Conformal Welding of Planar Trees

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Abstract

We pose a conformal welding problem for the continuum random tree, and two constructive methods for solving the analogous problem for discrete trees. The first is computable using Don Marshall's zipper algorithm, and may have applications in the computation of a special class of polynomials important in algebraic geometry. The second approach is better behaved with respect to composition, and allows us to define a limiting map and plane set that we conjecture to be a quasiconformal solution to the welding problem for the CRT. Submitted in preparation for the general exam.

1 Introduction

The aim of this project is to give a representation of the continuum random tree (CRT), first introduced by Aldous [2], as the solution to a conformal welding problem, and thus an embedding of the tree into the complex plane. The CRT is a compact real tree (a metric generalization of the tree graph) that is the scaling limit of the uniform finite trees with n vertices. The CRT forms the backbone of the Brownian map of LeGall [12], which is a sort of universal random surface, in the way that Brownian motion is a universal random function. The Brownian map is homeomorphic to the Riemann sphere, and its geodesic structure is tree-like. A closely related model, the Liouville Quantum Gravity of Duplantier and Sheffield [9], is also in some sense a universal random surface, but in this case represented as a measure on the complex plane. An important problem is to prove that LQG has a metric structure, which raises the question of finding a natural model for random trees in the plane, anticipating a tree-like structure for such a metric.

Conformal welding stands out as a possible approach for two reasons, but first we describe briefly what we mean by welding. Given a simply connected region and an equivalence relation on its boundary, we are looking for a conformal map that extends continuously to the boundary, which we call the welding map, such that the preimages of boundary points on the image respect the relation. Consider the map $z \mapsto z^2$, which maps the upper half-plane to the whole plane minus the positive real line. This is a conformal welding map for the relation $x \sim -x$. There is a natural way to pose the problem for the CRT, provided by the construction from a Brownian excursion. Furthermore, in [16], Sheffield shows that the LQG on the half plane is invariant under a welding map using its boundary values. Using conformal welding gives us a chance at obtaining an object with desirable conformal invariance properties.

In this paper we describe two approaches towards this goal in the form of welding algorithms for finite planar trees. We begin by giving two constructions of the CRT that relate to our algorithms, and give a statement of a welding problem for the CRT. Then, we describe sufficient conditions for existence of a solution to the welding problem for a simple curve, and describe the solution to this problem, which will be important to what follows. A few previous authors (see [6], [15]) have obtained the existence theorems for the conformal welding maps that we give here, with various levels of detail of proof. Christopher Bishop's proof is complete and his approach is very close to ours. However, each of our constructive approaches have unique applications. Our main tool for both algorithms is the theory of quasiconformal maps, following Ahlfors [1].

First we give our algorithm for computing welding maps for harmonically balanced trees, developed with Don Marshall using his zipper algorithm. The advantage to this approach is that the trees can be computed to high accuracy using software. This may have applications in algebraic geometry, as the trees produced by this operation are part of a well studied class of objects called dessins d'enfants (children's drawings), introduced by Grothendieck. For each tree, there is a polynomial mapping its vertices to just two critical values, and we can accurately estimate the coefficients of this polynomial using our algorithm. Finally, we describe an algorithm for coupling discrete trees to a scaling limit, using an arc-insertion method analogous to the stick-breaking construction of Aldous, developed with Steffen Rohde. We conjecture that this homeomophism (or an improved version) is a welding map for the CRT welding problem, and that it can be corrected to a conformal map using techniques from quasiconformal theory.

2 The continuum random tree

Since probability theory is not the main focus of this paper, we will give a mostly non-technical description of the CRT, for the purpose of defining our main problem and giving motivation for the constructions used later.

Definition 1. *A* graph *is a pair consisting of a finite set of vertices V, and a set of edges E, which are pairs of vertices* $e = (v_1, v_2)$*. A graph is* connected *if for any two vertices* $v, w \in V$*, there exist edges* $e_j = (v_1^j, v_2^j)$ *for* $j \in \{1, \dots, k\}$ *such that* $v = v_1^j$*,* $v_2^j = v_1^{j+1}$*, for* $j \in \{1, \dots, k-1\}$*, and* $v_2^k = w$*, which we call a* path *between* v *and* w*. A graph is* acyclic *if paths between vertices are unique, when they exist. A* tree *is a graph that is connected and acyclic.*

We can think of a graph G as a metric space by letting each edge be a copy of $[0, 1]$ and taking the quotient set with respect to the relation that identifies endpoints of edges where they meet at a vertex. Then isometric graphs are considered to be equal.

Definition 2. *A* planar graph *is an equivalence class of homeomorphic embeddings* Γ *of a graph* G *into* $\mathbb{C}$ *such that two embeddings* Γ *and* Γ' *are identified when there exists a orientation-preserving homeomorphism* $f : \mathbb{C} \to \mathbb{C}$ *such that* $f(\Gamma) = \Gamma'$*.*

Not every graph has such an embedding, but it is easy to construct one for a tree. However, different embeddings of a tree do not necessarily represent the same planar tree.

Definition 3. *A* real tree *is a metric space* Γ *such that for between any two points, there exists a unique path between those points, and its length is the distance between the points.*

Here, a path is the isometric image of a segment of $\mathbb{R}$. Trees are real trees, as described above, but in general real trees do not have finitely many edges. An important real tree is encoded by a continuous excursion. Let $h : [0, 1] \to \mathbb{R}$ be a continuous function with $h(0) = h(1) = 0$ and $h(x) \geq 0$. We can form a real tree Γ_h from h that is best visualized by drawing horizontal lines underneath the graph of h, and identifying points connected by unbroken horizontal lines. Evocatively, we think of applying glue to the underside of the function, and compressing the graph horizontally. We define a metric space (Γ_h, d_h) on $[0, 1]$ as follows:

Given points $0 < x < y < 1$, let $m_{x,y} = \inf_{x \le z \le y} h(z)$, then let

$$d_h(x, y) = h(x) + h(y) - 2m_{x,y}. \tag{1}$$

It is easy to verify that d_h satisfies the triangle inequality, so we let ρ be the equivalence relation on $[0, 1]$ such that $x \sim y$ if $d_h(x, y) = 0$, and let Γ_h be the set quotient $[0, 1]/\rho$. Then (Γ_h, d_h) is a metric space. It is also a real tree, but the proof is non-trivial, see [10].

Let $X.$ be a Brownian excursion on $[0, 1]$, that is, Brownian motion with $X_0 = 0$, conditioned to be positive on $(0, 1)$ and have $X_1 = 0$.

Definition 4. (Aldous [3]) *The* continuum random tree *is the tree* (Γ_X, d_X) *given by the procedure above.*

We can now state the welding problem for the CRT. The only difference between the welding problem and the construction above is that we seek a conformal map f that creates the identification given by ρ. Instead of $[0, 1]$, we define $X.$ on the unit circle.

Problem 1. (The welding problem for the CRT) *Given a Brownian excursion* $X.$ *on* $[0, 2\pi]$*, let* ρ *be the relation on* $\partial\mathbb{D}$ *defined by* $\zeta \sim \xi$ *when* $d_X(\arg(\zeta), \arg(\xi)) = 0$*. Find a conformal map* $f : \mathbb{D}^* \to \mathbb{C}^*$ *that extends continuously to the boundary such that* $f(\zeta) = f(\xi)$ *if and only if* $\zeta \sim \xi$*.*

There is another construction of the CRT which will be relevant to this work. Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of a random times, between points of a Poisson point process with intensity t in $[0, \infty)$ (the distribution is not important for this paper, it is only important to know that τ_i are real numbers greater than 0 tending to 0). On $[0, \infty)$ use the length metric within each segment $[\tau_i, \tau_{i+1}]$, but let distance be infinite between segments. Then let $\Gamma_0 = [0, \tau_1]$, and form the tree recursively as follows. Select a point p uniformly by length on Γ_n, then let

$$\Gamma_{n+1} = (\Gamma_n \cup [\tau_n, \tau_{n+1}])/(p \sim \tau_n). \tag{2}$$

Then $\Gamma = \overline{\cup_{n=0}^{\infty} \Gamma_n}$ is the CRT. This object is most naturally considered as a subset of l_1, with each segment occupying a new dimension. We will mirror this construction in the setting of conformal welding in the final section.

3 Welding a curve

We consider a first example of a welding map, between the upper and lower half planes. First we fix some notation. $\mathbb{C}$ is the complex plane, and $\mathbb{C}^*$ is the extended plane, with the topology of the sphere. $\mathbb{H}$ or $\mathbb{H}_+$ will be the upper half-plane $\{\operatorname{Im} z > 0\}$, and $\mathbb{H}_-$ the lower half-plane. $\mathbb{D}$ is the unit disc $\{|z| < 1\}$, and $\mathbb{D}^*$ is the complement of its closure. For the purpose of conformal maps, we consider $\mathbb{D}^*$ to be a simply-connected region including the point at infinity, so, by the Riemann mapping theorem, given a simply-connected region Ω containing a neighborhood of infinity, there exists a unique conformal map $f : \mathbb{D}^* \to \Omega$ such that $f(\infty) = \infty$ and such that f satisfies an asymptotic expansion at infinity given by

$$f(z) = z + O(|z|^{-1}). \tag{3}$$

This function will be referred to as the normalized conformal map for Ω.

Suppose

$$h : \mathbb{R} \to \mathbb{R} \tag{4}$$

is an increasing homeomorphism. We wish to find a Jordan curve J, dividing the plane into two regions Ω^+ and Ω^-, and conformal maps f_+ and f_- from $\mathbb{H}^+$ and $\mathbb{H}^-$ onto Ω^+ and Ω^-, respectively. A conformal map from the half-plane into a region bounded by a Jordan curve extends continuously to the boundary, and we require that

$$f_+(x) = f_-(y) \iff h(x) = y, \tag{5}$$

for all $x, y \in \mathbb{R}$. If f_+ and f_- satisfy this condition, we let f be the piecewise map, and we say that f is the *conformal welding map* for the homeomorphism h, which we call the *welding homeomorphism*. Uniqueness depends on removability of the curve J. If f and g are two conformal welding maps for curves J_f and J_g, and if $f \circ g^{-1}$ extends across J_g, then $f \circ g^{-1}$ is a linear map, and we see that the welding map is unique up to a normalization. Next, we identify a class of functions h for which a conformal welding map exists.

3.1 Quasisymmetry and quasiconformal maps

Quasiconformal maps are an important tool for constructing conformal maps. While conformal maps between given regions are uniquely determined by the images of three boundary points, quasiconformal maps may be constructed with a wide range of boundary conditions. Then composition with a solution to the Beltrami equation can be used to correct a quasiconformal map into a conformal map, obtaining the desired welding map. First, some definitions.

Given a function $f : \mathbb{C} \to \mathbb{C}$, $f(x+iy) = u(x+iy) + iv(x+iy)$, suppose f is absolutely continuous in x for almost every y, and vice versa, so that there exist partial derivatives almost everywhere,

$$f_x = u_x + iv_x \tag{6}$$

$$f_y = u_y + iv_y. \tag{7}$$

We define the Beltrami differentials,

$$f_z = f_x + \frac{1}{i} f_y \tag{8}$$

$$f_{\overline{z}} = f_x - \frac{1}{i} f_y. \tag{9}$$

Then to say that $f_{\overline{z}} = 0$ is equivalent to saying that f satisfies the Cauchy-Riemann equations, so f is analytic at $x+iy$ if and only if $f_{\overline{z}}(x+iy) = 0$. Next, let

$$\mu_f = \frac{f_{\overline{z}}}{f_z}, \tag{10}$$

and we say that μ_f is the *complex dilatation* of f. For a conformal map, $f_z = f' \neq 0$, so $\mu_f = 0$ if and only if f is conformal. We only consider orientation-preserving homeomorphisms, which are maps f such that

$$|f_{\overline{z}}| < |f_z|. \tag{11}$$

Finally, we say that a map is *K-quasiconformal* in a region D for $K \geq 1$ if

$$\|\mu_f\|_\infty \leq \frac{K-1}{K+1} < 1, \tag{12}$$

where the sup norm is taken over D. In order to construct a welding map, we need a solution to a certain boundary-value problem. Given an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$, we wish to find a quasiconformal map $g : \mathbb{H} \to \mathbb{H}$ with boundary values h. Such a map does not exist in general, so we place additional constraints on h.

For $M > 1$, we say that h is *M-quasisymmetric* if, for all x, y in $\mathbb{R}$, h satisfies

$$\frac{1}{M} < \frac{|h(x+y) - h(x)|}{|h(x) - h(x-y)|} < M. \tag{13}$$

Quasisymmetry is the necessary and sufficient condition for h to determine the boundary values of a quasiconformal map.

Theorem 1. *Given an M-quasisymmetric function $h : \mathbb{R} \to \mathbb{R}$, there exists a quasiconformal map $g : \mathbb{H} \to \mathbb{H}$, which extends continuously to $\mathbb{R}$, such that $g_{|\mathbb{R}} = h$. Conversely, if g is a quasiconformal map from $\mathbb{H}$ onto itself, then g extends continuously to the real line, and its restriction is a M-quasisymmetric function for some M.*

A proof can be found in [1], pp. 65 and 69. Now we can solve the welding problem for quasisymmetric welding homeomorphisms.

Theorem 2. *If h is a M-quasisymmetric welding homeomorphism, there exists a conformal welding map f for h, as defined above.*

Proof. Let g_+ be the quasiconformal map with boundary values h, and g_- the identity map. Then g, defined piecewise, welds the upper- and lower- half planes according to h. Suppose there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $f = \phi \circ g$ is conformal everywhere. Since ϕ is continuous on the real line, the composed map f will also weld according to h, and thus be the desired map.

In order for f to be conformal, we require that $\mu_{\phi \circ g} = 0$ at all points, which can be calculated with a chain rule on the total differential. One finds in [1] that the required complex dilatation is

$$\mu_\phi = \left[-\frac{g_z^2}{|g_z|^2} \mu_g \right] \circ g^{-1}. \tag{14}$$

Any ϕ satisfying the above equation will give a conformal map when composed with g. Now we have reduced the problem to finding the solution to a particular differential equation. A solution ϕ to the relation

$$\frac{\phi_{\overline{z}}}{\phi_z} = \mu, \tag{15}$$

for given $\mu : \mathbb{C} \to \mathbb{D}$, is called a solution to the *Beltrami equation* with data μ. The proof is completed by the measurable Riemann mapping theorem of Ahlfors and Bers.

Theorem 3. *For any measurable μ with $\|\mu\|_\infty < 1$, there exists a unique normalized quasiconformal mapping ϕ^μ with complex dilatation μ leaving $0, 1,$ and ∞ fixed.*

μ_ϕ from (14) is bounded away from 1, since g is quasiconformal, so if we let ϕ be the unique mapping with this dilatation, then the function $f = \phi \circ g$ is a conformal welding map. $\square$

Quasisymmetric boundary conditions yield image curves called quasicircles, and for such curves, the welding problem has a unique solution.

Theorem 4. *A welding homeomorphism from $\mathbb{R} \to \mathbb{R}$ has a unique conformal welding map such that the image of $\mathbb{R}$ is a quasicircle if and only if it is quasisymmetric.*

A natural example of a welding map not resulting from a quasisymmetric function is found in the Schramm-Loewner Evolution, or SLE. Given a curve γ_t in the half plane, starting at 0, there exists a canonical mapping f_t from $\mathbb{H}$ to $\mathbb{H} \setminus \gamma_t$, extending continuously to the boundary, and each point on γ_t has a pair of preimages on the real line. These pairs determine a decreasing homeomorphism from an interval $[x_t, 0]$ to $[0, y_t]$, and the map f_t is the unique conformal welding map for that function. In [16], Scott Sheffield shows how this homeomorphism arises from the boundary values of the Gaussian free field. However, the solution to the welding problem still depends on the construction of SLE, so there exists a class of interesting homeomorphisms such that conformal welding maps exist, but direct methods for constructing them do not. In [4], Astala, Jones, Kupiainen and Saksman give a construction of a random welded curve in the plane with a conformal invariance property using techniques from quasiconformal mapping theory, and conjecture a connection to SLE. In that paper, a locally quasiconformal welding map is constructed, then more advanced theorems than the one above are used to find a solution to the Beltrami equation and the conformal welding map. We propose a similar strategy for the construction of the embedded planar CRT, as we describe below.

4 Welded Planar Trees

One can easily extend the result of the previous section to a welding homeomorphism between the upper and lower halves of the unit circle by applying the map $z + 1/z$ and defining the homeomorphism to be the identity outside the interval $[-2, 2]$. Again, a quasisymmetric welding homeomorphism yields a conformal welding map and a curve in the plane as the image of the circle.

On the other hand, we can start with the curve, and it uniquely determines a conformal welding map by the Riemann mapping theorem, and induces its welding homeomorphism. As a natural starting point for the generalization to trees, we can consider the normalized conformal map f from the complement of $\mathbb{D}^*$ to the complement of a finite tree Γ embedded in $\mathbb{C}$. Let V be the set of vertices of the tree, and $\Gamma \setminus V$ is a pairwise disjoint union of open arcs in $\mathbb{C}$. Each arc γ has two preimages under f, and induces a welding homeomorphism w_Γ between them, with the property that $f(\zeta) = f(w(\zeta))$. The correspondence between the tree Γ and the welding homeomorphism w_Γ is our primary focus in what follows. Our main theorem gives sufficient conditions for the welding homeomorphism w to produce a conformal welding map to a planar tree Γ.

4.1 Laminations of the circle

Removing any point $z \in \Gamma$ divides Γ into one or more disjoint subtrees, each corresponding to a connected arc of the circle by f. All the preimages of another point $w \in \Gamma$ are then both contained in one of those arcs. Thus for two pairs of points matched in this way the chords of the circle joining the points of each pair do not intersect. Equivalence relations on the circle are called *laminations* and have been studied by Leung [13] and Gupta [11]. We will consider only what they call closed, flat laminations, omitting the modifiers.

Definition 5. *A (closed, flat) lamination is a closed subset of $\partial\mathbb{D} \times \partial\mathbb{D}$ whose elements are pairs of an equivalence relation $\sim$, with the property that if $\zeta_1 \sim \zeta_2$ and $\xi_1 \sim \xi_2$, and if ξ_1 and ξ_2 are in different components of $\partial\mathbb{D} \setminus (\zeta_1, \zeta_2)$, then $\xi_1 \sim \zeta_1$.*

The last property says that if two pairs of matched points are not all in the same class (i.e. not all mapped to the same point under the conformal map, for example), the chords joining them do not intersect. In fact, the relation resulting from a conformal map is always a lamination:

Theorem 5. *If $f : \overline{\mathbb{D}^*} \to \mathbb{C}^*$ is a continuous map, conformal on $\mathbb{D}^*$, then the relation given by $\zeta \sim \xi$ when $f(\zeta) = f(\xi)$ defines a lamination of the circle. A lamination resulting from such a map is called a* conformal lamination.

The primary thing we would like to know about laminations is which laminations are conformal laminations. The lamination of the upper and lower half circles by a quasisymmetric lamination can be shown to be a conformal lamination using the technique above, and Gupta ([11]) considered the case of laminations where the points with nontrivial relations form a set of logarithmic capacity zero. Here we consider a class of laminations corresponding to embedded planar trees.

Let $\mathcal{A} = \mathcal{A}_n$ be a set of $2n$ pairwise disjoint open arcs in $\partial\mathbb{D}$ such that $\overline{\cup_{\mathcal{A}} A} = \partial\mathbb{D}$. We will occasionally write an arc as an interval by the arguments of its endpoints. We denote the arc $\{e^{i\pi\theta}, \alpha < \theta < \beta\}$ as (α, β).

Definition 6. *A* non-crossing partition $\Pi = \Pi_n = \{\pi_j\}_{j=1}^n$ *of* $\mathcal{A}$ *is a partition of* $\mathcal{A}$ *into* n *pairs* $\pi_j = \{A_j, A'_j\}$ *such that for any two pairs* π_j *and* π_k*, the chord connecting the midpoints of* A_j *and* A'_j *does not intersect the chord connecting the midpoints of* A_k *and* A'_k*.*

Definition 7. *A function* $w : \cup_{\mathcal{A}} A \to \cup_{\mathcal{A}} A$ *is a* welding homeomorphism for Π *if, restricted to the arcs of a pair* $\pi_i \in \Pi$*,* w *is a homeomorphic involution that is decreasing in argument. That is, as* ζ *traverses* A_i *in the clockwise direction,* $w(\zeta)$ *traverses* A'_i *in the counter-clockwise direction, and vice-versa.*

Definition 8. *Let* L_0 *be the minimal equivalence relation such that* $\zeta \sim w(\zeta)$ *for all* $\zeta \in \cup_{\mathcal{A}} A$*, and let* $L = L(\Pi, w)$ *be the closure of* L_0 *in* $\partial\mathbb{D} \times \partial\mathbb{D}$*.* L *is a lamination, and we call such laminations* arc-pairing laminations.

The only relations added in taking the closure are the between endpoints of intervals, which correspond to vertices, and form classes of size equal to the degree of the vertex in the tree. It is easy to see that an embedded finite planar tree Γ with nice enough arcs yields an arc-pairing lamination. It will be important to have a canonical representation for a planar tree, so we define a distinguished class of arc-pairing laminations.

Definition 9. *Suppose* $L(\Pi, w)$ *is a, arc-pairing lamination such that the arcs of* Π *are the arcs between the* $2n$*th roots of unity, and* w *identifies points by arc length. That is, if* $\pi = \{(\pi j/n, \pi(j+1)/n), (\pi k/n, \pi(k+1)/n)\}$*, then* w *is given by*

$$w(\zeta) = \frac{e^{(k+j+1)\pi i/n}}{\zeta}.$$

Then we say that L *is a* balanced *arc-pairing lamination.*

Henceforth we omit "arc-pairing", since we only consider this type. For each lamination, there exists a balanced lamination, unique up to a rotation, by matching the combinatorial pattern of the non-crossing partition. The tree corresponding to a balanced lamination also has special properties, and will be called a balanced tree. Each edge of a balanced tree has equal harmonic measure, and furthermore, any subset of an edge has the same harmonic measure from each side, by the symmetry of the welding homeomorphism and the symmetry of harmonic measure on the circle, and the fact that harmonic measure is invariant under conformal map.

Given a planar tree, one can easily define the corresponding balanced lamination as follows. Starting from a vertex, label the edges 1 through $2n$, one label on each side of each edge, following the contour of the tree in the clockwise direction. Then label the intervals between the $2n$th roots of unity in the same way, and pair intervals whose labels are on opposite sides of an edge on the planar tree. See Figure 3 for a visualization.

5 Computable construction of the welding map for a balanced tree

Let L be a balanced lamination with n pairs. We label the endpoints of the arcs of the pair π_j as a_j, b_j, c_j, d_j, in order of increasing argument, with arcs (a_j, b_j) and (c_j, d_j), and we choose the ordering of the indices and labels so that π_1 is a pair of adjacent arcs with $b_1 = c_1$, and upon application of the jth welding map F_j, the next pair π_{j+1} will be taken to an adjacent pair of arcs with $F_j(b_{j+1}) = F_j(c_{j+1})$. Such an ordering is possible, since every lamination contains an adjacent pair, and when an adjacent pair is welded, a new lamination is induced on the remaining arcs. Our theorem follows from Lemma 2.3 in [7] with a bit of additional work, but we present a more constructive proof.

Theorem 6. *Given a balanced lamination L, there exists a conformal map H from the exterior of the unit circle to the complement of a finite planar tree Γ. This map, extended continuously to the boundary, has the property that for a point $z \in \Gamma$ not a vertex, the preimage of z under H is a pair of points ζ, ξ on $\partial\mathbb{D}$ such that $w(\zeta) = \xi$. Furthermore, restricted to an arc to one side of any root of unity ξ, H has the asymptotic expansion*

$$H(\zeta) - H(\xi) = a(\zeta - \xi)^{2/d} + O((\zeta - \xi)^{2/d+\epsilon}), \tag{16}$$

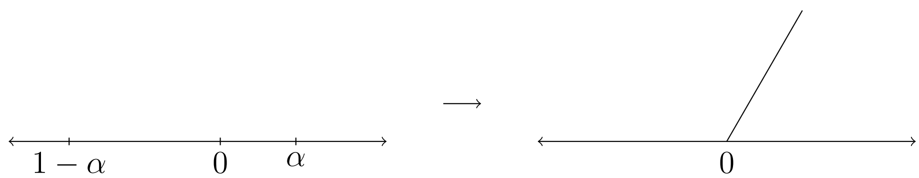

Figure 1: The boundary behavior of the conformal map ϕ_α. Here, $\alpha = 1/3$.

where d is the degree of the vertex of the image of ζ, for some $a \in \mathbb{C}$ and $\epsilon > 0$.

We will refer to the final property as the *welding property*. We will prove the theorem in two steps. First, we construct a conformal map from the exterior of the circle using two classes of elementary conformal maps that satisfies the welding condition for some welding map $\tilde{w}$ for the lamination Π (that is, for each j, $\tilde{w}_j$ is a homeomorphic involution on the pair π_j decreasing in angle). Then the function $h(\zeta) = \tilde{w}(w(\zeta))$ maps arcs into themselves, and can be extended continuously to the unit circle. In the second step, we use the function h to define a quasiconformal map of the plane G with boundary values on the circle such that $F \circ G$ satisfies the welding condition for w. Then a solution to the Beltrami equation corrects the quasiconformal modulus to produce the desired conformal map.

Let f_0 be the conformal map from the exterior of the circle to the upper half plane which takes the endpoints of the adjacent arcs of π_1 to $-1/2, 0$, and $1/2$. Define pairs of intervals π_i^0 with endpoints a_i^0, b_i^0, c_i^0, and d_i^0 as the images of the π_i under f_0. For $0 < \alpha < 1$, consider the function ϕ_α given by

$$\phi_\alpha(z) = (z-\alpha)^\alpha (z+1-\alpha)^{1-\alpha}. \tag{17}$$

Proposition 1. *The function ϕ_α has the following properties (see Figure 2).*

1. *It is a conformal map from the upper half plane into itself, and extends continuously to the boundary, mapping the interval $(-\infty, \alpha-1]$ to $(-\infty, 0]$, $[\alpha-1, 0]$ and $[0, \alpha]$ to the left and right sides, respectively, of a line segment from zero with argument $\pi\alpha$ in the upper half plane, and $[\alpha, \infty)$ to $[0, \infty)$. Note that 0 maps to the tip of the interval.*

2. *ϕ_α is analytic in a neighborhood of infinity, with expansion $z + b + O(1/z)$.*

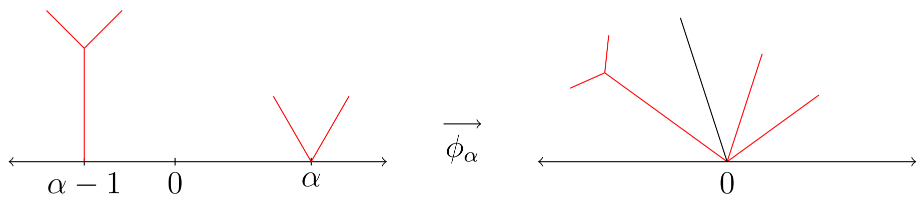

Figure 2: α is chosen to make the angles evenly spaced in the image. Here, $\alpha = 3/5$.

3. *ϕ_α is analytic in a neighborhood of* 0 *with expansion* $f(0) + O(z^2)$*.*

4. *ϕ_α extends analytically and one-to-one across all other boundary points except* α *and* $1 - \alpha$*.*

We now construct a sequence of conformal maps $\{f_k\}_{k=1}^{n-1}$ from the upper half plane into itself. The map f_k will be the composition of two maps: a Mӧbius transformation l_k sending the real line to itself, so that $\pi_k^{k-1} = \{[1 - \alpha_k, 0], [0, \alpha_k]\}$, for some α_k, and the map $\phi_k = \phi_{\alpha_k}$, which will take the pair π_k^{k-1} into the half plane and all other intervals back to the real line. For each map $f_k = \phi_k \circ l_k$, we carry the correspondences forward, letting $\pi_j^k = f_k(\pi_j^{k-1})$, $a_j^k = f_k(a_j^{k-1})$, and so on, for those segments with $j > k$. It remains to be shown how to choose α_k. The resulting map will not satisfy the welding property for w, but will be corrected in the following section.

5.1 Embedding with elementary functions

We start with $f_1 = \phi_{1/2}$, which sends a_0^1 and d_0^1 to 0 and b_0^1 to $i/2$. After applying f_1, the length of images of intervals near the zeros are asymptotic with the square root of their length. We will need to keep track of these asymptotics. We give a recursive definition of a function q_k on the endpoints of the π_j^k on the extended real line which corresponds to the degree of the tree under construction. Let $E_0 = \cup_{j=1}^n \{a_j^0, b_j^0, c_j^0, d_j^0\}$ be the set of endpoints of intervals, and let $E_k = f_k(E_{k-1})$ for $k > 0$. Let $q_0(x) = 1$ for $x \in E_0$. Then, for each $x \in E_k$ for $k > 0$, x is the image of one or two points, which we denote $E_{k-1}^x = \{y : f_k(y) = x\}$. Then define

$$q_k(x) = \sum_{y \in E_{k-1}^x} q_{k-1}(y).$$

For example, the point $x = 0$ will have $q_1(x) = 2$, since it is the image of $-1/2$ and $1/2$, and the rest will have $q_1(y) = 1$. Once a point has been mapped into the upper half plane, the value of q remains constant. We choose α so that, after welding, the angles on either side of the natural slit of the map will be divided equally by the branches of the tree from the point on that side. For example, if $q_1(a_1) = 1$ and $q_1(d_1) = 2$, let $\alpha_2 = 2/3$. See Figure 3 for a visual reference.

Lemma 1. *For $1 \leq k < n$, let*

$$\alpha_k := \frac{q_{k-1}(d_k^{k-1})}{q_{k-1}(a_k^{k-1}) + q_{k-1}(d_k^{k-1})} = \frac{q_{k-1}(F_{k-1}(d_k))}{q_k(F_k(d_k))},$$

and let $F_k = f_k \circ \cdots \circ f_1$ with $f_j = \phi_j \circ l_j$ for the α_k above. Then for each $1 < k < n$, and each point $x \in E_0$, in a neighborhood of x, F is one-to-one with series expansion

$$F_k(x \pm \delta) - F_k(x) = a_0\delta^p + \sum_{k=1}^{\infty} a_k \delta^{p+p_k}, \tag{18}$$

if $F_k(x) \in \mathbb{R}$, or

$$F_k(x \pm \delta) - F_k(x) = a_0\delta^{2p} + \sum_{k=1}^{\infty} a_k \delta^{p+p_k}, \tag{19}$$

if $F_k(x) \in \mathbb{H}$, for some increasing positive sequence p_k, $a_k \in \mathbb{C}$, and

$$p = \frac{1}{q_k(F_k(x))}.$$

The values of a_k and p_k depend on the sign of $\pm\delta$. The same condition applies at infinity, if necessary, after an inversion.

Proof. The induction hypothesis holds for $k = 0$. For $j < n - 1$, if the induction hypothesis holds for j and f_{j+1} is analytic and one-to-one near all e_k^j with $e_k^j \neq e_{j+1}^j$ for $e \in \{a, b, c, d\}$, and $q_{j+1}(F_{j+1}(e_k)) = q_j(F_j(e_k))$ so the hypothesis extends to $j+1$ for these points. The point $b_{j+1}^j = c_{j+1}^j$ maps into the half plane, and since $q_{k+1}(F_{k+1}(x)) = q_k(F_k(x))$ for this point, property (3) of the function ϕ_α gives the desired expansion. To the right of $x = d_{j+1}$, we have, by hypothesis, and conformality of l,

$$l_{j+1}(F_j(x + \delta)) - l_{j+1}(F_j(x)) = a_0\delta^{p_0} + \sum_{k=1}^{\infty} a_k \delta^{p_0+p_k},$$

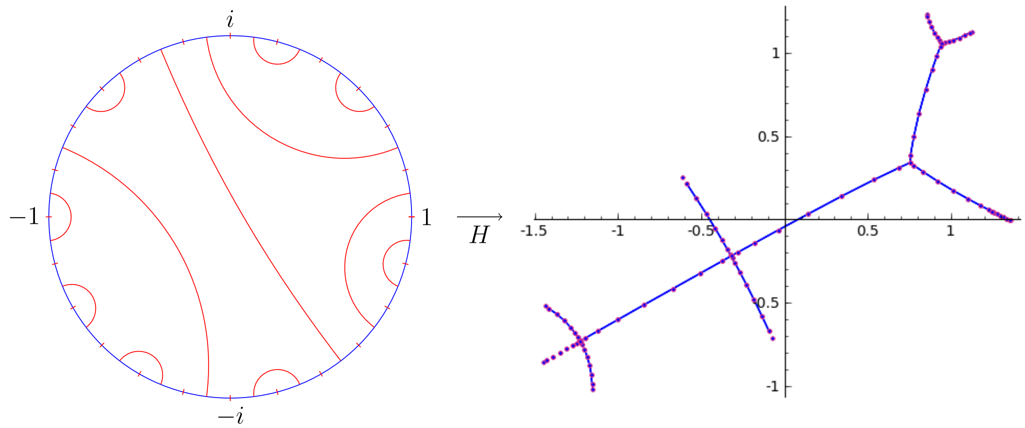

Figure 3: The plane tree resulting from the conformal welding map H for the balanced lamination on the left, which was sampled from the uniform distribution. The image was produced with an implementation of the algorithm by the author. More accurate images for trees of high degree can be produced using Donald Marshall's ziptree program.

where $p_0 = 1/q_k(F_k(x))$, and

$$\phi_\alpha(\alpha + \epsilon) = \epsilon^\alpha + \sum_{k=1}^{\infty} b_k \epsilon^{\alpha+k},$$

so

$$\begin{aligned} F_{j+1}(x+\delta) - F_{j+1}(x) &= \phi_{j+1}(\alpha_{j+1} + [l_{j+1}(F_j(x+\delta)) - l_{j+1}(F_j(x))]) \\ &= \left(a_0\delta^{p_0} + \sum_{k=1}^{\infty} a_k \delta^{p_0+p_k}\right)^{\alpha_{j+1}} + \\ &\quad \sum_{m=1}^{\infty} b_m \left(a_0\delta^{p_0} + \sum_{k=1}^{\infty} a_k \delta^{p_0+p_k}\right)^{\alpha_{j+1}+m} \\ &= a_0^{\alpha_{j+1}}\delta^p + \sum_{k=1}^{\infty} \tilde{a_k}\delta^{p+\tilde{p}_k}, \end{aligned}$$

where $p = 1/q_{k+1}(F_{k+1}(x))$, $\tilde{p}_1 = \min(p_1, p_0)$, and so on. To see that the resulting series expansions converge and represent locally invertible functions, consider the series to be expansions of analytic functions in δ^η, where η is a fractional power with lowest common denominator among the powers $p+p_k$. The same argument applies on the other side of d_{j+1} and on either side of a_{j+1}. $\square$

A composition of the maps f_k up to $n-1$ leaves the two points $a_n^{n-1} = d_n^{n-1}$ and $b_n^{n-1} = c_n^{n-1}$ on the real line. By shifting one point to zero and the other to infinity, we then apply a square map, and reset infinity (that is, compose with a fractional map such that the total composed map sends infinity to infinity), so that the resulting map f_n is such that $F_n = f_n \circ \cdots \circ f_0$ is a map from the exterior of the disk to the exterior of a tree, comprised of the union of the analytic images of the segments, and normalized to be approximately the identity in a neighborhood of infinity, that is,

$$F_n(z) = z + O(|z|^{-1}).$$

Together with Lemma 1, the finished construction gives a desirable property to the elementary embedding F_n:

Corollary 1. *The map F_n gives another embedding Γ' of the same planar tree as Γ, and, in a neighborhood to one side of a root of unity ξ, has the same asymptotic expansion as H,*

$$F_n(\zeta) - F_n(\xi) = a(\zeta - \xi)^{2/d} + O(|\zeta - \xi|^{2/d+\epsilon}), \tag{20}$$

with different constants.

To prove the corollary, simply follow the values of q_k in the proof above to see that they correspond to the degree of the planar tree as it is being constructed. In addition to providing the symmetry necessary in the following section, the choice of α_k ensures equal angles around each vertex in the image, and thus produces better results in the approximation.

5.2 Quasiconformal correction

Next, we turn to the quasiconformal function G which guarantees the welding property. Define $\tilde{w}$ on the union of the open arcs to be $\tilde{w}(\zeta) = F^{-1}(F(\zeta))$, where the inverse is chosen so that $F^{-1}(F(\zeta)) \neq \zeta$. Then F satisfies

$$F(\zeta) = F(\tilde{w}(\zeta)),$$

for all ζ except roots of unity. Let $\mathcal{A} = \mathcal{B} \cup \mathcal{W}$, a natural two-coloring of the arcs of the circle. On a given arc $[a, b]$, the function $\zeta \mapsto \tilde{w}(w(\zeta))$ is a homeomorphism of the arc into itself, increasing in angle. Define $h : \partial\mathbb{D} \to \partial\mathbb{D}$ as follows.

$$h(\zeta) = \begin{cases} \zeta & \text{for } \zeta \in \mathcal{W} \\ \tilde{w}(w(\zeta)) & \text{for } \zeta \in \mathcal{B}, \end{cases}$$

extended continuously to the roots of unity. We need a quasiconformal map H such that $H_{|\partial\mathbb{D}} = h$. The condition for the existence of such a map is quasisymmetry, which reduces to a bi-Lipschitz condition for the given h, because the map is linear on one side of each root of unity, the only places where it is not analytic. Thus we require that for each $x \in \mathcal{R}$, in a neighborhood of x, there exists $K > 1$ such that

$$\frac{1}{K} \leq \frac{|h(\zeta_1) - h(\zeta_0)|}{|\zeta_1 - \zeta_0|} < K,$$

or, equivalently, K such that for z_0, z_1 in the image of a neighborhood of x, if $F^{-1}(z_i) = \{\zeta_i, \xi_i\}$, for $i = 0, 1$, then take F^{-1} to be the inverse of F restricted to the neighborhood containing ξ. Then we need

$$\frac{1}{K} \leq \frac{|F^{-1}(F(\zeta_1)) - F^{-1}(F(\zeta_0))|}{|\zeta_1 - \zeta_0|} < K.$$

By inverting the expression for F in that neighborhood, it is easy to see that F^{-1} has an asymptotic expansion

$$F^{-1}(z) - x = a(z - F(x))^{d/2} + O((|z - F(x)|^{2/d+\epsilon}), \tag{21}$$

and therefore

$$F^{-1}(F(\zeta)) - x = ab(\zeta - x) + O(|\zeta - x|^{\epsilon}), \tag{22}$$

where ϵ is chosen as needed in each case. In both cases the error term has a convergent series expansion in an appropriate neighborhood, and is thus well-behaved with respect to composition and inversion. In a neighborhood on each side of x, h is thus approximately linear, and K can be chosen to satisfy the bi-Lipschitz condition above. Thus h is quasisymmetric on the circle, and there exists a quasiconformal map $H : \mathbb{C} \to \mathbb{C}$ with boundary values h on the unit circle. Consider $G_0 = F \circ H$. It is a quasiconformal map from the exterior of $\mathbb{D}$ to the exterior of a tree. For $z \in G_0(\partial\mathbb{D})$, $F^{-1}(z)$ is a pair of points $\{\zeta, \tilde{w}(\zeta)\}$ for $\zeta \in \mathcal{W}$. Then $\zeta = h(\zeta)$ and

$$\tilde{w}(\zeta) = \tilde{w}(w(w(\zeta))) = h(w(\zeta)),$$

so G_0 satisfies the welding condition for the desired welding map w.

Finally, there exists a quasiconformal map $B : \mathbb{C} \to \mathbb{C}$ such that $G = B \circ G_0$ is conformal and satisfies the normalization $G(z) = z + O(|z|^{-1})$ at infinity. The map B is again the solution to the Beltrami equation as outlined in the previous section. This nearly concludes the proof of Theorem 1, with G being the desired map. The asymptotic expansion will follow from the existence of the Shabat polynomial, as described in the following section.

6 Shabat polynomials and dessins d'enfants

Consider the function $q : \mathbb{D}^* \to \mathbb{D}^*$ given by

$$q(z) = \frac{1}{4}(z^n + z^{-n} + 2). \tag{23}$$

The power function z^n takes the $2n$th roots of unity to -1 and 1, and scales the points of the circle between them by arc length. Note that for any pair of a balanced lamination, one edge maps to the upper half-circle under z^n, and the other maps to the bottom, since there must be an even number of intervals between them. Since the welding homeomorphism for this pair is decreasing in angle, when it is carried forward by z^n, it becomes the relation $z \sim \overline{z}$. The conformal welding map for this relation is the map $z + z^{-1}$, which maps the circle to the interval $[-2, 2]$. Thus, $q(z)$ is an analytic function, with a pole of degree n at infinity that welds according to the welding homeomorphism of the balanced lamination. Let $p = q \circ H^{-1}$, an analytic function from Γ^c to $[0, 1]^c$. Because of the welding property of both maps, p extends continuously across the boundary, and thus extends to be analytic on the whole plane by Morera's theorem. Since the pole at infinity is of degree n, $p(z)$ must be a polynomial of degree n. This polynomial has a remarkable property. A tree with n edges has total degree $2n$, and each vertex with degree $d > 1$ is a critical point of p of order $d - 1$. Since there are $n + 1$ vertices, that leaves $n - 1$ critical points, counting multiplicity. Since p is only degree n, all of the critical points are vertices of the tree, and map to one of the points 0 or 1. Such polynomials, called Shabat polynomials, are objects of study in algebraic geometry. Using similar arguments to those above, it is not hard to see that all such p of degree n are obtained in this way from a balanced lamination, and correspond to a particular planar tree. The preimages of Shabat polynomials are the simplest examples of dessins d'enfants, or children's drawings, which are essentially planar graphs on a Riemann surface with a bipartite vertex structure. Grothendieck introduced dessins as part of the study of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Calculating the Shabat polynomials has been of some interest (see [5]), but exact solutions involve algebraic equations of impossibly high degree. Numeric methods have been proposed, such as in [8], but suffer from an inability to efficiently estimate the locations of critical points. In the cited paper, the authors compute the vertices for a tree of one less degree, then use those points as initial valuer for Newton's method in the next step. Our method can directly approximate the location of the vertices for any tree. Further exploration of this application will be part of our project going forward.

7 A recursive construction of an infinite planar tree with piecewise linear maps

An important construction of the CRT is the stick-breaking construction of Aldous. The idea is to build an infinite tree Γ starting with a segment Γ_0, which, in the nth step, adds a segment of decreasing length to a uniform point along the tree Γ_{n-1} to form the tree Γ_n. This construction can be imitated with quasiconformal maps, if we can find maps with the following property. We want a sequence of maps F_n from $\mathbb{D}^*$ to the complement of a tree Γ_n, such that Γ_n a finite planar tree with n edges, and $\Gamma_n \subset \Gamma_m$ for $n < m$. We can accomplish this with quasiconformal welding maps. Suppose we have a sequence of laminations $\{L_n\}$ corresponding to welded trees Γ_n as the images of the circle under a yet-undefined quasiconformal map. Given the lamination L_n, after one step of the welding process, we have a map f_n (we reverse the indexing of the maps for reasons that will become clear) , an induced lamination L_n^{n-1} on the circle, and a proto-tree P_{n-1}, the image of the welded arcs. Suppose $L_n^{n-1} = L_{n-1}$, that is, the induced lamination after one step towards creating the tree Γ_n is the same as the lamination for the tree Γ_{n-1}. Then we find that Γ_n is the union of Γ_{n-1} and $F_{n-1}(P_{n-1})$.

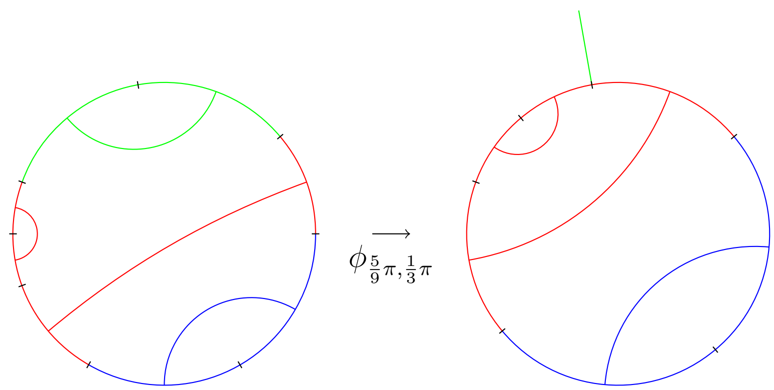

Figure 4: From left to right, the boundary values of the quasiconformal welding map $\phi_{\zeta,\alpha}$ with values $\zeta = \frac{5}{9}\pi$ and $\alpha = \frac{1}{3}\pi$. Also, an example of the arc-insertion sequence, showing L_2 and L_1. The red intervals form a single pair in the lamination L_1 on the right.

We will accomplish this goal with two related constructions. First, a sequence of laminations $\{L_n\}$ constructed recursively from a sequence of points $\{\zeta_n\}$, $\zeta_n \in \partial\mathbb{D}$, and a sequence of arc-lengths $\{\alpha_n\}$, $0 < \alpha_n < \pi$, $\alpha_n \to 0$. L_0 is the lamination that identifies the upper and lower halves of the circle by arc length, and L_n is formed from L_{n-1} by inserting adjacent, paired arcs of length α_n at the point ζ_n and shifting all other points, scaling linearly toward the antipodal point $\zeta_n^* = \zeta_n + \pi$. More specifically, let

$$T(\zeta) = \frac{\pi - \alpha_n}{\pi}(\zeta - \zeta_n^*) + \zeta_n^*, \tag{24}$$

where $-\pi < \zeta - \zeta^* < \pi$. Then, if $\pi = \{A, A'\} \in \Pi_{n-1}$ and $\zeta_n \notin A \cup A', \tilde{\pi} = \{T(A), T(A')\} \in \Pi_n$, and w_n identifies points by arc length as in the balanced lamination. Typically, ζ_n is in A for some interval of a pair π. Then split π into two pairs π_1 and π_2 on either side of ζ_n and $w_{n-1}(\zeta_n)$, and include pairs $\tilde{\pi}_1$ and $\tilde{\pi}_2$ in Π_n. The final pair of Π_n is $\{(\zeta_n - \alpha_n, \zeta_n), (\zeta_n, \zeta_n + \alpha_n)\}$. We call this sequence of laminations the *arc-insertion* sequence of laminations for the values $\{\zeta_n\}$ and $\{\alpha_n\}$.

Second, we need a two-parameter family of quasiconformal maps $\phi_{\zeta,\alpha}$ that welds $(\zeta - \alpha, \zeta)$ to $(\zeta, \zeta + \alpha)$ with boundary values $f(\zeta) = T^{-1}(\zeta)$. Figure 4 shows the desired image of $\phi_{\zeta,\alpha}$ on the boundary. We will construct ϕ as a piecewise linear map in polar coordinates. Without loss of generality, we assume $\zeta = 1$. Outside of the annulus $A_\alpha = \{z : 1 < |z| < e^\alpha\}$, ϕ is the

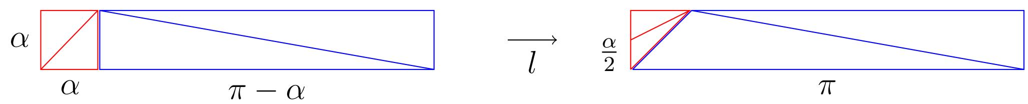

Figure 5: From left to right, T_1 and T_2 in red, and T_3 and T_4 in blue map to T_1', T_2', T_3', and T_4' under ϕ. The remainder of the rectangle and its image is the reflection of this image across the line $x = \pi$.

identity. The function $\psi(z) = i\log(z) + 2\pi$ maps $A_\alpha \setminus (1, e^\alpha)$ to the rectangle $R_\alpha = \{x + iy : 0 < x < 2\pi, 0 < y < \alpha\}$. We divide the rectangle into eight triangular regions $T_1, \ldots, T_8$ as follows.

$$T_1 = \triangle(0, \alpha + i\alpha, i\alpha), \tag{25}$$
$$T_2 = \triangle(0, \alpha, \alpha + i\alpha), \tag{26}$$
$$T_3 = \triangle(\alpha, \pi, \alpha + i\alpha), \tag{27}$$
$$T_4 = \triangle(\pi, \pi + i\alpha, \alpha + i\alpha). \tag{28}$$

T_5 through T_8 are the reflections of the others across the line $x = \pi$. Under ϕ, each triangle T_i maps onto a corresponding triangle T_i'. See Figure 5 for a visual reference.

$$T_1' = \triangle(i\alpha/2, \alpha + i\alpha, i\alpha), \tag{29}$$
$$T_2' = \triangle(i\alpha/2, 0, \alpha + i\alpha), \tag{30}$$
$$T_3' = \triangle(0, \pi, \alpha + i\alpha), \tag{31}$$
$$T_4' = \triangle(\pi, \pi + i\alpha, \alpha + i\alpha). \tag{32}$$

The map for each corresponding triangle is an affine linear transformation of $\mathbb{R}^2$. We can write a general form for the complex dilatation of such a map. Suppose $T = (0, x_1 + iy_1, x_2 + iy_2)$ and $T' = (0, u_1 + iv_1, u_2 + iv_2)$. Then if l is the linear transformation between them, we have

$$\mu_l = \frac{u_1y_2 - u_2y_1 - x_1v_2 + x_2v_1 + i(x_1u_2 - x_2u_1 + v_1y_2 - v_2y_1)}{u_1y_2 - u_2y_1 + x_1v_2 - x_2v_1 + i(x_1u_2 - x_2u_1 - v_1y_2 + v_2y_1)}. \tag{33}$$

For the triangles above, we have

$$\begin{aligned}
\mu_1 &= \frac{1}{3},\\
\mu_2 &= \frac{-1+2i}{3},\\
\mu_3 &= \frac{\alpha + i(\pi-\alpha)}{2\pi - \alpha + i(\pi-\alpha)},\\
\mu_4 &= 0.
\end{aligned}$$

Then, if l is the piecewise linear map on R_α, extended continuously across the inner boundaries, let

$$\phi = \psi^{-1} \circ l \circ \psi, \tag{34}$$

on A_α, and the identity outside. ϕ is a quasiconformal map of $\mathbb{D}^*$ to $\mathbb{D}^* \setminus [0, \alpha/2]$, since it is a homeomorphism with complex dilatation a.e. bounded away from 1, since composition with a conformal map does not change the magnitude of the complex dilatation. Also, $\phi(\zeta) = T^{-1}(\zeta)$ on $\partial\mathbb{D} \setminus [-\alpha, \alpha]$ as desired. Conjugation with a rotation gives the general map $\phi_{\zeta,\alpha}$.

Given sequences $\{\zeta_n\}$ and $\{\alpha_n\}$, it is now straightforward to construct a quasiconformal embedding of the tree corresponding to the lamination L_m for any $m > 0$. Let

$$\Phi_m = \rho \circ \phi_{\zeta_1,\alpha_1} \circ \phi_{\zeta_2,\alpha_2} \circ \cdots \circ \phi_{\zeta_m,\alpha_m}, \tag{35}$$

where $\rho(z) = z + z^{-1}$, the conformal map that welds the top and bottom of the circle. It is routine to check that Φ_m satisfies the welding condition for the arc-identifying welding homeomorphism for the lamination L_m. Then, using the same procedure as above, we can use the solution to the Beltrami equation for the composed complex dilatation to produce the conformal welding map. The advantage to the arc-insertion approach defined in this section is that we can easily produce infinite objects. The tree Γ_m is an increasing set, so let

$$\Gamma_\infty = \overline{\cup_{m=1}^{\infty} \Gamma_m}. \tag{36}$$

Of course, without additional analysis, there is no way to know whether or not Γ_∞ even has a tree structure. If $\alpha_n \to 0$, there is a limit function Φ.

Proposition 2. *If $\alpha_n \to 0$, Φ_n converges uniformly on compact subsest of $\mathbb{D}^*$ to a locally quasiconformal homeomorphism Φ, and $\Phi(\mathbb{D}^*) = \Gamma^c$.*

Proof. Since ϕ_{ζ_m,α_m} is the identity map for $|z| > e^{\alpha_m}$, If K is a compact neighborhood with $d(K, \mathbb{D}) = \epsilon$, let n_0 be such that $\alpha_n < \log(1+\epsilon)$ for $n > n_0$. Then for $m > n_0$, $\Phi_m(K)$ is constant, and the limit exists. Since Φ_m is quasiconformal on K, Φ is as well. To see that $\Phi(\mathbb{D}^*) = \Gamma^c$, consider $z \in \mathbb{D}^*$ with $B_\epsilon(z) \subset \mathbb{D}^*$. Eventually, Φ_m is constant on $B_\epsilon(z)$, so $d(\Phi(z), \Gamma_m)$ is fixed for all m large enough, and $\Phi(z) \notin \Gamma$, so $f(\mathbb{D}^*) \subset \Gamma^c$. On the other hand, suppose $z_n \to \partial\mathbb{D}$. Each map $\phi_{\alpha,\zeta}$ decreases the distance to the boundary, so $\Phi(z_n) \to \Gamma$, and $\Gamma^c \subset f(\mathbb{D}^*)$. □

In order to build the CRT, one uses a sequence of random times $\{\alpha_j\}$ firing at rate t, and i.i.d $\{\zeta_n\}$, uniform on the circle. Then we propose that Γ_∞ is a welding map for the CRT as defined in section 2.

8 Problems for further study

1. For a given planar tree, we can refine the lamination so that the first algorithm produces very accurate results. Prove that the sequence of computed welding maps (before quasiconformal correction) converges to the welding map for the balanced tree. Proving this fact may be difficult, however, see [14] for an example of this type of theorem.

2. The uniform measure on balanced laminations and their welding maps should have a weak limit. One approach could be to look at the coefficients of the Shabat polynomial as sequences random variables, and use their limits to characterize sequential limits.

3. Demonstrate a complete and efficient method for calculating the coefficients of the Shabat polynomial for a given tree.

4. Show that Γ_∞ is a real tree.

5. Show that Φ can be corrected into a conformal map.

6. Given the solution to the previous two problems, show that the resulting map and tree gives a solution to the conformal welding problem for the CRT.

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