Introduction to Finite Difference Methods

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Goals

- Fundamentals
  - Partial Differential Equations (PDEs)
  - Solution to a Partial Differential Equation
  - PDE Models
  - Classification of PDEs
  - Discrete Notation
- Finite Difference Method (FDM):
  - Details
  - General Concepts
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    - Boundary Conditions
  - Taylor's Theorem
  - Simple Finite Difference Approximation to a Derivative
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Fundamentals

Partial Differential Equations

The following equation is an example of a PDE:

$$a(t,x,y)\frac{\partial U(t,x,y)}{\partial t}+b(t,x,y)\frac{\partial^3 U(t,x,y)}{\partial x^3}+c(t,x,y)\frac{\partial^2 U(t,x,y)}{\partial y^2}=g(t,x,y)$$

Where

- t,x,y are the independent variables.
- a,b,c and f are know functions of the independent variables.
- U(t,x,y,) is the dependent variable and is an unknown function of the independent variables.

We will use the following notation:

$$\frac{\partial U(t,x,y)}{\partial t}=U_t \qquad \frac{\partial^2 U(t,x,y)}{\partial y^2}=U_{yy}$$

Partial Differential Equations

The order of a PDE is the order of its highest derivative.

$$a(t,x,y)\frac{\partial U(t,x,y)}{\partial t}+b(t,x,y)\frac{\partial^3 U(t,x,y)}{\partial x^3}+c(t,x,y)\frac{\partial^2 U(t,x,y)}{\partial y^2}=g(t,x,y)$$

A PDE is linear if U and all its partial derivatives occur to the first power only and there are no products involving more than one of these terms.

$$a(t,x,y)\frac{\partial U(t,x,y)}{\partial t}*\left(\frac{\partial^3 U(t,x,y)}{\partial x^3}\right)+b(t,x,y)\frac{\partial^2 U(t,x,y)}{\partial y^2}=g(t,x,y)$$

No-Linear

The dimension of a PDE is the number of independent spatial variables it contains.

2D

$$a(t,x,y)\frac{\partial U(t,x,y)}{\partial t}+b(t,x,y)\frac{\partial^3 U(t,x,y)}{\partial x^3}+c(t,x,y)\frac{\partial^2 U(t,x,y)}{\partial y^2}=g(t,x,y)$$

Solution to a Partial Differential Equation

Easy?

… Find U(t,x,y)

Analytical

An analytical (i.e. exact) solution of a PDE is a function that satisfies the PDE and also satisfies any **boundary** and/or **initial conditions** given with the PDE.

$$a(t,x,y)\frac{\partial U(t,x,y)}{\partial t}+b(t,x,y)\frac{\partial^3 U(t,x,y)}{\partial x^3}+c(t,x,y)\frac{\partial^2 U(t,x,y)}{\partial y^2}=g(t,x,y)$$

IV. NON-DETECTABILITY OF EVENT HORIZONS

Consider a spherical laboratory of radius R that collects data for some time duration T. Assume the equipment in the laboratory to have negligible mass, so that the gravitational self-field of the laboratory can be neglected. Surround the laboratory with a spherical dust shell; by the Birkhoff theorem all quasi-local experiments inside the laboratory will continue to detect a Minkowski spacetime.

Now choose the dust shell such that the mass parameter of the segment of Schwarzschild spacetime outside the dust shell satisfies

$$2m > R + T. \tag{3}$$

Most PDEs of interest **do not** have analytical solutions so a numerical procedure **must** be used to find an approximate solution.

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Solution to a Partial Differential Equation

Analytical

“The purpose of this article is to get mathematicians interested in studying a number of partial differential equations (PDEs) that naturally arise in macroeconomics.”

The equilibrium can be characterized in terms of an HJB equation for the value function v and a Fokker–Planck equation for the density of households g. In a *stationary* equilibrium, the unknown functions v and g and the unknown scalar r satisfy the following system of coupled PDEs (stationary mean field game) on $(\underline{a},\infty)\times(\underline{z},\bar{z})$:

$$\frac{1}{2}\sigma^2(z)\partial_{zz}v+\mu(z)\partial_z v+(z+ra)\partial_a v+H(\partial_a v)-\rho v=0, \tag{2.1}$$

$$-\frac{1}{2}\partial_{zz}(\sigma^2(z)g)+\partial_z(\mu(z)g)+\partial_a((z+ra)g)+\partial_a(\partial_p H(\partial_a v)g)=0, \tag{2.2}$$

$$\int g(a,z)\,\mathrm{d}a\,\mathrm{d}z=1,\quad g\geq 0 \tag{2.3}$$

and

$$\int a g(a,z)\,\mathrm{d}a\,\mathrm{d}z=0, \tag{2.4}$$

where the Hamiltonian H is given by

$$H(p)=\max_{c\geq 0}(-pc+u(c)). \tag{2.5}$$

The function v satisfies a state constraint boundary condition at $a=\underline{a}$ and Neumann boundary conditions at $z=\underline{z}$ and $z=\bar{z}$.

In general, the boundary value problem including the Bellman equation (2.1) and the boundary condition has to be understood in the sense of viscosity (see Bardi & Capuzzo [24], Crandall *et al.* [25], Barles [26]), whereas the boundary problem with the Fokker–Planck equation (2.3) is set in the sense of distributions. An important issue is to check that (2.1) actually yields an optimal control (verification theorem): this is a direct application of Itô’s formula if v is smooth enough; for general viscosity solutions, one may apply the results of Bouchard & Touzi [27] and Touzi [28] (this has not been done yet).

With well chosen initial and terminal conditions, solutions to the HJB equation (2.1) are expected to be smooth and we therefore look for such smooth solutions. If v is indeed smooth, the state constraint boundary condition can be shown to imply

$$(z+r\underline{a})\lambda+H(\lambda)\geq(w+r\underline{a})\partial_a v(\underline{a},z)+H(\partial_a v(\underline{a},z))\quad\forall\lambda\geq\partial_a v(\underline{a},z)$$

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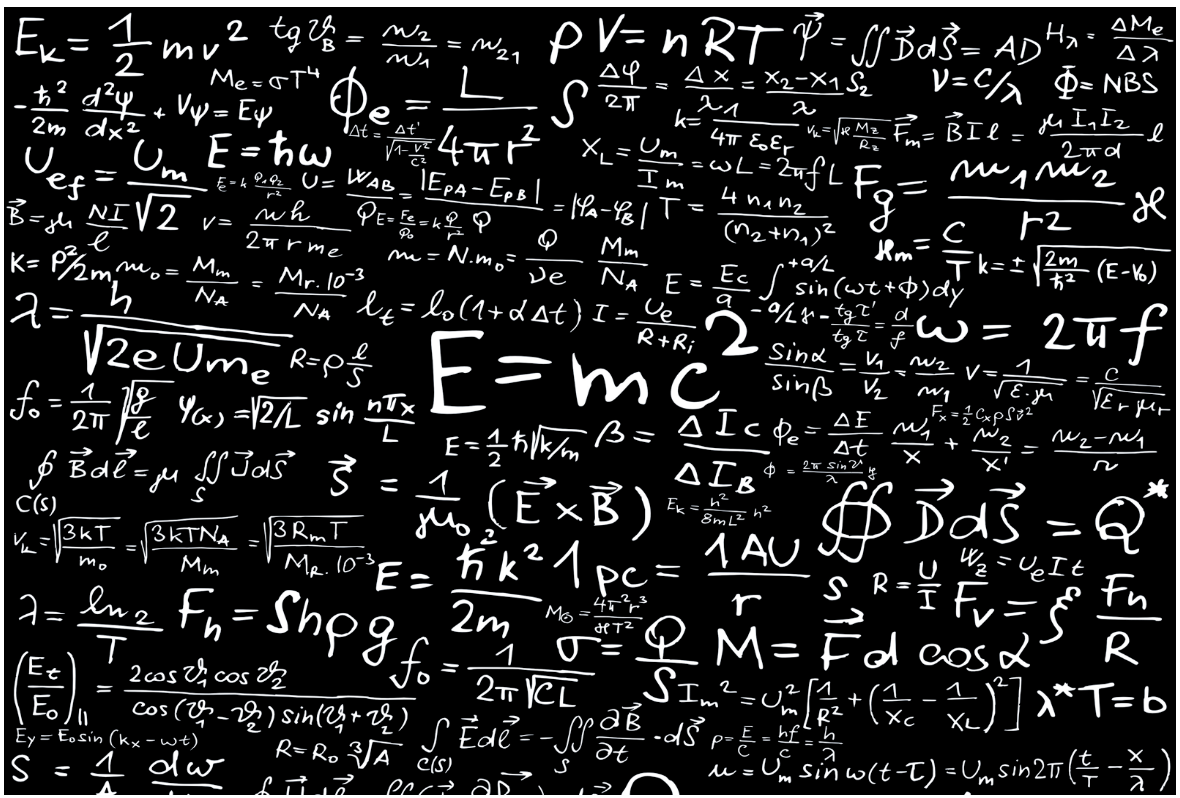
Partial Differential Equations

PDEs describe many of the fundamental natural laws



- Conservation of mass
- Laplace’s equation
- Maxwell’s equations
- Navier–Stokes equations

Solution to a Partial Differential Equation

Numerical

The approximation is made at discrete values of the independent variables and the approximation scheme is implemented via a computer program.

Finite Difference Method

The FDM replaces all partial derivatives and other terms in the PDE by approximations. After some manipulation, a finite difference scheme (FDS) is created from which the approximate solution is obtained. The FDM depends fundamentally on Taylor’s beautiful theorem (circa 1712!).

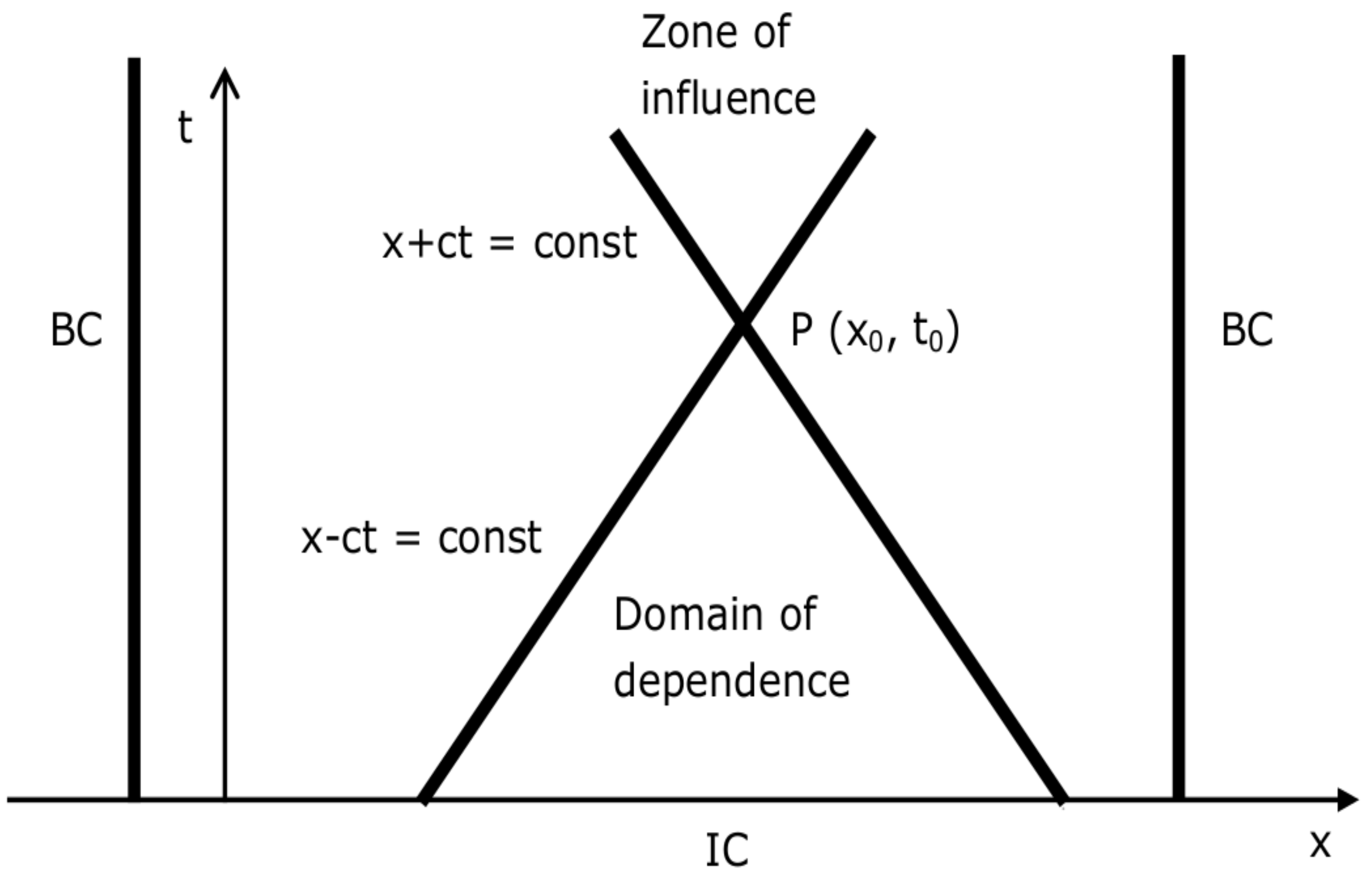
Classification of PDEs

Board I

Classification of PDEs

The differences between the types of PDEs can be illustrated by sketching their respective domains of dependence.

Hyperbolic case:

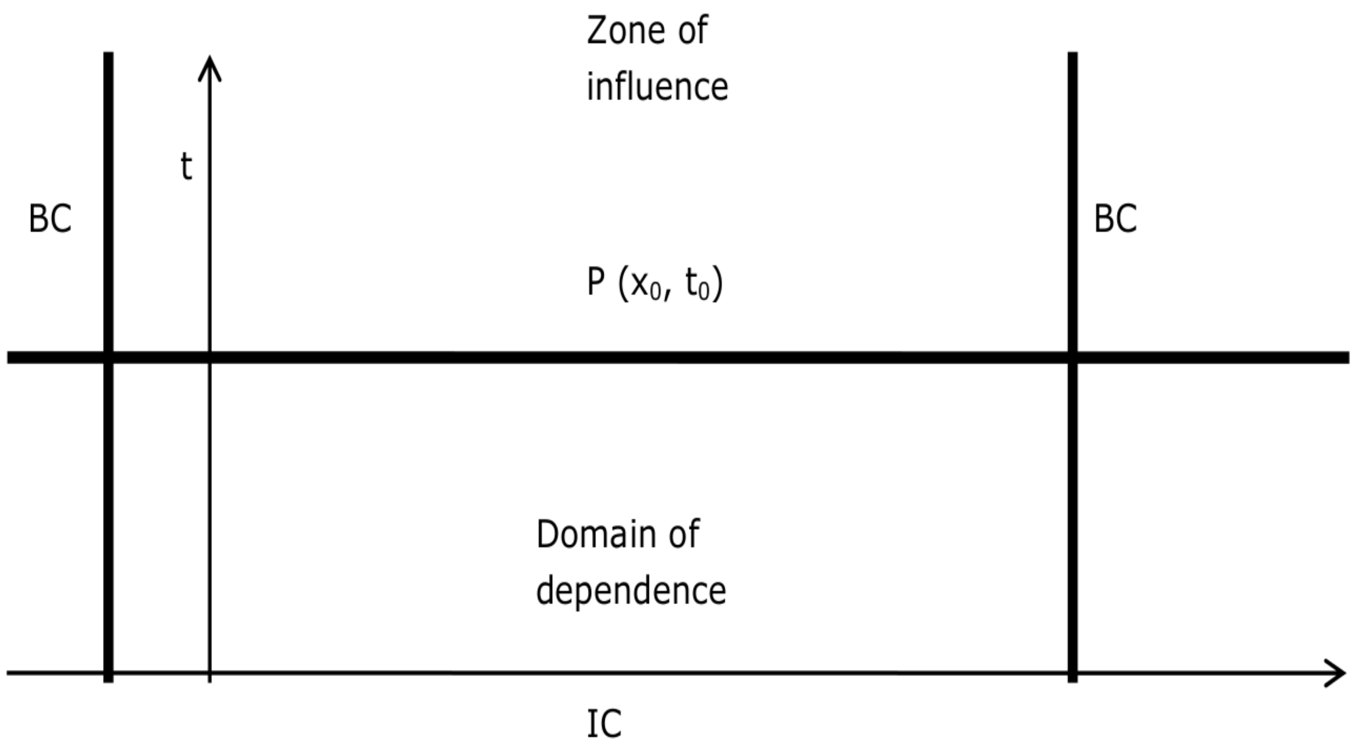


Point P (x_0,t_0) can only be influenced by points lying within the region bounded by the two characteristics x+ct = const and x-ct = const and $t < t_0$. This region is called the domain of dependence.

Classification of PDEs

The differences between the types of PDEs can be illustrated by sketching their respective domains of dependence.

Parabolic case:

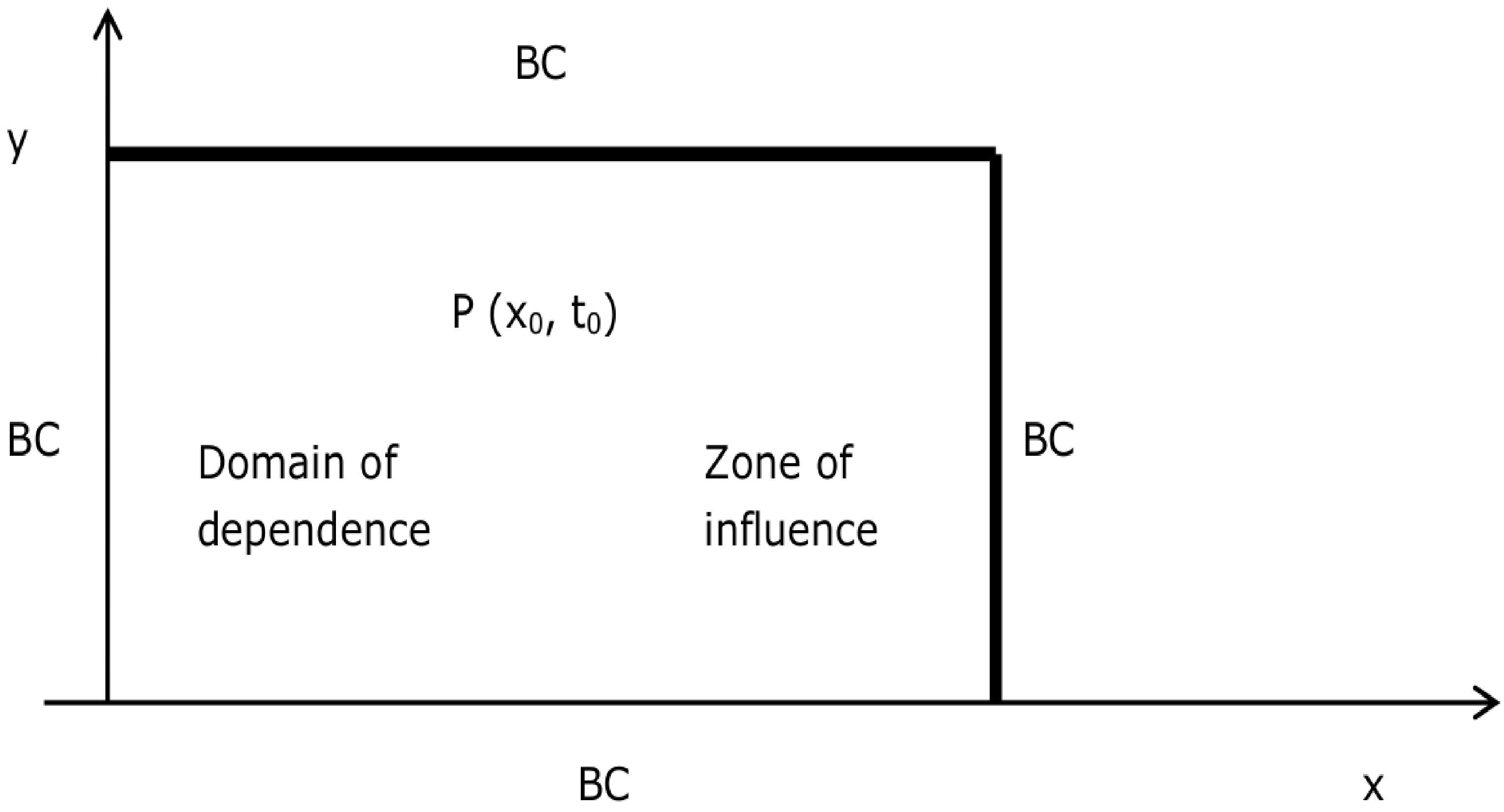


Information travels downstream (or forward in time) only and so the domain of dependence of point P (x_0,t_0) in this case is the region $t < t_0$ and the zone of influence is all points for which $t < t_0$

Classification of PDEs

The differences between the types of PDEs can be illustrated by sketching their respective domains of dependence.

Elliptic case:



Information travels in all directions at infinite speed so the solution at point $P(x_0,t_0)$ influences all points within the domain and vice versa.

Classification of PDEs

The type of PDE fundamentally influences the choice of solution strategy.

Time dependent hyperbolic problems and parabolic problems are solved numerically by time-marching methods which involves, as its name suggests, obtaining the numerical solution at a later time from that at an earlier time starting from given ICs.

Elliptic problems are solved numerically by so-called relaxation methods.

What do we need?

- Well-posedness
  - The solution exists
  - The solution is unique
  - The solution depends “continuously” on the initial and boundary conditions

Example:

$$u_t = u_x$$

Lets assume:

$$u(t, x) = f(t) * e^{ikx}$$

We should be REALLY careful about mathematical theory

Examples

- Find a solution to

Norm of the solutions:

$$u_t = u_x$$

Lets assume:

$$|u(t,x)| = 1$$

$$u(t,x) = f(t) * e^{ikx}$$

- Find a solution to

$$u_t = b\,u_{xx}$$

Lets assume:

$$u(t,x) = f(t) * e^{ikx}$$

$$|u(t,x)| = e^{bk^2 t}$$

Discrete Notation

We will use upper case U to denote the analytic (exact) solution of the PDE and lower case u to denote the numerical (approximate) solution.

Subscripts will denote discrete points in space and superscripts discrete levels in time.

e.g.

$$u_{i,j}^{n}$$

denotes the numerical solution at grid point (i, j) in a 2D region at time level n.

Checking Results

Verification:

The computer program implementing the scheme **must** be verified. This is a check to see if the program is doing what it is supposed to do. Comparing results from **pen and paper calculations** at a small number of points to equivalent computer output is a way to (partially) verify a program.

Validation:

Validation is really a **check** on whether the PDE is a good model for the real problem being studied. Validation means **comparing numerical results** with results from similar physical problems.

Taylor's Theorem

The finite difference method (FDM) works by replacing the region over which the independent variables in the PDE are defined by a finite grid (also called a *mesh*) of points at which the dependent variable is approximated.

The partial derivatives in the PDE at each grid point are approximated from neighbouring values by using Taylor's theorem.

Taylor's Theorem:

Let *U(x)* have *n* continuous derivatives over the interval *(a, b)*. Then for $a < x_o$, $x_o + h < b$,

$$U(x_0+h) = U(x_0) + hU_x(x_0) + h^2 \frac{U_{xx}(x_0)}{2!} + \ldots + h^{n-1} \frac{U_{n-1}(x_0)}{(n-1)!} + O(h^n)$$

where

$O(h^n)$ is an unknown error term

$U_x(x_0)$ is the derivative of U with respect to x evaluated at $x = x_o$.

"order h to the n"

References

D. M. Causon, C. G. Mingham, Introductory Finite Difference Methods for PDEs Ventus Publishing ApS, ISBN 978-87-7681-642-1 (2010).