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Forcing with Partial Functions and its Subsequent Iterations

A Step Towards Relative Consistencies

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Abstract

This paper will focus on the topic of forcing: a method used to construct larger models from a ground model by use of a generic set, allowing one to prove stronger hypotheses such as the negation of the Continuum Hypothesis or its general form GCH. Beginning with an overview of various related topics leading up to forcing, the main content will be about Cohen's forcing using finite partial functions and its role in constructing a model where CH fails. A natural development of such an extension is iterated forcing, enabling one to force as many times as one wishes to produce a whole range of desired results.

The Continuum Hypothesis

An introductory course in set theory will usually provide an exposition of cardinal numbers, and how the infinite cardinals can be easily classified using the aleph numbers. At first glance, this seems to paint a complete picture of every cardinal number out there, but in reality there is a loophole in this framework; they do not reveal where the cardinality of the power sets of infinite sets lie. The Generalised Continuum Hypothesis (GCH) asserts that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, with the Continuum Hypothesis (CH) asserting the case for $\alpha = 0$.

An unexpected result by the joint efforts of Kurt Gödel and Paul Cohen in the 20th century revealed that such a hypothesis cannot be decided with only our current axioms of ZFC. Gödel produced a model of ZFC where CH holds through the use of the Axiom of Constructibility (also known as $V = L$), and Cohen constructed one where $\neg$CH holds by inventing the idea of forcing with generic extensions. We will be focusing on Cohen's approach in this project.

Generic Extensions

To begin, we take M to be a countable transitive model for ZFC, and we will look towards constructing a generic extension $M[G]$, through the use of a $\mathbb{P}$-generic set G, for some partial order $(\mathbb{P}, \leqslant)$. $M[G]$ will have properties such as $M \subseteq M[G]$, $M \cap \mathbf{ON} = M[G] \cap \mathbf{ON}$, and $G \in M[G]$.

To work within M, we will need a method to describe and talk about the elements that may be in $M[G]$. However, we will not be able to pinpoint the exact elements of $M[G]$, as working in M means we do not have access to G. $\mathbb{P}$-names are defined recursively, with respect to the forcing conditions of $\mathbb{P}$ that the objects correspond to.

Definition 1. *τ is a $\mathbb{P}$-name iff τ is a relation and*

$$\forall \langle \sigma, p\rangle \in \tau \; (\sigma \text{ is a } \mathbb{P}\text{-name } \wedge p \in \mathbb{P})$$

We shall denote the class of $\mathbb{P}$-names to be $V^{\mathbb{P}}$, and $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$.

This allows us to talk about the objects that can be in $M[G]$ while working inside M. And of course, we can also do the opposite and talk about the the names of elements of $M[G]$ that lie within M.

Definition 2. $\check{x} = \{\langle \check{y}, \mathbb{1}_{\mathbb{P}}\rangle : y \in x\}$.

$\mathbb{P}$-names are used because we do not have knowledge of G within M. But if we work in $M[G]$, and we hence have knowledge of what G contains, it will be helpful to define a valuation function to bring up these names into $M[G]$.

Definition 3. $\mathrm{val}(\tau, G) = \{\mathrm{val}(\sigma, G) : \exists p \in G\; (\langle \sigma, p\rangle \in \tau)\} =: \tau_G$

One can then easily see that it makes sense to define $M[G]$ in the following manner. We can also further prove that this generic extension remains to be a model of ZFC.

Definition 4. $M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$.

Forcing Ideas

Forcing is formally defined as follows but in most forcing arguments we will be using the subsequent theorem instead.

Definition 5. *Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables among $x_1, \ldots, x_n$. Let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$, and let $p \in \mathbb{P}$. Then $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ iff*

$$\forall G\; [(p \in G) \rightarrow \phi^{M[G]}(\tau_{1G}, \ldots, \tau_{nG})]$$

Theorem 6. *Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables among $x_1, \ldots, x_n$, and let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Then,*

1. For every $p \in \mathbb{P}$,

$$p \Vdash \phi(\tau_1, \ldots, \tau_n) \leftrightarrow (p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M$$

2. For every G $\mathbb{P}$-generic over M,

$$\phi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]} \leftrightarrow \exists p \in G (p \Vdash \phi(\tau_1, \ldots, \tau_n))$$

Here, $\Vdash^*$ indicates forcing within M, which one can construct to sidestep the caveat that $G \notin M$.

Cohen Forcing

Definition 7. $\mathrm{Fn}(I, J) := \{p : |p| < \omega \wedge \mathrm{dom}(p) \subseteq I \wedge \mathrm{ran}(p) \subseteq J \wedge p$ *is a function*$\}$.
We order $\mathrm{Fn}(I, J)$ by $\leqslant$, with $p \leqslant q$ iff $p \supseteq q$.

Applying our definition of generic sets into this context, we can obtain a surjective function that will be useful for assessing the cardinalities of I and J.

Lemma 8. *For $I, J \in M$ such that I is infinite and J is nonempty, suppose that G is $\mathrm{Fn}(I, J)$-generic over M. Then $\bigcup G$ is a function from I onto J.*

Theorem 9. *Let κ be an uncountable cardinal in M and let G be $\mathrm{Fn}(\kappa \times \omega, 2)$-generic over M. Then* $[2^\omega \geqslant |\kappa|]^{M[G]}$.

Relative Consistency Results

Relative Consistency Result 1. $\mathrm{Con}(ZFC) \rightarrow \mathrm{Con}(ZFC + 2^\omega = \omega_2)$.

Relative Consistency Result 2. $\mathrm{Con}(ZFC) \rightarrow \mathrm{Con}(ZFC + CH + 2^{\omega_1} = \omega_2 + 2^{\omega_2} = \omega_{\omega_{2021}})$.

Starting with a ground model that satisfies GCH, we force using $(\mathrm{Fn}(\omega_{\omega_{2021}} \times \omega_2, 2, \omega_2))^M$ as our poset.

Relative Consistency Result 3. $\mathrm{Con}(ZFC) \rightarrow \mathrm{Con}(ZFC + CH + 2^{\omega_1} = \omega_3 + 2^{\omega_2} = \omega_4)$.

We force twice. First using $\mathbb{P} = (\mathrm{Fn}(\omega_4 \times \omega_2, 2, \omega_2))^M$ as the first poset and then with $\mathbb{Q} = (\mathrm{Fn}(\omega_3 \times \omega_1, 2, \omega_1))^{M[G]}$ as the second poset.

Iterated Forcing

With relative consistency results that we have to force over more than once, it is natural for us to generalise this so that we can easily force α many times, for some ordinal α. The main picture we will be looking at will be

$$\begin{array}{ccccccccc} M_0 & \subseteq & M_1 & \subseteq & \ldots & \subseteq & M_\xi & \subseteq & \ldots & \subseteq & M_\alpha \\ \parallel & & \parallel & & & & \parallel & & & & \parallel \\ M & & M[G_0] & & & & M_{\xi-1}[G_{\xi-1}] & & & & M_{\alpha-1}[G_{\alpha-1}] \end{array}$$

We can also perform two-step forcing in one step, as illustrated below.

$$\begin{tikzcd} K \arrow[r, dashed] \arrow[dd, dash] & M[K] \arrow[r, phantom, "="] & M[G][H] & & H \arrow[ll, dashed] \arrow[d, dash] \\ & & M[G] \arrow[u] & G \arrow[l, dashed] \arrow[d, dash] & \mathbb{Q} \arrow[d, squiggly, dash] \\ P * \pi & M \arrow[uu] \arrow[r, phantom, "="] & M \arrow[u] & \mathbb{P} & \pi \end{tikzcd}$$

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References

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