$Proposition.\ \forall x, y \in \mathbb{R}\ with\ x < y, \exists q \in \mathbb{Q}\ s.t.\ x < q < y.$

$Proof:$

$By\ Archimedean\ Principle, \exists n \in \mathbb{N}\ s.t.\ n > \dfrac{1}{y-x}, where\ x < y \Rightarrow y - x > 0.$

$\therefore nx < [nx] + 1 \leq nx + 1 < ny, x < q \coloneqq \dfrac{[nx]+1}{n} < y.$

$(Motivation: If\ y - x > 1, x < q \coloneqq [x] + 1 \leq x + 1 < y; thus, find\ an\ adequately\ large\ n \in \mathbb{N}\ to\ make\ n(y-x) > 1 \equiv ny - nx > 1.)$

$Corollary.\ \forall x, y \in \mathbb{R}\ with\ x < y, \exists r \in \mathbb{R} \setminus \mathbb{Q}\ s.t.\ x < r < y.$

$Proof:$

$By\ the\ preceding\ proposition, \exists q \in \mathbb{Q} \cap (x - \sqrt{2}, y - \sqrt{2})\ s.t.\ x - \sqrt{2} < q < y - \sqrt{2}, \therefore x < r \coloneqq q + \sqrt{2} < y.$