$Lemma:\ Each\ infinite\ sequence\ (x_n)_{n=1}^{\infty}\ in\ \mathbb{R}\ has\ a\ monotone\ subsequence.$

$Proof:\ B \coloneqq \{n \in \mathbb{N} \mid x_n < x_m\ for\ all\ m \in \mathbb{N} \cap (n, \infty)\} \subseteq \mathbb{N}.$

$(i)\ If\ B = \emptyset, \mathbb{N} \cap B = \emptyset, i_1 \coloneqq 1 \notin B.\ \exists i_2 \in \mathbb{N} \cap (i_1, \infty)\ s.t.\ x_{i_1} \geq x_{i_2}.$

$Suppose\ \forall n \in \mathbb{N} \cap [3, \infty), \exists i_n \in \mathbb{N} \cap (i_{n-1}, \infty)\ s.t.\ x_{i_{n-1}} \geq x_{i_n}.$

$Analogously, i_n \notin B, \exists i_{n+1} \in \mathbb{N} \cap (i_n, \infty)\ s.t.\ x_{i_n} \geq x_{i_{n+1}}\ by\ inductive\ hypothesis.$

$Consequently, \forall n \in \mathbb{N} \cap [2, \infty), \exists i_n \in \mathbb{N} \cap (i_{n-1}, \infty)\ s.t.\ x_{i_{n-1}} \geq x_{i_n}\ by\ induction$

$and\ hence\ the\ subsequence\ (x_{i_n})_{n=1}^{\infty}\ decreases.$

$(ii)\ If\ \emptyset \neq B\ is\ finite, maxB\ exists\ in\ B \cap \mathbb{N}, i_1 \coloneqq maxB + 1, [i_1, \infty) \cap B = \emptyset.$

$Similar\ to\ (i), there\ exists\ a\ decreasing\ subsequence\ (x_{i_n})_{n=1}^{\infty}.$

$(iii)\ If\ B\ is\ denumerable, there\ exists\ an\ i_1 \in B, \therefore \exists i_2 \in B\ s.t.\ i_1 < i_2; otherwise, B\ is\ finite.$

$Suppose\ \forall n \in \mathbb{N} \cap [3, \infty), \exists i_n \in B\ s.t.\ i_{n-1} < i_n.$

$Likewise, \exists i_{n+1} \in B\ s.t.\ i_n < i_{n+1}\ by\ inductive\ hypothesis; otherwise, B\ is\ finite$

$Thus, \exists (i_n)_{n=1}^{\infty}\ in\ B\ s.t.\ (i_n)_{n=1}^{\infty}\ strictly\ increases\ by\ induction.$

$Accordingly, subsequence\ (x_{i_n})_{n=1}^{\infty}\ is\ strictly\ increasing.$