Quantum Mechanics II Cheat Sheet

Ritz Theorem

$R[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$. **Step 1: Expand $|\psi\rangle$ in energy eigenbasis**. $|\psi\rangle = \sum_n c_n|n\rangle$, $c_n = \langle n|\psi\rangle$. Then $\langle\psi|\psi\rangle = \sum_n |c_n|^2$. Using $H|n\rangle = E_n|n\rangle$, $\langle\psi|H|\psi\rangle = \sum_{n,m} c_m^* c_n \langle m|H|n\rangle = \sum_n |c_n|^2 E_n$. So $R[\psi] = \frac{\sum_n |c_n|^2 E_n}{\sum_n |c_n|^2}$. **Step 2: Prove $R[\psi] \geq E_0$**. Since $E_n \geq E_0$, $\sum_n |c_n|^2 E_n \geq E_0 \sum_n |c_n|^2$. $R[\psi] \geq E_0$.

Thm 3.3.2 Schur's Lemma for Hermitian Ops

Let T be an IRREP of a finite group G on V. If Hermitian operator $C : V \to V$ implies $\forall g \in G, T(g)C = CT(g)$, then $C \propto I$, i.e., $C = \lambda I, \lambda \in \mathbb{C}$. *Proof:* C Hermitian $\implies \exists$ eigenstate $v \in V$ s.t. $Cv = \lambda v$. Subspace *span*$\{T(g)v\}$ is invariant. Since T is IRREP, must be V. $\forall w \in V$, $Cw = C\sum a_g T(g)v = \sum a_g T(g)Cv = \lambda w$. Thus $C = \lambda I$.

Thm 4.2.1 Spectrum of Hamiltonian

$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{a}{r^s}$ with $a > 0, s > 2$ is unbounded from below. *Proof:* Scaled state $\Psi(\mathbf{r}) = Ne^{-r^2/r_0^2}$. Expectation values: $\langle T\rangle \sim \frac{\hbar^2}{2mr_0^2}, \langle V\rangle \sim -\frac{a}{r_0^s}$. $E(r_0) \approx \frac{\alpha}{r_0^2} - \frac{\beta}{r_0^s}$. For $s > 2$, as $r_0 \to 0$, $E \to -\infty$.

Thm 3.1.1 Operators equal iff expectations equal

$\langle a|A|a\rangle = \langle a|B|a\rangle \forall |a\rangle \iff A = B$. *Proof:* Use $|\psi\rangle = |a\rangle + |b\rangle$ and $|\psi\rangle = |a\rangle + i|b\rangle$ to show $\langle a|A|b\rangle = \langle a|B|b\rangle$.

Thm 4.1.1 Ritz Theorem

$E = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} \geq E_1$. Equality iff Ψ is ground state.

Thm 4.1.2 Generalized Ritz

Expectation value of H is stationary in neighborhood of eigenvalues. $\delta E(\Psi) = 0 \iff H\Psi = E\Psi$.

Thm 4.1.3 Variance Theorem

$\sigma^2 = \frac{\langle\Psi|(H-E)^2|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \langle H^2\rangle - E^2$. There is at least one eigenval in $[E - \sigma, E + \sigma]$.

Thm 5.4.1 Upper bound on j_0

Given Hilbert space with basis $\{|ab\rangle\}$ of J^2, J_0 with eigenvalues a, b. If $a \geq b^2$. *Proof:* $J^2 = J_0^2 + \frac{1}{2}(J_-J_+ + J_+J_-) \implies a - b^2 \geq 0$.

Thm 3.1.2 Ops equal within phase

$A = e^{i\theta}B \iff |\langle a|A|b\rangle| = |\langle a|B|b\rangle|$. *Proof* "$\Leftarrow$": $A|b_j\rangle = e^{i\theta_j}B|b_j\rangle$. Apply to $|b_1\rangle + |b_2\rangle$. $A(|b_1\rangle + |b_2\rangle) = e^{i\theta_{12}}B(|b_1\rangle + |b_2\rangle) = e^{i\theta_1}B|b_1\rangle + e^{i\theta_2}B|b_2\rangle$. Linearity $\implies B(e^{i\theta_{12}} - e^{i\theta_1})|b_1\rangle + B(e^{i\theta_{12}} - e^{i\theta_2})|b_2\rangle = 0$. Inner product with $B|b_i\rangle \implies e^{i\theta_1} = e^{i\theta_2}$. Phase is global.

Thm 3.1.3 Scalar Product Preserving

If $T : \mathcal{V} \to \mathcal{V}$ preserves scalar product magnitude $|\langle\phi|\psi\rangle| = |\langle T\phi|T\psi\rangle|$, then T is unitary or anti-unitary. (Wigner's Theorem).

Thm 4.2.2 For $s < 2$ spectrum of H

$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{a}{r^s}$ $(a > 0)$ contains infinite bound states. *Proof:* Trial $\Psi(r) = Ne^{-(r-r_0)^2/\beta^2 r_0^2}$. For large r_0, $E < 0$ is possible.

Thm 8.2.2 Triangular Rule

Admissible j are $|j_1 - j_2| \leq j \leq j_1 + j_2$.

Thm 12.3.1 Ops S and A

Operators $\mathcal{S}$ and $\mathcal{A}$ satisfy: (a) $\mathcal{S}^\dagger = \mathcal{S}, \mathcal{A}^\dagger = \mathcal{A}$. (b) Commute with P_g for all $g \in S_N$. $P_g\mathcal{S} = \mathcal{S}P_g = \mathcal{S}$, $P_g\mathcal{A} = \mathcal{A}P_g = \text{sign}(g)\mathcal{A}$. (c) Orthogonal projectors of $\mathcal{H}_{so}$. $\mathcal{S}^2 = \mathcal{S}, \mathcal{A}^2 = \mathcal{A}, \mathcal{S}\mathcal{A} = \mathcal{A}\mathcal{S} = 0$. *Proof:* $\mathcal{S} = \frac{1}{N!}\sum_g P_g$. $\mathcal{A} = \frac{1}{N!}\sum_g \text{sign}(g)P_g$. $P_g\mathcal{A} = \text{sign}(g)\mathcal{A}$.

Landau Levels Derivation (Handwritten)

$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2$. $B = B\hat{z}$. Gauge $\mathbf{A} = (-By, 0, 0)$. $H = \frac{1}{2m}((p_x + qBy)^2 + p_y^2 + p_z^2)$. $[p_x, H] = 0 \implies p_x \to \hbar k$. $H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_c^2(y - y_0)^2 + \frac{\hbar^2 k_z^2}{2m}$. Harmonic oscillator centered at $y_0 = -\frac{\hbar k}{qB}$. $\omega_c = \frac{qB}{m}$. $E = \hbar\omega_c(n + \frac{1}{2}) + \frac{\hbar^2 k_z^2}{2m}$.

Thm 13.9.1 Optical Theorem

$\sigma_{tot}(k) = \frac{4\pi}{k}\text{Im}f_k(0)$. *Proof:* $f_k(\theta) = f(k, k') = -\frac{m}{2\pi\hbar^2}(2\pi\hbar)^3\langle\mathbf{k}|T|\mathbf{k}\rangle$. Use Lippmann-Schwinger: $\text{Im}\langle\mathbf{k}|T|\mathbf{k}\rangle = \text{Im}\langle\mathbf{k}|V|\Psi_{\mathbf{k}}^{in}\rangle$. Principal value integral contour (semicircle over pole). $\frac{1}{E - H_0 + i\epsilon} = \text{Pr}\frac{1}{E - H_0} - i\pi\delta(E - H_0)$. Result: $\text{Im}f_k(0) = \frac{k}{4\pi}\int d\Omega'|f_k(\theta')|^2 = \frac{k}{4\pi}\sigma_{tot}$.

Thm 8.5.2 Eigenstates of J^2, J_0

If $\{|\Phi_{ajm}\rangle\}$ and $\{|\Psi_{\beta jm}\rangle\}$ are basis states of subspace V (eigenstates of J^2, J_0), then transformation matrix elements only depend on a, β (independent of j, m). $|\Psi_{\beta jm}\rangle = \sum_a \langle\Phi_{ajm}|\Psi_{\beta jm}\rangle|\Phi_{ajm}\rangle$. Elements $\langle\Phi_{ajm}|\Psi_{\beta jm}\rangle = \delta_{jj'}\delta_{mm'}\langle\Phi_{aj}||\Psi_{\beta j}\rangle$. (Reduced matrix element).

CG Coefficients Example (Handwritten)

Addition of angular momentum $J = J_1 + J_2$. Max $m = j_1 + j_2$ is unique: $|j_1 j_2; j_1 + j_2, j_1 + j_2\rangle = |j_1, j_1\rangle|j_2, j_2\rangle$. Apply $J_- = J_{1-} + J_{2-}$ to find lower m states. Example calculation: $|j_1 j_2; j, m\rangle = \alpha|\ldots\rangle + \beta|\ldots\rangle$. Orthogonality used to find other states (like $j = j_1 + j_2 - 1$). Condition $\alpha^2 + \beta^2 = 1$. Values like $\alpha = \sqrt{\frac{j_1}{j_1 + j_2}}, \beta = -\sqrt{\frac{j_2}{j_1 + j_2}}$.

Hydrogen-like Half-Space

$V(z) = -Ze^2/z(z > 0), \infty(z < 0)$. Solving SE gives Rydberg states for odd parity (wavefunc must vanish at $z = 0$). $E_n = -\frac{Z^2 e^4 m}{2\hbar^2 n^2}$ (Same as H-atom).

Spin-Orbit Coupling

$H = \frac{\alpha}{\hbar^2}\mathbf{L} \cdot \mathbf{S} = \frac{\alpha}{2\hbar^2}(J^2 - L^2 - S^2)$. Energy shift: $\Delta E = \frac{\alpha}{2}(j(j+1) - l(l+1) - s(s+1))$. Splitting between states $j = l \pm 1/2$.

Angular Momentum Operators

$L_z|l, m\rangle = \hbar m|l, m\rangle$, $L^2|l, m\rangle = \hbar^2 l(l+1)|l, m\rangle$. $L_\pm|l, m\rangle = \hbar\sqrt{l(l+1) - m(m \pm 1)}|l, m \pm 1\rangle$. $L_x = \frac{1}{2}(L_+ + L_-)$, $L_y = \frac{1}{2i}(L_+ - L_-)$.

Infinite Spherical Well

$V(r) = 0(r < R), \infty(r > R)$. Radial eq becomes spherical Bessel eq. Sol: $R_{nl}(r) = Aj_l(k_n r)$. Boundary $j_l(k_n R) = 0$. Energy: $E_{nl} = \frac{\hbar^2 k_{nl}^2}{2m}$ where $k_{nl} = x_{nl}/R$ (x_{nl} is n-th zero of j_l).

Permutation Group Representations (S_3)

Matrices for basis vectors (possibly defining specific rep): $T(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $T(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Eigenvalues of $T(123)$: $\det(T - \lambda I) = -\lambda^3 + 1 = 0 \implies \lambda = 1, e^{\pm i2\pi/3}$. Vectors: $|1\rangle = \frac{1}{\sqrt{3}}(1, 1, 1)^T$ (invariant).

Spin in Magnetic Field (Rabi)

$H = -\gamma\mathbf{S} \cdot \mathbf{B}$. Let $\mathbf{B} = B_0\hat{z} + B_1(\cos\omega t\hat{x} + \sin\omega t\hat{y})$. Rotating frame approx. Transition probability (Rabi formula): $P(t) = \frac{\Omega^2}{\Omega^2 + \Delta^2}\sin^2\left(\frac{\sqrt{\Omega^2 + \Delta^2}t}{2}\right)$. Where $\Delta = \omega - \omega_0$ (detuning), $\Omega = \gamma B_1$ (Rabi freq). Spin precession: $\frac{d\langle\mathbf{S}\rangle}{dt} = \gamma\langle\mathbf{S}\rangle \times \mathbf{B}$.

Tensor Operators Example

Consider $V \propto (x^2 - y^2)$. Commutators: $[L_z, z^2] = 0$. Rewrite x, y using $L_\pm$? No, using spherical harmonics. $x \pm iy \propto Y_{1,\pm 1}$. $(x \pm iy)^2 \propto Y_{2,\pm 2}$. Selection rules for matrix elements $\langle l', m'|V|l, m\rangle$: If V transforms like T_k^q (here $k = 2, q = \pm 2$), then $m' = m + q$. So $\Delta m = \pm 2$. Matrix form example: $V \doteq \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha^* & 0 & 0 \end{pmatrix}$ in basis corresponding to m values ordered. $\alpha = \langle 2, 1|x^2 - y^2|2, -1\rangle$.

Time-Dependent Perturbation (HO)

$V(t) = F_0 x (0 < t < T)$. First order amp: $c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' \langle n|V(t')|i\rangle e^{i\omega_{ni}t'}$. For HO $x \propto (a + a^\dagger)$. Selection: $\langle n|x|m\rangle \neq 0$ only if $n = m \pm 1$. Transition $0 \to 1$: $c_1^{(1)}(t) = -\frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega}}\int_0^t dt' e^{i\omega t'}$. Result involves $\frac{e^{i\omega t}-1}{i\omega}$.

Identical Particles

Two particles in 1D box. $E = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2)$. Wavefunc $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) \pm \psi_{n_1}(x_2)\psi_{n_2}(x_1)]\chi_{spin}$. Total Ψ must be: - Symmetric for Bosons. - Antisymmetric for Fermions. Spin States: - Triplet ($S = 1$): Symmetric. Requires Antisym spatial (Fermions) or Sym spatial (Bosons). - Singlet ($S = 0$): Antisymmetric. Requires Sym spatial (Fermions) or Antisym spatial (Bosons).

3 Particles in Harmonic Oscillator

Hamiltonian $H = \sum_{i=1}^{3}\frac{p_i^2}{2m} + \frac{1}{2}m\omega x_i^2$. Energy $E = \hbar\omega(n_1 + n_2 + n_3 + \frac{3}{2})$. Example: 3 identical fermions ($s = 1/2$, polarized spin). Spatial part must be totally antisymmetric. Ground State: distinct n's required. $n_1 = 0, n_2 = 1, n_3 = 2$. $E_0 = \hbar\omega(0 + 1 + 2 + \frac{3}{2}) = \frac{9}{2}\hbar\omega$. $\Psi_{GS} = \frac{1}{\sqrt{3!}}\det|\psi_{n_i}(x_j)|$.

Slater Determinant

N-fermion state: $\Psi(1, \ldots, N) = \frac{1}{\sqrt{N!}}\begin{vmatrix}\psi_1(1) & \ldots & \psi_1(N) \\ \vdots & \ddots & \vdots \\ \psi_N(1) & \ldots & \psi_N(N)\end{vmatrix}$.

Zero if two particles in same state (Pauli exclusion).