$In\ this\ document, the\ Cantor - Bernstein\ Theorem\ is\ frequently\ applied,$
$and\ duducing\ subsequent\ conclusions\ adopts\ these\ symbols\ that\ \mathbb{N}\ denotes\ \mathbb{Z}^+\ and$
$\mathcal{P}^n(R)\ denotes\ \underbrace{\mathcal{P}(\mathcal{P}(\mathcal{P}(\cdots(\mathcal{P}(\mathbb{R}))\cdots)))}_{n-times}, \forall n \in \mathbb{N} \cup \{0\}.$

$(i)\ Bijective\ f:$

$$\prod_{i=1}^{\infty} \mathbb{N} = \mathbb{N}^{\mathbb{N}} \longrightarrow (0,1) \setminus \mathbb{Q} : f((a_n)_{n=1}^{\infty}) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

$(ii)\ Bijective\ \theta:$

$$\mathbb{N}^{\mathbb{N}} \longrightarrow \mathcal{P}(\mathbb{N}) \setminus \{\mathcal{S} \subseteq \mathbb{N} \mid |\mathcal{S}| < \infty\} : \theta((a_n)_{n=1}^{\infty}) = \left\{ \sum_{i=1}^{n} a_i \mid n, a_i's \in \mathbb{N} \right\}.$$

$In\ actuality, |[0,1)| = |\mathcal{P}(\mathbb{N}) \setminus K| \leq |\mathcal{P}(\mathbb{N})| = |\{0,1\}^{\mathbb{N}}| \leq |A^{\mathbb{N}}| \leq |\mathbb{N}^{\mathbb{N}}| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})| = |\mathcal{P}(\mathbb{N})| \leq |[0,1)|.$
$(where\ 2 \leq |A| < \infty\ and\ K = \{\mathcal{S} \subseteq \mathbb{N} \mid \exists k \in \mathbb{N}\ s.t.\ [k, \infty) \cap \mathbb{N} \subseteq \mathcal{S}\};$
$consider\ the\ bijection\ f : [0,1) \longrightarrow \mathcal{P}(\mathbb{N}) \setminus K : f(x) = \{j \in \mathbb{N} \mid b_j = 1\}$
$via\ x's\ binary\ decimal\ expansion\ \sum_{j=1}^{\infty} \frac{b_j}{2^j}\ with\ b_j's \in \{0,1\},$
$where\ the\ binary\ representation\ ending\ in\ infinitely\ consecutive\ (repeating)\ 1's\ is\ excluded$
$so\ that\ the\ binary\ representation\ is\ unique,$
$and\ the\ injection\ g : \mathcal{P}(\mathbb{N}) \longrightarrow [0,1) : g(T) = \sum_{i=1}^{\infty} \frac{d_i}{10^i}, where\ d_i = \mathcal{X}_T(i),$
$and\ in\ reality, g\ is\ injective \Rightarrow |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| \leq |[0,1)| \leq |\mathbb{R}| \Rightarrow |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = |(0,1)|$
$by\ the\ uncountability\ of\ \mathcal{P}(\mathbb{N})\ and\ the\ Continuum\ Hypothesis.)$

$(iii)\ |\mathcal{P}(\mathbb{R})| = |\{0,1\}^{\mathbb{R}}| \leq |A^{\mathbb{R}}| \leq |\mathbb{Z}^{\mathbb{R}}| \left(= \left| \prod_{i \in \mathbb{R}} \mathbb{Z} \right| \right) \leq |\mathcal{P}(\mathbb{R} \times \mathbb{Z})| = |\mathcal{P}(\mathbb{R})|.$
$(where\ 2 \leq |A| < \infty.\ \{0,1\}^{\mathbb{R}} \hookrightarrow \mathbb{Z}^{\mathbb{R}}\ as\ monoids\ with\ common\ product.)$

$(iv)\ |\mathbb{R}^{\mathbb{N}}| = |(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}| = |\{0,1\}^{\mathbb{N} \times \mathbb{N}}| = |\{0,1\}^{\mathbb{N}}| = |\mathbb{R}|.$

$(v)\ |\mathbb{R}^{\mathbb{R}}| = |(\{0,1\}^{\mathbb{N}})^{\mathbb{R}}| = |\{0,1\}^{\mathbb{N} \times \mathbb{R}}| = |\{0,1\}^{\mathbb{R}}| = |\mathcal{P}(\mathbb{R})|.$

$(vi)\ |\mathcal{P}(\mathbb{R})| \leq |\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})| \leq |\mathcal{P}(\mathbb{R})^{\mathbb{N}}| \leq |\mathcal{P}(\mathbb{R})^{\mathbb{R}}| = |(\{0,1\}^{\mathbb{R}})^{\mathbb{R}}| = |\{0,1\}^{\mathbb{R} \times \mathbb{R}}| = |\{0,1\}^{\mathbb{R}}| = |\mathcal{P}(\mathbb{R})|.$

$(vii)\ Bijective\ \pi : (\{0,1\}^X)^Y \longrightarrow \{0,1\}^{X \times Y} (or\ \{0,1\}^{Y \times X}) : f \mapsto \pi(f).$
$(where\ \forall f \in (\{0,1\}^X)^Y, \forall (x,y) \in X \times Y, f(y) \in \{0,1\}^X, \pi(f)(x,y) \coloneqq f(y)(x) \in \{0,1\}, \pi(f) \in \{0,1\}^{X \times Y}.)$

$(viii)\ (supplement)\ Bijective\ f:[0,1]\times[0,1]\longrightarrow[0,1]:f(x,y)=\begin{cases}\sum\limits_{i=1}^{\infty}\dfrac{c_i}{10^i}, if\ x^2+y^2\neq 0;\\ 0, if\ x=y=0;\end{cases}$

$where\ c_i=0\ for\ all\ odd\ (or\ even)\ i\ and\ c_j\neq 0\ for\ some\ even\ (or\ odd)\ j$

$if\ x=0\neq y\ (or\ y=0\neq x), respectively; x=\sum\limits_{i=1}^{\infty}\dfrac{a_i}{10^i}, y=\sum\limits_{i=1}^{\infty}\dfrac{b_i}{10^i}, c_i=\begin{cases}a_{\frac{i+1}{2}}, if\ i\ is\ odd;\\ b_{\frac{i}{2}}, otherwise;\end{cases}$

$for\ some\ a_i's, b_i's\ in\ \{0,1,2,3,\ldots,9\}\ and\ all\ i\in\mathbb{N}\ if\ x\neq 0\neq y.$

$Proof:$

$\forall t\in(0,1], the\ decimal\ representation\ of\ t\ ending\ in$

$infinitely\ consecutive\ (repeating)\ 0's\ is\ excluded\ so\ that\ the\ decimal\ representation\ is\ unique.$

$\forall(x,y),(x',y'),r\in[0,1], x=0\ or\sum\limits_{i=1}^{\infty}\dfrac{a_i}{10^i}, y=0\ or\sum\limits_{i=1}^{\infty}\dfrac{b_i}{10^i}, x'=0\ or\sum\limits_{i=1}^{\infty}\dfrac{a_i'}{10^i}, y'=0\ or\sum\limits_{i=1}^{\infty}\dfrac{b_i'}{10^i},$

$f(x',y')=0\ or\sum\limits_{i=1}^{\infty}\dfrac{c_i'}{10^i}, r=0\ or\sum\limits_{i=1}^{\infty}\dfrac{s_i}{10^i}\ for\ some\ a_i's, b_i's,(a_i')'s,(b_i')'s, s_i's,(c_i')'s\ in\ \{0,1,2,3,\ldots,9\}.$

$(1)\ Suppose\ (x,y)=(x',y'). If\ x=y=0, x'=y'=0, f(x,y)=f(x',y')=0; if\ x=0\neq y, x'=0\neq y',$

$c_i=0=c_i'\ for\ all\ odd\ i\ and\ c_i=b_{\frac{i}{2}}=b'_{\frac{i}{2}}=c_i'\ for\ all\ even\ i, \therefore c_i=c_i', \forall i\in\mathbb{N}, f(x,y)=f(x',y');$

$if\ y=0\neq x, y'=0\neq x', c_i=0=c_i'\ for\ all\ even\ i\ and\ c_i=a_{\frac{i+1}{2}}=a'_{\frac{i+1}{2}}=c_i'\ for\ all\ odd\ i,$

$\therefore c_i=c_i', \forall i\in\mathbb{N}, f(x,y)=f(x',y'); if\ x\neq 0\neq y, x'\neq 0\neq y',$

$c_i=a_{\frac{i+1}{2}}=a'_{\frac{i+1}{2}}=c_i'\ for\ all\ odd\ i\ and\ c_i=b_{\frac{i}{2}}=b'_{\frac{i}{2}}=c_i'\ for\ all\ even\ i,$

$\therefore c_i=c_i', \forall i\in\mathbb{N}, f(x,y)=f(x',y'), f\ is\ well-defined.$

$(2)\ Suppose\ f(x,y)=f(x',y'). If\ f(x,y)=0=f(x',y'), x=y=0=x'=y', (x,y)=0=(x',y'); if\ not,$

$f(x,y)=f(x',y')\neq 0, c_i=c_i', \forall i\in\mathbb{N}.$

$\therefore x=0=x'\wedge b_k=c_{2k}=c_{2k}'=b_k', \forall k\in\mathbb{N}$

$if\ c_i=0=c_i'\ for\ all\ odd\ i\wedge c_j=c_j'\neq 0\ for\ some\ even\ j\Rightarrow(x,y)=(x',y');$

$or\ y=0=y'\wedge a_k=c_{2k-1}=c_{2k-1}'=a_k', \forall k\in\mathbb{N}$

$if\ c_i=0=c_i'\ for\ all\ even\ i\wedge c_j=c_j'\neq 0\ for\ some\ odd\ j\Rightarrow(x,y)=(x',y');$

$or\ a_i=c_{2k-1}=c_{2k-1}'=a_i'\wedge b_i=c_{2k}=c_{2k}'=b_i', \forall i\in\mathbb{N}$

$if\ c_i=c_i'\neq 0\ for\ some\ odd\ i\wedge c_j=c_j'\neq 0\ for\ some\ even\ j\Rightarrow(x,y)=(x',y'), f\ is\ injective.$

$(3)\ If\ r=0, f(0,0)=r; if\ s_i=0\ for\ all\ odd\ i\wedge s_j\neq 0\ for\ some\ even\ j, f\left(0,\sum\limits_{i=1}^{\infty}\dfrac{s_{2i}}{10^i}\right)=r;$

$if\ s_i=0\ for\ all\ even\ i\wedge s_j\neq 0\ for\ some\ odd\ j, f\left(\sum\limits_{i=1}^{\infty}\dfrac{s_{2i-1}}{10^i},0\right)=r;$

$if\ s_i\neq 0\ for\ some\ odd\ i\wedge s_j\neq 0\ for\ some\ even\ j, f\left(\sum\limits_{i=1}^{\infty}\dfrac{s_{2i-1}}{10^i},\sum\limits_{i=1}^{\infty}\dfrac{s_{2i}}{10^i}\right)=r, f\ is\ surjective.$

$(ix)\ (supplement)\ Bijective\ f:(0,1)\longrightarrow[0,1]:f(x)=\begin{cases}0, if\ x=\dfrac{1}{2};\\ \dfrac{1}{n-2}, if\ x=\dfrac{1}{n}\ for\ some\ n\in\mathbb{N}\cap[3,\infty);\\ x, otherwise.\end{cases}$

(x) $(supplement)$ $To\ prove\ |\mathbb{R}| = |(0,1)|.$

$Proof\ 1: bijective\ f: (-1,1) \longrightarrow \mathbb{R}: f(x) = \begin{cases} \dfrac{x}{1-x}, if\ x \in (0,1); \\ \dfrac{x}{1+x}, otherwise. \end{cases}$ $\quad (consider\ y = \dfrac{1}{x}, offset\ and\ symmetry.)$

$Proof\ 2: bijective\ f: \mathbb{R} \longrightarrow (0,1): f(x) = \dfrac{\exp(x)}{\exp(x)+1}.$

(xi) $(supplement)$ $To\ prove\ |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|.$

$Proof\ 1: Claim\ bijective\ f: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}: f(i,j) = 2^{i-1}(2j-1).$

$(1)\ f, trivially, is\ well-defined.$

$(2)\ \forall (i,j),(k,l) \in \mathbb{N} \times \mathbb{N}\ with\ f(i,j) = f(k,l), 2^{i-1}(2j-1) = 2^{k-1}(2l-1). If\ k = 1, 2^{i-1}(2j-1) = 2l-1\ is\ odd,$
$\therefore i = 1 = k, j = l.\ If\ k \geq 2, k-1 \geq 1, 2^{k-1} \geq 2, \dfrac{2^{i-1}(2j-1)}{2^{k-1}} = 2l-1 \in \mathbb{N} \wedge 2^{k-1} \nmid 2j-1\ is\ odd,$
$\therefore 2^{k-1} \mid 2^{i-1}, and\ note\ 2^{k-1}, 2^{i-1} \in \mathbb{N}, \therefore 2^{k-1} \leq 2^{i-1};$
$analogously, 2^{k-1} \geq 2^{i-1}, \therefore 2^{k-1} = 2^{i-1}, i = k, j = l, f\ is\ injective.$

$(3)\ \forall m \in \mathbb{N}, if\ m\ is\ even, \exists n \in \mathbb{N}, primes\ p_1, p_2, p_3, \ldots, p_n\ with\ p_1 < p_2 < p_3 < \cdots < p_n,$
$\{r_1, r_2, r_3, \ldots, r_n\} \subseteq \mathbb{N}\ s.t.\ m = p_1^{r_1} p_2^{r_2} p_3^{r_3} \ldots p_n^{r_n}\ by\ the\ Fundamental\ Thm.\ of\ Arithmetic,$
$\therefore p_1 = 2 \wedge p_2, p_3, \ldots, p_n\ is\ prime, \therefore a := p_2^{r_2} p_3^{r_3} \ldots p_n^{r_n}\ is\ odd, take\ i = r_1 + 1 \in \mathbb{N}, j = \dfrac{a+1}{2} \in \mathbb{N},$
$then\ f(i,j) = m; if\ m\ is\ odd, take\ i = 1, j = \dfrac{m+1}{2} \in \mathbb{N}, then\ f(i,j) = m.\ Accordingly, f\ is\ surjective.$

$Proof\ 2: Claim\ bijective\ f: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}: f(i,j) = \dfrac{\big(1 + (i + (j-1) - 1)\big)(i + (j-1) - 1)}{2} + j \in \mathbb{N},$
$\forall i \in \mathbb{N}, j \in \{1, 2, 3, \ldots, i + (j-1)\}.\ See\ the\ following\ attachment.$

$Proof\ 3:$

$(1)\ \forall n \in \mathbb{N}, A_n := \{(n,1), (n,2), (n,3), \ldots, (n,n)\}, A_i's\ are\ mutually\ disjoint,$
$D := \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i \geq j\} = \bigcup_{n \in \mathbb{N}} A_n, define\ f: \mathbb{N} \times \mathbb{N} \longrightarrow D: f(i,j) = \big(i + (j-1), j\big), f\ is\ obviously\ bijective.$
$(i.e.\ \forall j, horizontally\ push\ (i,j)\ rightward\ by\ (j-1)\ units.)$
$Define\ g: D \longrightarrow \mathbb{N}: g(i,j) = \dfrac{(1 + (i-1))(i-1)}{2} + j = \dfrac{i(i-1)}{2} + j \in \mathbb{N}.$

$(2)(a)\ \forall (i,j_1),(i,j_2) \in D\ with\ j_1 \leq j_2, g(i,j_1) = \dfrac{i(i-2)}{2} + j_1 \leq \dfrac{i(i-1)}{2} + j_2 = g(i,j_2);$
$(b)\ g(i,i) = \dfrac{i(i-1)}{2} + i = \dfrac{(i+1)i}{2} + 1 - 1 = g(i+1,1) - 1.$

$(3)\ Claim\ g\ is\ bijective.\ (a)\ Trivially, g\ is\ well-defined.$
$(b)\ \forall (i,j),(k,l) \in D\ with\ g(i,j) = g(k,l).\ If\ i \neq k, we\ may\ assume\ i > k. \therefore 1 \leq l \leq k \leq i-1, 0 \leq k-1 \leq i-2,$
$g(k,l) = \dfrac{k(k-1)}{2} + l \leq \dfrac{(i-1)(i-2)}{2} + (i-1) = g(i-1,i-1)$
$< g(i,1) \leq g(i,j)\ by\ (2)-(b)\ and\ (2)-(a), C!.$
$Hence, i = k.\ Note\ g(i,j) = g(k,l) \Rightarrow \dfrac{i(i-1)}{2} + j = \dfrac{k(k-1)}{2} + l \Rightarrow j = l \Rightarrow (i,j) = (k,l).\ Thus, g\ is\ injective.$
$(c)\ \forall p, i \in \mathbb{N}, g(i,1) = \dfrac{i(i-1)}{2} + 1 = \dfrac{1}{2}\left(\left(i - \dfrac{1}{2}\right)^2 + \dfrac{7}{4}\right) \to \infty\ as\ i \to \infty,$
$I := \{i \in \mathbb{N} \mid p < g(i,1)\} \neq \emptyset, v := minI \in \mathbb{N}\ by\ well-ordering\ principle, v-1 \notin I,$

$g(v-1,1) \le p < g(v,1) = g(v-1,v-1)+1$ *by* $(2)-(b), g(v-1,1) \le p \le g(v-1,v-1).$

$\therefore 1 \le m \coloneqq p - g(v-1,1)+1 \le g(v-1,v-1) - g(v-1,1)+1 = v-1,$

$\therefore (v-1,m) \in D \wedge g(v-1,m) = \dfrac{(v-1)(v-2)}{2} + m = (g(v-1,1)-1)+m = p, g$ *is surjective.*

(4) *It follows from* (3) *that* $g \circ f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ *is bijective.*

Proof 4 :

(1) $\forall n \in \mathbb{N}, A_n \coloneqq \{(n+1,1),(n+1,2),(n+1,3),\ldots,(n+1,n)\}, A_i's$ *are mutually disjoint,*

$D_1 \coloneqq \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid i > j\} = \bigcup_{n\in\mathbb{N}} A_n, D_2 \coloneqq \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid i < j\},$

$D_3 \coloneqq \{(i,i) \in \mathbb{N}\times\mathbb{N} \mid i \in \mathbb{N}\}, D_1 \cup D_2 \cup D_3 = \mathbb{N}\times\mathbb{N}, D_i's$ *are pairwise disjoint;*

$E_1 \coloneqq \{3t-2 \mid t \in \mathbb{N}\}, E_2 \coloneqq \{3t-1 \mid t \in \mathbb{N}\}, E_3 \coloneqq \{3t \mid t \in \mathbb{N}\}, E_1 \cup E_2 \cup E_3 = \mathbb{N}, E_i's$ *are pairwise disjoint.*

Define $f_1 : D_1 \longrightarrow \mathbb{N} : f_1(i,j) = \dfrac{\big(1+((i-1)-1)\big)((i-1)-2+1)}{2} + j = \dfrac{(i-1)(i-2)}{2} + j \in \mathbb{N}.$

(2)(a) $\forall (i,j_1),(i,j_2) \in D_1$ *with* $j_1 \le j_2, f_1(i,j_1) = \dfrac{(i-1)(i-2)}{2} + j_1 \le \dfrac{(i-1)(i-2)}{2} + j_2 = f_1(i,j_2);$

(b) *Note* $i \ge 2, f_1(i,i-1) = \dfrac{(i-1)(i-2)}{2} + (i-1) = \dfrac{i(i-1)}{2} + 1 - 1 = f_1(i+1,1) - 1.$

(3) *Claim* f_1 *is bijective.* (a) *Trivially,* f_1 *is well − defined.*

(b) $\forall (i,j),(k,l) \in D_1$ *with* $f_1(i,j) = f_1(k,l)$. *If* $i \ne k$, *we may assume* $i > k$. $\therefore 2 \le k \le i-1, 0 \le k-2 \le i-3,$

$f_1(k,l) = \dfrac{(k-1)(k-2)}{2} + l \le \dfrac{((i-1)-1)((i-1)-2)}{2} + (k-1) \le \dfrac{(i-2)(i-3)}{2} + (i-2) = f_1(i-1,i-2)$

$< f_1(i,1) \le f_1(i,j)$ *by* $(2)-(b)$ *and* $(2)-(a), C!$. *Hence,* $i = k$. *Note* $f_1(i,j) = f_1(k,l)$

$\Rightarrow \dfrac{(i-1)(i-2)}{2} + j = \dfrac{(k-1)(k-2)}{2} + l \Rightarrow j = l \Rightarrow (i,j) = (k,l)$. *Thus,* f_1 *is injective.*

(c) $\forall p, i \in \mathbb{N}$ *with* $i \ge 2, f_1(i,1) = \dfrac{(i-1)(i-2)}{2} + 1 = \dfrac{1}{2}\left(\left(i - \dfrac{3}{2}\right)^2 + \dfrac{7}{4}\right) \to \infty$ *as* $i \to \infty,$

$I \coloneqq \{i \in \mathbb{N} \cap [2,\infty) \mid p < f_1(i,1)\} \ne \emptyset, 2 \le v \coloneqq minI \in \mathbb{N}$ *by well − ordering principle,* $v-1 \notin I,$

$f_1(v-1,1) \le p < f_1(v,1) = f_1(v-1,v-2)+1$ *by* $(2)-(b), f_1(v-1,1) \le p \le f_1(v-1,v-2).$

$\therefore 1 \le m \coloneqq p - f_1(v-1,1)+1 \le f_1(v-1,v-2) - f_1(v-1,1)+1 = v-2, v-1 \ge 2,$

$\therefore (v-1,m) \in D_1 \wedge f_1(v-1,m) = \dfrac{(v-2)(v-3)}{2} + m = (f_1(v-1,1)-1)+m = p, f_1$ *is surjective.*

(4) *Likewise, define the following bijections* $f_2 : D_2 \longrightarrow \mathbb{N} : f_2(i,j) = \dfrac{(j-1)(j-2)}{2} + i \in \mathbb{N},$

$f_3 : D_3 \longrightarrow \mathbb{N} : f_3(i,i) = i, g_1 : \mathbb{N} \longrightarrow E_1 : g_1(t) = 3t-2, g_2 : \mathbb{N} \longrightarrow E_2 : g_2(t) = 3t-1, g_3 : \mathbb{N} \longrightarrow E_3 : g_3(t) = 3t.$

(5) *Define* $h : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N} : h(i,j) = \begin{cases} (g_1 \circ f_1)(i,j), if\ x \in D_1; \\ (g_2 \circ f_2)(i,j), if\ x \in D_2; \\ (g_3 \circ f_3)(i,j), if\ x \in D_3. \end{cases}$ *It follows from* (3) *and* (4) *that* h *is bijective.*

Furthermore, accordingly, $\forall n \in \mathbb{N}, A_n$ *is denumerable* $\Rightarrow \bigcup\limits_{n\in\mathbb{N}} A_n$ *is denumerable. Proof :*

(1) $\forall n \in \mathbb{N}, B_1 \coloneqq A_1, B_n \coloneqq A_n \Big\backslash \bigcup\limits_{i=1}^{n-1} A_i, B_i's$ *are pairwise disjoint* $\wedge \bigcup\limits_{n\in\mathbb{N}} B_n = \bigcup\limits_{n\in\mathbb{N}} A_n.$

$(2)\ \forall n \in \mathbb{N}, I_n \coloneqq \{(n,m) \mid m \in \mathbb{N}\} \sim \mathbb{N}, I_i' s\ are\ mutually\ disjoint, B_n\ is\ countable, \exists\ injective\ f_n : B_n \longrightarrow I_n.$

$Define\ f : \bigcup_{n\in\mathbb{N}} B_n \longrightarrow \mathbb{N}\times\mathbb{N} = \bigcup_{n\in\mathbb{N}} I_n : f(x) = f_n(x), if\ x \in B_n\ for\ some\ n \in \mathbb{N}.$

$Then\ f\ is\ injective, |A_1| \leq \left|\bigcup_{n\in\mathbb{N}} A_n\right| = \left|\bigcup_{n\in\mathbb{N}} B_n\right| \leq |\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|,$

$\therefore \bigcup_{n\in\mathbb{N}} A_n\ is\ infinite\ and\ countable; that\ is, \bigcup_{n\in\mathbb{N}} A_n\ is\ denumerable.$

$(xii)\ (supplement)\ \forall n \in \mathbb{N}\cup\{0\}, |\mathcal{P}^n(\mathbb{R}) \times \mathcal{P}^n(\mathbb{R})| = |\mathcal{P}^n(\mathbb{R})|.$

$Proof : |\mathbb{R}| = |\mathbb{R}\times\{1\}| \leq |\mathbb{R}\times\mathbb{N}| \leq |\mathbb{R}\times\mathbb{R}| \leq |\mathbb{R}^{\mathbb{N}}| = |(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}| = |\{0,1\}^{\mathbb{N}\times\mathbb{N}}| = |\{0,1\}^{\mathbb{N}}| = |\mathbb{R}|;$

$suppose\ |\mathcal{P}^{n-1}(\mathbb{R})\times\mathbb{N}| = |\mathcal{P}^{n-1}(\mathbb{R})|\ for\ n \in \mathbb{N};$

$|\mathcal{P}^n(\mathbb{R})| = |\mathcal{P}^n(\mathbb{R})\times\{1\}| \leq |\mathcal{P}^n(\mathbb{R})\times\mathbb{N}| \leq |\mathcal{P}^n(\mathbb{R})\times\mathcal{P}^n(\mathbb{R})| \leq |(\mathcal{P}^n(\mathbb{R}))^{\mathbb{N}}| = |(\{0,1\}^{\mathcal{P}^{n-1}(\mathbb{R})})^{\mathbb{N}}|$

$= |\{0,1\}^{\mathcal{P}^{n-1}(\mathbb{R})\times\mathbb{N}}| = |\{0,1\}^{\mathcal{P}^{n-1}(\mathbb{R})}|\ (inductive\ hypothesis) = |\mathcal{P}(\mathcal{P}^{n-1}(\mathbb{R}))| = |\mathcal{P}^n(\mathbb{R})|.$

$(xiii)\ (supplement)\ \forall n \in \mathbb{N}, m \in \mathbb{N}\cup\{0\}\ with\ n \geq m, |\mathcal{P}^n(\mathbb{R})\times\mathcal{P}^m(\mathbb{R})| = |\mathcal{P}^n(\mathbb{R})|.$

$Proof : |\mathcal{P}^n(\mathbb{R})| = |\mathcal{P}^n(\mathbb{R})\times\{1\}| \leq |\mathcal{P}^n(\mathbb{R})\times\mathbb{N}| \leq |\mathcal{P}^n(\mathbb{R})\times\mathcal{P}^m(\mathbb{R})| \leq |\mathcal{P}^n(\mathbb{R})\times\mathcal{P}^n(\mathbb{R})| \leq |(\mathcal{P}^n(\mathbb{R}))^{\mathbb{N}}|$

$= |(\{0,1\}^{\mathcal{P}^{n-1}(\mathbb{R})})^{\mathbb{N}}| = |\{0,1\}^{\mathcal{P}^{n-1}(\mathbb{R})\times\mathbb{N}}| = |\{0,1\}^{\mathcal{P}^{n-1}(\mathbb{R})}|\ (by\ (xii)) = |\mathcal{P}(\mathcal{P}^{n-1}(\mathbb{R}))| = |\mathcal{P}^n(\mathbb{R})|.$