Department of physics

- Swinging Attwood's machine -

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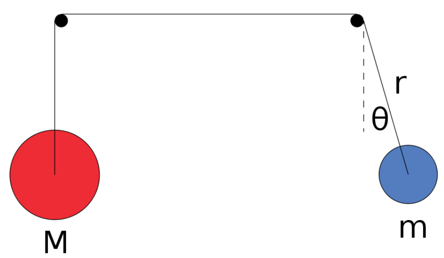

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Introduction

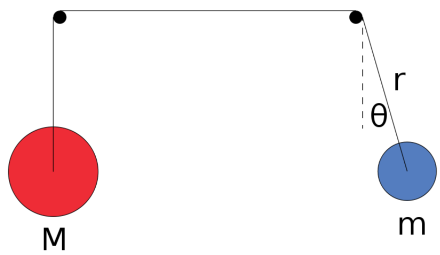
This is the result of the work done in the context of an honour program (PAF) at the Department of Physics of the University of Trento during the fall semester of 2018.

The *Swinging Atwood's machine* is a system similar to the *Atwood's pulley*, except for the degrees of freedom of one of its two masses. Indeed it can swing in a bi-dimensional plane, producing a system that, for some values of parameters and initial conditions, turns out to be chaotic. This machine consists of two masses (m and M, with $M > m$) connected by an inextensible and massless rope, that can move on two radiusless and massless pulleys, moreover the two masses can not collide. In order to study the dynamics of this system, we can not use Newtonian mechanics, because the problem would become unsolvable, but we have to use Lagrangian/Hamiltonian mechanics. In the following report we find the equations that determine the dynamics of the system and study the solutions while solving them numerically.

1 Equations of motion

1.1 Coordinate system

First of all it is necessary to establish a frame of reference $\mathscr{I}$, and, for simplicity, we assume that it is the frame of an observer outside the system that we can consider as inertial. Hence, we set the fixed axes of this cartesian coordinate system, and we call them (X, Y).
In order to consider the generalized coordinates, it is convenient to choose the coordinate r, that represents the distance between the pivot and the mass m free to swing, and the coordinate θ, that describes the angle between r and a hypothetical and ground-perpendicular line. To be precise, the other two coordinates for the mass M should be considered as well, but as we will see later, these two can be written as a function of r and θ.



Now we convert cartesian coordinates to Lagrangian ones:

$$\text{For the mass } m: \begin{cases} x_m = r\sin\theta \\ y_m = -r\cos\theta \end{cases} \qquad \text{For the mass } M: \begin{cases} x_M = 0 \\ y_M = +v \end{cases}$$

where the coordinate v represents the vertical movement of the mass M. Considering that we assume the rope is inextensible we can write $L = r + v + k$ (k is the distance between the two pivots, that stays unchanged for during the motion of the system). The coordinate x for the mass M is always zero for hypothesis, indeed this mass can move only vertically.

1.2 Position, speed and mechanical energy

In order to apply the physics of Lagrange, we have, now, to express the position and speed vectors in the lagrangian coordinates. We denote the versors of the reference system $\mathscr{I}$ as $(\hat{e}_x,\ \hat{e}_y)$, and we call as O, P, O', P' respectively the points in space occupied by the pivot of the mass m, by the mass m, by the pivot of the mass M and by the mass M. We found the relations between them to be

$$P - O = r\sin\theta\,\hat{e}_x - r\cos\theta\,\hat{e}_y$$
$$P' - O' = +v\,\hat{e}_y = +(r+k)\,\hat{e}_y$$

Pursuing the speed of m in the reference frame $\mathscr{I}$, we have to derive the previous relation with respect to time, considering that both r and θ are functions of time and that the versors $(\hat{e}_x,\ \hat{e}_y)$ are not time-dependant, but they are fixed in the reference system $\mathscr{I}$, and therefore they need to be treated as constants during the integration.

$$\vec{v}_m|_{\mathscr{I}} = (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\,\hat{e}_x + (-\dot{r}\cos\theta + r\dot{\theta}\sin\theta)\,\hat{e}_y$$

$$\vec{v}_M|_{\mathscr{I}} = -\dot{r}\,\hat{e}_y$$

From this relation, we can obtain $\vec{v}_m|^2_{\mathscr{I}}$, that is essential for the calculation of the kinetic energy.

$$\vec{v}_m|^2_{\mathscr{I}} =< \vec{v}_m|_{\mathscr{I}}\,, \vec{v}_m|_{\mathscr{I}} >= (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 + (-\dot{r}\cos\theta + r\dot{\theta}\sin\theta)^2 = \dot{r}^2 + \dot{\theta}^2\,r^2$$

$$\vec{v}_M|^2_{\mathscr{I}} =< \vec{v}_M|_{\mathscr{I}}\,, \vec{v}_M|_{\mathscr{I}} >= \dot{r}^2$$

Once these relationships are obtained, the only thing left to do is to calculate the energies involved. The forces present in the system are the gravitational one, the binding reactions of the pivots and the tension exerted by the rope, but the last two forces do no work, so they are not considered in the calculation of energy. Therefore the energies that play a crucial role in the system are the kinetic ones, due to the velocity of the two masses, and the gravitational ones, still due to the two masses. So we can calculate the kinetic energy $\mathscr{T}|_{\mathscr{I}}$ of the system

$$\mathscr{T}_m|_{\mathscr{I}} = \frac{1}{2}m\vec{v}_m|^2_{\mathscr{I}} = \frac{1}{2}m(\dot{r}^2 + \dot{\theta}^2\,r^2)$$

$$\mathscr{T}_M|_{\mathscr{I}} = \frac{1}{2}M\vec{v}_M|^2_{\mathscr{I}} = \frac{1}{2}M(\dot{r}^2)$$

$$\mathscr{T}|_{\mathscr{I}} = \mathscr{T}_m|_{\mathscr{I}} + \mathscr{T}_M|_{\mathscr{I}} = \frac{1}{2}m(\dot{r}^2 + \dot{\theta}^2\,r^2) + \frac{1}{2}M(\dot{r}^2)$$

Now, in the same way, we evaluate the potential gravitational energy $\mathscr{U}|_{\mathscr{I}}$ of the system

$$\mathscr{U}_m|_{\mathscr{I}} = mgy_m = -mgr\cos\theta$$

$$\mathscr{U}_M|_{\mathscr{I}} = Mgy_M = Mg(r+k)$$

$$\mathscr{U}|_{\mathscr{I}} = \mathscr{U}_m|_{\mathscr{I}} + \mathscr{U}_M|_{\mathscr{I}} = -mgr\cos\theta + Mg(r+k)$$

1.3 Lagrangian and equations of motion

Let's name the time variable t, the lagrangian coordinates $q = (q^1, \ldots, q^n)$ and the velocity of these last ones $\dot{q} = (\dot{q}^1, \ldots, \dot{q}^n)$, so the Lagrangian function is as follows

$$\mathscr{L}|_{\mathscr{I}}(t,q,\dot{q}) = \mathscr{T}|_{\mathscr{I}}(t,q,\dot{q}) - \mathscr{U}|_{\mathscr{I}}(t,q)$$

From this we can obtain the motion's equations by solving the following system (Eulero-Lagrange equations)

$$\begin{cases} \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \frac{\partial \mathscr{L}}{\partial q} = 0 \\ \dot{q} = \frac{dq}{dt} \end{cases}$$

This system of equations allows us to obtain the equations of motion only if the forces involved in the system are conservative, otherwise it is necessary to use another extension of the last formula; that is because it is impossible to define a potential if the forces are not conservative. In our case, the lagrangian of the system is

$$\mathscr{L}|_{\mathscr{I}} = \frac{1}{2}m(\dot{r}^2 + \dot{\theta}^2 r^2) + \frac{1}{2}M(\dot{r}^2) + mgr\cos\theta - Mg(r+k)$$

Now we derive the Lagrangian, in order to obtain two systems: each of them is referred to the lagrangian coordinate considered.

$$\begin{cases} \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{r}} - \frac{\partial \mathscr{L}}{\partial r} = 0 \\ \dot{r} = \frac{dr}{dt} \end{cases} \qquad \begin{cases} \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\theta}} - \frac{\partial \mathscr{L}}{\partial \theta} = 0 \\ \dot{\theta} = \frac{d\theta}{dt} \end{cases}$$

First we make the derivation with respect to the lagrange coordinates $r, \dot{r}$.

$$\frac{\partial \mathscr{L}}{\partial \dot{r}} = m\dot{r} + M\dot{r} \qquad \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{r}} = (m+M)\ddot{r}$$
$$\frac{\partial \mathscr{L}}{\partial r} = m\dot{\theta}^2 r + mg\cos\theta - Mg$$

Then we derive with respect to $\theta, \dot{\theta}$.

$$\frac{\partial \mathscr{L}}{\partial \dot{\theta}} = m\dot{\theta}r^2 \qquad \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\theta}} = m\ddot{\theta}r^2 + 2m\dot{\theta}r\dot{r}$$
$$\frac{\partial \mathscr{L}}{\partial \theta} = -mgr\sin\theta$$

Now we just replace all these relations within the systems:

$$\begin{cases} (m+M)\ddot{r} - m\dot{\theta}^2 r - mg\cos\theta + Mg = 0 \\ \dot{r} = \frac{dr}{dt} \end{cases} \qquad \begin{cases} m\ddot{\theta}r^2 + 2m\dot{\theta}r\dot{r} + mgr\sin\theta = 0 \\ \dot{\theta} = \frac{d\theta}{dt} \end{cases}$$

These are, as expected, two second order differential equations (just like Newton's equations). In particular, these are two systems of coupled differential equation, that means that it is not possible to solve these two independently.
Joining the two systems, we obtain a four-coupled-differential-equations system, analytically unsolvable, but numerically solvable.
In order to simplify the notation, we do the following replacement $\mu = \frac{M}{m}$.

Our system becomes

$$\begin{cases} (1+\mu)\,\ddot{r} - \dot{\theta}^2 r + g(\mu - \cos\theta) = 0 \\ \dot{r} = \frac{dr}{dt} \\ \ddot{\theta}\, r + 2\dot{\theta}\,\dot{r} + g\sin\theta = 0 \\ \dot{\theta} = \frac{d\theta}{dt} \end{cases}$$

or, written in its normal form,

$$\begin{cases} \ddot{r} = \frac{\dot{\theta}^2 r - g(\mu-\cos\theta)}{(1+\mu)} \\ \dot{r} = \frac{dr}{dt} \\ \ddot{\theta} = -\frac{2\dot{\theta}\dot{r}+g\sin\theta}{r} \\ \dot{\theta} = \frac{d\theta}{dt} \end{cases}$$

Please note that, once we fix the acceleration, the angular velocity and the angle to zero ($\ddot{\theta} = 0,\ \dot{\theta} = 0,\ \theta = 0$), we obtain the equations of the mere *Atwood's pulley*, that consists of two masses, subject to weight force, connected together by a rope supported by two pulleys (in first approximation two rheonomous). Indeed, the equation of motion for the mass m is

$$\ddot{r} = -g\frac{\mu-1}{\mu+1}$$

1.4 Hamiltonian

Now we consider the Hamiltonian of the system, that correspond to the mechanical energy, and in this particular case it turns out to be a conserved quantity, in fact $\mathscr{L}$ does not explicitly depend on time.

$$\mathscr{H}|_{\mathscr{I}} = \mathscr{T}|_{\mathscr{I}} + \mathscr{U}|_{\mathscr{I}} = \frac{1}{2}m(\dot{r}^2 + \dot{\theta}^2\, r^2) + \frac{1}{2}M\dot{r}^2 - mgr\cos\theta + Mg(r+k)$$

At this point, if we want to carry on the Hamiltonian analysis, we have to convert the coordinates, through the Legendre transformations:

$$\begin{cases} t = t \\ q^k = q^k \\ p_k = \frac{\partial\mathscr{L}}{\partial\dot{q}^k} \end{cases}$$

In our case, θ e r remain the same; but we have to introduce the two new coordinates p_r e p_θ, that are defined through the previous transformations

$$p_r = \frac{\partial\mathscr{L}}{\partial\dot{r}} = \dot{r}\,(m+M)$$
$$p_\theta = \frac{\partial\mathscr{L}}{\partial\dot{\theta}} = m\dot{\theta}r^2$$

Now we have to convert the coordinates in order to obtain a function that depends only on the new coordinates, so we want to move from the form $\mathscr{H}|_{\mathscr{I}}(t,q,\dot{q})$ to $\mathscr{H}|_{\mathscr{I}}(t,q,p)$. We reach this purpose just replacing the new coordinates p_r and p_θ with the old ones $\dot{r}$ and $\dot{\theta}$. The new Hamiltonian is

$$\mathscr{H}|_{\mathscr{I}} = \frac{1}{2}\frac{p_r^2}{m+M} + \frac{1}{2}\frac{p_\theta^2}{mr^2} - mgr\cos\theta + Mgr$$

The following system now allows us to obtain the required differential equations:

$$\begin{cases} \frac{dq^k}{dt} = \frac{\partial \mathscr{H}}{\partial p_k} \\ \frac{dp_k}{dt} = -\frac{\partial \mathscr{H}}{\partial q^k} \end{cases}$$

Using this system, we obtain a first-order-coupled-differential-equations system (Eulero-Lagrange equations are second order equations). This system is analytically unsolvable as well, but it is easier to solve numerically than the Eulero-Lagrange-system. In our computation we use this system. Thus we can express the differential-equations-system in the following way:

$$\begin{cases} \dot{r} = \frac{\partial \mathscr{H}}{\partial p_r} = \frac{p_r}{m+M} \\ \dot{p}_r = -\frac{\partial \mathscr{H}}{\partial r} = \frac{p_\theta^2}{mr^3} + mg\cos\theta - Mg \\ \dot{\theta} = \frac{\partial \mathscr{H}}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_\theta = -\frac{\partial \mathscr{H}}{\partial \theta} = -mgr\sin\theta \end{cases}$$

1.5 Equilibrium points

We want to find the equilibrium points of the problem. Here, in our discussion, we pursue Lyapunov's stability. We can find these points, considering the potential energy; in particular we have to solve the following system

$$\begin{cases} \frac{\partial \mathscr{U}}{\partial r} = 0 \\ \frac{\partial \mathscr{U}}{\partial \theta} = 0 \end{cases}$$

The potential energy of the system is only the gravitational one $\mathscr{U}|_{\mathscr{I}} = -mgr\cos\theta + Mg(r + cost)$, thus

$$\begin{cases} -mg\cos\theta + Mg = 0 \\ mgr\sin\theta = 0 \end{cases}$$

From the system, we obtain the expression $m^2g^2r^2(1-\cos^2\theta) = m^2g^2r^2(1-\mu^2) = 0$, and this vanish for $r = 0$ and for any angle, or when $\cos^2\theta = 1 \Rightarrow \mu = 1 \Rightarrow M = m$. The first solution means, physically, that the distance between the mass m and the pivot is zero, so it can not move as a pendulum. The second solution, instead, means that the system is at equilibrium for any rope length, as long as the masses M and m are equal and, in addition, the angle θ is 0.
Now we have to check whether these two situations represent points of stable or unstable equilibrium. Therefore we can study whether the Hessian is positive or negative

for the two configurations

$$H = \begin{pmatrix} \frac{\partial^2 \mathscr{U}}{\partial r^2} & \frac{\partial^2 \mathscr{U}}{\partial r \partial \theta} \\ \frac{\partial^2 \mathscr{U}}{\partial \theta \partial r} & \frac{\partial^2 \mathscr{U}}{\partial \theta^2} \end{pmatrix} = \begin{pmatrix} 0 & mg \sin\theta \\ mg \sin\theta & mgr \cos\theta \end{pmatrix}$$

For the first point, in which we have $r = 0$, we obtain the determinant of the Hessian equal to $-m^2 g^2 \sin^2\theta$ which is negative, therefore this is a saddle point, so unstable equilibrium. For the second point we have $\det(H) = 0$; hence the Hessian matrix is undefined and for this reason we can conclude nothing mathematically.

2 System configurations

In order to solve the equation of motion, obtained previously, we need four initial conditions and we consider $r(0)$, $\dot{r}(0)$, $\theta(0)$, $\dot{\theta}(0)$. We also consider the ratio between the masses as a known and fixed parameter during the numerical solution of the problem. Once the equation is solved, the calculator returns the hourly laws for r, $\dot{r}$, θ, $\dot{\theta}$, and from this we can obtain the trajectories of the two masses. Obviously these trajectories change as the initial conditions change, so we decide to solve the different trajectories of the system depending on the ratio of the masses μ, keeping the initial conditions unchanged.

For the computation of the differential equations, we use the Runge-Kutta algorithm, implemented by MATLAB exploiting the function *ode45*, keeping a relative error less than 10^{-12}. The two figures below show respectively the evolution of the variables $r(t)$ and $\theta(t)$ as a function of time (for $20s$) with $\mu = 2$.

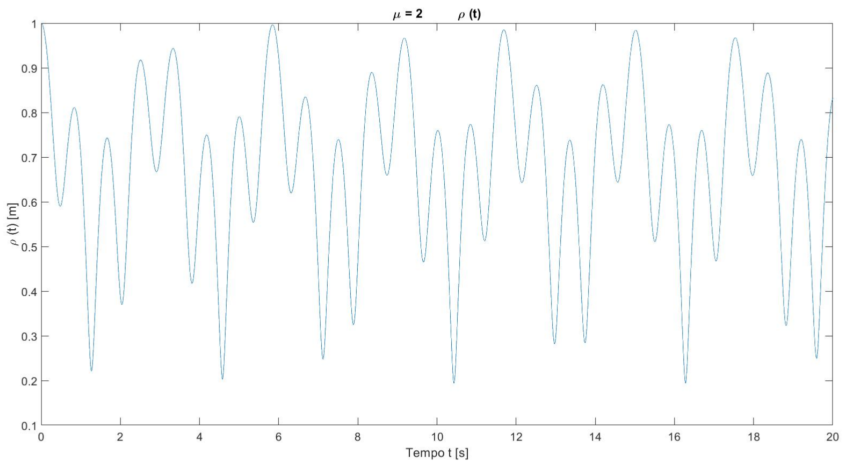


Figure 1: Graph of r as a function of time, for $20s$.

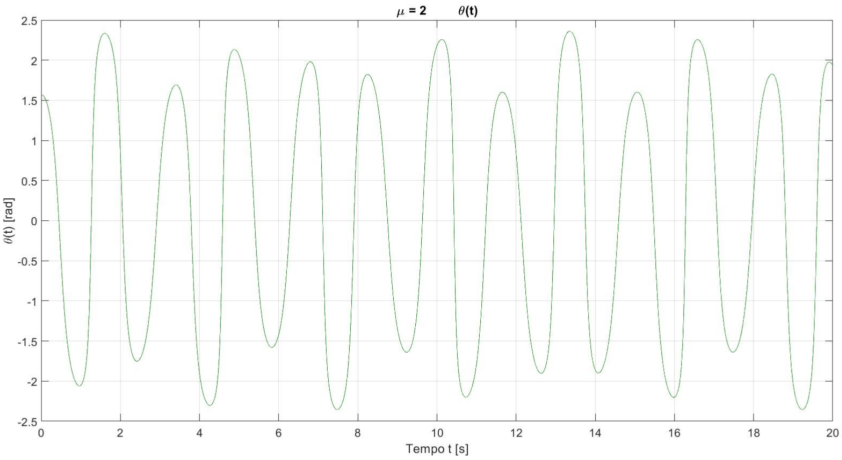


Figure 2: Graph of θ as a function of time, for $20s$.

The following figure represents, instead, the trajectory of the mass m on the x-y plane.

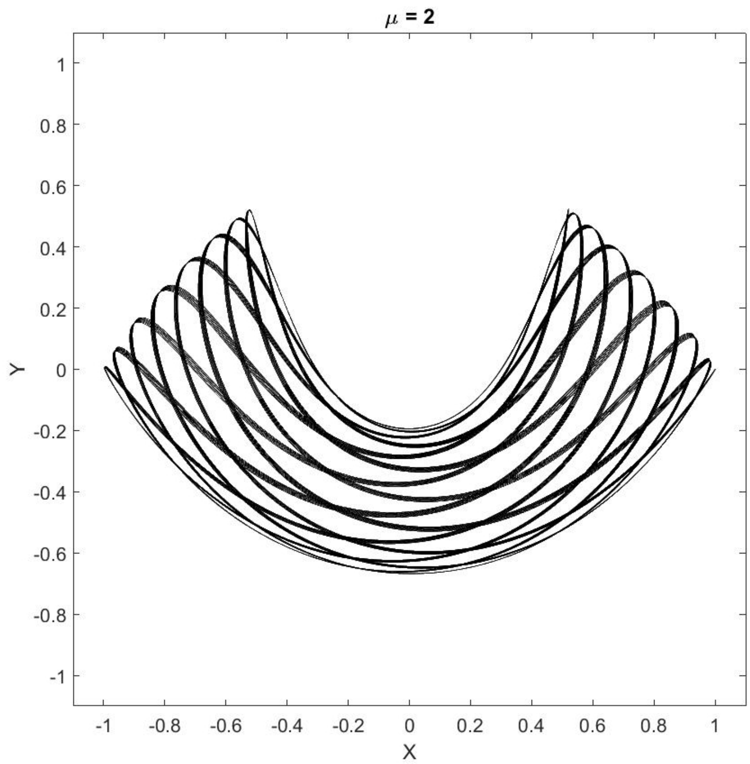


Figure 3: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 2$, for a time of $t = 100s$.

Please find below other plots that show some examples of trajectories. In all the simulations only the ratio of the masses μ vary and the initial conditions were set as follow

$$\rho(0) = 1\,(m), \quad \dot{\rho}(0) = 0\,(m/s), \quad \theta(0) = \pi/2\,(rad) \quad \dot{\theta}(0) = 0\,(rad/s)$$

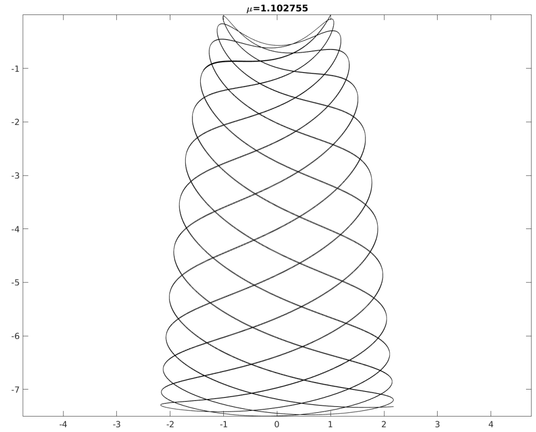


Figure 4: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 1.102755$, for a time of $t = 100\,s$.

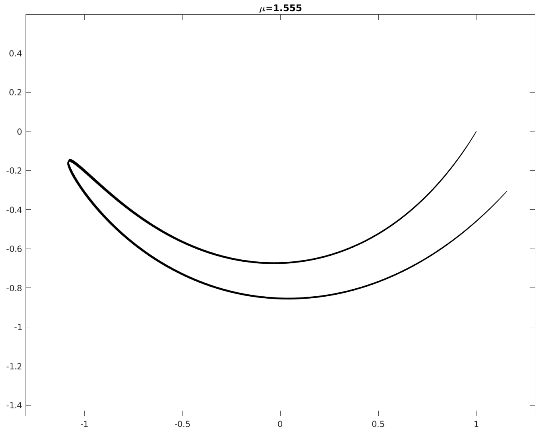


Figure 5: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 1.555$, for a time of $t = 100\,s$.

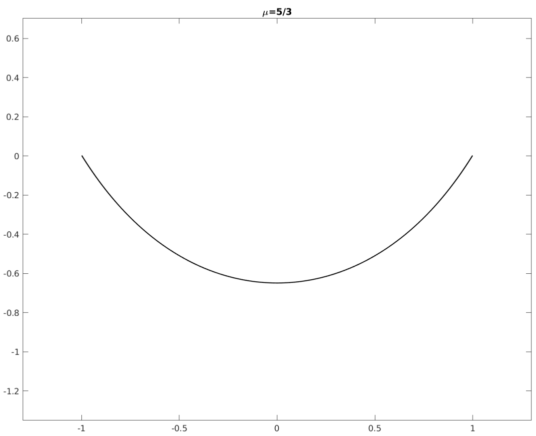


Figure 6: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 5/3$, for a time of $t = 100\,s$, smile trajectory.

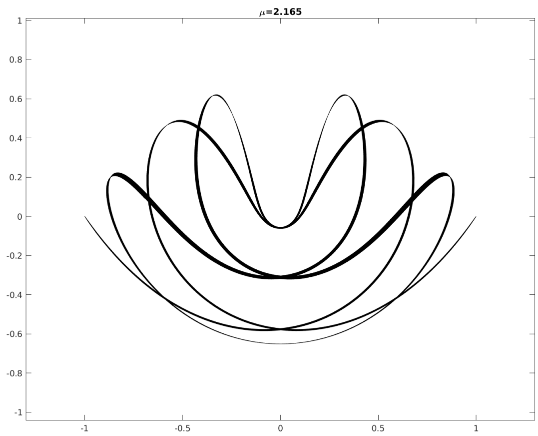


Figure 7: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 2.165$, for a time of $t = 100\,s$.

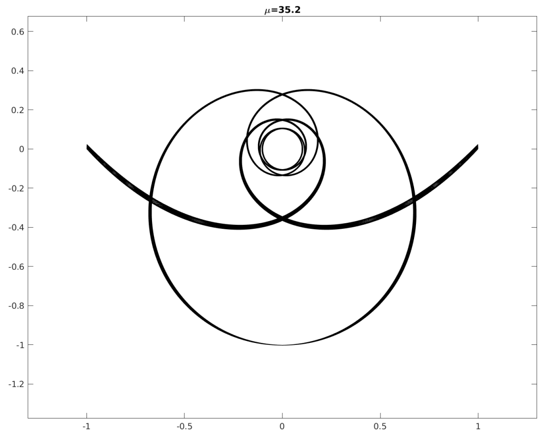


Figure 8: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 35.2$, for a time of $t = 100\,s$.

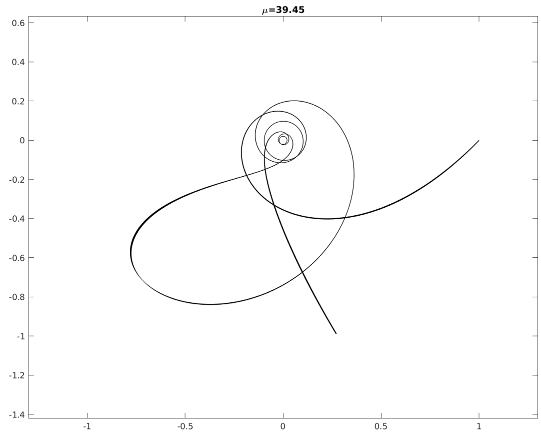


Figure 9: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 39.45$, for a time of $t = 100\,s$.

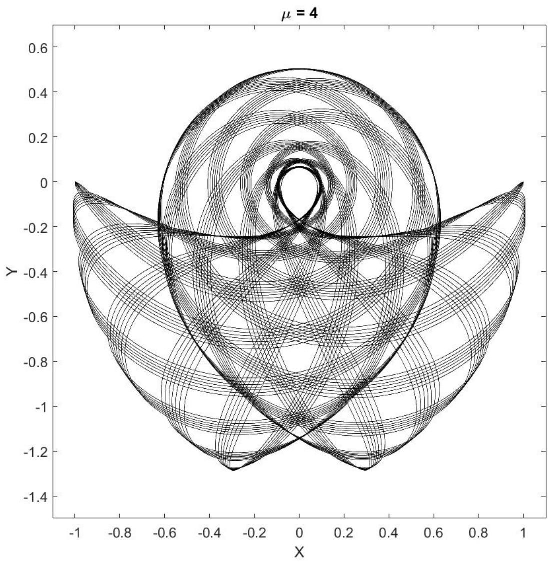


Figure 10: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 4$, for a time of $t = 100\,s$.

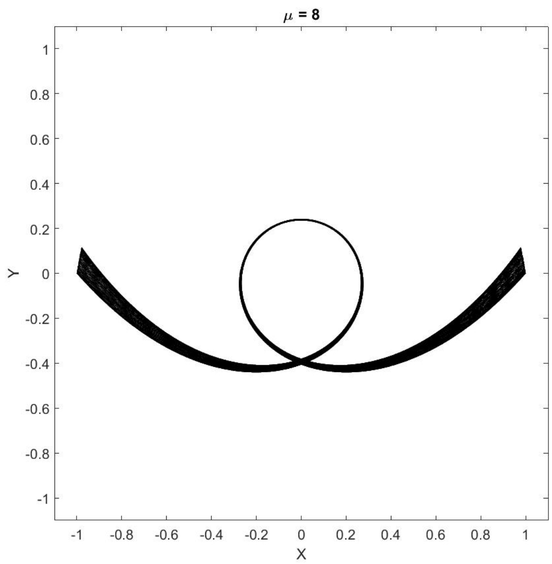


Figure 11: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 8$, for a time of $t = 100\,s$.

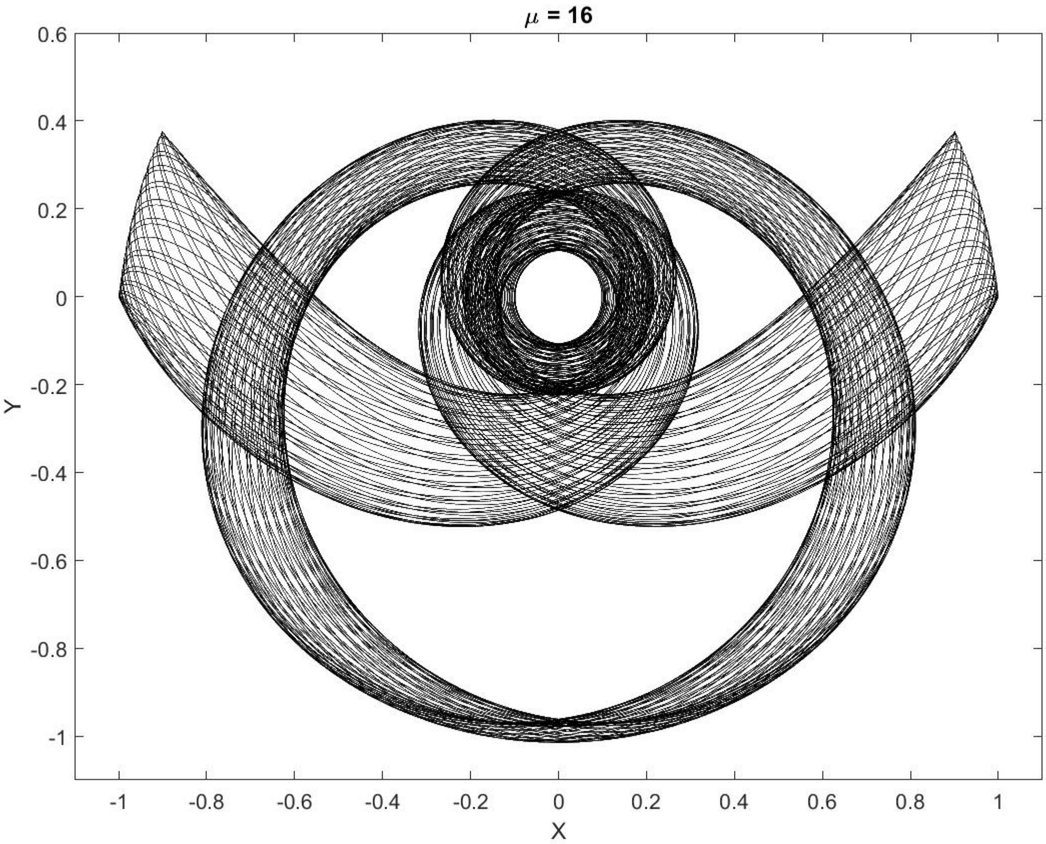


Figure 12: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 16$, for a time of $t = 100\,s$.

3 Limit

One can demonstrate that, for an initial set of values for r and θ, the trajectories of the mass m can be enclosed by conic sections dependent on the initial parameters. These conic sections can be obtained considering the conservation of energy. The total energy, calculated previously, is

$$E|_{\mathscr{I}} = \frac{1}{2}m(\dot{r}^2 + \dot{\theta}^2 r^2) + \frac{1}{2}M\dot{r}^2 - mgr\cos\theta + Mgr$$

We start considering the initial conditions $r = r_0$ and $\theta = \theta_0$. We are in the approximation in which the velocities are zero (starting situation), with $\dot{r} = 0$ and $\dot{\theta} = 0$. In this situation the expression of the total energy become:

$$E|_{\mathscr{I}} = -mgr_0\cos\theta_0 + Mgr_0$$

After making some appropriate substitutions we can obtain the following form for the energy:

$$E|_{\mathscr{I}} = mgr_0(-\cos\theta + \mu) \Leftrightarrow \frac{E}{mg\mu} = r_0\left(1 - \frac{\cos\theta_0}{\mu}\right)$$

At this point we can compare it to the conic-sections' equations in polar coordinates

$$\rho(1 + e\cos\varphi) = l$$

where we can use the following substitution in order to adapt this equation to our case:

$$\rho \equiv r_0, \qquad \varphi \equiv \theta_0, \qquad e \equiv -\frac{1}{\mu}, \qquad l \equiv \frac{E}{mg\mu}$$

The e represents the eccentricity, this parameter determines the shape of the conic sections, that could be a circle ($e = 0$), an ellipse ($0 < e < 1$), a parabola ($e = 1$) or an hyperbola ($e > 1$). Note that we are in the case $e = 0$ (a circle) only if $\mu \to \infty$, so either if M has an infinite mass and $m \to 0$ (degenerate circumference) or if m is a finite mass and $M \to \infty$; physically this situation represents a pendulum with r fixed and constant (simple pendulum case).
We plot below some examples of trajectories of the system for different configurations, with their limit-conic-sections.

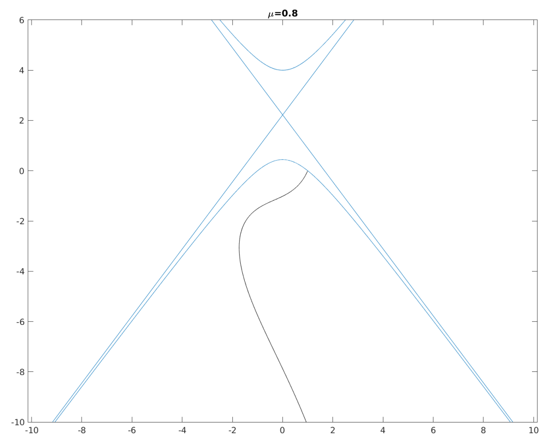


Figure 13: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 0.8$, for a time of $t = 100\,s$, hyperbola.

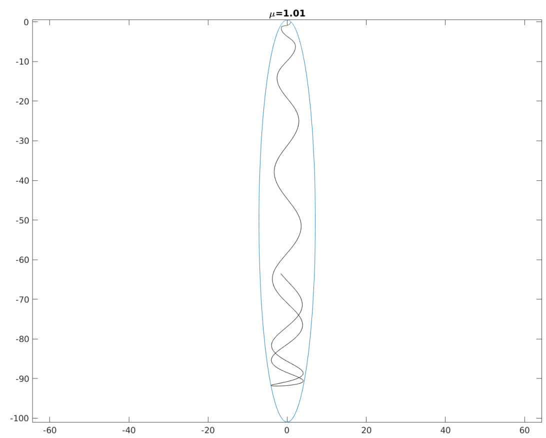


Figure 14: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 1.01$, for a time of $t = 100\,s$, ellipse.

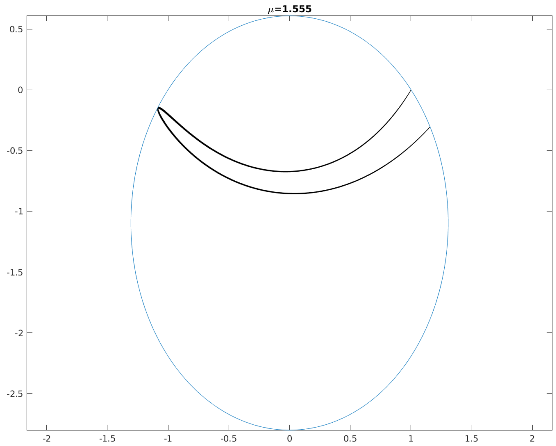


Figure 15: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 1.555$, for a time of $t = 100\,s$.

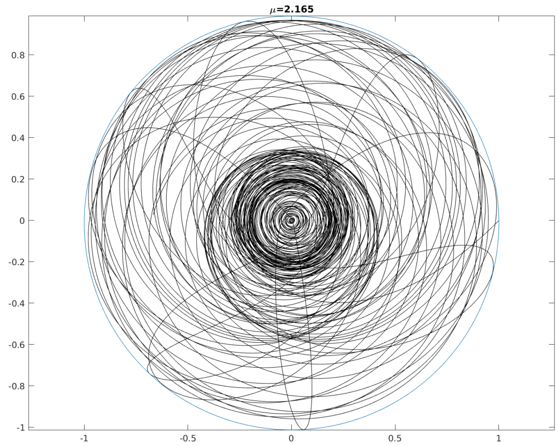


Figure 16: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 2.165$, for a time of $t = 100\,s$.

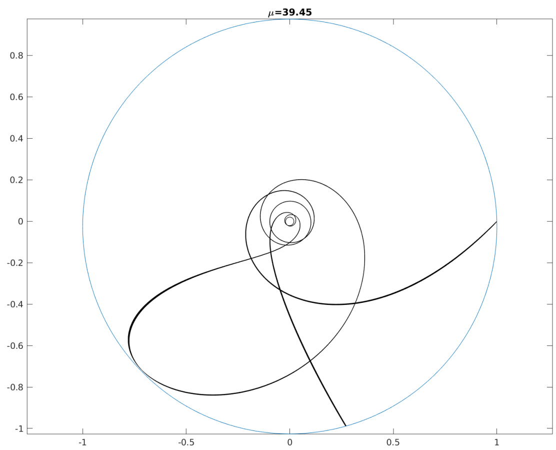


Figure 17: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 39.45$, for a time of $t = 100\,s$.

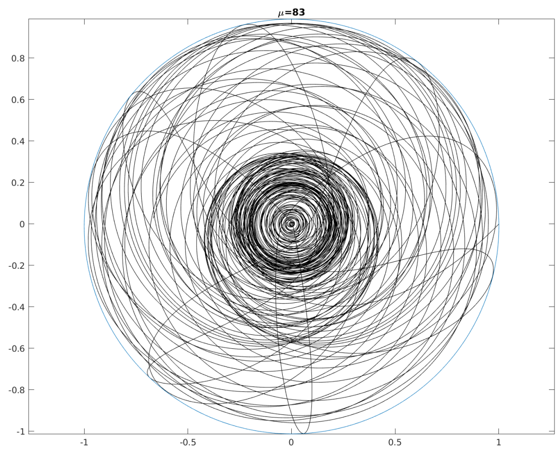


Figure 18: Trajectory of the mass m on the plane x-y, in the configuration $\mu = 83$, for a time of $t = 100\,s$.

4 Chaotic system

The *Swinging-Atwood's machine* turns out to be a chaotic system for some configurations. In fact, starting from slightly different initial conditions, the system develops over time in a completely different and unpredictable way. In order to determine how much chaotic a system is, with respect to the parameters, we consider the Lyapunov coefficient.
The following analysis is done in order to determine the degree of chaos in the system. Firstly we examine the motion of the system considering some initial conditions, then we re-examine the same system configuration, but considering other slightly different initial conditions. Next we plot the difference between these two hourly laws of the lagrangian coordinates as a function of time (different initial conditions). We note

that the trend of this difference $\delta(t)$ turns out to be exponential with respect to time, following a law like this

$$\delta(t) \approx \delta(0)e^{\lambda t},$$

where λ is precisely the exponent of Lyapunov. Once we find $\delta(t)$, we can obtain λ making a linear regression of the following equation

$$log\left(\frac{\delta(t)}{\delta(0)}\right) = \lambda t.$$

Some graphs show the trajectory of the mass m, on the bi-dimensional plane x-y, starting from the two different initial conditions of $\delta(0) = 2 \times 10^{-8}$, in order to show the differences in trajectories when the system is or is not in a chaotic configuration. The evolution time-span is about 35 seconds.

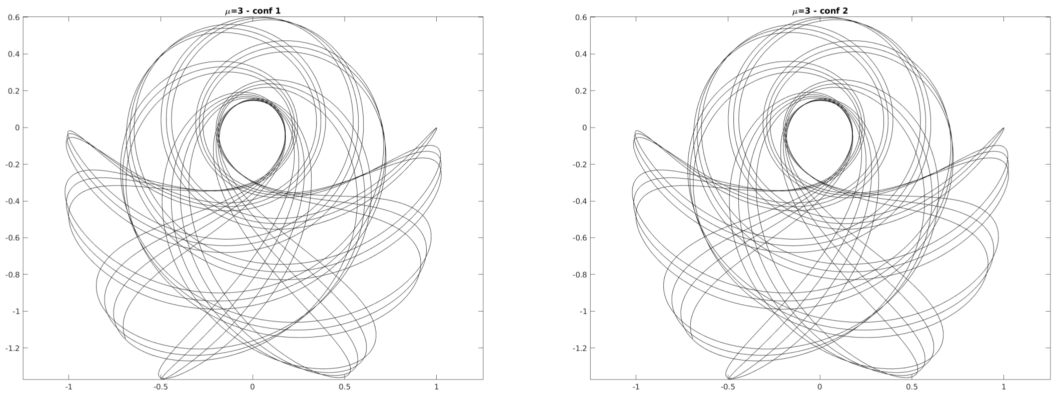


Figure 19: System comparison in the two configurations with $\mu = 3$ which difference in the initial conditions is about $\delta(0) = 2 \times 10^{-8}\,s$.

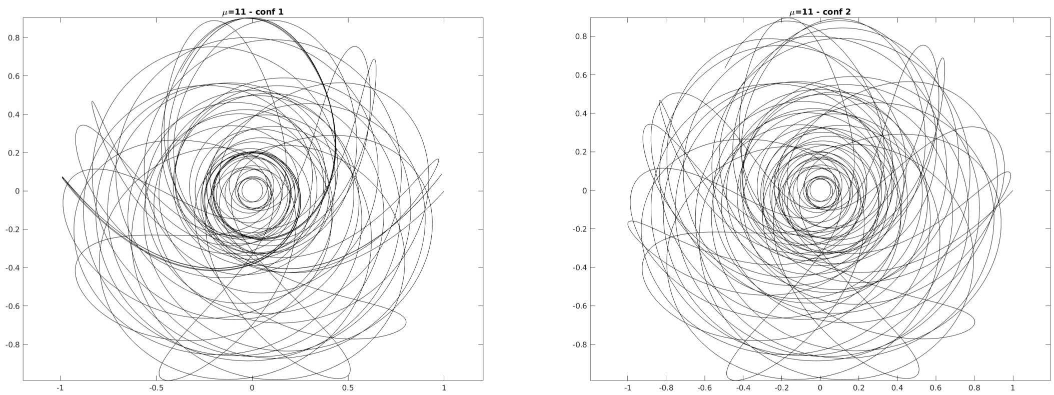


Figure 20: System comparison in the two configurations with $\mu = 11$ which difference in the initial conditions is about $\delta(0) = 2 \times 10^{-8}\,s$.

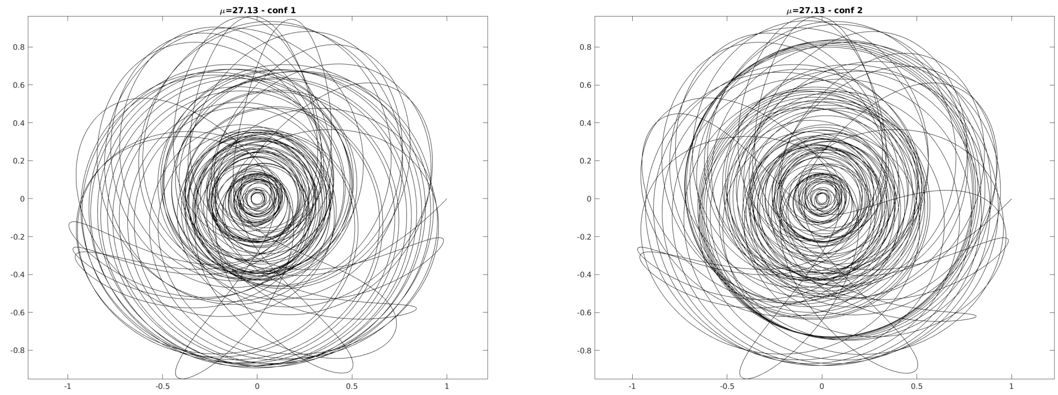


Figure 21: System comparison in the two configurations with $\mu = 27.13$ which difference in the initial conditions is about $\delta(0) = 2 \times 10^{-8}\,s$.

The figures below show respectively to the mass values shown above, the evolution of δ over time, with the estimated linear regression (the vertical scale is logarithmic). We note that from a precise moment onward, typically after about $25 - 30$ seconds, δ remains constant; this is due to the fact that our system is limited (by the total rope's lenght L), so the difference between the systems cannot diverge over time.

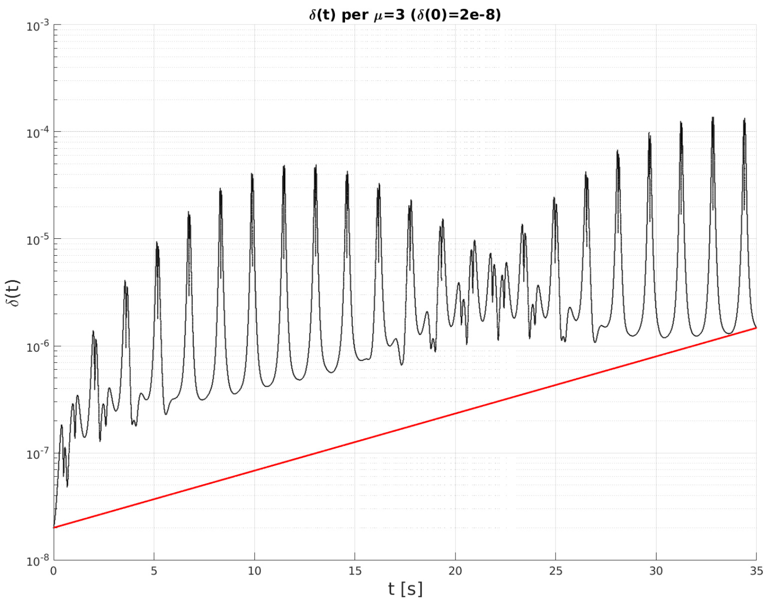


Figure 22: δ as a function of time for $\mu = 3$ with the respective linear regression.

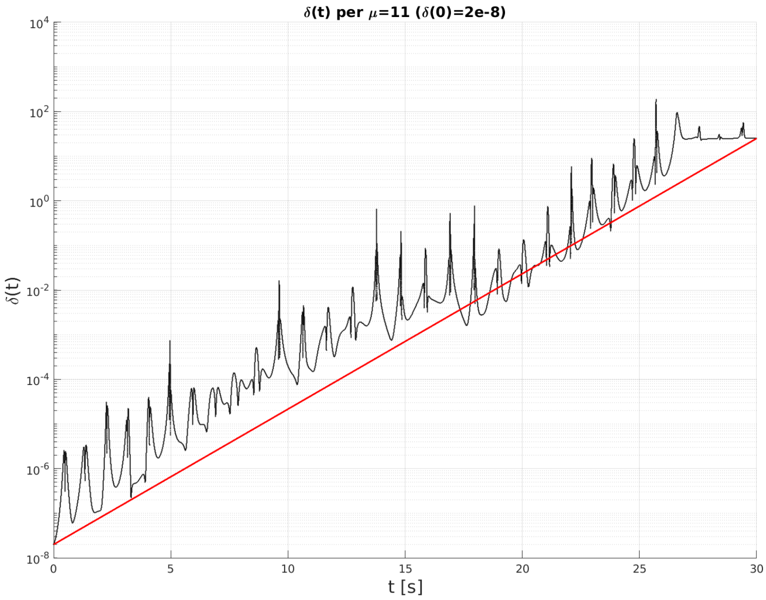


Figure 23: δ as a function of time for $\mu = 11$ with the respective linear regression.

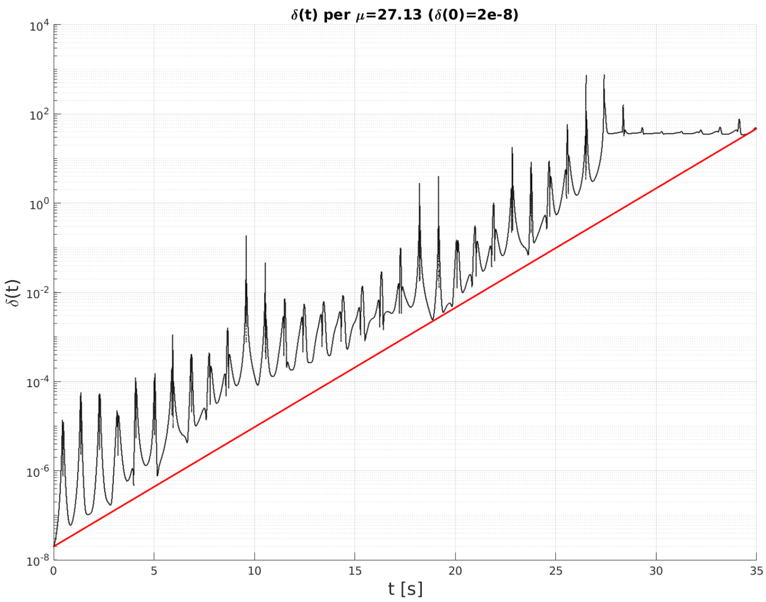


Figure 24: δ as a function of time for $\mu = 27.13$ with the respective linear regression.

We can observe that sometimes the trajectory differences are very important, whereas some other times they are not that big; this depends on the relationship between the masses, which apparently randomly influences the chaotic nature of our dynamic system. We can better highlight this fact observing the following graphs that represent the Lyapunov exponent of the system as a function of the ratio between the masses. For the estimation of the exponent, the timespan is set to 35 seconds, for all the values of μ. We consider 1000 values for the ratio of the masses: from 0.1 to 100, evenly

distributed.
In the figures below, in the first case we consider $\delta(0) = 2 \times 10^{-8}$ uniform for all the lagrangian coordinates, in the second case $\delta(0) = 2 \times 10^{-6}$, and in the last two graphs $\delta(0) = 10^{-8}$, keeping the initial angle and angular velocity unchanged, but varying at first only the radius, and then only the radial velocity.

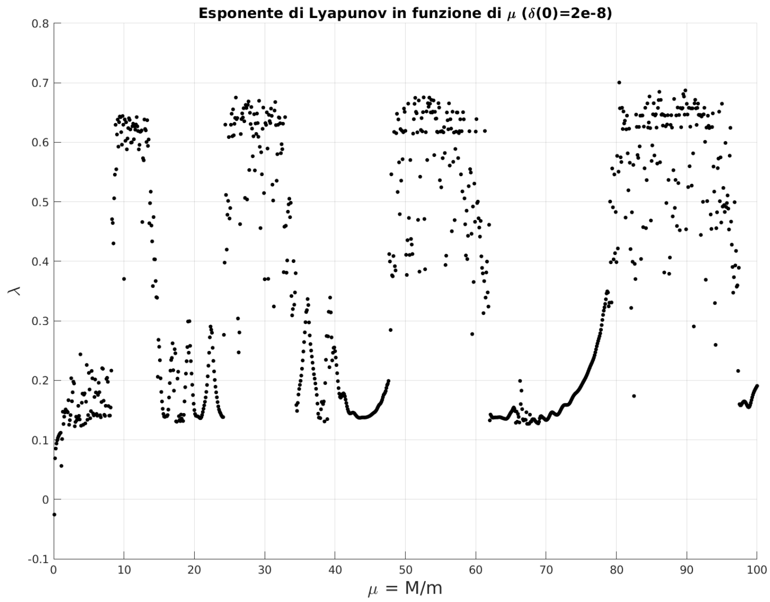


Figure 25: Lyapunov exponent as a function of masses ratio, the timespan of the initial conditions is about 2×10^{-8}.

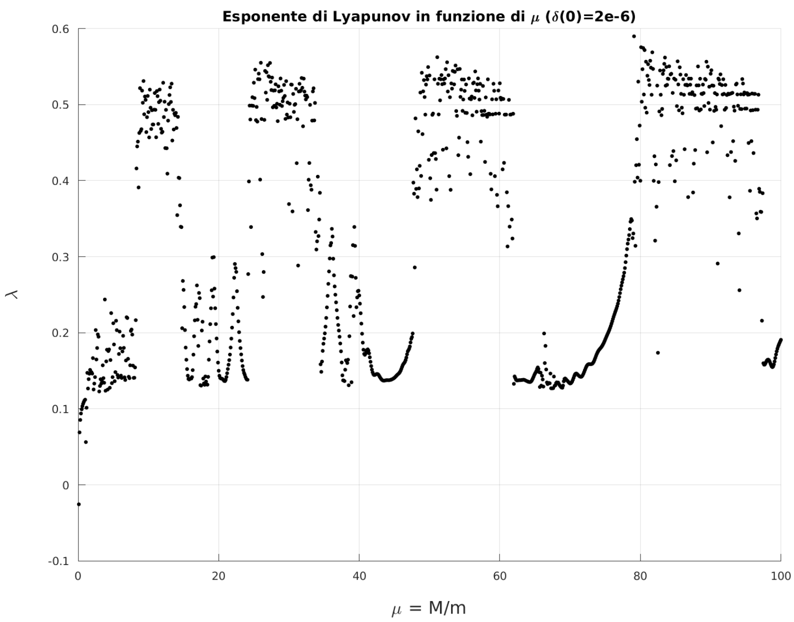


Figure 26: Lyapunov exponent as a function of masses ratio, the timespan of the initial conditions is about 2×10^{-6}.

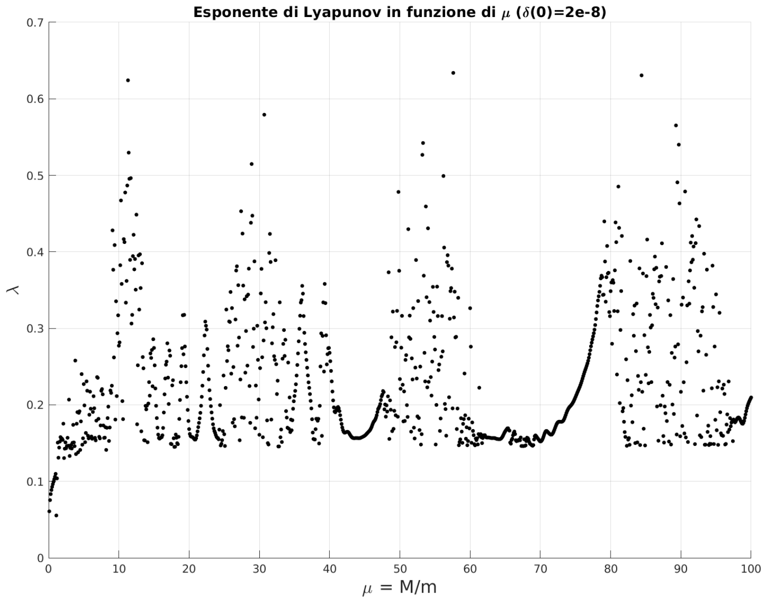


Figure 27: Lyapunov exponent as a function of masses ratio, the timespan of the initial radius is about 1×10^{-8}.

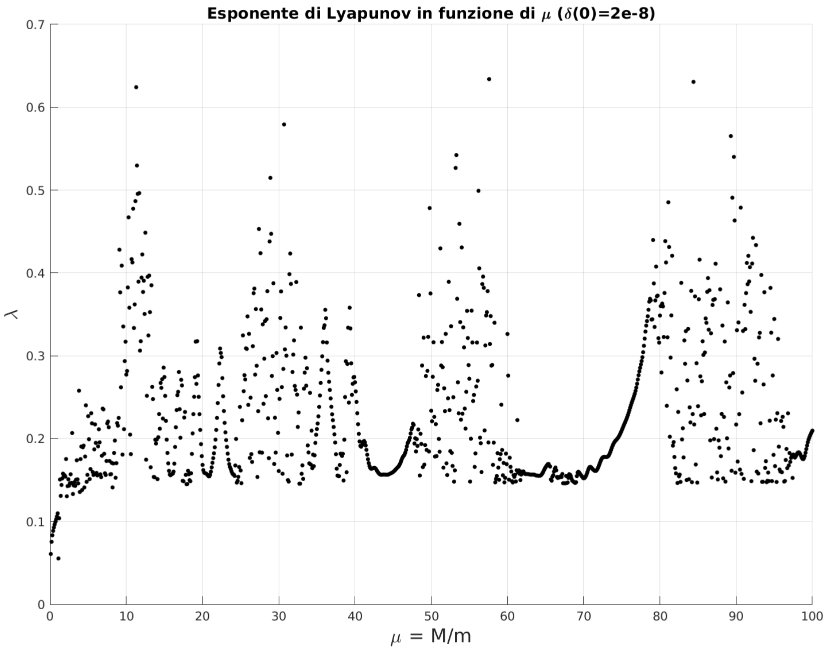


Figure 28: Lyapunov exponent as a function of masses ratio, the timespan of the initial radial velocity is about 1×10^{-8}.

We note that, for values of $\mu < 1$, the system is not so unpredictable (the exponent is negative), indeed if the mass m is bigger than M (or slightly bigger), the tension of the rope, generated by the centripetal force, is immediately overcome by the gravity, causing the system degeneration into one of the equilibrium configurations described

above.
As for the other values of μ, we see the system is very chaotic for some sets of values that change slightly as we change the configuration of the $\delta(0)$.