The Notebook rel.nb

In this notebook we arrive at the result in Eq. (30) (based on which Eq, (16) was also found), which is the generalized entanglement measure to leading order in 1/c for the case of two gaussians. The calculation proceeds as follows: (1) solving the implicit definition of the retarded time from the paragraph after Eq. (24) to the desired order, (2) defining the relativistic action Eq. (21) , (2) finding the classical solutions of the system, (3) computing the gaussian integrals of which Eq. (9) consists, (4) calculating for this wavefunction the generalized entanglement measure [Swain et al, Phys. Rev. A 105, 052441 (2022)] defined in Eq. (28), to get Eq. (30).

In[ ]:=

```
<< VariationalMethods`

$Assumptions = {m > 0, d > 0, ω > 0, t > 0, ti > 0,
   tii > 0, ϕ > 0 , ϵ > 0 , Ω > 0 , y1 ∈ Reals, y2 ∈ Reals, y1p ∈ Reals,
   y2p ∈ Reals, dtN > 0, 4 ϵ^2 ϕ < 1, trp ≥ 0, kin ≥ 0, ret ≥ 0};

(*defining the dimensionless quantities ϵ,
ϕ and Ω which fully define the problem.*)
ℏ = ϵ^2 d^2 m ω;
(*ϵ = α/d is a measure of the "quantumness" of the particles*)
G = ϕ ℏ d ω / (m^2 ) ;
(*ϕ = Gm²/ℏωd is a measure of the effect of gravity on the system*)
c = d ω / (Ω ) ;
(*Ω = ωd/c is a measure of the effect of retardation in the system*)

α = Sqrt[ℏ / (m ω)]; (*the width of the gaussian*)

(*a function calculating an integral of the form fPre Exp[fExp],
where fPre is a polynomial and fExp a 2nd order polynomial. Taken
  from https://mathematica.stackexchange.com/a/6846.*)
gaussMoment[fPre_, fExp_, vars_] := Module[{coeff, dist, ai, μ, norm},
  coeff = CoefficientArrays[fExp, vars, "Symmetric" → True];
  ai = Inverse[2 coeff⟦3⟧];
  μ = -ai.coeff⟦2⟧;
  dist = MultinormalDistribution[μ, -ai];
  norm = 1 / PDF[dist, vars] /. Thread[vars → μ];
  norm Exp[1 / 2 coeff⟦2⟧.μ + coeff⟦1⟧] ×
    Distribute@Expectation[fPre, vars ≈ dist]]

(*functions replacing coefficients of given polynomials with
 single symbols. This makes mathematica work much faster.*)
myMatrix[symb_, d1_, d2_, d3_, d4_] :=
  Table[Subscript[symb, i1, i2, i3, i4], {i1, d1}, {i2, d2}, {i3, d3}, {i4, d4}];
symbolize[poly_, vars2_, symb_] :=
```

```
  Module[{lst, flst, symbolized, dims, symbCoeffs},
   lst = CoefficientList[poly, vars2];
   dims = Dimensions[lst];
   symbCoeffs = myMatrix[symb, dims[[1]], dims[[2]], dims[[3]], dims[[4]]];
   Do[
    If[lst[[i1, i2, i3, i4]] == 0, symbCoeffs[[i1, i2, i3, i4]] = 0];
    , {i1, 1, dims[[1]]}, {i2, 1, dims[[2]]}, {i3, 1, dims[[3]]}, {i4, 1, dims[[4]]}];

   symbolized = Internal`FromCoefficientList[symbCoeffs, vars2];
   symbolized
  ];

(*a function wrapping gaussMoment,
replacing coefficients with symbols before integrating.*)
myGauss[fPre_, fExp_, vars1_] :=
 Module[{a0exp, a0pre, aexp, apre, sympre, symexp, res},
  a0exp = CoefficientList[fExp, vars];
  symexp = symbolize[fExp, vars, aexp];
  a0pre = CoefficientList[fPre, vars];
  sympre = symbolize[fPre, vars, apre];
  res = gaussMoment[ sympre , symexp, vars1] /.
    {aexp → a0exp, apre → a0pre, Subscript → Part};
  res
 ]

(*operations on complex numbers done in a custom way
 which for some reason works faster, at least in this code.*)
myRe[var_] := Module[{var1, var1C, re, im},
  var1 = var // ComplexExpand // Evaluate;
  var1C = var1 // Conjugate // Refine;
  re = (var1 + var1C) / 2 // Simplify;
  re
 ]

myIm[var_] := Module[{var1, var1C, re, im},
  var1 = var // ComplexExpand // Evaluate;
  var1C = var1 // Conjugate // Refine;
  im = -ⅈ (var1 - var1C) / 2 // Simplify;
  im
 ]

myConj[var_] := Module[{var1, var1C, re, im},
  var1 = var // ComplexExpand // Evaluate;
  var1C = var1 // Conjugate // Refine;
  var1C
 ]
```

```
(*the series expansion to second order in ϵ and
 second order in Ω. The second function is for quantities
 which will be taken with a square root later.*)
mySeries[expr_] := Series[ expr, {Ω, 0, 2}, {ϵ, 0, 2}] // Normal;
mySeriesSq[expr_] := Series[expr, {Ω, 0, 4}, {ϵ, 0, 4}] // Normal;
```

Find the retarded time to relevant order

Expand the definition of the retarded time t_{ab} to second order in $\Omega = \omega d/c$ and solve:

```
dtNequals = 1 + (α / d ) q1[ti] + (α / d ) q2[ti - dtN d / c] // Simplify;
(*the implicit expression of the retarded time difference from the paragraph
 after Eq.(24). We work with normalized time, in units of d/c.*)
dtNequalsSer = dtNequals // mySeries // Expand (*expand the expression*)
lst = Roots[dtNequalsSer - dtN == 0 , dtN] ;
(*solve the polynomial equation obtained by equating the
 retarded time difference to the expanded implicit expression.*)
If[Length[lst⟦1⟧] < 2, lst = List[lst]];

(*loop to find the right solution dtNsol,
i.e. positive and 1 in the non-relativistic limit*)
For[i = 1, i ≤ Length[lst], i++,
  Print["checking solution number ", i];
  this = lst⟦i⟧ /. Equal → List;
  sol = this⟦2⟧ // Simplify;
  Print[sol];
  ldng = Series[sol, ϵ → 0, Ω → 0] // Normal // Simplify;
  If[Not[ldng == 1], Continue[]];
  If[Not[sol > 0], Continue[]];
  Print["Found"];
  dtNsol = sol;
  Break[];
 ];
```

Out[ ]=

$$1 + \epsilon\, q1[ti] + \epsilon\, q2[ti] - \frac{dtN\, \epsilon\, \Omega\, q2'[ti]}{\omega} + \frac{dtN^2\, \epsilon\, \Omega^2\, q2''[ti]}{2\, \omega^2}$$

checking solution number 1

$$\frac{\omega \left(\omega + \epsilon\, \Omega\, q2'[ti] - \sqrt{(\omega + \epsilon\, \Omega\, q2'[ti])^2 - 2\, \epsilon\, \Omega^2\, (1 + \epsilon\, q1[ti] + \epsilon\, q2[ti])\, q2''[ti]}\right)}{\epsilon\, \Omega^2\, q2''[ti]}$$

Found

Action

The action of the gravitational interaction from Eq. (21): SGhb = $S_G / \hbar$. Below the retarded quantities in it are defined. ret and kin are bookkeeping variables to keep track of contributions from the retardation part and kinetic part of the GR correction.

In[∘]:=
```
tab = ti - ret dtNsol d / c; (*the retarded time*)
dab = d dtNsol; (*the retarded distance*)
γ[v_] := (1 - v^2 / c^2) ^ (-1 / 2) (*Lorentz factor*)
VV[ti_, xa_, xaDot_, xb_, xbDot_, a_, b_] :=
 c^4 γ[α xaDot[tab]] × γ[α xbDot[ti]] ( (1 - α xaDot[tab] α xbDot[ti] / c^2) ^2 -
    1/2 (1 - α^2 xaDot[tab] ^2 / c^2) (1 - α^2 xbDot[ti] ^2 / c^2) )
LG12 =
  (G m^2 ((Series[VV[ti, q1, q1', q2, q2', a, b] / c^4, Ω → 0] // Normal) (1 - kin) +
        kin VV[ti, q1, q1', q2, q2', a, b] / c^4)) /
     ((Series[dab - dab * (-α q1'[tab]) / c, Ω → 0] // Normal) (1 - ret) +
       ret (dab - dab * (-α q1'[tab]) / c))   / ħ // Simplify;
LGhb = (LG12 + (LG12 /. {q1 → q2, q2 → q1}) ) // mySeries //
  Simplify (*the lagrangian of the on-shell gravitational action,
  we sum over the potential induced by 1 on 2 and 2 on 1.*)
```

Out[∘]=

$$\phi\,\omega-\epsilon\,\phi\,\omega\,(q1[ti]+q2[ti])+\epsilon^2\,\phi\,\omega\,(q1[ti]+q2[ti])^2+\frac{1}{4\,\omega}\epsilon\,\phi\,\Omega^2\Big(2\,(3\,kin+ret\,(-1+2\,ret))\,\epsilon\,q1'[ti]^2-4\,(4\,kin+ret\,(-1+2\,ret))\,\epsilon\,q1'[ti]\,q2'[ti]+2\,(3\,kin+ret\,(-1+2\,ret))\,\epsilon\,q2'[ti]^2+ret\,(-1+2\,ret)\,(q1''[ti]+q2''[ti])\Big)$$

the full gravity lagrangian for kin=ret=1:

In[∘]:=
```
LGhb /. {kin → 1, ret → 1} // Simplify
```

Out[∘]=

$$\phi\,\omega-\epsilon\,\phi\,\omega\,(q1[ti]+q2[ti])+\epsilon^2\,\phi\,\omega\,(q1[ti]+q2[ti])^2+\frac{\epsilon\,\phi\,\Omega^2\,\left(8\,\epsilon\,q1'[ti]^2-20\,\epsilon\,q1'[ti]\,q2'[ti]+8\,\epsilon\,q2'[ti]^2+q1''[ti]+q2''[ti]\right)}{4\,\omega}$$

The free, uncoupled action: Sunc = $\Sigma_{a=1,2}\,S_a\,/\,\hbar$, its lagrangian:

In[∘]:=
```
Lunc = ( -m c^2 / γ[α q1'[ti]] - m c^2 / γ[α q2'[ti]] ) / ħ // mySeries // Simplify
```

Out[∘]=

$$-\frac{2\,\omega}{\epsilon^2\,\Omega^2}+\frac{q1'[ti]^2+q2'[ti]^2}{2\,\omega}+\frac{\epsilon^2\,\Omega^2\,\left(q1'[ti]^4+q2'[ti]^4\right)}{8\,\omega^3}$$

Classical Solutions

We find the classical trajectories of the particles for initial locations y1p, y2p = y'_1/α, y'_2/α and final locations y1, y2 = y_1/α, y_2/α. Here, the y's are in units of α. We do it by solving the system under the second order approximation in ϵ and in Ω. The problem becomes two harmonic oscillators with some frequency trp×ω, which are coupled by another spring, which is gravity. The variable trp is just a bookkeeping variable which will be taken to zero to achieve the trajectories of the two otherwise free particles coupled by gravity.

In[∘]:=

```
L[ti] := (LGhb + Lunc) ;
L[ti] // Simplify
```

Out[∘]=

$$\phi\,\omega - \frac{2\,\omega}{\epsilon^2\,\Omega^2} - \epsilon\,\phi\,\omega\,(\text{q1}[\text{ti}] + \text{q2}[\text{ti}]) + \epsilon^2\,\phi\,\omega\,(\text{q1}[\text{ti}] + \text{q2}[\text{ti}])^2 + \frac{\text{q1}'[\text{ti}]^2 + \text{q2}'[\text{ti}]^2}{2\,\omega} + \frac{\epsilon^2\,\Omega^2\left(\text{q1}'[\text{ti}]^4 + \text{q2}'[\text{ti}]^4\right)}{8\,\omega^3} + \frac{1}{4\,\omega}\,\epsilon\,\phi\,\Omega^2 \left(2\,(3\,\text{kin} + \text{ret}\,(-1 + 2\,\text{ret}))\,\epsilon\,\text{q1}'[\text{ti}]^2 - 4\,(4\,\text{kin} + \text{ret}\,(-1 + 2\,\text{ret}))\,\epsilon\,\text{q1}'[\text{ti}]\,\text{q2}'[\text{ti}] + 2\,(3\,\text{kin} + \text{ret}\,(-1 + 2\,\text{ret}))\,\epsilon\,\text{q2}'[\text{ti}]^2 + \text{ret}\,(-1 + 2\,\text{ret})\,(\text{q1}''[\text{ti}] + \text{q2}''[\text{ti}])\right)$$

In[∘]:=

```
eqs = EulerEquations[L[ti], {q1[ti], q2[ti]}, ti] // Expand;
(*the classical equations of motion*)
Collect[eqs, Ω]
```

Out[∘]=

$$\Big\{-\epsilon\,\phi\,\omega + 2\,\epsilon^2\,\phi\,\omega\,\text{q1}[\text{ti}] + 2\,\epsilon^2\,\phi\,\omega\,\text{q2}[\text{ti}] - \frac{\text{q1}''[\text{ti}]}{\omega} + \Omega^2\left(-\frac{3\,\text{kin}\,\epsilon^2\,\phi\,\text{q1}''[\text{ti}]}{\omega} + \frac{\text{ret}\,\epsilon^2\,\phi\,\text{q1}''[\text{ti}]}{\omega} - \frac{2\,\text{ret}^2\,\epsilon^2\,\phi\,\text{q1}''[\text{ti}]}{\omega} - \frac{3\,\epsilon^2\,\text{q1}'[\text{ti}]^2\,\text{q1}''[\text{ti}]}{2\,\omega^3} + \frac{4\,\text{kin}\,\epsilon^2\,\phi\,\text{q2}''[\text{ti}]}{\omega} - \frac{\text{ret}\,\epsilon^2\,\phi\,\text{q2}''[\text{ti}]}{\omega} + \frac{2\,\text{ret}^2\,\epsilon^2\,\phi\,\text{q2}''[\text{ti}]}{\omega}\right) == 0,$$

$$-\epsilon\,\phi\,\omega + 2\,\epsilon^2\,\phi\,\omega\,\text{q1}[\text{ti}] + 2\,\epsilon^2\,\phi\,\omega\,\text{q2}[\text{ti}] - \frac{\text{q2}''[\text{ti}]}{\omega} + \Omega^2\left(\frac{4\,\text{kin}\,\epsilon^2\,\phi\,\text{q1}''[\text{ti}]}{\omega} - \frac{\text{ret}\,\epsilon^2\,\phi\,\text{q1}''[\text{ti}]}{\omega} + \frac{2\,\text{ret}^2\,\epsilon^2\,\phi\,\text{q1}''[\text{ti}]}{\omega} - \frac{3\,\text{kin}\,\epsilon^2\,\phi\,\text{q2}''[\text{ti}]}{\omega} + \frac{\text{ret}\,\epsilon^2\,\phi\,\text{q2}''[\text{ti}]}{\omega} - \frac{2\,\text{ret}^2\,\epsilon^2\,\phi\,\text{q2}''[\text{ti}]}{\omega} - \frac{3\,\epsilon^2\,\text{q2}'[\text{ti}]^2\,\text{q2}''[\text{ti}]}{2\,\omega^3}\right) == 0\Big\}$$

we solve the non-relativistic equations and then find the correction

In[ ]:=
```
eqsNonrel =
  Collect[eqs // Series[#, {Ω, 0, 0}] & // Normal, Ω] /. {q1 → q1nr, q2 → q2nr}
(*the non-relativistic equations*)
solNonrel =
   DSolve[{eqsNonrel[[1]], eqsNonrel[[2]], q1nr[0] == y1p, q2nr[0] == y2p, q1nr[t] == y1,
       q2nr[t] == y2}, {q1nr[ti], q2nr[ti]}, {ti}] /.
     ti → tii // Simplify; (*their solution*)
g1[ti_] := (q1nr[tii] /. solNonrel[[1]] /. tii → ti) + Ω c1[ti];
(*the solution for the relativistic equations to the relevant
   order would be the non-relativistic solution plus a correction*)
g2[ti_] := (q2nr[tii] /. solNonrel[[1]] /. tii → ti) + Ω c2[ti];
eqs2 = eqs /. {q1 → g1, q2 → g2} // mySeries // Refine // Simplify
(*equation for the corrections*)
solCor = DSolve[{eqs2[[1]], eqs2[[2]], c1[0] == 0, c2[0] == 0, c1[t] == 0, c2[t] == 0},
   {c1[ti], c2[ti]}, {ti}] (*their solution*)
```

Out[ ]=

$$\left\{-\epsilon\,\phi\,\omega+2\,\epsilon^2\,\phi\,\omega\,\text{q1nr}[\text{ti}]+2\,\epsilon^2\,\phi\,\omega\,\text{q2nr}[\text{ti}]-\frac{\text{q1nr}''[\text{ti}]}{\omega}==0,\right.$$
$$\left.-\epsilon\,\phi\,\omega+2\,\epsilon^2\,\phi\,\omega\,\text{q1nr}[\text{ti}]+2\,\epsilon^2\,\phi\,\omega\,\text{q2nr}[\text{ti}]-\frac{\text{q2nr}''[\text{ti}]}{\omega}==0\right\}$$

Out[ ]=

$$\left\{2\,\epsilon^2\,\phi\,\omega^2\,(\text{c1}[\text{ti}]+\text{c2}[\text{ti}])==\text{c1}''[\text{ti}],\ 2\,\epsilon^2\,\phi\,\omega^2\,(\text{c1}[\text{ti}]+\text{c2}[\text{ti}])==\text{c2}''[\text{ti}]\right\}$$

Out[ ]=

$$\{\{\text{c1}[\text{ti}]\to 0,\ \text{c2}[\text{ti}]\to 0\}\}$$

turns out that at this order there is no correction to the classical solutions.

In[ ]:=
```
solCor = solCor /. ti → tii // Simplify;
x1tii = g1[tii] /. solCor[[1]] // Simplify;
x2tii = g2[tii] /. solCor[[1]] // Simplify;
x1[ti_] := x1tii /. tii → ti ;
x2[ti_] := x2tii /. tii → ti ;
```

Sanity check: initial and final locations are actually the y’s, the trajectories actually solve the equations:

In[ ]:=
```
eqs /. {q1 → x1, q2 → x2} // mySeries // Simplify
x1[0] // mySeries // Simplify
x2[0] // mySeries // Simplify
x1[t] // mySeries // Simplify
x2[t] // mySeries // Simplify
```

Out[ ]=
```
{True, True}
```

Out[ ]=
```
y1p
```

Out[ ]=
```
y2p
```

Out[ ]=
```
y1
```

Out[ ]=
```
y2
```

Wavefunction

We would now like to compute the wavefunction $\tilde{\psi}^{\text{fi}}(y1, y2)$ as given in Eq. (9). We start with achieving the expression for the on-shell (= stationary-phase approximated) action $\tilde{S}_a + \tilde{S}_G$, here Shb.

```
In[◦]:= Lhb = LGhb + Lunc /. {q1 → x1, q2 → x2} // mySeries // Simplify;
Shb1 = Integrate[Lhb, {ti, 0, t}] ;
Shb = Shb1 - (Shb1 /. {y1 → 0, y2 → 0, y1p → 0, y2p → 0}) // Expand // Refine //
   Simplify // Collect[#, {y1, y2, y1p, y2p}] &
```

Out[◦]=

$$-\frac{1}{2} t\, y2p \in \phi\, \omega + \frac{y1^4 \in^2 \Omega^2}{8 t^3 \omega^3} - \frac{y1^3\, y1p \in^2 \Omega^2}{2 t^3 \omega^3} + \frac{y1p^4 \in^2 \Omega^2}{8 t^3 \omega^3} + \frac{y2^4 \in^2 \Omega^2}{8 t^3 \omega^3} - \frac{y2^3\, y2p \in^2 \Omega^2}{2 t^3 \omega^3} +$$
$$\frac{y2p^4 \in^2 \Omega^2}{8 t^3 \omega^3} + y1p^2 \left(\frac{1}{3} t \in^2 \phi\, \omega + \frac{1 + 3\, kin \in^2 \phi\, \Omega^2 - ret \in^2 \phi\, \Omega^2 + 2\, ret^2 \in^2 \phi\, \Omega^2}{2 t\, \omega}\right) +$$
$$y2p^2 \left(\frac{1}{3} t \in^2 \phi\, \omega + \frac{1 + 3\, kin \in^2 \phi\, \Omega^2 - ret \in^2 \phi\, \Omega^2 + 2\, ret^2 \in^2 \phi\, \Omega^2}{2 t\, \omega}\right) +$$
$$y1^2 \left(\frac{1}{3} t \in^2 \phi\, \omega + \frac{3\, y1p^2 \in^2 \Omega^2}{4 t^3 \omega^3} + \frac{1 + 3\, kin \in^2 \phi\, \Omega^2 - ret \in^2 \phi\, \Omega^2 + 2\, ret^2 \in^2 \phi\, \Omega^2}{2 t\, \omega}\right) +$$
$$y2^2 \left(\frac{1}{3} t \in^2 \phi\, \omega + \frac{3\, y2p^2 \in^2 \Omega^2}{4 t^3 \omega^3} + \frac{1 + 3\, kin \in^2 \phi\, \Omega^2 - ret \in^2 \phi\, \Omega^2 + 2\, ret^2 \in^2 \phi\, \Omega^2}{2 t\, \omega}\right) +$$
$$y1p \left(-\frac{1}{2} t \in \phi\, \omega + y2p \left(\frac{2}{3} t \in^2 \phi\, \omega + \frac{(-4\, kin - ret\, (-1 + 2\, ret)) \in^2 \phi\, \Omega^2}{t\, \omega}\right)\right) +$$
$$y1 \left(-\frac{1}{2} t \in \phi\, \omega - \frac{y1p^3 \in^2 \Omega^2}{2 t^3 \omega^3} + y2 \left(\frac{2}{3} t \in^2 \phi\, \omega + \frac{(-4\, kin - ret\, (-1 + 2\, ret)) \in^2 \phi\, \Omega^2}{t\, \omega}\right) +\right.$$
$$y2p \left(\frac{1}{3} t \in^2 \phi\, \omega + \frac{(4\, kin + ret\, (-1 + 2\, ret)) \in^2 \phi\, \Omega^2}{t\, \omega}\right) +$$
$$\left. y1p \left(\frac{1}{3} t \in^2 \phi\, \omega - \frac{1 + 3\, kin \in^2 \phi\, \Omega^2 - ret \in^2 \phi\, \Omega^2 + 2\, ret^2 \in^2 \phi\, \Omega^2}{t\, \omega}\right)\right) +$$
$$y2 \left(-\frac{1}{2} t \in \phi\, \omega - \frac{y2p^3 \in^2 \Omega^2}{2 t^3 \omega^3} + y1p \left(\frac{1}{3} t \in^2 \phi\, \omega + \frac{(4\, kin + ret\, (-1 + 2\, ret)) \in^2 \phi\, \Omega^2}{t\, \omega}\right) +\right.$$
$$\left. y2p \left(\frac{1}{3} t \in^2 \phi\, \omega - \frac{1 + 3\, kin \in^2 \phi\, \Omega^2 - ret \in^2 \phi\, \Omega^2 + 2\, ret^2 \in^2 \phi\, \Omega^2}{t\, \omega}\right)\right)$$

We now solve the y1p and y2p integrals while also performing the approximations. We denote for any function f the parts fExp and fPre such that f(x) = fPre(x) × $e^{\text{fExp}(x)}$.
The initial state wfIn is a gaussian.

```
vars = {y1, y2, y1p, y2p};
wfIn = Exp[ - (y1p) ^2 / 2 - (y2p) ^2 / 2];

wf4vars = wfIn Exp[ⅈ Shb];(*the integrand in Eq. (9).*)
wf4vars = wf4vars // mySeries;
wf4varsExp = Exponent[wf4vars, ⅇ];
wf4varsPre = wf4vars Exp[-wf4varsExp] ;
wf = myGauss[ wf4varsPre , wf4varsExp, {y1p, y2p}] // Simplify;
(*This is ψ~ fi(y1, y2)*)
wfExp = Exponent[wf, ⅇ] // Simplify
wfPre = wf Exp[-wfExp]
wfExpC = wfExp // myConj // Simplify;(*complex conjugate*)
wfPreC = wfPre // myConj // Simplify;
```

Out[∘]=

$$\frac{i\left(y1^2+y2^2\right)}{-2\,i+2\,t\,\omega}$$

Out[∘]=

$$
\begin{aligned}
&-\frac{1}{12\,(-i+t\,\omega)^5}\\
&\pi\Big(-t^5\left(24-60\,y2\,\epsilon\,\phi+6\,\epsilon^2\,\phi\,(8+3\,\phi)+y1^2\,\epsilon^2\,\phi\,(40+39\,\phi)+y2^2\,\epsilon^2\,\phi\,(40+39\,\phi)+\right.\\
&\qquad\left.2\,y1\,\epsilon\,\phi\,(-30+y2\,\epsilon\,(40+39\,\phi))\right)\omega^5-2\,i\,t^6\,\epsilon\,\phi\\
&\quad\left(-6\,y2+y1^2\,\epsilon\,(4+9\,\phi)+y2^2\,\epsilon\,(4+9\,\phi)+\epsilon\,(8+9\,\phi)+2\,y1\,(-3+y2\,\epsilon\,(4+9\,\phi))\right)\omega^6+\\
&\quad 3\,t^7\left(2+y1^2+2\,y1\,y2+y2^2\right)\epsilon^2\,\phi^2\,\omega^7+18\,i\,\epsilon^2\,\Omega^2-\\
&\quad 6\,t\,\omega\left(4+\epsilon^2\left(6+3\,y1^2+3\,y2^2-12\,\mathrm{kin}\,\phi+4\,\mathrm{ret}\,\phi-8\,\mathrm{ret}^2\,\phi\right)\Omega^2\right)-\\
&\quad 3\,i\,t^2\,\omega^2\Big(32-8\,y2\,\epsilon\,\phi+6\,\epsilon^2\,\Omega^2+y1^4\,\epsilon^2\,\Omega^2+y2^4\,\epsilon^2\,\Omega^2-72\,\mathrm{kin}\,\epsilon^2\,\phi\,\Omega^2+24\,\mathrm{ret}\,\epsilon^2\,\phi\,\Omega^2-\\
&\qquad 48\,\mathrm{ret}^2\,\epsilon^2\,\phi\,\Omega^2+2\,y1^2\,\epsilon^2\left(3\,\Omega^2+\phi\left(4-6\,\mathrm{kin}\,\Omega^2+2\,\mathrm{ret}\,\Omega^2-4\,\mathrm{ret}^2\,\Omega^2\right)\right)+\\
&\qquad 2\,y2^2\,\epsilon^2\left(3\,\Omega^2+\phi\left(4-6\,\mathrm{kin}\,\Omega^2+2\,\mathrm{ret}\,\Omega^2-4\,\mathrm{ret}^2\,\Omega^2\right)\right)+\\
&\qquad 8\,y1\,\epsilon\,\phi\left(-1+y2\,\epsilon\left(2+4\,\mathrm{kin}\,\Omega^2-\mathrm{ret}\,\Omega^2+2\,\mathrm{ret}^2\,\Omega^2\right)\right)\Big)+\\
&\quad 4\,t^3\,\omega^3\Big(36-21\,y2\,\epsilon\,\phi+4\,\epsilon^2\,\phi-54\,\mathrm{kin}\,\epsilon^2\,\phi\,\Omega^2+18\,\mathrm{ret}\,\epsilon^2\,\phi\,\Omega^2-\\
&\qquad 36\,\mathrm{ret}^2\,\epsilon^2\,\phi\,\Omega^2+3\,y1^2\,\epsilon^2\,\phi\left(6+\phi-6\,\mathrm{kin}\,\Omega^2+2\,\mathrm{ret}\,\Omega^2-4\,\mathrm{ret}^2\,\Omega^2\right)+\\
&\qquad 3\,y2^2\,\epsilon^2\,\phi\left(6+\phi-6\,\mathrm{kin}\,\Omega^2+2\,\mathrm{ret}\,\Omega^2-4\,\mathrm{ret}^2\,\Omega^2\right)+\\
&\qquad 3\,y1\,\epsilon\,\phi\left(-7+2\,y2\,\epsilon\left(6+\phi+8\,\mathrm{kin}\,\Omega^2-2\,\mathrm{ret}\,\Omega^2+4\,\mathrm{ret}^2\,\Omega^2\right)\right)\Big)+\\
&\quad 2\,i\,t^4\,\omega^4\Big(48-54\,y2\,\epsilon\,\phi+3\,\epsilon^2\,\phi\left(8+\phi-12\,\mathrm{kin}\,\Omega^2+4\,\mathrm{ret}\,\Omega^2-8\,\mathrm{ret}^2\,\Omega^2\right)+\\
&\qquad 2\,y1^2\,\epsilon^2\,\phi\left(20+9\,\phi-9\,\mathrm{kin}\,\Omega^2+3\,\mathrm{ret}\,\Omega^2-6\,\mathrm{ret}^2\,\Omega^2\right)+\\
&\qquad 2\,y2^2\,\epsilon^2\,\phi\left(20+9\,\phi-9\,\mathrm{kin}\,\Omega^2+3\,\mathrm{ret}\,\Omega^2-6\,\mathrm{ret}^2\,\Omega^2\right)+\\
&\qquad 2\,y1\,\epsilon\,\phi\left(-27+2\,y2\,\epsilon\left(20+9\,\phi+12\,\mathrm{kin}\,\Omega^2-3\,\mathrm{ret}\,\Omega^2+6\,\mathrm{ret}^2\,\Omega^2\right)\right)\Big)\Big)
\end{aligned}
$$

calculate the norm squared of the state:

In[∘]:=

```
preee = wfPreC wfPre // mySeriesSq // Simplify;
normsq = myGauss[preee, wfExpC + wfExp, {y1, y2}];
normsq = normsq // Simplify;
```

Generalized Entanglement Measure

we now calculate the entanglement measure in Eq. (28) after replacing the norm square in the formula by multiplication with the conjugate:

$$|\int\psi\,(y1,\,y2)\,\psi^*\,(y'1,\,y2)\,dy2|^2 = \int\psi^*\,(y1,\,y2)\,\psi\,(y'1,\,y2)\,\psi\,(y1,\,y'2)\,\psi^*\,(y'1,\,y'2)\,dy2\,dy'2$$

In[ ]:=
```
(*each of the four multipicants is slightly
 different in where is the ' and whether it is conjugate*)
wfPre1p = wfPre /. y1 → y1p;
wfExp1p = wfExp /. y1 → y1p;
wfPre2p = wfPre /. y2 → y2p;
wfExp2p = wfExp /. y2 → y2p;
wfPreC12p = wfPreC /. {y1 → y1p, y2 → y2p};
wfExpC12p = wfExpC /. {y1 → y1p, y2 → y2p};

(*again, preform the integral after getting
  to the form myGauss accepts: f(x)=fPre(x)×e^fExp(x)*)
thisPre = wfPre1p * wfPreC * wfPreC12p * wfPre2p // mySeriesSq;
Print["pre series"];
thisExp = wfExp1p + wfExpC + wfExpC12p + wfExp2p // mySeriesSq // FullSimplify;
Print["exp simp"];
int = myGauss[thisPre, thisExp, {y1, y2, y1p, y2p}];
Print["int"];
int = int // Simplify ;
Print["int simp"];
int = int // mySeriesSq;
Print["int ser"];
```

```
pre series
exp simp
int
int simp
int ser
```

finally, the generalized entanglement measure from Eq. (28) is calculated to obtain:

In[ ]:=
```
gem = 2 (1 - int / normsq^2 ) // Sqrt // mySeries // FullSimplify;
gem = gem // Collect[#, {Ω, kin}] &
```

Out[ ]=

$$\frac{2}{3}\,t\,\epsilon^2\,\phi\,\omega\,\sqrt{9+3\,t^2\,\omega^2+t^4\,\omega^4}+\left(-\frac{4\,\text{kin}\,t\,\epsilon^2\,\phi\,\omega\,\left(-3+t^2\,\omega^2\right)}{\sqrt{9+3\,t^2\,\omega^2+t^4\,\omega^4}}-\frac{\text{ret}\,(-1+2\,\text{ret})\,t\,\epsilon^2\,\phi\,\omega\,\left(-3+t^2\,\omega^2\right)}{\sqrt{9+3\,t^2\,\omega^2+t^4\,\omega^4}}\right)\Omega^2$$

the different contributions, see Supplementary Material:

In[ ]:=
```
gem /. {kin → 1, ret → 0}
  (*with correction due to gravitation of kinetic energy only*)
```

Out[ ]=

$$\frac{2}{3}\, t\, \epsilon^2\, \phi\, \omega\, \sqrt{9 + 3\, t^2\, \omega^2 + t^4\, \omega^4} - \frac{4\, t\, \epsilon^2\, \phi\, \omega\, \left(-3 + t^2\, \omega^2\right)\, \Omega^2}{\sqrt{9 + 3\, t^2\, \omega^2 + t^4\, \omega^4}}$$

In[ ]:=
```
gem /. {kin → 0, ret → 1} (*with correction due to retardation only*)
```

Out[ ]=

$$\frac{2}{3}\, t\, \epsilon^2\, \phi\, \omega\, \sqrt{9 + 3\, t^2\, \omega^2 + t^4\, \omega^4} - \frac{t\, \epsilon^2\, \phi\, \omega\, \left(-3 + t^2\, \omega^2\right)\, \Omega^2}{\sqrt{9 + 3\, t^2\, \omega^2 + t^4\, \omega^4}}$$

In[ ]:=
```
gem /. {kin → 1, ret → 1}
  (*with both corrections: the first GR correction to the entanglement*)
```

Out[ ]=

$$\frac{2}{3}\, t\, \epsilon^2\, \phi\, \omega\, \sqrt{9 + 3\, t^2\, \omega^2 + t^4\, \omega^4} - \frac{5\, t\, \epsilon^2\, \phi\, \omega\, \left(-3 + t^2\, \omega^2\right)\, \Omega^2}{\sqrt{9 + 3\, t^2\, \omega^2 + t^4\, \omega^4}}$$