Cramer's Rule

Omar Fayed / 900191831
Mohamed Bedda / 900161918
Mohamed A.Soliman / 900182638

May 17, 2022

Abstract

The paper will first introduce the life and career of Gabriel Cramer. It will be followed by proof of the rule and solution of two examples using Cramer's rule.

1 Brief Introduction

Gabriel Cramer was born in Geneva in 1704 in an academically successful family. Gabriel's father, Jean Isaac Cramer, and his brother were medical doctors and his other sibling was a professor of law. Cramer was a prodigy as he earned his first doctorate in theory of sound in1722 while he was 18 years of age. At the age of 20 he was a candidate for the chair of philosophy at the Académic de Calvin in Geneva. However, Cramer was offered co-chair of Mathematics accompanied by Giovanni Calandrini. This role allowed Cramer to travel across Europe meeting the best scientists in the continent back then like Euler,Bernoulli,Fontenelle. Cramer published multiple in journals such as Memoirs of the Paris Academy and Berlin Academy specially in geometry and history of mathematics. The most known book in Cramer's career is entitle dIntroduction à l'analyse des lignes courbes algébraique which focused on modeling Newton's memoir on cubic curves. In the third chapter of the book, Cramer discusses the famous " Cramers rule".

2 Definitions and axioms needed for the proof

2.1 The determinant of a matrix

Every matrix is a value that is defined for square matrices. The Determinant of a matrix A is written as $|A|$ or $det(A)$. The determinant is calculated in multiple ways depending on the size of the matrix.

2.2 Transpose of a matrix

The transpose of a matrix A is denoted as A^T and has the same dimensions (reversed if not square) of matrix A but the rows of the matrix A written as columns in the matrix A^T. For example,

$$if A = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix} \; then \;\; A^T = \begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{matrix}$$

2.3 Inverse of a matrix

An inverse matrix B exists if $AB = I_n$ and $AB = BA$. Then matrix B is the multiplicative inverse of the matrix A. Matrix B can also be written as A^{-1}. The matrix A is then called invertible. If a matrix does not have an inverse matrix is called singular or noninvertible.

$$if AB = I_n \;\; then \;\; B = A^{-1}$$

2.4 The Minor of an element of a matrix

To get the minor M_{ij} of a certain entry a_{ij} in a square matrix we delete the i^{th} row and the j^{th} column in the matrix and find the determinant of the new matrix

2.5 The Cofactor of an element of a matrix

The cofactor is calculated by the following formula

$$C_{ij} = (\text{-}1)^{i+j} * M_{ij}.$$

2.6 The Adjoint matrix

The adjoint of a matrix the transpose of the matrix of cofactors. In other words, we obtain first the matrix of cofactors which we assign each entry a_{ij} its equivalent cofactor c_{ij} and then transpose this matrix to obtain the adjoint matrix which is denoted by $adj(A)$.

2.7 Formulas and Axioms that will be needed

- If $AX = B$ then $X = A^{-1}B$
- $A^{-1} = (1/|A|) * adj(A)$

3 Cramer's rule

If a coefficient matrix of a system of linear equation is A and the system has n variables and $|A|$ is not equal to zero, then the system has a solution as follows:

$$x_1 = \frac{det(A_1)}{det(A)} \;\; , \;\; x_2 = \frac{det(A_2)}{det(A)} \;\; , \;\; \ldots \, x_n = \frac{det(A_n)}{det(A)}$$

keeping in consideration that the column of constants of A_i in the system is the i^{th} column.

4 Proof

Suppose we have an invertible matrix A, then the inverse A^{-1} exists. Using the identity mentioned above then,

$$A^{-1} = (1/|A|) * adj(A)$$

Therefore

$$x = A^{-1}B$$

We substitute A^{-1} with $(1/|A|) * adj(A)$ to get

$$x = (1/|A|) * adj(A) * B$$

As we know the entries in adjoint matrix of A is as follows

$$\text{adj(A)} = \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

We Then multiply the adjoint matrix of A with B to get

$$\sum_{j=1}^{n} C_{ij} B_j \ = \ det(A_i)$$

Then

$$x_i = (1/|A|) * \sum_{j=1}^{n} C_{ij} B_j$$

Therefore,

$$x_i = \frac{det(A_i)}{det(A)}$$

4.1 Example

Use Cramer's Rule to solve the system of linear equations for x.
$2x + 3y - z = 2$
$2x–y - 2z = 0$
$y + 3z = 3$

Solution:
$|A| = -22$
As $|A|$ is non zero then it has a unique solution.

$$|A_1| = \begin{vmatrix} 2 & 3 & -1 \\ 0 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix} = 13$$

$x = -13/22$

4.2 Exercise 27

$kx \ + \ (1-k)y \ = \ 1$
$(1-k)x \ + \ ky = 3$

$$\begin{vmatrix} k & 1-k \\ 1-k & k \end{vmatrix} = k^2 - (1 - 2k + k^2) \quad = \ 2k + 1$$

$$|A_1| \ = \ \begin{vmatrix} k & 1 \\ 1-k & 3 \end{vmatrix}$$
$= \ 4k - 1$

$$|A_2| \ = \ \begin{vmatrix} 1 & 1-k \\ 3 & k \end{vmatrix}$$
$= \dot{4}k - 3$

$x_1 \ = \ \frac{4k-1}{2k-1}$

$4k - 1 \ \neq 0$, $2k - 1 \ \neq 0$, $4k - 3 \ \neq 0$
$k \ = \ \frac{1}{4}$, $k \ = \ \frac{3}{4}$, $k \ = \ \frac{1}{2}$

References

"Gabriel Cramer - Biography." Maths History, https://mathshistory.st-andrews.ac.uk/Biographies/Cramer/.

Larson, Ron, and Alta Jooste. Elementary Linear Algebra: WTW146. Cengage Learning, EMEA, 2019.

Siniora, Daoud. "Linear Algebra Notes." 2019.