Proposition. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. $\forall n \in \mathbb{N}, a_n = \left(1+\frac{1}{n}\right)^n$.

Show that $(a_n)_{n=1}^{\infty}$ *strictly increases* $\wedge\, \forall n \in \mathbb{N}, a_n \leq 3 - \frac{1}{n}$.

Proof :

(i)

$$\forall n \in \mathbb{N}, a_{n+1} - a_n = \left(1+\frac{1}{n+1}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{1}{n+1}\right)^k - \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= \sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{n+1}\right)^k - \binom{n}{k}\left(\frac{1}{n}\right)^k\right] + \left(\frac{1}{n+1}\right)^{n+1}$$

$$= \begin{cases} 0+0+\frac{1}{4} > 0, if\ n = 1; \\ 0+0+\sum_{k=2}^{n}\left[\binom{n+1}{k}\left(\frac{1}{n+1}\right)^k - \binom{n}{k}\left(\frac{1}{n}\right)^k\right] + \left(\frac{1}{n+1}\right)^{n+1}, otherwise. \end{cases}$$

(ii)

$$\forall n \in \mathbb{N}\cap[2,\infty), k \in \mathbb{N}\cap[2,n], \binom{n+1}{k}\left(\frac{1}{n+1}\right)^k - \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= \frac{(n+1)(n+1-1)\cdots[n+1-(k-1)]}{k!}\left(\frac{1}{n+1}\right)^k - \frac{n(n-1)\cdots[n-(k-1)]}{k!}\left(\frac{1}{n}\right)^k$$

$$= \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(1-\frac{i}{n+1}\right) - \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)\right] > \frac{1}{k!}\left[\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) - \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)\right] = 0.$$

$\therefore \forall n \in \mathbb{N}, a_{n+1} - a_n > 0$ *by(i)(ii)*, $(a_n)_{n=1}^{\infty}$ *is strictly increasing.*

(iii)

$a_1 = 2 = 3 - \frac{1}{1}$. $\forall n \in \mathbb{N}\cap[2,\infty)$,

$$a_n = 1 + \binom{n}{1}\frac{1}{n} + \sum_{k=2}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k = 2 + \sum_{k=2}^{n}\frac{n(n-1)\cdots[n-(k-1)]}{k!\cdot n^k} = 2 + \sum_{k=2}^{n}\frac{1}{k!}\prod_{i=0}^{k-1}\frac{n-i}{n}$$

$$< 2 + \sum_{k=2}^{n}\frac{1}{k!} < 2 + \sum_{k=2}^{n}\frac{1}{(k-1)k} = 2 + (1-\frac{1}{n}) = 3 - \frac{1}{n}.$$

$Proposition.\forall a, b, p, n \in \mathbb{Z}\ with\ p \neq 0 \neq n, ap \equiv bp \pmod{n}, gcd(p, n) = 1 \Rightarrow a \equiv b \pmod{n}$.

$Proof:$

$n|ap - bp = p(a - b)$. $\because gcd(p, n) = 1, \therefore n|a - b\ by\ Fundamental\ Thm.\ of\ Arithmetic, a \equiv b \pmod{n}$.

$(Generally, the\ division\ of\ congruence\ does\ not\ hold.\ E.g.\ 6 \equiv 4 \pmod{2}, 2 \equiv 2 \pmod{2}, \frac{6}{2} = 3 \not\equiv 2 = \frac{4}{2} \pmod{2}.\ E.g.\ 16 \equiv 12 \pmod{4}, 8 \equiv 4 \pmod{4}, \dfrac{16}{8} = 2 \not\equiv 3 = \dfrac{12}{4}.)$