

# A Safety-Critical Model Predictive Controller for a Quadrotor with Barrier States

MinGyu Kim

December 5, 2022

## 1 Intro - Quadrotor Dynamics

- Rotor 1 Thrust:  $f_1$
- Rotor 2 Thrust:  $f_2$
- Rotor 3 Thrust:  $f_2$
- Rotor 4 Thrust:  $f_2$
- Rotation matrix from body to inertial coordinates:  $R$
- Quadrotor mass:  $m = 0.5$  kg
- Quadrotor inertia in axis (x):  $I_{xx} = 0.0032$  kgm<sup>2</sup>
- Quadrotor inertia in axis (y):  $I_{yy} = 0.0032$  kgm<sup>2</sup>
- Quadrotor inertia in axis (z):  $I_{zz} = 0.0055$  kgm<sup>2</sup>
- Rotor torque constant:  $k_t = 0.01691$ /m
- Rotor moment arm:  $l = 0.17$ m
- Gravity constant:  $g = 9.81$  m/s<sup>2</sup>
- Body roll rate:  $p$
- Body pitch rate:  $q$
- Body yaw rate:  $r$
- Euler angle roll:  $\phi$
- Euler angle pitch:  $\theta$
- Euler angle yaw:  $\psi$

$$statex = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \phi \\ \theta \\ \psi \\ p \\ q \\ r \end{bmatrix} \quad (1)$$

$$thrust\ control\ u = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \quad (2)$$

## 1.1 Translational Equation of Motion

$$m * \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \mathbf{R} * \begin{bmatrix} 0 \\ 0 \\ f_1 + f_2 + f_3 + f_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \quad (3)$$

where

$$R = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \sin(\theta)\sin(\phi)\cos(\psi) - \cos(\phi)\sin(\psi) & \sin(\theta)\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi) & \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\psi) + \cos(\phi)\sin(\theta)\cos(\psi) & \sin(\theta)\sin(\psi)\cos(\phi) - \sin(\phi)\sin(\psi) & \cos(\theta)\cos(\phi) \end{bmatrix}$$

## 1.2 Rotational Equations of Motion

$$\begin{bmatrix} I_{xx}\dot{p} \\ I_{yy}\dot{q} \\ I_{zz}\dot{r} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}(f_1 + f_3 - f_2 - f_4) * l - (I_{zz} - I_{yy}) * q * r \\ \frac{\sqrt{2}}{2}(f_3 + f_4 - f_1 - f_2) * l - (I_{zz} - I_{xx}) * p * r \\ k_t(f_1 + f_4 - f_2 - f_3) \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \frac{\sin(\phi)}{\cos(\theta)} & \frac{\cos(\phi)}{\cos(\theta)} \end{bmatrix} \quad (5)$$

## 1.3 Safety Constraints (obstacle locations)

$$h_1 = (x - 2.2)^2 + (y - 2.2)^2 + (z - 1)^2 - 1$$

$$h_2 = (x - 0)^2 + (y + 0.2)^2 + (z)^2 - 1$$

$$h_3 = (x - 3)^2 + (y)^2 + (z - 0.5)^2 - 1$$

## 2 METHODS

### 2.1 Differential Dynamic Programming (DDP) and Backpropagation

---

**Algorithm 1** Differential Dynamic Programming (DDP)

---

```

1: procedure FNDDP
2:   INITIALIZE VARIABLES: initial state value, desired state value,  $\bar{u}, \bar{x}$ , t, dt, horizon, iteration,
   learning rate and WEIGHTS: Q, S, R
3:   find dynamics  $\dot{x}$ 
4:   for 1:iteration do
5:     for 1:horizon-1 (forward propagation) do
6:       Calculate running cost  $L(x, u)$ 
7:       Calculate the linearized matrices A and B
8:       Obtain the value function matrices,  $V_{xx}, V_x, V$ 
9:     for horizon-1:1 (backpropagation) do
10:      Calculate Q matrices from the value function
11:      Obtain feedback matrices
12:    update control u
13:  end

```

---

### 2.2 Model Predictive Control (a.k.a. MPC or Receding Horizon Control)

---

**Algorithm 2** Model Predictive Control (MPC or Receding Horizon Control)

---

```

1: procedure FNMP
2:   INITIALIZE VARIABLES: initial state value, desired state value,  $\bar{u}, u_{mpc}, \bar{x}, x$ , cHorizon, pHorizon, t, dt, horizon, iteration, learning rate and WEIGHTS: Q, S, R
3:   for 1:pHorizon-1 do
4:     Obtain  $\bar{x}, \bar{u} \leftarrow$  Run fnDDP function
5:     update  $u_{mpc} \leftarrow \bar{u}$ 
6:     update trajectory
7:     update  $\bar{u}$  (shift horizon)
8:     update  $\bar{x}$  (shift horizon)
9:   end

```

---

### 2.3 Safety Constraint - Barrier State (BaS)

$h(x)$  = constraint equation

The barrier state operator  $\mathbf{B}$  is expressed as

$$\mathbf{B} = \frac{1}{h(x)} \quad (6)$$

$$\dot{\mathbf{B}} = \frac{\delta \mathbf{B}}{\delta h} \frac{\delta h}{\delta x} \frac{\delta x}{\delta t} = \dot{\mathbf{B}} h_x \dot{x} = \dot{\mathbf{B}} h_x (f(x) + g(x)u) \quad (7)$$

where  $\dot{x} = f(x) + g(x)u$  (control affine system).

The safety constraint equation derived from the barrier state operator is written as

$\beta = \mathbf{B}$

$$\dot{z} = \dot{\beta} h_x \dot{x} - \gamma * (z - (\beta - \beta_o)) \quad (8)$$

If this term  $\dot{z}$  is added to the original system  $\dot{x}$ , the constraint can be managed by a controller.

When multiple constraints are considered, you may integrate all constraints into one equation like below

$$\mathbf{B} = \beta = \sum_{i=1}^k \frac{1}{h_i} \quad (9)$$

However, from experience in this project, you might have to spend more time adjusting the weight matrices if you use that method (Equation 9).

## 3 RESULTS

### 3.1 Initial Configuration

time = 8

dt = 0.01

pHorizon = t/dt

cHorizon (for MPC) = 100

learning rate = 0.5

Qsafe = [1, 1, 1]

Ssafe = [10, 10, 10]

Q = diag([0, 0, 0, 0, 0, 0, 1, 1, 1, .1, .1, .1, Qsafe])

S = diag([10, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, Ssafe])

R = diag([1, 1, 1, 1].\*0.01)

### 3.2 DDP application

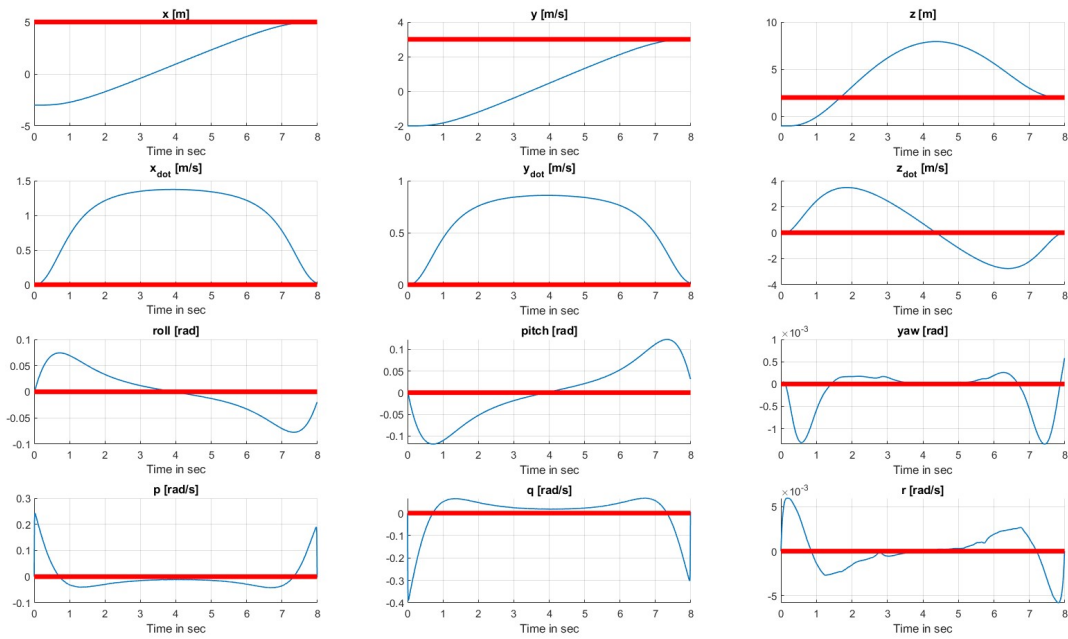


Figure 1: Optimal State Paths Determined Through DDP

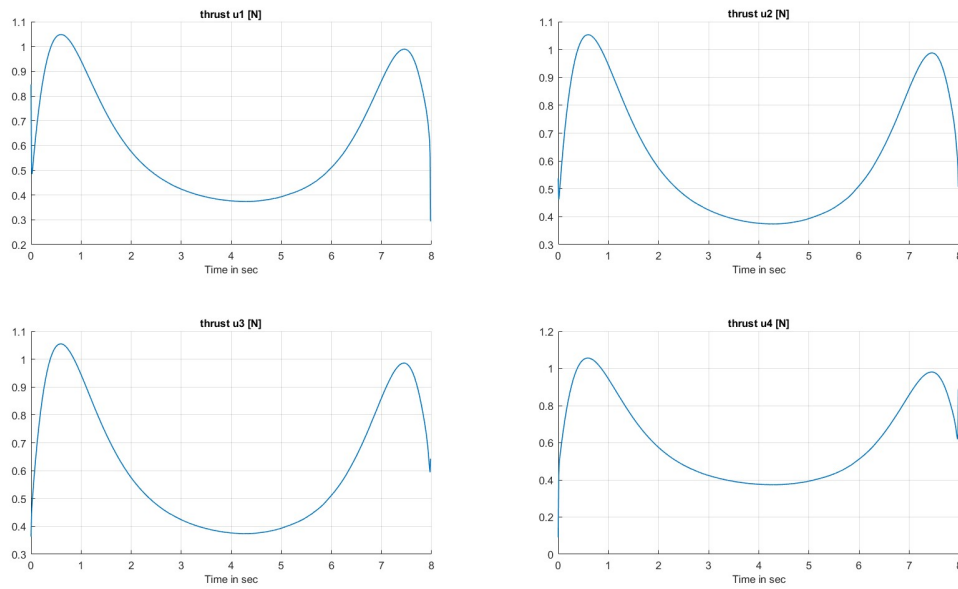


Figure 2: Optimal Control Determined Through DDP

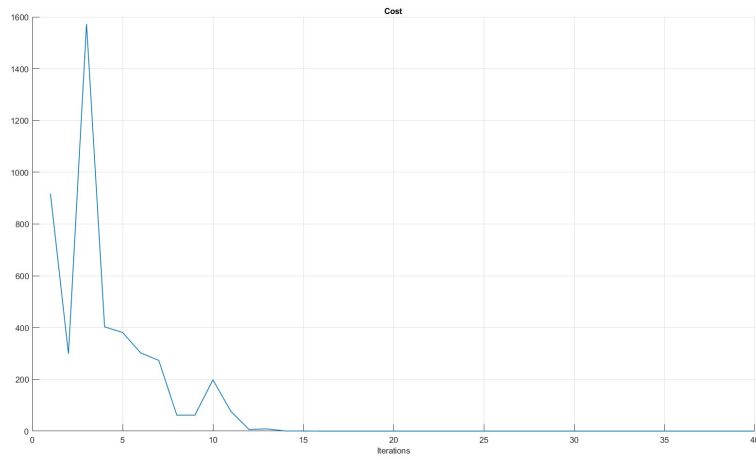


Figure 3: Corresponding Cost Through DDP

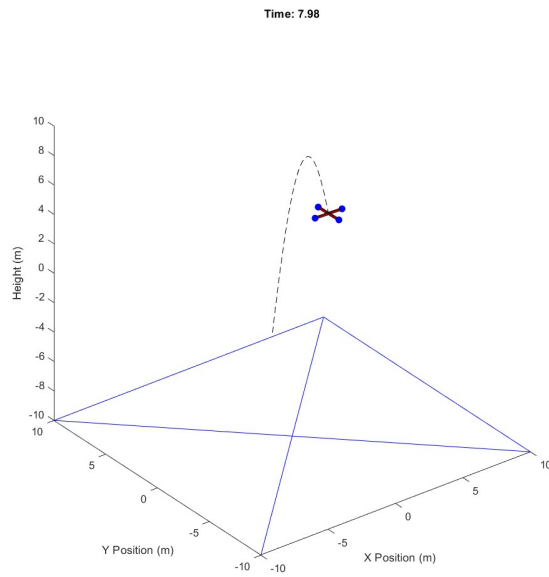


Figure 4: Trajectory Simulated by DDP

### 3.3 MPC application

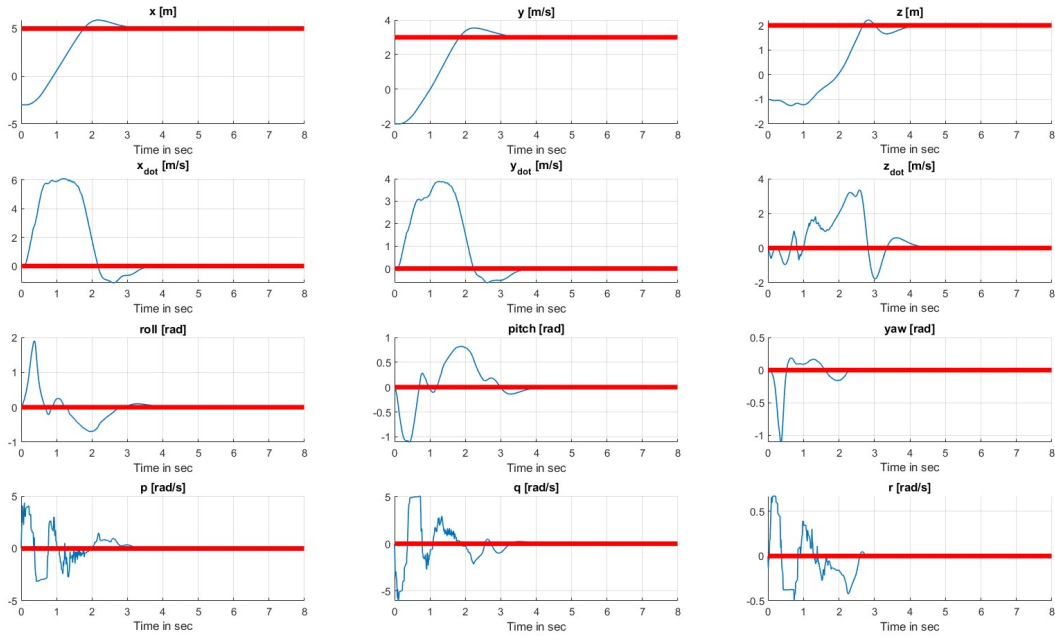


Figure 5: Optimal State Paths Determined Through MPC

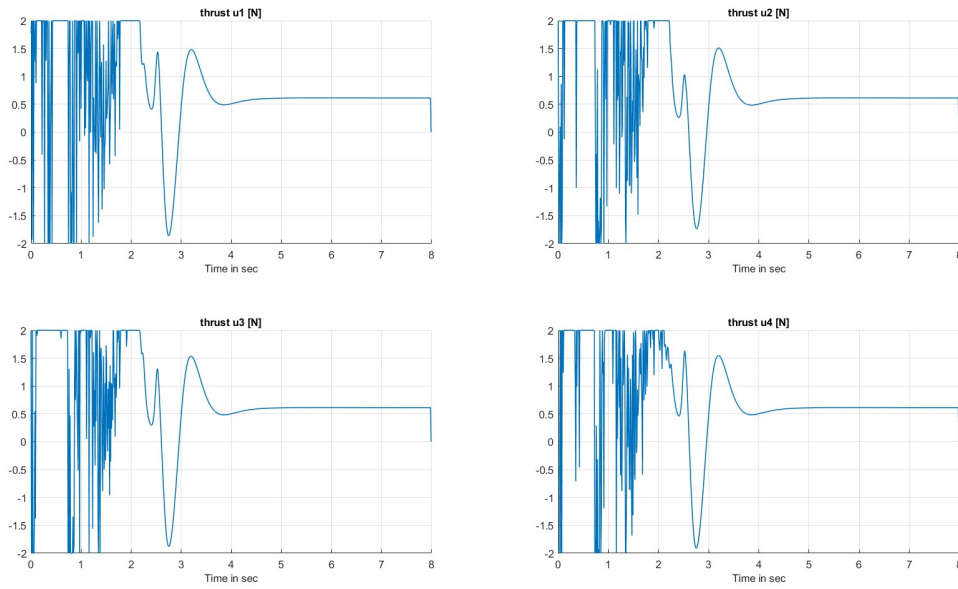


Figure 6: Optimal Control Determined Through MPC

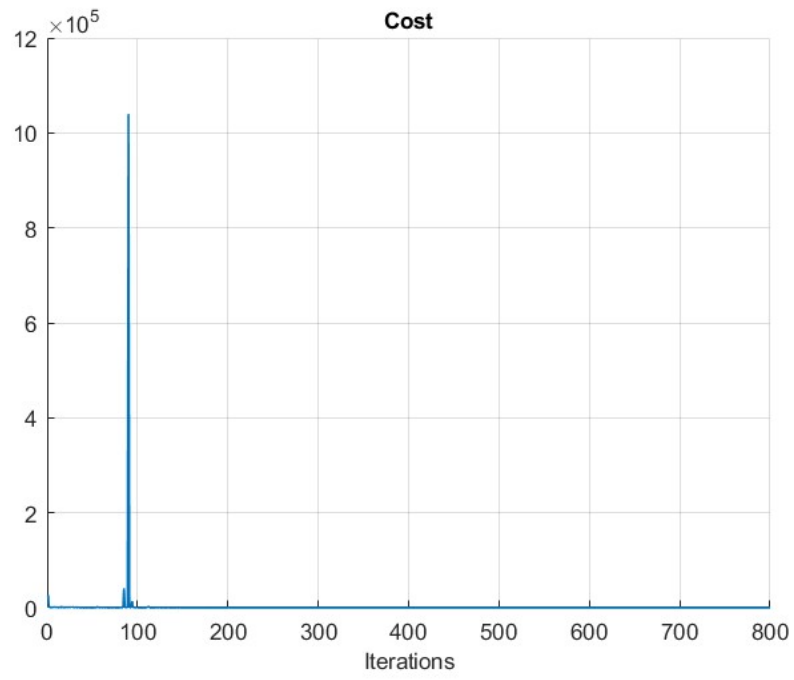


Figure 7: Corresponding Cost Through MPC

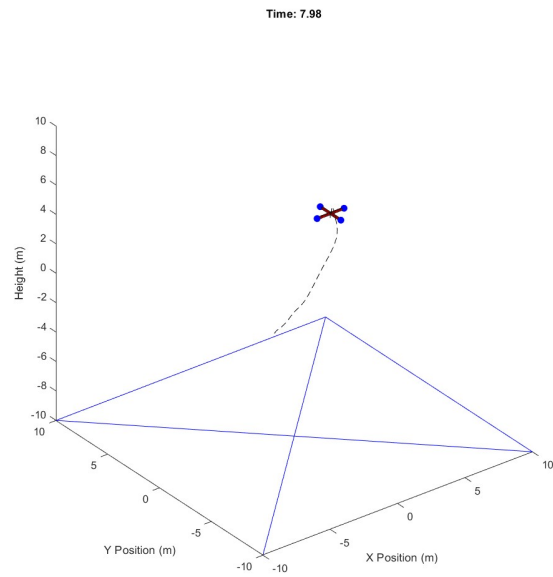


Figure 8: Trajectory Simulated by MPC

### 3.4 obstacle avoidance using MPC with BaS



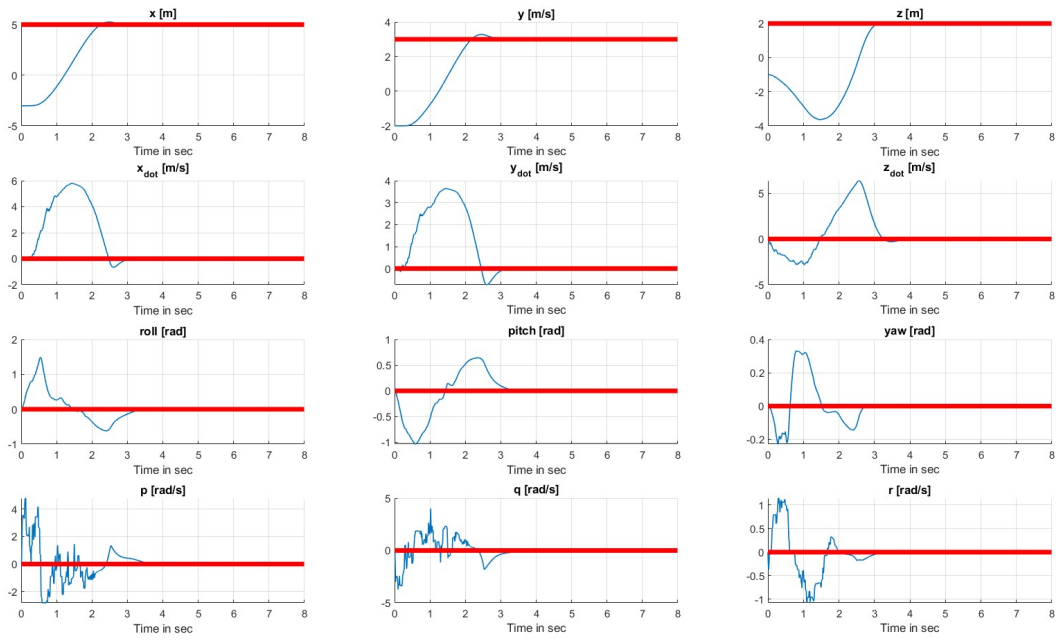


Figure 9: Optimal State Paths Determined Through MPC-BaS

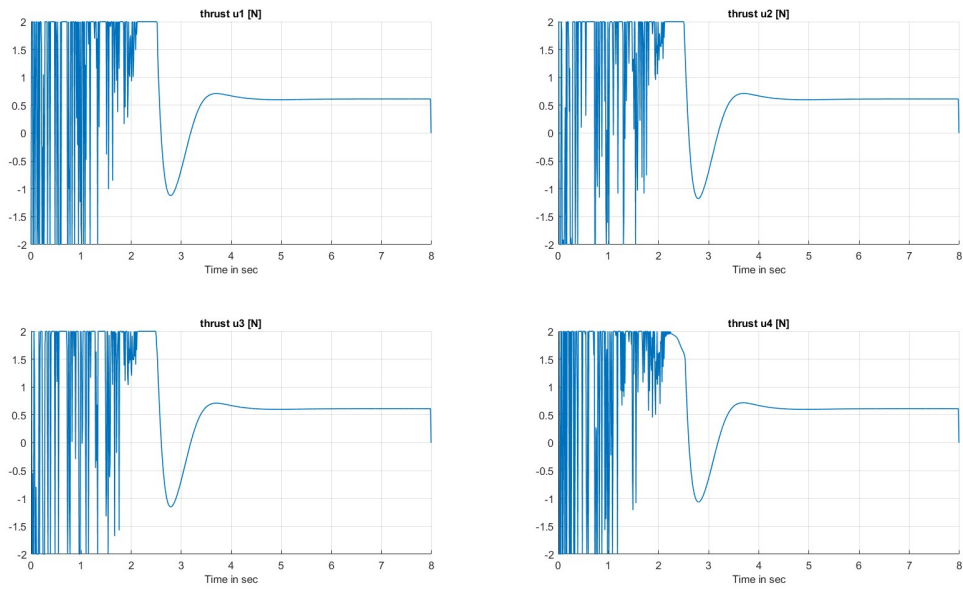


Figure 10: Optimal Control Determined Through MPC-BaS

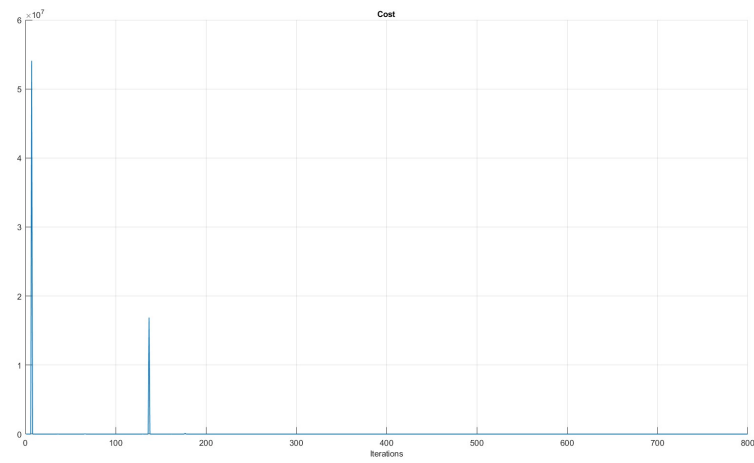


Figure 11: Corresponding Cost Through MPC-BaS

**Time: 7.98**

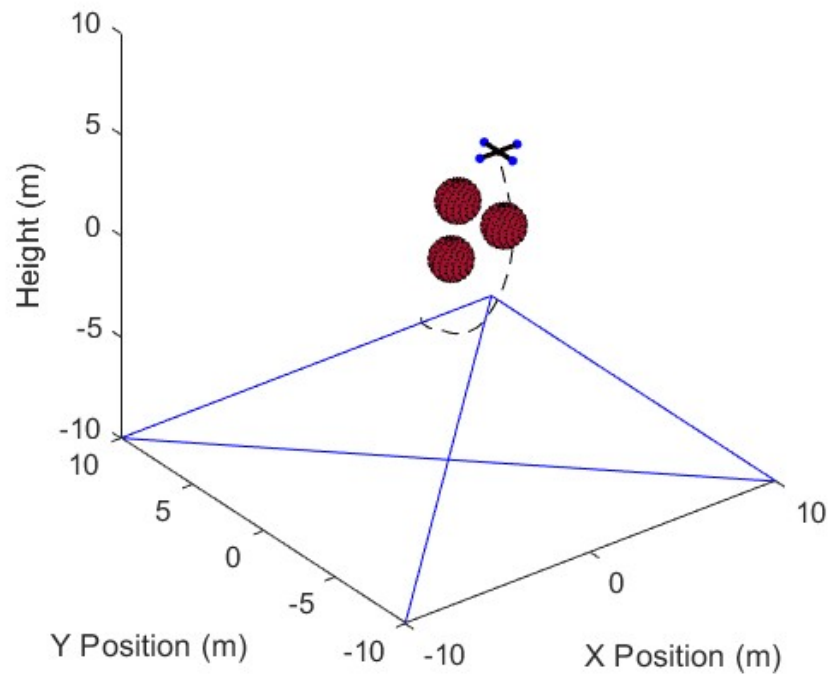


Figure 12: Trajectory Simulated by MPC-BaS

## 4 DISCUSSION

### 4.1 vanishing gradient $Q_{uu}$ in backpropagation

While implementing the DDP function, one gradient of cost matrix  $Q_{uu}$  was vanishing, which caused zero feedback to the system. In other words, the quadrotor received a feedback matrix of zeros at a specific time step. To solve this issue, I tried to force the  $Q_{uu}$  to have as minimum values as it can contain so that it can keep sending small feedback to the system.

---

**Algorithm 3** Force Backpropagation

---

```
1: procedure WITHIN BACKPROPAGATION OF FNDDP
2:    $\lambda \leftarrow ||\min(\text{eig}(Q_{uu}))|| + \text{rand}$ 
3:   if  $\text{any}(\text{eig}(Q_{uu})) < 1e - 10$  then
4:      $Q_{uu} += \lambda I$  where  $I$  is an identity matrix
5:   end
```

---

### 4.2 restricted controls

The first simulation was implemented without clamp of the control, however, the quadrotor initially flew too much fast and spent much time going back to the convergence. This was shown as a very unstable control. Therefore, to stabilize the motion of the drone, I gave the system a control restriction, for example, maximum control value is 4 N and minimum control value is -2.

## 5 Appendix: MATLAB CODE

```
1  clc; clear; close all;
2  %% INITIAL CONFIGURATION
3  % global m; % drone mass
4  % global Ixx; % MOI in x axis
5  % global Iyy; % MOI in y axis
6  % global Izz; % MOI in z axis
7  % global kt; % rotor torque constant
8  % global l; % rotor moment arm length
9  % global g; % gravity constant
10
11 % global p; % body roll rate
12 % global q; % body pitch rate
13 % global r; % body yaw rate
14 % global phi; %roll
15 % global theta; %pitch
16 % global psi; %yaw
17
18 m = 0.5;
19 Ixx = 0.0032;
20 Iyy = 0.0032;
21 Izz = 0.0055;
22 kt = 0.01691;
23 l = 0.17;
24 g = 9.81;
25
26 %% MODEL DEFINITION
27 % rotation matrix R123 = R1*R2*R3 = R.transpose;
28 syms x y z xdot ydot zdot roll pitch yaw p q r u1 u2 u3 u4
29 %R = R1(roll)*R2(pitch)*R3(yaw);
30 R = [cos(pitch)*cos(yaw), cos(pitch)*sin(yaw), -sin(pitch);
31      sin(pitch)*sin(yaw) - cos(roll)*sin(yaw), sin(pitch)*sin(roll)*
32      sin(yaw) + cos(roll)*cos(yaw), sin(roll)*cos(pitch);
33      sin(roll)*sin(yaw) + cos(roll)*sin(pitch)*cos(yaw), sin(pitch)*
34      sin(yaw)*cos(roll) - sin(roll)*sin(yaw), cos(pitch)*cos(roll)
35      ];
36
37 % translational motion
38 % f1 = [xddot; yddot; zddot]
39 f1 = 1/m.*R*[0; 0; u1+u2+u3+u4] + [0; 0; -m*g];
40 % f2 = [rolldot, pitchdot, yawdot] http://www.stengel.mycpanel.princeton.edu/Quaternions.pdf
41 R_f2 = [1, tan(pitch)*sin(roll), tan(pitch)*cos(roll);
42         0, cos(roll), -sin(roll);
43         0, sin(roll)/cos(pitch), cos(roll)/cos(pitch)];
44 f2 = R_f2*[p; q; r];
```

```

42 % rotational motion
43 % f3 = [pdot; qdot; rdot]
44 MOI = [Ixx; Iyy; Izz];
45 f3 = 1./MOI.*[ sqrt(2)/2*(u1 + u3 - u2 - u4)*l - (Izz - Iyy)*q*r;
46      sqrt(2)/2*(u3 + u4 - u1 - u2)*l + (Izz - Ixx)*p*r;
47      kt*(u1 + u4 - u2 - u3)];
48
49 dyn = [xdot; ydot; zdot; f1; f2; f3];
50 x = [x; y; z; xdot; ydot; zdot; roll; pitch; yaw; p; q; r];
51 u = [u1; u2; u3; u4];
52 A = jacobian(dyn, x);
53 B = jacobian(dyn, u);
54 %% safety constraints
55 syms x y z xdot ydot zdot roll pitch yaw p q r z1 z2 z3
56
57 xc1 = 2.2;
58 yc1 = 2.2;
59 zc1 = 1;
60 rc1 = 1;
61
62 xc2 = 0;
63 yc2 = -0.2;
64 zc2 = 0;
65 rc2 = 1;
66
67 xc3 = 3;
68 yc3 = 0;
69 zc3 = 0.5;
70 rc3 = 1;
71
72 xvec = [x; y; z; xdot; ydot; zdot; roll; pitch; yaw; p; q; r];
73 h1 = (x - xc1)^2 + (y - yc1)^2 + (z - zc1)^2 - rc1;
74 hx1 = jacobian(h1, xvec);
75 b1 = 1/h1;
76
77 h2 = (x - xc2)^2 + (y - yc2)^2 + (z - zc2)^2 - rc2;
78 hx2 = jacobian(h2, xvec);
79 b2 = 1/h2;
80
81 h3 = (x - xc3)^2 + (y - yc3)^2 + (z - zc3)^2 - rc3;
82 hx3 = jacobian(h3, xvec);
83 b3 = 1/h3;
84 %% state shift
85 x0 = [-3; -2; -1; 0; 0; 0; 0; 0; 0; 0; 0; 0];
86 xd = [5; 3; 2; 0; 0; 0; 0; 0; 0; 0; 0; 0];
87
88 bnut1 = double(subs(b1, {x, y, z}, {x0(1), x0(2), x0(3)}));

```

```

89 bnut2 = double(subs(b2, {x, y, z}, {x0(1), x0(2), x0(3)}));
90 bnut3 = double(subs(b3, {x, y, z}, {x0(1), x0(2), x0(3)}));
91 zz1 = b1 - bnut1;
92 zz2 = b2 - bnut2;
93 zz3 = b3 - bnut3;
94 %% safety constraint states z
95 gamma = 1;
96 zdot1 = diff(b1)*hx1*dyn - gamma*(z1 - zz1);
97 zdot2 = diff(b2)*hx2*dyn - gamma*(z2 - zz2);
98 zdot3 = diff(b3)*hx3*dyn - gamma*(z3 - zz3);
99
100 %% safety constraints states' initial/desired values
101 xsafe = [xvec; z1; z2; z3];
102
103 z10 = double(subs(zz1, {x, y, z}, {x0(1), x0(2), x0(3)}));
104 z1d = double(subs(zz1, {x, y, z}, {xd(1), xd(2), xd(3)}));
105
106 z20 = double(subs(zz2, {x, y, z}, {x0(1), x0(2), x0(3)}));
107 z2d = double(subs(zz2, {x, y, z}, {xd(1), xd(2), xd(3)}));
108
109 z30 = double(subs(zz3, {x, y, z}, {x0(1), x0(2), x0(3)}));
110 z3d = double(subs(zz3, {x, y, z}, {xd(1), xd(2), xd(3)}));
111 %% the whole system's initial / desired values
112 x0safe = [x0; z10; z20; z30]; % z0 = subs(z, x1, 5);
113 xd safe = [xd; z1d; z2d; z3d]; % z-des = subs(z, x1, 0);
114
115 %% system update
116 dyn_obs = [dyn; zdot1; zdot2; zdot3];
117 fxSafe = jacobian(dyn_obs, xsafe);
118 fuSafe = jacobian(dyn_obs, u);
119 %% DDP-MPC
120 %% initial configuration
121 t = 8;
122 dt = 0.01;
123 pHorizon = t/dt;
124 cHorizon = 100;
125 iter = 5;
126 lr = 0.5;
127 Qsafe = [1, 1, 1];
128 Ssafe = [10, 10, 10];
129 Q = diag([0, 0, 0, 0, 0, 0, 1, 1, 1, .1, .1, .1, Qsafe]);
130 S = diag([10, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, Ssafe].*1);
131 R = diag([1, 1, 1, 1].*0.01);
132 ddp_message = false;
133 safe_mode = true;
134 ddp_cost = false;
135 %% nominal state update

```

```

136 ubar = zeros(4, cHorizon - 1);
137 ubar(:, 1) = [1, 1, 1, 1].*5;
138 x_des = zeros(12 + 3, cHorizon);
139 x_des(:, 1) = x0safe;
140 x_des(:, end) = xdsafe;
141 xbar = zeros(12 + 3, cHorizon);
142 xbar(:, 1) = x0safe;
143 for i = 1:cHorizon-1
144     [x_, ~, ~] = fnStateDroneSafe(xbar(:, i), ubar(:, i));
145     xbar(:, i+1) = xbar(:, i) + x_.*dt;
146 end
147 umpc = zeros(4, pHorizon);
148 traj = xbar;
149 cost_ = zeros(1, pHorizon);
150 %% simulation
151 for h = 1:pHorizon - 1
152     % find optimal control u at horizon h or step t
153     [xbar, u, cost] = fnDDPdrone(xbar, ubar, x_des, cHorizon, iter,
154         lr, Q, S, R, ddp_message, safe_mode, ddp_cost); %(horizon + 1
155         - h)*dt,
156
157     % save the first element of the control at step t where we
158     generate a
159     % new trajectory. ==>> will be the optimal control of this MPC
160     model
161     umpc(:, h) = u(:, 1);
162     % find the next state by calculating it with the initial control
163     input
164     % and current state
165     [x_, ~, ~] = fnStateDroneSafe(traj(:, h), u(:, 1));
166     x_ = x_.*dt;
167
168     % update the next state
169     traj(:, h + 1) = x_ + traj(:, h);
170
171     % update & shift nominal state/control by 1
172     % this's aka receding horizon
173     % At the next state (first elements of the updated state/control)
174     % , the model will re-optimize control
175     ubar(:, 1:end-1) = u(:, 2:end);
176     xbar(:, 1) = traj(:, h + 1);
177
178     % cost display
179     cost_(h) = cost;
180     fprintf('%d DDP-MPC cost: %.4f with lr = %.4f \n', h, cost_(h),
181         lr);
182 end

```

```

177 %% MPC trajectory
178 mpc_traj = traj;
179 figure(101)
180 plot3(mpc_traj(1, :), mpc_traj(2, :), mpc_traj(3, :), '—')
181 hold on
182 plot3(x0(1), x0(2), x0(3), 'rx')
183 plot3(xd(1), xd(2), xd(3), 'bx')
184 hold off
185 %% MPC animation
186 xd = x_des(1:3, end);
187 obs_info = [xc1, yc1, zc1, rc1; xc2, yc2, zc2, rc2; xc3, yc3, zc3,
    rc3];
188 [h,MMmpc] = quadrotor_visualize_w_obstacles(mpc_traj, u, 0:dt:t-dt,
    obs_info, xd);
189 %%
190 figure(2)
191 movie(MMmpc,1,length(mpc_traj(1,:))/t/2)
192
193 %% PERFORMANCE
194 time = 0:dt:t-dt;
195 horizon = pHorizon;
196
197 figure(11);
198 subplot(4,3,1)
199 hold on
200 plot(time, mpc_traj(1,:), 'linewidth',1);
201 plot(time, x_des(1,end)*ones(1,horizon), 'red', 'linewidth',4)
202 title('x [m]', 'fontsize',10);
203 xlabel('Time in sec', 'fontsize',10)
204 %hold off;
205 grid on
206
207 subplot(4,3,2)
208 hold on;
209 plot(time, mpc_traj(2,:), 'linewidth',1);
210 plot(time, x_des(2,end)*ones(1,horizon), 'red', 'linewidth',4)
211 title('y [m/s]', 'fontsize',10);
212 xlabel('Time in sec', 'fontsize',10)
213 grid on
214 %hold off;
215
216 subplot(4,3,3)
217 hold on
218 plot(time, mpc_traj(3,:), 'linewidth',1);
219 plot(time, x_des(3,end)*ones(1,horizon), 'red', 'linewidth',4)
220 title('z [m]', 'fontsize',10)
221 xlabel('Time in sec', 'fontsize',10)

```



```

222 grid on
223 %hold off;$\theta$
224
225 subplot(4,3,4)
226 hold on
227 plot(time,mpc_traj(4,:), 'linewidth',1);
228 plot(time,x_des(4,end)*ones(1,horizon), 'red', 'linewidth',4)
229 title('x_{dot} [m/s]', 'fontsize',10)
230 xlabel('Time in sec', 'fontsize',10)
231 grid on
232
233 subplot(4,3,5)
234 hold on
235 plot(time,mpc_traj(5,:), 'linewidth',1);
236 plot(time,x_des(5,end)*ones(1,horizon), 'red', 'linewidth',4)
237 title('y_{dot} [m/s]', 'fontsize',10)
238 xlabel('Time in sec', 'fontsize',10)
239 grid on
240 %hold off
241
242 subplot(4,3,6)
243 hold on
244 plot(time,mpc_traj(6,:), 'linewidth',1);
245 plot(time,x_des(6,end)*ones(1,horizon), 'red', 'linewidth',4)
246 title('z_{dot} [m/s]', 'fontsize',10)
247 xlabel('Time in sec', 'fontsize',10)
248 grid on
249
250 subplot(4,3,7)
251 hold on
252 plot(time,mpc_traj(7,:), 'linewidth',1);
253 plot(time,x_des(7,end)*ones(1,horizon), 'red', 'linewidth',4)
254 title('roll [rad]', 'fontsize',10)
255 xlabel('Time in sec', 'fontsize',10)
256 grid on
257
258 subplot(4,3,8)
259 hold on
260 plot(time,mpc_traj(8,:), 'linewidth',1);
261 plot(time,x_des(8,end)*ones(1,horizon), 'red', 'linewidth',4)
262 title('pitch [rad]', 'fontsize',10)
263 xlabel('Time in sec', 'fontsize',10)
264 grid on
265
266 subplot(4,3,9)
267 hold on
268 plot(time,mpc_traj(9,:), 'linewidth',1);

```

```

269 plot(time , x_des(9 ,end)*ones(1 ,horizon) , 'red' , 'linewidth' ,4)
270 title('yaw [rad]' , 'fontsize' ,10)
271 xlabel('Time in sec' , 'fontsize' ,10)
272 grid on
273
274 subplot(4 ,3 ,10)
275 hold on
276 plot(time , mpc_traj(10 ,:) , 'linewidth' ,1);
277 plot(time , x_des(10 ,end)*ones(1 ,horizon) , 'red' , 'linewidth' ,4)
278 title('p [rad/s]' , 'fontsize' ,10)
279 xlabel('Time in sec' , 'fontsize' ,10)
280 grid on
281
282 subplot(4 ,3 ,11)
283 hold on
284 plot(time , mpc_traj(11 ,:) , 'linewidth' ,1);
285 plot(time , x_des(11 ,end)*ones(1 ,horizon) , 'red' , 'linewidth' ,4)
286 title('q [rad/s]' , 'fontsize' ,10)
287 xlabel('Time in sec' , 'fontsize' ,10)
288 grid on
289
290 subplot(4 ,3 ,12)
291 hold on
292 plot(time , mpc_traj(12 ,:) , 'linewidth' ,1);
293 plot(time , x_des(12 ,end)*ones(1 ,horizon) , 'red' , 'linewidth' ,4)
294 title('r [rad/s]' , 'fontsize' ,10)
295 xlabel('Time in sec' , 'fontsize' ,10)
296 grid on
297 %%%
298 figure(13)
299 subplot(2 , 2 , 1)
300 hold on
301 plot(time , umpc(1 ,:) , 'linewidth' ,1);
302 title('thrust u1 [N]' , 'fontsize' ,10)
303 xlabel('Time in sec' , 'fontsize' ,10)
304 grid on
305
306 subplot(2 , 2 , 2)
307 hold on
308 plot(time , umpc(2 ,:) , 'linewidth' ,1);
309 title('thrust u2 [N]' , 'fontsize' ,10)
310 xlabel('Time in sec' , 'fontsize' ,10)
311 grid on
312
313 subplot(2 , 2 , 3)
314 hold on
315 plot(time , umpc(3 ,:) , 'linewidth' ,1);

```

```

316 title('thrust u3 [N]','fontsize',10)
317 xlabel('Time in sec','fontsize',10)
318 grid on
319
320 subplot(2, 2, 4)
321 hold on
322 plot(time,umpc(4,:), 'linewidth',1);
323 title('thrust u4 [N]','fontsize',10)
324 xlabel('Time in sec','fontsize',10)
325 grid on
326
327 %%
328 figure(14)
329 hold on
330 plot(cost_,'linewidth',1);
331 xlabel('Iterations','fontsize',10)
332 title('Cost','fontsize',10);
333 grid on
334 %save('DDP_Data');
335
336
337
338 %% functions
339 function r = R1(rad)
340     r = [1 0 0; 0 cos(rad) sin(rad); 0 -sin(rad) cos(rad)];
341 end
342 function r = R2(rad)
343     r = [cos(rad) 0 -sin(rad); 0 1 0; sin(rad) 0 cos(rad)];
344 end
345 function r = R3(rad)
346     r = [cos(rad) sin(rad) 0; -sin(rad) cos(rad) 0; 0 0 1];
347 end

```

```

1  clc; clear; close all;
2  %% INITIAL CONFIGURATION
3  global m; % drone mass
4  global Ixx; % MOI in x axis
5  global Iyy; % MOI in y axis
6  global Izz; % MOI in z axis
7  global kt; % rotor torque constant
8  global l; % rotor moment arm length
9  global g; % gravity constant
10
11 % global p; % body roll rate
12 % global q; % body pitch rate
13 % global r; % body yaw rate
14 % global phi; %roll

```

```

15 % global theta; %pitch
16 % global psi; %yaw
17
18 m = 0.5;
19 Ixx = 0.0032;
20 Iyy = 0.0032;
21 Izz = 0.0055;
22 kt = 0.01691;
23 l = 0.17;
24 g = 9.81;
25
26 %% MODEL DEFINITION
27 % rotation matrix R123 = R1*R2*R3 = R.transpose;
28 syms x y z xdot ydot zdot roll pitch yaw p q r u1 u2 u3 u4
29 R = R1(roll)*R2(pitch)*R3(yaw);
30 R = [cos(pitch)*cos(yaw), cos(pitch)*sin(yaw), -sin(pitch);
31      sin(pitch)*sin(yaw) - cos(roll)*sin(yaw), sin(pitch)*sin(roll)*
32      sin(yaw) + cos(roll)*cos(yaw), sin(roll)*cos(pitch);
33      sin(roll)*sin(yaw) + cos(roll)*sin(pitch)*cos(yaw), sin(pitch)*
34      sin(yaw)*cos(roll) - sin(roll)*sin(yaw), cos(pitch)*cos(roll)
35      ];
36
37 % translational motion
38 % f1 = [xddot; yddot; zddot]
39 f1 = 1/m.*R*[0; 0; u1+u2+u3+u4] + [0; 0; -m*g];
40 % f2 = [rolldot, pitchdot, yawdot] http://www.stengel.mycpanel.princeton.edu/Quaternions.pdf
41 R_f2 = [1, tan(pitch)*sin(roll), tan(pitch)*cos(roll);
42         0, cos(roll), -sin(roll);
43         0, sin(roll)/cos(pitch), cos(roll)/cos(pitch)];
44 f2 = R_f2*[p; q; r];
45 % rotational motion
46 % f3 = [pdot; qdot; rdot]
47 MOI = [Ixx; Iyy; Izz];
48 f3 = 1./MOI.*[sqrt(2)/2*(u1 + u3 - u2 - u4)*l - (Izz - Iyy)*q*r;
49              sqrt(2)/2*(u3 + u4 - u1 - u2)*l + (Izz - Ixx)*p*r;
50              kt*(u1 + u4 - u2 - u3)];
51
52 dyn = [xdot; ydot; zdot; f1; f2; f3];
53 x = [x; y; z; xdot; ydot; zdot; roll; pitch; yaw; p; q; r];
54 u = [u1; u2; u3; u4];
55 A = jacobian(dyn, x)
56 B = jacobian(dyn, u)
57
58 %% DDP-MPC
59 t = 8;
60 dt = 0.01;

```

```

58 pHorizon = t/dt;
59 cHorizon = 100;
60 iter = 5;
61 lr = 0.5;
62 Q = diag([0, 0, 0, 0, 0, 0, 1, 1, 1, .1, .1, .1]);
63 S = diag([10, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1].*1);
64 R = diag([1, 1, 1, 1].*0.01);
65
66 ubar = zeros(4, cHorizon - 1);
67 ubar(:, 1) = [1, 1, 1, 1].*5;
68 x0 = [-3; -2; -1; 0; 0; 0; 0; 0; 0; 0; 0; 0];
69 xd = [5; 3; 2; 0; 0; 0; 0; 0; 0; 0; 0; 0];
70
71 x_des = zeros(12, cHorizon);
72 x_des(:, 1) = x0;
73 x_des(:, end) = xd;
74 xbar = zeros(12, cHorizon);
75 xbar(:, 1) = x0;
76 for i = 1:cHorizon-1
77     [x_, ~, ~] = fnStateDrone(xbar(:, i), ubar(:, i));
78     xbar(:, i+1) = xbar(:, i) + x_.*dt;
79 end
80
81 umpc = zeros(4, pHorizon);
82 traj = xbar;
83 cost_ = zeros(1, pHorizon);
84 ddp_message = false;
85 safe_mode = false;
86 ddp_cost = false;
87 for h = 1:pHorizon - 1
88     % find optimal control u at horizon h or step t
89     [xbar, u, cost] = fnDDPdrone(xbar, ubar, x_des, cHorizon, iter,
90         lr, Q, S, R, ddp_message, safe_mode, ddp_cost); %(horizon + 1
91         - h)*dt,
92
93     % save the first element of the control at step t where we
94     % generate a
95     % new trajectory. ==>> will be the optimal control of this MPC
96     % model
97     umpc(:, h) = u(:, 1);
98     % find the next state by calculating it with the initial control
99     % input
100    % and current state
101    [x_, ~, ~] = fnStateDrone(traj(:, h), u(:, 1));
102    x_ = x_.*dt;
103
104    % update the next state

```

```

100     traj(:, h + 1) = x_ + traj(:, h);
101
102     % update & shift nominal state/control by 1
103     % this 's aka receding horizon
104     % At the next state (first elements of the updated state/control)
105     % , the model will re-optimize control
106     ubar(:, 1:end-1) = u(:, 2:end);
107     %     ubar(:, end) = 0;
108     %     xbar(:, 1:end-1) = xbar(:, 2:end);
109     xbar(:, 1) = traj(:, h + 1);
110
111     % cost display
112     cost_(h) = cost;
113     fprintf("%d DDP-MPC cost: %.4f with lr = %.4f \n", h, cost_(h),
114             lr);
115 end
116 %% MPC trajectory
117 mpc_traj = traj;
118 figure(101)
119 plot3(mpc_traj(1, :), mpc_traj(2, :), mpc_traj(3, :), '—')
120 hold on
121 plot3(x0(1), x0(2), x0(3), 'rx')
122 plot3(xd(1), xd(2), xd(3), 'bx')
123 hold off
124 %% MPC animation
125 xd = x_des(1:3, end);
126 [h, MMmpc] = quadrotor_visualize(mpc_traj, u, 0:dt:t-dt, xd)
127 %%
128 figure(2)
129 movie(MMmpc, 1, length(mpc_traj(1, :))/t/2)
130
131 %% PERFORMANCE
132 time = 0:dt:t-dt;
133 horizon = pHorizon;
134
135 figure(11);
136 subplot(4,3,1)
137 hold on
138 plot(time, mpc_traj(1, :), 'linewidth', 1);
139 plot(time, x_des(1, end)*ones(1, horizon), 'red', 'linewidth', 4)
140 title('x [m]', 'fontsize', 10);
141 xlabel('Time in sec', 'fontsize', 10)
142 %hold off;
143 grid on
144
145 subplot(4,3,2)
146 hold on;

```

```

146 plot(time , mpc_traj(2,:) , 'linewidth' ,1);
147 plot(time , x_des(2,end)*ones(1,horizon) , 'red' , 'linewidth' ,4)
148 title('y [m/s]' , 'fontsize' ,10);
149 xlabel('Time in sec' , 'fontsize' ,10)
150 grid on
151 %hold off;
152
153 subplot(4,3,3)
154 hold on
155 plot(time , mpc_traj(3,:) , 'linewidth' ,1);
156 plot(time , x_des(3,end)*ones(1,horizon) , 'red' , 'linewidth' ,4)
157 title('z [m]' , 'fontsize' ,10)
158 xlabel('Time in sec' , 'fontsize' ,10)
159 grid on
160 %hold off;$\theta$
161
162 subplot(4,3,4)
163 hold on
164 plot(time , mpc_traj(4,:) , 'linewidth' ,1);
165 plot(time , x_des(4,end)*ones(1,horizon) , 'red' , 'linewidth' ,4)
166 title('x_{dot} [m/s]' , 'fontsize' ,10)
167 xlabel('Time in sec' , 'fontsize' ,10)
168 grid on
169
170 subplot(4,3,5)
171 hold on
172 plot(time , mpc_traj(5,:) , 'linewidth' ,1);
173 plot(time , x_des(5,end)*ones(1,horizon) , 'red' , 'linewidth' ,4)
174 title('y_{dot} [m/s]' , 'fontsize' ,10)
175 xlabel('Time in sec' , 'fontsize' ,10)
176 grid on
177 %hold off
178
179 subplot(4,3,6)
180 hold on
181 plot(time , mpc_traj(6,:) , 'linewidth' ,1);
182 plot(time , x_des(6,end)*ones(1,horizon) , 'red' , 'linewidth' ,4)
183 title('z_{dot} [m/s]' , 'fontsize' ,10)
184 xlabel('Time in sec' , 'fontsize' ,10)
185 grid on
186
187 subplot(4,3,7)
188 hold on
189 plot(time , mpc_traj(7,:) , 'linewidth' ,1);
190 plot(time , x_des(7,end)*ones(1,horizon) , 'red' , 'linewidth' ,4)
191 title('roll [rad]' , 'fontsize' ,10)
192 xlabel('Time in sec' , 'fontsize' ,10)

```

```

193 grid on
194
195 subplot(4,3,8)
196 hold on
197 plot(time, mpc_traj(8,:), 'linewidth', 1);
198 plot(time, x_des(8, end)*ones(1, horizon), 'red', 'linewidth', 4)
199 title('pitch [rad]', 'fontsize', 10)
200 xlabel('Time in sec', 'fontsize', 10)
201 grid on
202
203 subplot(4,3,9)
204 hold on
205 plot(time, mpc_traj(9,:), 'linewidth', 1);
206 plot(time, x_des(9, end)*ones(1, horizon), 'red', 'linewidth', 4)
207 title('yaw [rad]', 'fontsize', 10)
208 xlabel('Time in sec', 'fontsize', 10)
209 grid on
210
211 subplot(4,3,10)
212 hold on
213 plot(time, mpc_traj(10,:), 'linewidth', 1);
214 plot(time, x_des(10, end)*ones(1, horizon), 'red', 'linewidth', 4)
215 title('p [rad/s]', 'fontsize', 10)
216 xlabel('Time in sec', 'fontsize', 10)
217 grid on
218
219 subplot(4,3,11)
220 hold on
221 plot(time, mpc_traj(11,:), 'linewidth', 1);
222 plot(time, x_des(11, end)*ones(1, horizon), 'red', 'linewidth', 4)
223 title('q [rad/s]', 'fontsize', 10)
224 xlabel('Time in sec', 'fontsize', 10)
225 grid on
226
227 subplot(4,3,12)
228 hold on
229 plot(time, mpc_traj(12,:), 'linewidth', 1);
230 plot(time, x_des(12, end)*ones(1, horizon), 'red', 'linewidth', 4)
231 title('r [rad/s]', 'fontsize', 10)
232 xlabel('Time in sec', 'fontsize', 10)
233 grid on
234 %%
235 figure(13)
236 subplot(2, 2, 1)
237 hold on
238 plot(time, umpc(1,:), 'linewidth', 1);
239 title('thrust u1 [N]', 'fontsize', 10)

```



```

240 xlabel('Time in sec','fontsize',10)
241 grid on
242
243 subplot(2, 2, 2)
244 hold on
245 plot(time,umpc(2,:), 'linewidth',1);
246 title('thrust u2 [N]','fontsize',10)
247 xlabel('Time in sec','fontsize',10)
248 grid on
249
250 subplot(2, 2, 3)
251 hold on
252 plot(time,umpc(3,:), 'linewidth',1);
253 title('thrust u3 [N]','fontsize',10)
254 xlabel('Time in sec','fontsize',10)
255 grid on
256
257 subplot(2, 2, 4)
258 hold on
259 plot(time,umpc(4,:), 'linewidth',1);
260 title('thrust u4 [N]','fontsize',10)
261 xlabel('Time in sec','fontsize',10)
262 grid on
263
264 %%
265 figure(14)
266 hold on
267 plot(cost_,'linewidth',1);
268 xlabel('Iterations','fontsize',10)
269 title('Cost','fontsize',10);
270 grid on
271 %save('DDP_Data');
272 %% functions
273 function r = R1(rad)
274     r = [1 0 0; 0 cos(rad) sin(rad); 0 -sin(rad) cos(rad)];
275 end
276 function r = R2(rad)
277     r = [cos(rad) 0 -sin(rad); 0 1 0; sin(rad) 0 cos(rad)];
278 end
279 function r = R3(rad)
280     r = [cos(rad) sin(rad) 0; -sin(rad) cos(rad) 0; 0 0 1];
281 end

1 clc; clear; close all;
2 %% INITIAL CONFIGURATION
3 global m; % drone mass
4 global Ixx; % MOI in x axis

```

```

5 global Iyy; % MOI in y axis
6 global Izz; % MOI in z axis
7 global kt; % rotor torque constant
8 global l; % rotor moment arm length
9 global g; % gravity constant
10
11 % global p; % body roll rate
12 % global q; % body pitch rate
13 % global r; % body yaw rate
14 % global phi; %roll
15 % global theta; %pitch
16 % global psi; %yaw
17
18 m = 0.5;
19 Ixx = 0.0032;
20 Iyy = 0.0032;
21 Izz = 0.0055;
22 kt = 0.01691;
23 l = 0.17;
24 g = 9.81;
25
26 %% MODEL DEFINITION
27 % rotation matrix R123 = R1*R2*R3 = R.transpose;
28 syms x y z xdot ydot zdot roll pitch yaw p q r u1 u2 u3 u4
29 R = R1(roll)*R2(pitch)*R3(yaw);
30 R = [cos(pitch)*cos(yaw), cos(pitch)*sin(yaw), -sin(pitch);
31      sin(pitch)*sin(yaw) - cos(roll)*sin(yaw), sin(pitch)*sin(roll)*
32      sin(yaw) + cos(roll)*cos(yaw), sin(roll)*cos(pitch);
33      sin(roll)*sin(yaw) + cos(roll)*sin(pitch)*cos(yaw), sin(pitch)*
34      sin(yaw)*cos(roll) - sin(roll)*sin(yaw), cos(pitch)*cos(roll)
35      ];
36
37 % translational motion
38 % f1 = [xddot; yddot; zddot]
39 f1 = 1/m.*R*[0; 0; u1+u2+u3+u4] + [0; 0; -m*g];
40 % f2 = [rolldot, pitchdot, yawdot] http://www.stengel.mycpanel.princeton.edu/Quaternions.pdf
41 R_f2 = [1, tan(pitch)*sin(roll), tan(pitch)*cos(roll);
42         0, cos(roll), -sin(roll);
43         0, sin(roll)/cos(pitch), cos(roll)/cos(pitch)];
44 f2 = R_f2*[p; q; r];
45 % rotational motion
46 % f3 = [pdot; qdot; rdot]
47 MOI = [Ixx; Iyy; Izz];
48 f3 = 1./MOI.*[sqrt(2)/2*(u1 + u3 - u2 - u4)*l - (Izz - Iyy)*q*r;
49              sqrt(2)/2*(u3 + u4 - u1 - u2)*l + (Izz - Ixx)*p*r;
50              kt*(u1 + u4 - u2 - u3)];

```

```

48
49 dyn = [xdot; ydot; zdot; f1; f2; f3];
50 x = [x; y; z; xdot; ydot; zdot; roll; pitch; yaw; p; q; r];
51 u = [u1; u2; u3; u4];
52 A = jacobian(dyn, x)
53 B = jacobian(dyn, u)
54
55 %% DDP
56 % Q = diag([0, 0, 0, 0, 0, 0, 1, 1, 1, .1, .1, .1]);
57 % S = diag([10, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1].*1);
58 % %S = diag([1, 1, 1, 10, 10, 10, 1, 1, 1, 10, 10, 10].*1);
59 % R = diag([1, 1, 1, 1].*0.01);
60 % %lr = 0.3:-0.01:0.2;
61 % lr = 0.6
62 Q = diag([0, 0, 0, 0, 0, 0, 1, 1, 1, .1, .1, .1]);
63 S = diag([10, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1].*1);
64 R = diag([1, 1, 1, 1].*0.01);
65 lr = 0.5;
66 t = 8;
67 dt = 0.01;
68 cHorizon = t/dt;
69 iter = 40;
70
71 ubar = zeros(4, cHorizon - 1);
72 ubar(:, 1) = [1, 1, 1, 1].*5;
73 x0 = [-3; -2; -1; 0; 0; 0; 0; 0; 0; 0; 0; 0];
74 xbar = zeros(12, cHorizon);
75 xbar(:, 1) = x0;
76 for i = 1:cHorizon-1
77     [x_, ~, ~] = fnStateDrone(xbar(:, i), ubar(:, i));
78     xbar(:, i+1) = xbar(:, i) + x_.*dt;
79 end
80 x_des = zeros(12, cHorizon);
81 x_des(:, 1) = x0;
82 xd = [5; 3; 2; 0; 0; 0; 0; 0; 0; 0; 0; 0];
83 x_des(:, end) = xd;
84 %% DDP
85 ddp_message = true;
86 safe_mode = false;
87 ddp_cost = true;
88 [x_traj, u, cost] = fnDDPdrone(xbar, ubar, x_des, cHorizon, iter, lr,
    Q, S, R, ddp_message, safe_mode, ddp_cost);
89 %%
90 figure(100)
91 plot3(x_traj(1, :), x_traj(2, :), x_traj(3, :), '—')
92 hold on
93 plot3(x0(1), x0(2), x0(3), 'rx')

```

```

94 plot3(xd(1), xd(2), xd(3), 'bx')
95 hold off
96 %%
97 xd = x_des(1:3, end);
98 [h,MMddp] = quadrotor_visualize(x_traj,u, 0:dt:t-dt, xd)
99 %%
100 figure(2)
101 movie(MMddp,1,length(x_traj(1,:))/t/2)
102
103 %% PERFORMANCE
104 time = 0:dt:t-dt;
105 horizon = cHorizon;
106 figure(11);
107 subplot(4,3,1)
108 hold on
109 plot(time, x_traj(1,:), 'linewidth',1);
110 plot(time, x_des(1,end)*ones(1,horizon), 'red', 'linewidth',4)
111 title('x [m]', 'fontsize',10);
112 xlabel('Time in sec', 'fontsize',10)
113 %hold off;
114 grid on
115
116 subplot(4,3,2)
117 hold on;
118 plot(time, x_traj(2,:), 'linewidth',1);
119 plot(time, x_des(2,end)*ones(1,horizon), 'red', 'linewidth',4)
120 title('y [m/s]', 'fontsize',10);
121 xlabel('Time in sec', 'fontsize',10)
122 grid on
123 %hold off;
124
125 subplot(4,3,3)
126 hold on
127 plot(time, x_traj(3,:), 'linewidth',1);
128 plot(time, x_des(3,end)*ones(1,horizon), 'red', 'linewidth',4)
129 title('z [m]', 'fontsize',10)
130 xlabel('Time in sec', 'fontsize',10)
131 grid on
132 %hold off;$\theta$
133
134 subplot(4,3,4)
135 hold on
136 plot(time, x_traj(4,:), 'linewidth',1);
137 plot(time, x_des(4,end)*ones(1,horizon), 'red', 'linewidth',4)
138 title('x_{dot} [m/s]', 'fontsize',10)
139 xlabel('Time in sec', 'fontsize',10)
140 grid on

```

```

141
142 subplot(4,3,5)
143 hold on
144 plot(time, x_traj(5,:), 'linewidth', 1);
145 plot(time, x_des(5, end)*ones(1, horizon), 'red', 'linewidth', 4)
146 title('y_{dot} [m/s]', 'fontsize', 10)
147 xlabel('Time in sec', 'fontsize', 10)
148 grid on
149 %hold off
150
151 subplot(4,3,6)
152 hold on
153 plot(time, x_traj(6,:), 'linewidth', 1);
154 plot(time, x_des(6, end)*ones(1, horizon), 'red', 'linewidth', 4)
155 title('z_{dot} [m/s]', 'fontsize', 10)
156 xlabel('Time in sec', 'fontsize', 10)
157 grid on
158
159 subplot(4,3,7)
160 hold on
161 plot(time, x_traj(7,:), 'linewidth', 1);
162 plot(time, x_des(7, end)*ones(1, horizon), 'red', 'linewidth', 4)
163 title('roll [rad]', 'fontsize', 10)
164 xlabel('Time in sec', 'fontsize', 10)
165 grid on
166
167 subplot(4,3,8)
168 hold on
169 plot(time, x_traj(8,:), 'linewidth', 1);
170 plot(time, x_des(8, end)*ones(1, horizon), 'red', 'linewidth', 4)
171 title('pitch [rad]', 'fontsize', 10)
172 xlabel('Time in sec', 'fontsize', 10)
173 grid on
174
175 subplot(4,3,9)
176 hold on
177 plot(time, x_traj(9,:), 'linewidth', 1);
178 plot(time, x_des(9, end)*ones(1, horizon), 'red', 'linewidth', 4)
179 title('yaw [rad]', 'fontsize', 10)
180 xlabel('Time in sec', 'fontsize', 10)
181 grid on
182
183 subplot(4,3,10)
184 hold on
185 plot(time, x_traj(10,:), 'linewidth', 1);
186 plot(time, x_des(10, end)*ones(1, horizon), 'red', 'linewidth', 4)
187 title('p [rad/s]', 'fontsize', 10)

```

```

188 xlabel('Time in sec','fontsize',10)
189 grid on
190
191 subplot(4,3,11)
192 hold on
193 plot(time,x_traj(11,:), 'linewidth',1);
194 plot(time,x_des(11,end)*ones(1,horizon), 'red', 'linewidth',4)
195 title('q [rad/s]','fontsize',10)
196 xlabel('Time in sec','fontsize',10)
197 grid on
198
199 subplot(4,3,12)
200 hold on
201 plot(time,x_traj(12,:), 'linewidth',1);
202 plot(time,x_des(12,end)*ones(1,horizon), 'red', 'linewidth',4)
203 title('r [rad/s]','fontsize',10)
204 xlabel('Time in sec','fontsize',10)
205 grid on
206 %%
207 figure(13)
208 subplot(2, 2, 1)
209 hold on
210 plot(time(1:end-1),u(1,:), 'linewidth',1);
211 title('thrust u1 [N]','fontsize',10)
212 xlabel('Time in sec','fontsize',10)
213 grid on
214
215 subplot(2, 2, 2)
216 hold on
217 plot(time(1:end-1),u(2,:), 'linewidth',1);
218 title('thrust u2 [N]','fontsize',10)
219 xlabel('Time in sec','fontsize',10)
220 grid on
221
222 subplot(2, 2, 3)
223 hold on
224 plot(time(1:end-1),u(3,:), 'linewidth',1);
225 title('thrust u3 [N]','fontsize',10)
226 xlabel('Time in sec','fontsize',10)
227 grid on
228
229 subplot(2, 2, 4)
230 hold on
231 plot(time(1:end-1),u(4,:), 'linewidth',1);
232 title('thrust u4 [N]','fontsize',10)
233 xlabel('Time in sec','fontsize',10)
234 grid on

```

```

235
236 %%
237 figure(14)
238 hold on
239 plot(cost,'linewidth',1);
240 xlabel('Iterations','fontsize',10)
241 title('Cost','fontsize',10);
242 grid on
243 %save('DDP_Data');
244
245
246 %% functions
247 function r = R1(rad)
248     r = [1 0 0; 0 cos(rad) sin(rad); 0 -sin(rad) cos(rad)];
249 end
250 function r = R2(rad)
251     r = [cos(rad) 0 -sin(rad); 0 1 0; sin(rad) 0 cos(rad)];
252 end
253 function r = R3(rad)
254     r = [cos(rad) sin(rad) 0; -sin(rad) cos(rad) 0; 0 0 1];
255 end

1 function [x_traj, u, cost] = fnDDPdrone(xbar, ubar, x_des, h2, iter,
    gamma, Q, S, R, ddp_message, safe_mode, ddp_cost)
2
3     %% initial configuration
4     dt = 0.01;
5     horizon = h2;
6     u = ubar;
7     x_traj = xbar;
8     cost = [];
9     %%
10    for p = gamma
11        lr = p; % learning rate
12        for k = 1:iter
13            %% Linearization of the dynamics
14            for i = 1:(horizon-1)
15                % Running cost L(x, u)
16                [l_x, l_xx, l_u, l_uu, l_ux] = fnCost(x_traj(:,i) -
                    x_des(:,i), u(:,i), Q, R, dt);
17                L0(i) = l0;
18                Lx(:,i) = l_x;
19                Lxx(:, :, i) = l_xx;
20                Lu(:,i) = l_u;
21                Luu(:, :, i) = l_uu;
22                Lux(:, :, i) = l_ux; % M = N^T
23

```

```

24         if safe_mode
25             [~, dfx, dfu] = fnStateDroneSafe(x_traj(:, i), u
26                 (:, i));
27         else
28             [~, dfx, dfu] = fnStateDrone(x_traj(:, i), u(:, i
29                 ));
30         end
31
32         % linearized form in discrete time
33         A(:, :, i) = eye(size(xbar, 1)) + dfx*dt; % Phi
34         B(:, :, i) = dfu*dt; % B of linearized equation in
35         the classnote.
36     end
37
38     %—————Search the control —————%
39     Vxx(:, :, horizon) = S;
40     Vx(:, horizon) = S * (x_traj(:, horizon) - x_des(:,
41         horizon));
42     V(horizon) = 0.5*(x_traj(:, horizon) - x_des(:, horizon))
43         '*S*(x_traj(:, horizon) - x_des(:, horizon));
44
45     %————— Backpropagation of the Value Function —————(
46     Chapter 1/ p.7)—%
47     for j = (horizon-1):-1:1
48         Qu(:, j) = Lu(:, j) + B(:, :, j)'*Vx(:, j+1);
49         Qx(:, j) = Lx(:, j) + A(:, :, j)'*Vx(:, j+1); %%
50         different Vx
51         Quu(:, :, j) = B(:, :, j)'*Vxx(:, :, j+1)*B(:, :, j)
52             + Luu(:, :, j);
53         % Quu becomes dead in the middle. Its value ~0 or
54         negative, which means
55         % it will never update the propagation / no feedback.
56         % For example, imagine the zero or negative slope in
57         the optimization graph.
58         lambda = abs(min(eig(Quu(:, :, j)))) + rand;
59         if any(eig(Quu(:, :, j)) < 0.0001)
60             Quu(:, :, j) = Quu(:, :, j) + lambda *eye(length(
61                 Quu(:, :, j)));
62         end
63         Qxx(:, :, j) = A(:, :, j)'*Vxx(:, :, j+1)*A(:, :, j)
64             + Lxx(:, :, j);
65         Qux(:, :, j) = B(:, :, j)'*Vxx(:, :, j+1)*A(:, :, j)
66             + Lux(:, :, j)';
67
68         %feedback
69         Kfb(:, :, j) = -Quu(:, :, j)\Qux(:, :, j);
70         %feedforward

```



```

58     Kff(:, j) = -Quu(:, :, j)\Qu(:, j);
59
60     Vxx(:, :, j) = round(Qxx(:, :, j), 6) + round(Kfb(:, :, j)'*Quu(:, :, j)*Kfb(:, :, j), 6) + round(2*Qux(:, :, j)'*Kfb(:, :, j), 6);
61     Vx(:, j) = round(Qx(:, j), 6) + round(Kfb(:, :, j)'*Qu(:, j), 6) + round(Kfb(:, :, j)'*Quu(:, :, j)*Kff(:, j), 6) + round(Qux(:, :, j)'*Kff(:, j), 6);
62     % disp(j)
63     % disp([Vx(:, j)])
64     % disp([Vxx(:, :, j)])
65     % disp([Qx(:, j)])
66     % disp([Qxx(:, :, j)])
67     % disp([Quu(:, :, j)]) % becoming large at 763
68     % disp([Qux(:, :, j)])
69     end
70
71     %-----> Forward
72     Propagaion:
73     %-----> Find the controls/ forward
74     if safe_mode
75         for i=1:(horizon-1)
76             du = Kff(:, i) + Kfb(:, :, i)*(x_traj(:, i)-xbar(:, i));
77             u(:, i) = ubar(:, i) + lr * du;
78             u(:, i) = min(ones(4, 1).*2, u(:, i));
79             u(:, i) = max(-ones(4, 1).*2, u(:, i));
80             [x_, ~, ~] = fnStateDroneSafe(x_traj(:, i), u(:, i));
81             x_traj(:, i+1) = x_traj(:, i) + x_.*dt;
82         end
83     else
84         for i=1:(horizon-1)
85             du = Kff(:, i) + Kfb(:, :, i)*(x_traj(:, i)-xbar(:, i));
86             u(:, i) = ubar(:, i) + lr * du;
87             u(:, i) = min(ones(4, 1).*2, u(:, i));
88             u(:, i) = max(-ones(4, 1).*2, u(:, i));
89             [x_, ~, ~] = fnStateDrone(x_traj(:, i), u(:, i));
90             x_traj(:, i+1) = x_traj(:, i) + x_.*dt;
91         end
92     end
93     ubar = u;
94     %-----> Simulation of the
95     Nonlinear System
96     xbar = x_traj;
97     [Cost(:, k)] = fnCostComputation(x_traj, u, x_des, dt, Q, S

```

```

    , R);
96     if ddp_message
97         fprintf('DDP Iteration %d, Current Cost = %.6f,
                Learning rate = %.4f\n',k, Cost(1,k), lr);
98     end
99     if ddp_cost
100         cost = [cost Cost(1,k)];
101     end
102 end %% end iterating over the algorithm
103 if ddp_cost == 0
104     cost = [cost Cost(:, end)];
105 end
106 %figure(1);
107 % subplot(2,4,1)
108 % hold on
109 % plot(time, x_traj(1,:), 'linewidth',1);
110 % plot(time, x_des(1,1)*ones(1,horizon), 'red', 'linewidth',4)
111 % title('x [m]', 'fontsize',10);
112 % xlabel('Time in sec', 'fontsize',10)
113 % %hold off;
114 % grid on
115 %
116 %
117 % subplot(2,4,2);hold on;
118 % plot(time, x_traj(2,:), 'linewidth',1);
119 % plot(time, x_des(2,1)*ones(1,horizon), 'red', 'linewidth',4)
120 % title('x_{dot} [m/s]', 'fontsize',10);
121 % xlabel('Time in sec', 'fontsize',10)
122 % grid on
123 % %hold off;
124 %
125 % subplot(2,4,3);hold on
126 % plot(time, x_traj(3,:), 'linewidth',1);
127 % plot(time, x_des(3,1)*ones(1,horizon), 'red', 'linewidth',4)
128 % title('\theta [rad]', 'fontsize',10)
129 % xlabel('Time in sec', 'fontsize',10)
130 % grid on
131 % %hold off;$\theta$
132 %
133 % subplot(2,4,4);hold on
134 % plot(time, x_traj(4,:), 'linewidth',1);
135 % plot(time, x_des(4,1)*ones(1,horizon), 'red', 'linewidth',4)
136 % title('\theta_{dot} [rad/s]', 'fontsize',10)
137 % xlabel('Time in sec', 'fontsize',10)
138 % grid on
139 %
140 % subplot(2,4,5);hold on

```

```

141 %      plot(time(1:end-1), u(:, 1:horizon-1), 'linewidth',1)
142 %      title('optimal control','fontsize',10)
143 %      xlabel('Time in sec','fontsize',10)
144 %      grid on;
145 %      %hold off
146 %
147 %
148 %      subplot(2,4,6);hold on
149 %      plot(Cost,'linewidth',1);
150 %      xlabel('Iterations','fontsize',10)
151 %      title('Cost','fontsize',10);
152 %      grid on
153 %      %save('DDP_Data');
154 end
155 end

```

```

1 function [x_, dfx, dfu] = fnStateDroneSafe(xhat, u)
2
3 x = xhat(1);
4 y = xhat(2);
5 z = xhat(3);
6 xdot = xhat(4);
7 ydot = xhat(5);
8 zdot = xhat(6);
9 roll = xhat(7);
10 pitch = xhat(8);
11 yaw = xhat(9);
12 p = xhat(10);
13 q = xhat(11);
14 r = xhat(12);
15 z1 = xhat(13);
16 z2 = xhat(14);
17 z3 = xhat(15);
18 u1 = u(1);
19 u2 = u(2);
20 u3 = u(3);
21 u4 = u(4);
22
23 dfx = [0,0,0,1,0,0,0,0,0,0,0,0,0,0,0;
24        0,0,0,0,1,0,0,0,0,0,0,0,0,0,0;
25        0,0,0,0,0,1,0,0,0,0,0,0,0,0,0;
26        0,0,0,0,0,0,0,-2*cos(pitch)*(u1 + u2 + u3 + u4), 0,0,0,0,0,0,0;
27        0,0,0,0,0,0,2*cos(pitch)*cos(roll)*(u1 + u2 + u3 + u4),-2*sin(
28            pitch)*sin(roll)*(u1 + u2 + u3 + u4), 0,0,0,0, 0, 0, 0;
29        0,0,0,0,0,0,-2*cos(pitch)*sin(roll)*(u1 + u2 + u3 + u4),-2*cos(
30            roll)*sin(pitch)*(u1 + u2 + u3 + u4), 0,0,0,0, 0, 0, 0;
31        0,0,0,0,0,0,q*cos(roll)*tan(pitch) - r*tan(pitch)*sin(roll),r*cos

```



$$(8*x^2*x\dot{y}*(2*y + 2/5))/((y + 1/5)^2 + x^2 + z^2 - 1)^3 - (4*x*y\dot{y})/((y + 1/5)^2 + x^2 + z^2 - 1)^2 - (2*y + 2/5)/((y + 1/5)^2 + x^2 + z^2 - 1)^2 + (4*x*y\dot{y}*(2*y + 2/5)^2)/((y + 1/5)^2 + x^2 + z^2 - 1)^3 + (8*x*z*z\dot{y}*(2*y + 2/5))/((y + 1/5)^2 + x^2 + z^2 - 1)^3,$$

$$(16*x^2*x\dot{y}*z)/((y + 1/5)^2 + x^2 + z^2 - 1)^3 - (4*x*z\dot{y})/((y + 1/5)^2 + x^2 + z^2 - 1)^2 - (2*z)/((y + 1/5)^2 + x^2 + z^2 - 1)^2 + (16*x*z^2*z\dot{y})/((y + 1/5)^2 + x^2 + z^2 - 1)^3 + (8*x*y\dot{y}*z*(2*y + 2/5))/((y + 1/5)^2 + x^2 + z^2 - 1)^3, \\ - (4*x^2)/((y + 1/5)^2 + x^2 + z^2 - 1)^2, \\ - (2*x*(2*y + 2/5))/((y + 1/5)^2 + x^2 + z^2 - 1)^2, \\ - (4*x*z)/((y + 1/5)^2 + x^2 + z^2 - 1)^2,$$

0,

$$0, 0, 0, 0, \\ 0, -1, 0;$$

$$(2*x\dot{y}*(2*x - 6)^3)/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3 - (x\dot{y}*(8*x - 24))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - (2*z\dot{y}*(2*z - 1))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - (4*y*y\dot{y})/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - (2*x - 6)/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 + (2*z\dot{y}*(2*x - 6)^2*(2*z - 1))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3 + (4*y*y\dot{y}*(2*x - 6)^2)/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3,$$

$$(4*x\dot{y}*y*(2*x - 6)^2)/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3 - (2*y\dot{y}*(2*x - 6))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - (2*y)/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 + (8*y^2*y\dot{y}*(2*x - 6))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3 + (4*y*z\dot{y}*(2*x - 6)*(2*z - 1))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3,$$

$$(2*x\dot{y}*(2*x - 6)^2*(2*z - 1))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3 - (2*z\dot{y}*(2*x - 6))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - (2*z - 1)/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 + (2*z\dot{y}*(2*x - 6)*(2*z - 1)^2)/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3 + (4*y*y\dot{y}*(2*x - 6)*(2*z - 1))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^3, \\ - (2*x - 6)^2/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2, \\ - (2*y*(2*x - 6))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2, \\ - ((2*x - 6)*(2*z - 1))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2,$$

0,

$$0, 0, 0, 0, \\ 0, 0, -1];$$

37

38

39

```

40
41 dfu = [0,0,0,0;
42         0,0,0,0;
43         0,0,0,0;
44         -2*sin(pitch), -2*sin(pitch), -2*sin(pitch),
45         -2*sin(pitch);
46         2*cos(pitch)*sin(roll), 2*cos(pitch)*sin(roll), 2*cos(pitch)*sin(
47         roll), 2*cos(pitch)*sin(roll);
48         2*cos(pitch)*cos(roll), 2*cos(pitch)*cos(roll), 2*cos(pitch)*cos(
49         roll), 2*cos(pitch)*cos(roll);
50         0, 0,
51         0, 0;
52         0, 0;
53         0, 0;
54         0, 0;
55         0, 0;
56         (425*2^(1/2))/16, -(425*2^(1/2))/16, (425*2^(1/2))
57         /16, -(425*2^(1/2))/16;
58         -(425*2^(1/2))/16, -(425*2^(1/2))/16, (425*2^(1/2))
59         /16, (425*2^(1/2))/16;
60         1691/550, -1691/550,
61         -1691/550, 1691/550
62         0, 0,
63         0, 0;
64         0, 0,
65         0, 0;
66         0, 0,
67         0, 0;
68         0, 0;
69         0, 0;];
70
71 x_ = [xdot;
72       ydot;
73       zdot;
74       -2*sin(pitch)*(u1 + u2 + u3 + u4);
75       2*cos(pitch)*sin(roll)*(u1 + u2 + u3 + u4);
76       2*cos(roll)*cos(pitch)*(u1 + u2 + u3 + u4) - 981/200;
77       p + r*cos(roll)*tan(pitch) + q*sin(roll)*tan(pitch);
78       q*cos(roll) - r*sin(roll);
79       (r*cos(roll))/cos(pitch) + (q*sin(roll))/cos(pitch);
80       (425*2^(1/2)*(conj(u1) - conj(u2) + conj(u3) - conj(u4)))/16 -
81       (23*conj(q)*conj(r))/32;
82       (23*conj(p)*conj(r))/32 - (425*2^(1/2)*(conj(u1) + conj(u2) -
83       conj(u3) - conj(u4)))/16;
84       (1691*conj(u1))/550 - (1691*conj(u2))/550 - (1691*conj(u3))/550 +
85       (1691*conj(u4))/550;
86       1/((x - 11/5)^2 + (y - 11/5)^2 + (z - 1)^2 - 1) - z1 - (xdot*(2*x
87       - 22/5)^2)/((x - 11/5)^2 + (y - 11/5)^2 + (z - 1)^2 -
88       (ydot*(2*x - 22/5)*(2*y - 22/5))/((x - 11/5)^2 + (y - 11/5)^2

```

```

70      + (z - 1)^2 - 1)^2 - (zdot*(2*x - 22/5)*(2*z - 2))/((x - 11/5)
      ^2 + (y - 11/5)^2 + (z - 1)^2 - 1)^2 - 25/1192;
      1/((y + 1/5)^2 + x^2 + z^2 - 1) - z2 - (4*x^2*xdot)/((y + 1/5)^2
      + x^2 + z^2 - 1)^2 - (2*x*ydot*(2*y + 2/5))/((y + 1/5)^2 + x^2
      + z^2 - 1)^2 - (4*x*z*zdot)/((y + 1/5)^2 + x^2 + z^2 - 1)^2 -
      25/306;
71      1/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1) - z3 - (xdot*(2*x - 6)^2)
      /((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - (zdot*(2*x - 6)*(2*z
      - 1))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - (2*y*ydot*(2*x -
      6))/((x - 3)^2 + (z - 1/2)^2 + y^2 - 1)^2 - 4/165];
72 end

```

```

1  function [x_, dfx, dfu] = fnStateDrone(xhat, u)
2
3  x = xhat(1);
4  y = xhat(2);
5  z = xhat(3);
6  xdot = xhat(4);
7  ydot = xhat(5);
8  zdot = xhat(6);
9  roll = xhat(7);
10 pitch = xhat(8);
11 yaw = xhat(9);
12 p = xhat(10);
13 q = xhat(11);
14 r = xhat(12);
15 u1 = u(1);
16 u2 = u(2);
17 u3 = u(3);
18 u4 = u(4);
19
20 dfx = [0, 0, 0, 1, 0, 0,
        0,
        0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 1, 0,
        0,
        0, 0, 0, 0, 0, 1,
        0,
        0, 0, 0, 0, 0, 0,
        0,
        0, 0, 0, 0, 0, 0,
        0,
        u2 + u3 + u4), 0, 0, 0,
        -2*cos(pitch)*(u1 +
        0,
        0;

```

```

24 0, 0, 0, 0, 0, 0,      2*cos(pitch)*cos(roll)*(u1 + u2 + u3 + u4)
    ,                    -2*sin(pitch)*sin(roll)*(u1 +
    u2 + u3 + u4), 0,      0,      0,
    0;
25 0, 0, 0, 0, 0, 0,      -2*cos(pitch)*sin(roll)*(u1 + u2 + u3 + u4)
    ,                    -2*cos(roll)*sin(pitch)*(u1 +
    u2 + u3 + u4), 0,      0,      0,
    0;
26 0, 0, 0, 0, 0, 0,      q*cos(roll)*tan(pitch) - r*tan(pitch)*sin(roll)
    ,                    r*cos(roll)*(tan(pitch)^2 + 1) + q*sin(roll)*(tan(
    pitch)^2 + 1), 0,      1, tan(pitch)*sin(roll), cos(roll)*tan(
    pitch);
27 0, 0, 0, 0, 0, 0,      - r*cos(roll) - q*sin(roll)
    ,
    0, 0,      0,      cos(roll),      -sin(roll);
28 0, 0, 0, 0, 0, 0, (q*cos(roll))/cos(pitch) - (r*sin(roll))/cos(pitch)
    , (r*cos(roll)*sin(pitch))/cos(pitch)^2 + (q*sin(pitch)*sin(roll))
    /cos(pitch)^2, 0,      0, sin(roll)/cos(pitch), cos(roll)/cos(
    pitch);
29 0, 0, 0, 0, 0, 0,
    0,
    0, 0,      0,      -(23*r)/32,      -(23*q)/32;
30 0, 0, 0, 0, 0, 0,
    0,
    0, 0, (23*r)/32,      0,      (23*p)/32;
31 0, 0, 0, 0, 0, 0,
    0,
    0, 0,      0,      0,      0];
32 dfu = [
33      0,      0,
34      0,      0,
35      0,      0,
36      0,      0,
37      -2*sin(pitch),      -2*sin(pitch),      -2*sin(pitch)
    ),      -2*sin(pitch);
38 2*cos(pitch)*sin(roll), 2*cos(pitch)*sin(roll), 2*cos(pitch)*sin(roll)
    ), 2*cos(pitch)*sin(roll);
39 2*cos(pitch)*cos(roll), 2*cos(pitch)*cos(roll), 2*cos(pitch)*cos(roll)
    ), 2*cos(pitch)*cos(roll);
    0,      0,
    0,      0;

```



```

40         0,                                0,
41         0,                                0;
42         0,                                0,
43         0,                                0;
44         (425*2^(1/2))/16,      -(425*2^(1/2))/16,      (425*2^(1/2))
         /16,      -(425*2^(1/2))/16;
45         -(425*2^(1/2))/16,      -(425*2^(1/2))/16,      (425*2^(1/2))
         /16,      (425*2^(1/2))/16;
46         1691/550,      -1691/550,
         -1691/550,      1691/550];
47
48 x_ = [xdot;...
49       ydot;...
50       zdot;...
51       -2*sin(pitch)*(u1 + u2 + u3 + u4);...
52       2*cos(pitch)*sin(roll)*(u1 + u2 + u3 + u4);...
53       2*cos(roll)*cos(pitch)*(u1 + u2 + u3 + u4) - 981/200;...
54       p + r*cos(roll)*tan(pitch) + q*sin(roll)*tan(pitch);...
55       q*cos(roll) - r*sin(roll);...
56       (r*cos(roll))/cos(pitch) + (q*sin(roll))/cos(pitch);...
57       (425*2^(1/2)*(conj(u1) - conj(u2) + conj(u3) - conj(u4)))/16 -
         (23*conj(q)*conj(r))/32;...
58       (23*conj(p)*conj(r))/32 - (425*2^(1/2)*(conj(u1) + conj(u2) -
         conj(u3) - conj(u4)))/16;...
59       (1691*conj(u1))/550 - (1691*conj(u2))/550 - (1691*conj(u3))/550 +
         (1691*conj(u4))/550];
60 end

```

```

1 function [l0,l_x,l_xx,l_u,l_uu,l_ux] = fnCost(x, u, Q, R, dt)
2
3 l0 = 0.5.*(u' *R *u + x'*Q*x) .* dt;
4 l_x = Q * x .* dt;
5 l_xx = Q .* dt;
6 l_u = R * u .* dt;
7 l_uu = R .* dt;
8 l_ux = zeros(length(Q), length(R)) .* dt;
9 end

```

```

1 function [Cost] = fnCostComputation(x_traj, u_new, x_des, dt, Q, S, R)
2
3 [~, Horizon] = size(x_traj);
4 Cost = 0;
5 for j =1:(Horizon-1)
6     Cost = Cost + (u_new(:,j)' * R * u_new(:,j) + (x_traj(:, j) -
         x_des(:, j))'*Q*(x_traj(:, j) - x_des(:, j)))*dt;
7 end
8 TerminalCost= (x_traj(:, Horizon) - x_des(:, end))'*S * (x_traj(:,

```

```
9         Horizon) - x_des(:, end));  
10     Cost = Cost + TerminalCost;  
11  
12 end
```