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Relativistic Quantum Field Theories in Standard Model of Particle Physics

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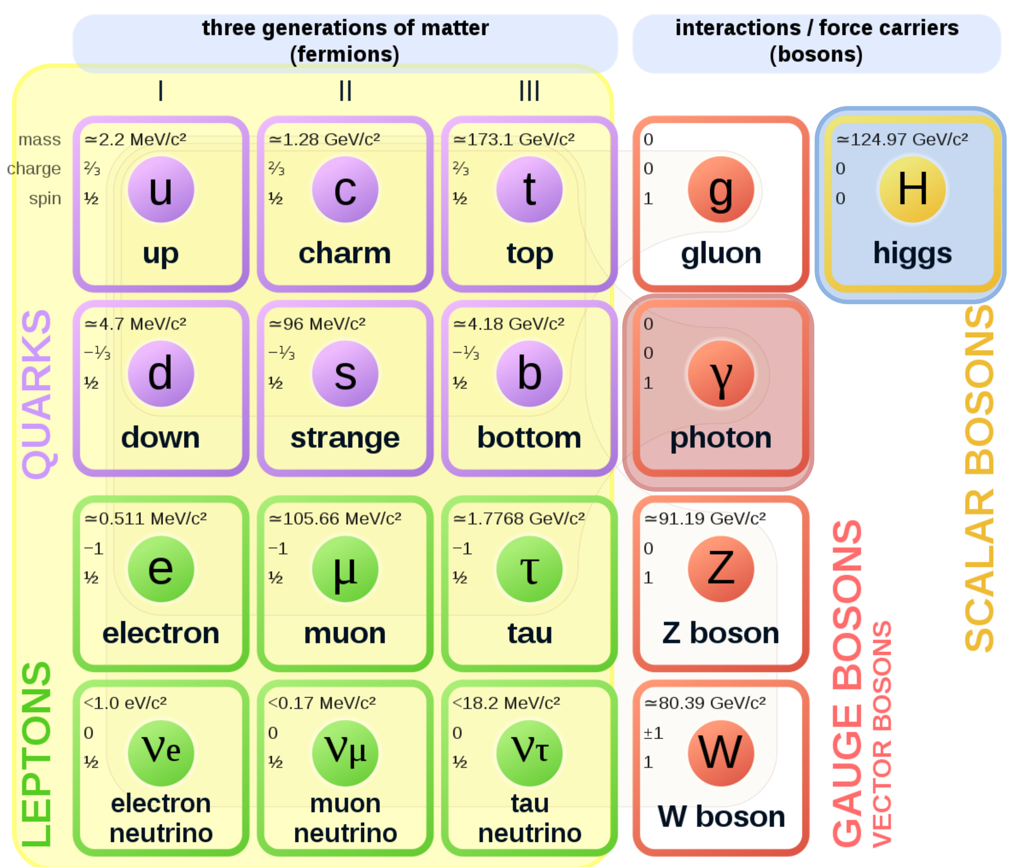
Introduction

Quantum Field Theory is one of the pioneering theories used to describe the physics behind the elementary particles in the Standard Model of Particle Physics. It is a combination of Classical Field theory (Electromagnetic field), Quantum Mechanics and Special Relativity. This project aims at reviewing the various ingredients in the Quantum Field theory and show its application in quantum Electrodynamics.

Theory

Standard Model of Elementary Particles



Klein Gordon Scalar Field

Klein Gordon Equation: $-\Box\,\Phi + m^2\,\Phi = 0$, where Φ is a multi-particle field operator. The field is further described using the Lagrangian Density and Hamiltonian Density.

- **Lagrangian Density:** $\mathcal{L}_0^{\mathcal{KG}} = -\frac{1}{2}\left(\partial_\mu \Phi\right)\left(\partial^\mu \Phi\right) - \frac{1}{2} m^2 \Phi^2$
- **Hamiltonian Density:** $\mathcal{H}_0^{\mathcal{K}\mathrm{G}} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\Phi)\cdot(\vec{\nabla}\Phi) + \frac{1}{2}m^2\Phi^2$

Note we require density for fields. Also, the Lagrangian is a Lorentz scalar.

To understand the field, we Fourier decompose and interpret in the momentum space.

→**Field operator:** $\Phi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\left(e^{ik\cdot x}A_{\vec{k}} + e^{-ik\cdot x}A_{\vec{k}}^\dagger\right)$

→**Conjugate operator:** $\Pi(x) = \dot{\Phi} = \frac{i}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\left(-e^{ik\cdot x}A_{\vec{k}} + e^{-ik\cdot x}A_{\vec{k}}^\dagger\right)$

→**Canonical Quantization Rule:** $[\Phi(x^0,\vec{x}),\Pi(x^0,\vec{y})] = i\delta^3(\vec{x}-\vec{y})$

→**Hamiltonian:** $H_0^{\mathrm{KG}} = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\omega_{\vec{k}}\left(A_{\vec{k}}^\dagger A_{\vec{k}} + A_{\vec{k}}A_{\vec{k}}^\dagger\right)$

→**Commutator relations:** $\left[H_0^{\mathrm{KG}}, A_{\vec{k'}}^\dagger\right] = \omega_{\vec{k'}}A_{\vec{k'}}^\dagger$, $\left[H_0^{\mathrm{KG}}, A_{\vec{k'}}\right] = -\omega_{\vec{k'}}A_{\vec{k'}}$

The commutator relations show $A_{\vec{k'}}^\dagger$ and $A_{\vec{k'}}$ as creation and annihilation operators. Thus, the vacuum state is interpreted as $|0\rangle$ state having no particles and particles of momentum $\vec{k}$ are created and destroyed based on the operators.

Using the Noether's theorem, we determine the conserved quantity to be Charge and the Normal-ordered charge operator is

$$\mathcal{N}(Q) = \int d^3k\,\left(A_{\vec{k}}^\dagger A_{\vec{k}} - B_{\vec{k}}^\dagger B_{\vec{k}}\right)$$

where A and B represent particles with opposite charges.

Dirac-Fermionic Field

Dirac Fermionic Equation: $(-i\gamma^\mu\partial_\mu + m)\psi = 0$, where γ^μ are gamma objects defined as: $\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{1}_4$, where $\mathbb{1}_4$ is identity matrix in the spinor space.

Again, the field is described as:

- **Lagrangian Density:** $\mathcal{L}_0^{\mathrm{Dirac}} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$
- **Hamiltonian Density:** $H_0^{\mathrm{Dirac}} = \int d^3\vec{x}\,\overline{\Psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\Psi$

The plane wave solution is $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$ $\begin{matrix}\left.\begin{matrix}\psi_1\\ \psi_2\end{matrix}\right\}\text{Particle spin }\uparrow\downarrow\\ \left.\begin{matrix}\psi_3\\ \psi_4\end{matrix}\right\}\text{Anti-particle spin}\uparrow\downarrow\end{matrix}$

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Using the plane wave solution, we perform Fourier decomposition and interpret in the momentum space

→**Field Operator :** $\Psi(x) = \sum_{s=\uparrow,\downarrow}\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\left[C_{\vec{k}s}u_{(s)}(\vec{k})\,e^{ik\cdot x} + D_{\vec{k}s}^\dagger v_{(s)}(\vec{k})\,e^{-ik\cdot x}\right]$

→**Canonical quantization rule (preserving exchange symmetry):**

- **In real space :** $\{\Psi_\alpha(x^0,\vec{x}),(\Pi_\Psi)_\beta(x^0,\vec{y})\} = i\delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y})$, α,β are spinor indices
- **In momentum space :**

$$\{C_{\vec{k}s}, C_{\overrightarrow{k's'}}\} = 0 \qquad \{C_{\vec{k}s}^\dagger, C_{\overrightarrow{k's'}}^\dagger\} = 0 \qquad \{D_{\vec{k}s}^\dagger, D_{\overrightarrow{k's'}}^\dagger\} = 0$$
$$\{D_{\vec{k}s}, D_{\overrightarrow{k's'}}\} = 0 \qquad \{C_{\vec{k}s}^\dagger, D_{\overrightarrow{k's'}}\} = 0 \qquad \{C_{\vec{k}s}, C_{\overrightarrow{k's'}}^\dagger\} = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k'})2\omega_{\vec{k}}\delta_{ss'}$$
$$\{C_{\vec{k}s}, D_{\overrightarrow{k's'}}^\dagger\} = 0 \qquad \{C_{\vec{k}s}, D_{\overrightarrow{k's'}}\} = 0 \qquad \{D_{\vec{k}s}^\dagger, D_{\overrightarrow{k's'}}\} = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k'})2\omega_{\vec{k}}\delta_{ss'}$$

Based on the above relations, we derive that the $C_{\vec{k}s}$ and $D_{\vec{k}s}$ act as annihilation operators while $C_{\vec{k}s}^\dagger$ and $D_{\vec{k}s}^\dagger$ act as creation operators. Also, we deduce the Pauli's exclusion Principle which states that fermions with same quantum number cannot exist.

From the Noether's Theorem, we derive the Charge operator as:

$$\mathcal{N}(Q) = \sum_{s=\uparrow,\downarrow}\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\left(C_{\vec{k}s}^\dagger C_{\vec{k}s} - D_{\vec{k}s}^\dagger D_{\vec{k}s}\right)$$

where $C_{\vec{k}s}$ and $D_{\vec{k}s}$ are opposite charges (basically particles and anti-particles)

Maxwell's Electrodynamics For Photons

Maxwell's Equation(Lorentz Covariant form) : $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$; $\partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} = 0$

where Field strength 2-tensor ($F^{\mu\nu}$), Potential 4-vector (A^μ) and Current Density (J^μ)

- **Lagrangian Density:** $\mathcal{L}_0^{\mathrm{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\mu A^\mu\right)^2 = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu)$
- **Hamiltonian Density:** $\mathcal{H}_0^{EM} = \Pi^\mu \dot{A}_\mu - \mathcal{L}_0^{\mathrm{EM}} = \frac{1}{2}\Pi^\mu\Pi_\mu + \frac{1}{2}(\partial_i A_\nu)(\partial^i A^\nu)$

The Maxwell field (Potential 4-vector (A^μ)) is Fourier decomposed in the momentum space.

→**Potential 4-vector :** $A^\mu(x) = \sum_{\lambda=0}^{3}\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}}\left[\varepsilon_{\vec{k}\lambda}^{*\mu}e^{ik\cdot x}\mathcal{A}_{\vec{k}\lambda} + \varepsilon_{\vec{k}\lambda}^{\mu}e^{-ik\cdot x}\mathcal{A}_{\vec{k}\lambda}^\dagger\right]$, where $\omega_{\vec{k}} = k^0$ for photons. $\varepsilon_{\vec{k}\lambda}^{\mu}$ are the 4D polarisation vectors with index running through $\lambda = 0,1,2,3$.

$\mathcal{A}_{\vec{k}\lambda}$- annihilation, $\mathcal{A}_{\vec{k}\lambda}^\dagger$- creation and have the following commutator relations:

$$\left[\mathcal{A}_{\vec{k}\lambda}, \mathcal{A}_{\overrightarrow{k'\lambda'}}\right] = 0 = \left[\mathcal{A}_{\vec{k}\lambda}^\dagger, \mathcal{A}_{\overrightarrow{k'\lambda'}}^\dagger\right],\ \left[\mathcal{A}_{\vec{k}\lambda}, \mathcal{A}_{\overrightarrow{k'\lambda'}}^\dagger\right] = -(2\pi)^3 2\omega_{\vec{k}}\eta^{\lambda\lambda'}\delta^3(\vec{k}-\vec{k'})$$

→**Normal-ordered Hamiltonian :**

$$\mathcal{N}(H_0^{\mathrm{EM}}) = \int d^3\vec{k}\ \omega_{\vec{k}}\left[-\mathcal{A}_{\vec{k},\lambda=0}^\dagger\mathcal{A}_{\vec{k},\lambda=0} + \mathcal{A}_{\vec{k},\lambda=1}^\dagger\mathcal{A}_{\vec{k},\lambda=1} + \mathcal{A}_{\vec{k},\lambda=2}^\dagger\mathcal{A}_{\vec{k},\lambda=2} + \mathcal{A}_{\vec{k},\lambda=3}^\dagger\mathcal{A}_{\vec{k},\lambda=3}\right]$$

is not positive definite. This is resolved by applying the Lorenz gauge via the Gupta-Bleuler Condition: $\partial_\mu A^{\mu(+)}(x)|\text{physical}\rangle = 0$ on a general linear combination of states : $|\psi\rangle = \sum_{n_0,n_1,n_2,n_3} C_{n_0,n_1,n_2 n_3}|n_0,n_1,n_2,n_3\rangle$. As a result, we pick up the physical Hilbert space (indices = 1,2) while preserving Lorentz invariance.

Lagrangian for an electron-photon interaction

$$\mathcal{L}^{QED} = \mathcal{L}_0^{EM} + \mathcal{L}_0^{Dirac} + \mathcal{L}_{int}^{QED}$$

where $\mathcal{L}_{int}^{QED}$ is the interaction term

Principle of Local Gauge Invariance:

Used to determine the interaction by preserving invariance where the local transformation is

$$\Psi' = e^{\{i\theta(x)\}}\Psi$$

$$\boxed{\mathcal{L}^{Q\mathcal{E}\mathcal{D}} = -\frac{1}{4}\mathrm{F}^{\alpha\beta}\mathrm{F}_{\alpha\beta} + \mathrm{J}^\mu\mathrm{A}_\mu + \overline{\Psi}(\mathrm{i}\gamma^\mu\partial_\mu - \mathrm{m})\Psi = -\frac{1}{4}\mathrm{F}^{\alpha\beta}\mathrm{F}_{\alpha\beta} + \overline{\Psi}(\mathrm{i}\gamma^\mu\mathrm{D}_\mu - \mathrm{m})}$$

where $D_\mu$is covariant derivative defined as $D_\mu = \partial_\mu - ieA_\mu$