B. “Multilevel path simulation for jump-diffusion SDEs”. In: *Monte Carlo and Quasi-Monte Carlo Methods 2010*. Springer, 2012, pp. 695–708 (cit. on p. 3).