Odyssey Research Programme
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On the convergence of the Collatz sequence for all positive integer inputs

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Introduction

In 1937, German mathematician Lothar Collatz introduced a conjecture which states that for all inputs $a_1 \in \mathbb{Z}^+$, the Collatz sequence (a_n), $n \in \mathbb{Z}^+$ reaches 1 with a finite minimum length $k \in \mathbb{Z}^+$. We define (a_n) as follows.

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & a_n = 2m \\ 3a_n + 1 & a_n = 2m+1 \end{cases},$$

for all $n \in \mathbb{Z}^+$ and some $m \in \mathbb{Z}^+$, where a_1 is the input.

In 2015, Maya showed that any sequence (a_n), with an odd input that is divisible by 3, converges to 1 using inductive arguments. Moreover, in 2021, Farzali showed that any Collatz sequence, whose input is an odd positive remainder modulo 6, 12, ..., 108, converges to 1 by first proving the Syracuse conjecture, which if proven implies the Collatz conjecture. In addition, Clark and Lewis (1995, 1998) considered difference equations, each of whose initial conditions is a pair of coprime integers. In one of their papers published in 1998, Clark and Lewis simplified the initial first-order system of non-linear difference equations to that of the second order with the aid of linear and geometric mappings in $\mathbb{R}^2$. Tao (2020) utilised probabilistic and algorithmic approaches to investigate the convergence of the Collatz conjecture, analysing the stabilities of Collatz orbits in the process. Tao used such approaches to specifically establish a stabilisation property for a first passage random variable associated with the Collatz iteration by estimating a characteristic function of a skew random walk on a 3-adic cyclic group at high frequencies.

While there has been no successful and documented proof, we provide a different approach by studying the relationship between the minimum length, k, of the sequence (a_n) and the corresponding input a_1.

Results

We first found by induction that the smallest possible minimum lengths of (a_n) with respect to the number of digits of integers in base 2 correspond to inputs of the form 2^m which have lengths $(m + 1)$. All other inputs satisfying the inequality $2^m < a_1 < 2^{m+1}$, for all $m \in \mathbb{Z}^+$, do not give the minimum length of (a_n) for a particular m.

Next, we made use of some of the inputs that correspond to the largest possible minimum lengths of (a_n) with respect to the number of digits in decimal notation. Gary and Mike studied several search algorithms which they used to calculate such inputs up to 5.6×10^{13}. These values are shown in **Table 1**.

m	c_m	k_m
1	9	20
2	97	119
3	871	179
4	6,171	262
5	77,031	351
6	837,799	525
7	8,400,511	686
8	63,728,127	950
9	670,617,279	987
10	9,780,657,630	1,133
11	75,128,138,247	1,229
12	989,345,275,647	1,349
13	7,887,663,552,367	1,564
14	51,173,735,510,107	1,652

Table 1 Inputs c_m corresponding to the minimum lengths k_m of (a_n), with respect to the number of digits $m \in \mathbb{Z}^+$ in decimal notation.

We defined a sequence (b_m) whose general term is given by $b_m = \frac{k_m}{c_m}$ for all $m \in \mathbb{Z}^+$. Using Table 1, we can numerically determine the first fourteen terms of (b_m). More work is needed to determine the values of c_m and k_m for subsequent values of m.

From the abstract, we chose a subsequence (u_n) whose general term is given by $u_n = c_n$ for all positive integers n.

We used the above subsequence in our subsequent partial proof that the sequence (b_m) is contractive if and only if there is a constant c, with $0 < c \le 1$ such that $|b_{m+2} - b_{m+1}| \le c\,|b_{m+1} - b_m|$ for all $m \ge 2$.

Since we have not yet found the values of b_m for all $m \ge 1$, we set $c = 1$. While it is easy to show that for $m = 1, 2, ..., 12$, the ratio

$$\frac{|b_{m+2} - b_{m+1}|}{|b_{m+1} - b_m|} < 1$$

more work is needed to show that this ratio converges to 1 from below for large values of m.

While we have not yet proven that the above lemma holds true so far, we aspire to make use of various previously published works, such as those mentioned in the abstract, to justify our proof.

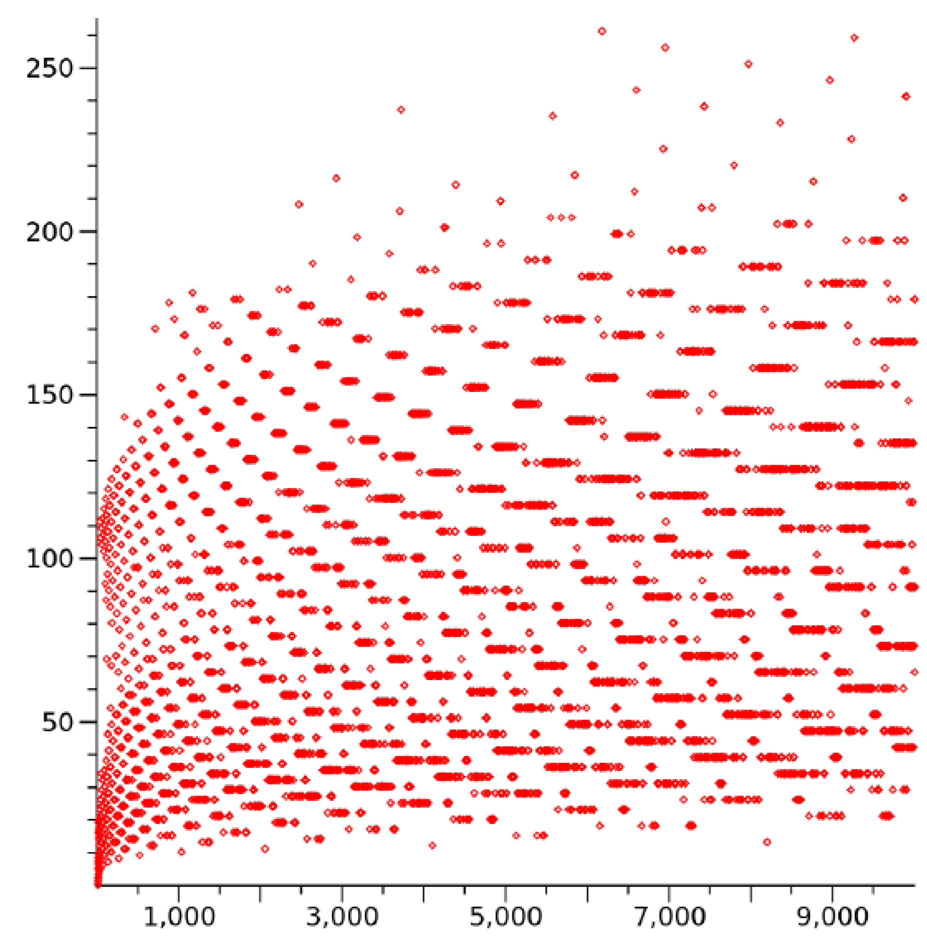

Figure 1: Values of $k - 1$ corresponding to inputs between 1 and 9,999 inclusive

Conclusion

More work is needed to show that the Collatz sequence converges to 1 for any odd input greater than 1, and consequently the proof of the lemma described above. In addition, more work is needed to show that the ratio of the minimum length of the Collatz sequence to the corresponding u_1 converges to 0.

References

Maya M. Ahmed, *A window to the Convergence of a Collatz sequence*, arXiv:1510.01274v1 [math.GM].

Farzali I., *Complete Proof of the Collatz conjecture*, arXiv:2101.06107v4 [math.GM].

Leavens, Gary T.; Vermeulen, Mike, *3x + 1 Search Programs*, Computers & Mathematics with Applications. 24 (11) (1992), 79-99.

Figure 1: Plot of numbers 1-9999 against their Collatz total stopping time. (2010, March 5). In Wikipedia. https://en.wikipedia.org/wiki/Collatz_conjecture#/media/File:Collatz-stopping-time.svg