A Primer on Geometric Mechanics and Symmetry

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Abstract

My project focused on building the foundations to read the following papers in the area of Geometric Mechanics: "Discrete Mechanics and Variational Integrators" by J. E. Marsden and M. West and "Variational Integrators" by Matthew West. In "Variational Integrators" by Mathew West, the basic theory of discrete variational mechanics for ordinary differential equations is developed in depth, and is used as the basis for constructing variational integrators and analyzing their numerical properties. In "Discrete Mechanics and Variational Integrators", the paper gives a review of integration algorithms for finite dimensional mechanical systems that are based on discrete variational principles. To understand these 2 papers, I focused on learning the basics of Manifold Theory, Lagrangian and Hamiltonian mechanical systems, Calculus of Variations, Discretization of ordinary differential equations and the numerical methods to solve these differential equations. In this poster, I will give a brief overview of the definition, concepts and numerical algorithms I have learnt over the course of 2 months.

What are Smooth Manifolds?

Geometric mechanics takes the viewpoint that classical mechanics is a fundamentally geometric theory and best treated with the mathematical machinery of differential geometry and modern algebra.Moreover, Geometric mechanics applies principally to systems for which the configuration space is a Lie group, or a group of diffeomorphisms, or more generally where some aspect of the configuration space has this group structure. Therefore, I naturally started with the basics of differential geometry and manifold theory. Some of the key definition that will prove to be useful when reading the papers later on are listed below.

Suppose M is a topological space. Then, M is a topological manifold of dimension n or a topological n-manifold if it has the following properties: M is a hausdorff space, M is second countable and M is locally euclidean of dimension n.

A coordinate chart on M is a pair (U, ϕ), where U is an open subet of M and $\phi : U \to V$ is a homeomorphism from U to an open subset $V = \phi(U) \subset \mathbb{R}^n$.

If U and V are open subsets of Euclidean Spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, a map $F : U \to V$ is said to be smooth if each of the component functions of F has continuous partial derivatives of all orders. If F is bijective and has a smooth inverse map, it is called a diffeomorphism.

2 charts (U, ϕ) and (V, Φ) are said to be smoothly compatible if either the intersection of U and V is empty or the transition map is a diffeomorphism.

We define an atlas for M to be a collection of charts whose domain cover M. An atlas A is called smooth if any two charts in A are smoothly compatible.

A smooth atlas A on M is maximal if it is not contained in any strictly larger smooth atlas.

A smooth structure on a topological manifold M is a maximal smooth atlas. A smooth manifold is a pair (M, A) where M is a topological manifold and A is a smooth structure.

Discretization of ODEs and Numerical Methods

Many differential equations we encounter will not be easily solvable. Therefore, we require numerical methods to estimate the solution of the ODE. Note that when we use the computer to solve the ODE, we need to discretize the continuous variable time into step size h.

For the euler method, given $\frac{dy}{dx} = f(x, y)$ and the initial condition (x_0, y_0), we can calculate the slope at $f(x,y_0)$. As illustrated in the figure below, we can proceed from (x_0, y_0) in the direction of the slope until the abscissa is $x_0 + h = x_1$, where the solution will be $y_1 = y_0 + hf(x_0, y_0)$. At this new point (x_1, y_1) the new slope can be calculated and this process can be continued indefinitely. This method is called the Euler Method.

As the Euler Method may prove to be unstable, that is to say, even if the step size is infinitesimally small, there may be errors that grow exponentially as the solution is advanced, stability analysis is required.

To overcome this, we can use the Runge-Kutta method, which is a stable numerical method. The Runge-Kutta method is a weighted version of the improved Euler method.

D'Alembert Principle and Lagrangian Equation

When we restrict ourselves to systems for which the virtual work of the forces of constraint vanishes, we obtain

$$\sum_i (F_i^{(a)} - \frac{dp}{dt}) \cdot \delta r_i = 0$$

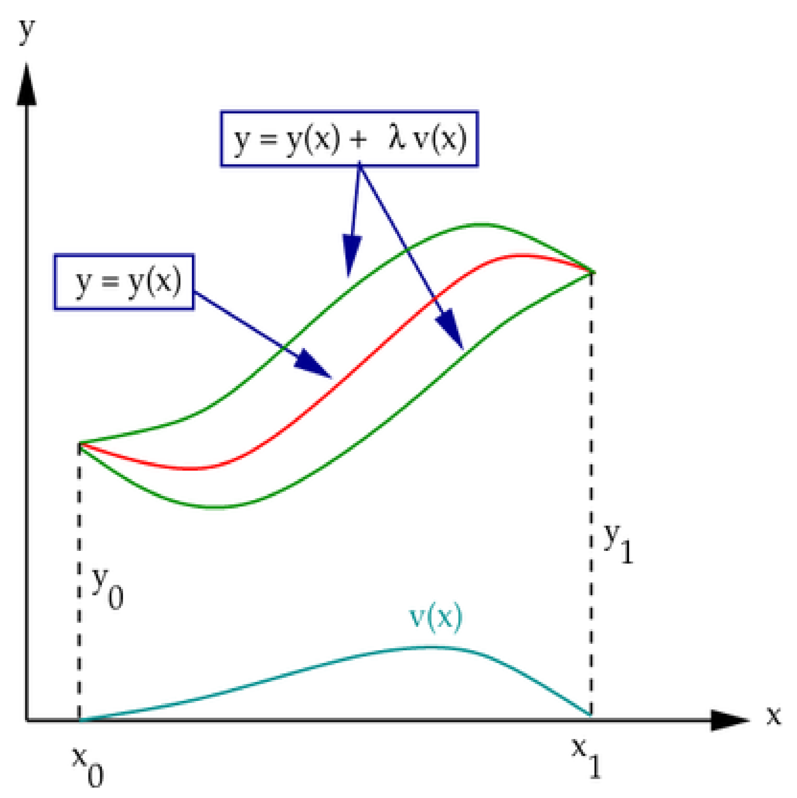
which is called the D'Alembert Prinicple.

We can further develop this principle into the Lagrange's Equation:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q}.$$

Calculus of Variation

The Calculus of variation has been one of the major branches of analysis for more than two centuries. We use variations, which are small changes in functions and functionals, to find the maxima and minima of functionals: mappings from a set of functions to the real numbers. Firstly, we find admissible $y(x)$ that satisfies the Euler's equation above. At this point, $y(x)$ may not necessarily be the maxima or minima functional. At this point, $y(x)$ is just the stationary functional or curve. In calculus, we use the second derivative to give sufficient conditions distinguishing one type of stationary value from another. Similar sufficient conditions are availbale in the calculus of variations, but in actual practical scenarios like in mechanics, the problem under discussion often makes it possible to determine whether a particular stationary function maximises or minimises an integral (or neither).



How it all comes together

The thesis by Mathew West begin with a simple overview of variational integrators for ODEs. Then,they develop discrete variational mechanics for ODEs, including extensive comparisons with continuous-time Lagrangian and Hamiltonian mechanics. This is then used later on as the basis for variational integrators, whose numerical properties are investigated in detail. Next, both discrete mechanics and variational integrators for systems with forcing and constraints is considered. The final two chapters deal with variational mechanics and integrators for PDEs. In the "Discrete Mechanics and Variational Integrators", the variational approach gives a comprehensive and unified view of much of the literature on both discrete mechanics as well as integration methods for mechanical systems. So, we can see that the concepts studied above builds up to these two papers.