$In\ this\ document, \mathbb{N}\ denotes\ \mathbb{Z}^+\ instead\ of\ \mathbb{Z}^+ \cup \{0\}.$

$Theorem\ 8.7.\ The\ class\ of\ all\ cardinal\ numbers\ is\ linearly\ ordered\ by \leq.\ If\ \alpha\ and\ \beta\ are\ cardinal\ numbers,$
$then\ exactly\ one\ of\ the\ following\ is\ true: \alpha < \beta; \alpha = \beta; \beta < \alpha\ (Trichotomy\ Law).$

$Sketch\ of\ proof:$

$(i)\ It\ is\ easy\ to\ verify \leq is\ a\ partial\ order.\ Let\ A, B\ be\ sets\ s.t.\ |A| = \alpha, |B| = \beta\ and$
$\emptyset \in \mathcal{F} \coloneqq \{S \subseteq A \mid |S| \leq |B|\} \neq \emptyset.\ Obviously, \subseteq is\ a\ partial\ order\ of\ \mathcal{F}.$
$Suppose\ L \coloneqq \{X_i \mid i \in I\}\ is\ a\ chain\ in\ \mathcal{F}, M \coloneqq \bigcup_{i\in I} X_i.$
$\forall i \in I, \exists\ injective\ f_i : X_i \longrightarrow B.\ Define\ f : M \longrightarrow B : f(x) = f_i(x)\ for\ x \in X_i\ for\ some\ i \in I.$

$(ii)\ X_i's\ are\ pairwise\ comparable\ in\ L \wedge f_i's\ are\ well-defined\ injections, \therefore f\ is\ injective,$
$|M| \leq |B|, X_i \subseteq M \in \mathcal{F}\ for\ all\ i \in I, M\ is\ an\ upper\ bound\ of\ L\ in\ \mathcal{F}.$
$Therefore\ by\ Zorn's\ Lemma\ there\ is\ a\ maximal\ element\ X\ of\ \mathcal{F}, |X| \leq |B|, \exists\ injective\ g : X \longrightarrow B.$

$(iii)\ Claim\ either\ X = A\ or\ Im(g) = B.\ For\ if\ both\ of\ these\ statements\ were\ false,$
$we\ could\ find\ a \in A \backslash X\ and\ b \in B \backslash Im(g)\ and\ define\ an\ injective\ map$
$h : X \cup \{a\} \longrightarrow B\ by\ h(x) = g(x)\ for\ x \in X\ and\ h(a) = b\ because$
$g\ is\ a\ well-defined\ injective\ map \wedge a \in X^c \wedge b \in (Im(g))^c, |X \cup \{a\}| \leq |B|, X \cup \{a\} \in \mathcal{F},$
$X \subsetneqq X \cup \{a\} \subseteq X, which\ contradicts\ the\ maximality\ of\ X.$

$(iv)\ Hence\ either\ X = A\ so\ that\ |A| \leq |B|\ or\ Im(g) = B\ in\ which\ case$
$the\ injective\ map\ B \xrightarrow{g^{-1}} X \subseteq A\ (the\ Axiom\ of\ Choice\ is\ required)\ shows\ that\ |B| \leq |A|.$
$Thus, if\ |A| = |B|, \alpha = \beta; if\ not, \nexists\ bijection\ \theta : A \longrightarrow B, \therefore \alpha = |A| \lneqq |B| = \beta \vee \beta = |B| \lneqq |A| = \alpha.$

$Theorem\ 8.10.\ If\ \alpha\ and\ \beta\ are\ cardinal\ numbers\ s.t.\ \beta \leq \alpha\ and\ \alpha\ is\ infinite, then\ \alpha + \beta = \alpha.$

$Sketch\ of\ proof:$

$(i)\ It\ suffices\ to\ prove\ \alpha + \alpha = \alpha\ (simply\ verify\ \alpha \leq \alpha + \beta \leq \alpha + \alpha = \alpha\ and\ apply$
$the\ Schroeder-Bernstein\ Theorem\ to\ conclude\ \alpha + \beta = \alpha).\ Let\ A\ be\ a\ set\ with\ |A| = \alpha\ and$
$\mathcal{F} \coloneqq \{S \subseteq A \mid S\ is\ infinite \wedge |S \times \{0,1\}| = |S|\}.$

$(ii)\ A\ is\ infinite\ by\ hypothesis, \exists\ D \subseteq A\ s.t.\ D\ is\ denumerable\ by\ Theorem\ 8.8,$
$D \coloneqq \{d_n \mid n \in \mathbb{N} \wedge d_i's\ are\ all\ distinct\}.$
$The\ map\ D \times \{0,1\} \longrightarrow D\ given\ by\ (d_n, 0) \mapsto 2n\ and\ (d_n, 1) \mapsto 2n-1\ for\ all\ n \in \mathbb{N}\ is\ bijective,$
$\therefore D \in \mathcal{F} \neq \emptyset.\ Furthermore, obviously, \subseteq is\ a\ partial\ order\ of\ \mathcal{F}.$
$Suppose\ L \coloneqq \{X_i \mid i \in I\}\ is\ a\ chain\ in\ \mathcal{F}, M \coloneqq \bigcup_{i\in I} X_i.\ \forall i \in I, \exists\ bijective\ f_i : X_i \times \{0,1\} \longrightarrow X_i.$
$Define\ f : M \times \{0,1\} \longrightarrow M : f(x,y) = f_i(x,y)\ for\ x \in X_i\ for\ some\ i \in I.$

$(iii)\ X_i's\ are\ pairwise\ comparable\ in\ L \wedge f_i's\ are\ well-defined\ bijections, \therefore f\ is\ bijective,$
$|M \times \{0,1\}| = |M|, X_i \subseteq M \in \mathcal{F}\ for\ all\ i \in I, M\ is\ an\ upper\ bound\ of\ L\ in\ \mathcal{F}.$

$Thus\ by\ Zorn's\ Lemma\ there\ is\ a\ maximal\ element\ C\ of\ \mathcal{F}, |C \times \{0,1\}| = |C|, \exists\ bijective\ g : C \times \{0,1\} \longrightarrow C.$

$(iv)\ Claim\ |A| = |C|.\ If\ not, note\ |A| = |C \cup (A \backslash C)|, \therefore A \backslash C\ is\ infinite\ by\ Lemma\ 8.9,$
$\exists\ B \subseteq (A \backslash C)\ s.t.\ B\ is\ denumerable\ by\ Theorem\ 8.8, and\ as\ above,$
$there\ would\ be\ a\ bijection\ \zeta : B \times \{0,1\} \longrightarrow B.\ By\ combining\ \zeta\ with\ g, we\ could\ then\ construct\ a\ bijection$
$$h : (C \cup B) \times \{0,1\} \longrightarrow (C \cup B) : h(x,y) = \begin{cases} g(x,y), if\ x \in C; \\ \zeta(x,y), if\ x \in B; \end{cases} \quad because\ C \cap B = \emptyset$$
$\wedge g\ and\ \zeta\ are\ both\ well-defined\ bijective\ maps.$
$\therefore (C \cup B) \in \mathcal{F}, C \subsetneqq (C \cup B) \subseteq C, which\ would\ contradict\ the\ maximality\ of\ C.\ Therefore$
$\alpha + \alpha = |A| + |A| = |C| + |C| = |C \times \{0\}| + |C \times \{1\}| = |(C \times \{0\}) \cup (C \times \{1\})| = |C \times \{0,1\}| = |C| = |A| = \alpha.$

$Theorem\ 8.11.\ If\ \alpha\ and\ \beta\ are\ cardinal\ numbers\ s.t.\ 0 \neq \beta \leq \alpha\ and\ \alpha\ is\ infinite, then\ \alpha\beta = \alpha;$
$in\ particular, \alpha\aleph_0 = \alpha\ and\ if\ \beta\ is\ finite\ \aleph_0\beta = \aleph_0.$
$Sketch\ of\ proof:$
$(i)\ Since\ \alpha \leq \alpha\beta \leq \alpha\alpha\ it\ suffices\ (as\ in\ the\ proof\ of\ Theorem\ 8.10)\ to\ prove\ \alpha\alpha = \alpha.$
$Let\ A\ be\ an\ infinite\ set\ with\ |A| = \alpha\ and\ \mathcal{F} \coloneqq \{X \subseteq A \mid X\ is\ infinite \wedge |X \times X| = |X|\}.$

$(ii)\ A\ is\ infinite\ by\ hypothesis, \exists\ D \subseteq A\ s.t.\ D\ is\ denumerable\ by\ Theorem\ 8.8,$
$D \coloneqq \{d_n \mid n \in \mathbb{N} \wedge d_i's\ are\ all\ distinct\}.$
$The\ map\ D \times D \longrightarrow D\ given\ by\ (d_m, d_n) \mapsto 2^{m-1}(2n-1)\ for\ all\ d_i's\ in\ D\ is\ bijective,$
$\therefore D \in \mathcal{F} \neq \emptyset.\ Furthermore, obviously, \subseteq is\ a\ partial\ order\ of\ \mathcal{F}.$
$Suppose\ L \coloneqq \{X_i \mid i \in I\}\ is\ a\ chain\ in\ \mathcal{F}, M \coloneqq \bigcup_{i \in I} X_i.\ \forall i \in I, \exists\ bijective\ f_i : X_i \times X_i \longrightarrow X_i.$
$$Define\ f : M \times M \longrightarrow M : f(x,y) = \begin{cases} f_i(x,y), if\ X_i \supseteq X_j; \\ f_j(x,y), if\ X_i \subseteq X_j; \end{cases} \quad where\ x \in X_i, y \in X_j\ for\ some\ i,j \in I.$$

$(iii)\ X_i's\ are\ pairwise\ comparable\ in\ L \wedge f_i's\ are\ well-defined\ bijections, \therefore f\ is\ bijective, |M \times M| = |M|,$
$X_i \subseteq M \in \mathcal{F}\ for\ all\ i \in I, M\ is\ an\ upper\ bound\ of\ L\ in\ \mathcal{F}.$
$Thus\ by\ Zorn's\ Lemma\ there\ is\ a\ maximal\ element\ B\ of\ \mathcal{F}, and\ recall\ Definition\ 8.3, |B||B| = |B \times B| = |B|.$

$(iv)\ Claim\ |A| = |B|.\ If\ not, note\ |A| = |B \cup (A \backslash B)| = |B| + |A \backslash B|, \therefore |A \backslash B| > |B|\ by\ Theorem\ 8.10\ and\ 8.7.$
$Then\ by\ Definition\ 8.4\ there\ is\ a\ subset\ C\ of\ A \backslash B\ s.t.\ |C| = |B|.\ Accordingly$
$|C| = |B| = |B \times B| = |B \times C| = |C \times B| = |C \times C|\ and\ these\ sets\ are\ mutually\ disjoint\ because\ C \cap B = \emptyset.$
$Consequently\ by\ Definition\ 8.3\ and\ Theorem\ 8.10\ |(B \cup C) \times (B \cup C)| = |(B \times B) \cup (B \times C) \cup (C \times B) \cup (C \times C)|$
$= |B \times B| + |B \times C| + |C \times B| + |C \times C| = (|B| + |B|) + (|C| + |C|) = |B| + |C| = |B \cup C|, (B \cup C) \in \mathcal{F},$
$B \subsetneqq (B \cup C) \subseteq B, which\ contradicts\ the\ maximality\ of\ B.$
$Hence\ also\ by\ Definition\ 8.3, \alpha\alpha = |A||A| = |B||B| = |B \times B| = |B| = |A| = \alpha.$