Ritz Theorem $$R[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}.$$

Step 1: Expand $|\psi\rangle$ in the energy eigenbasis

Because the eigenvectors $\{|n\rangle\}$ form an orthonormal basis (for the discrete part),

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \qquad c_n = \langle n|\psi\rangle.$$

Then

$$\langle\psi|\psi\rangle = \sum_n |c_n|^2,$$

and using $H|n\rangle = E_n|n\rangle$,

$$\langle\psi|H|\psi\rangle = \sum_{m,n} c_m^* c_n \langle m|H|n\rangle = \sum_{m,n} c_m^* c_n E_n \langle m|n\rangle = \sum_n |c_n|^2 E_n.$$

So

$$R[\psi] = \frac{\sum_n |c_n|^2 E_n}{\sum_n |c_n|^2}.$$

Step 2: Prove the inequality $R[\psi] \geq E_0$

Since $E_n \geq E_0$ for all n,

$$\sum_n |c_n|^2 E_n \;\geq\; \sum_n |c_n|^2 E_0 \;=\; E_0 \sum_n |c_n|^2.$$

Divide both sides by $\sum_n |c_n|^2 = \langle\psi|\psi\rangle > 0$:

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0.$$

■

Theorem 3.3.2 — Schur's lemma for hermitian operators. Let T be an IRREP of a finite group G on a linear subspace $\mathcal{V}$. If there is a hermitian operator $C : \mathcal{V} \to \mathcal{V}$ such that

$$\forall g \in G \qquad T(g)C = CT(g)$$

then C is proportional to the unit operator I, i.e. $C = \lambda I$, $\lambda \in \mathbb{C}$.

Proof. Given that C is an hermitian operator we can find an eigenstate $v \in \mathcal{V}$ of C such that

$$Cv = \lambda v. \tag{3.39}$$

The subspace span$\{T(G)v\}$ is an invariant subspace. We can easily see that for any vector $\boldsymbol{\alpha} = \sum_g \alpha_g T(g)v \in \mathcal{V}$

$$\begin{aligned} C\boldsymbol{\alpha} &= C\sum_g \alpha_g T(g)v = \sum_g \alpha_g T(g)Cv = \sum_g \alpha_g T(g)\lambda v \\ &= \lambda\boldsymbol{\alpha}\,. \end{aligned} \tag{3.40}$$

Since T is an IRREP of G, span$\{T(G)v\}$ must be $\mathcal{V}$ or the null space. As $v = T(I)v$ we conclude that span$\{T(G)v\}$ must be equal to $\mathcal{V}$. ■

Two comments: (a) This theorem is valid to any operator commuting with the group. (b) Altough the proof is given for finite groups, it can be generalized for continuous but compact groups.

Conclusion, if $\mathcal{V}$ is an IRREP invariant under the action of the hamiltonian H, and if

$$\forall g \in G \qquad T(g)H = HT(g) \Leftrightarrow [T(g), H] = 0, \tag{3.41}$$

then all the states in the subspace $\mathcal{V}$ have the same enenrgy.

Theorem 4.2.1 The spectrum of the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{\alpha}{r^s} \tag{4.11}$$

with $\alpha > 0$, is unbounded from below if $s > 2$.

Proof. To find the constraint on s, we prepare the system in a simple spherically symmetric state characterized by radius r_0,

$$\Psi(\mathbf{r}) = Ne^{-r^2/r_0^2} \tag{4.12}$$

with

$$N = \left(\frac{2}{\pi}\right)^{3/4} r_0^{-3/2}\,, \tag{4.13}$$

a normalization factor ensuring the normalization $\langle\Psi|\Psi\rangle = 1$.

As a result of the spherical symmetry

$$\langle x_i\rangle = \langle p_i\rangle = 0, \quad \text{for } i = 1,2,3\,, \tag{4.14}$$

while uncertainty relations give

$$\Delta x_i = \sqrt{\langle x_i^2\rangle - \langle x_i\rangle^2} \sim r_0 \quad\Rightarrow\quad \Delta p_i = \sqrt{\langle p_i^2\rangle - \langle p_i\rangle^2} = \sqrt{\langle p_i^2\rangle} \sim \frac{\hbar}{r_0}\,. \tag{4.15}$$

Hence, we can write

$$\langle H\rangle = \frac{\langle \boldsymbol{p}^2\rangle}{2m} + \langle V\rangle \sim \frac{\hbar^2}{2mr_0^2} - \frac{\alpha}{r_0^s}\,. \tag{4.16}$$

In the above qualitative estimation, we were not careful about order one numbers as they are not going to have any impact on our discussion. Thus, for instance, we used[1]

$$\begin{aligned} \langle V\rangle &= \int d\mathbf{r}\,\Psi^*(\mathbf{r})V(r)\Psi(\mathbf{r}) = -\frac{\alpha}{r_0^s}\left(\frac{2}{\pi}\right)^{\frac{3}{2}} 4\pi \underbrace{\int_0^\infty dx\,\frac{e^{-2x^2}}{x^{s-2}}}_{2^{(s-5)/2}\Gamma(\frac{3-s}{2})} \\ &= -\frac{2^{(s+2)/2}\Gamma\left(\frac{3-s}{2}\right)}{\sqrt{\pi}}\frac{\alpha}{r_0^s} \sim -\frac{\alpha}{r_0^s}\,. \end{aligned} \tag{4.17}$$

If $s > 2$, then one can make $\langle H\rangle$ arbitrarily negative by choosing r_0 sufficiently small. This clearly indicates that the spectrum of the Hamiltonian is unbounded from below and the system is unstable. A different point of view is to regard Ψ as a variational trial function, by definition $\langle H\rangle \geq E_0$, where E_0 is the lowest eigenstate of H. Thus, we deduce that $s \leq 2$. Henceforth, we take $s < 2$. The case $s = 2$ requires separate consideration, and we leave it outside the scope of these lectures. ■

Theorem 4.2.2 For $s < 2$ the spectrum of the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{\alpha}{r^s} \quad \alpha > 0 \tag{4.18}$$

contains infinite number of bound states.

Proof. To prove this statement we cook up a state represented by a spherically symmetric shell centred at $r = r_0$ with thickness $\Delta r = \beta r_0$. We assume that the state is very narrow, i.e. $\beta \ll 1$. As a concrete example of such state, we may consider

$$\Psi(\mathbf{r}) = Ne^{-(r-r_0)^2/\beta^2 r_0^2} \tag{4.19}$$

where N is a normalization factor. Following our previous arguments, we may conclude this time that

$$\langle H\rangle = \frac{\langle \boldsymbol{p}^2\rangle}{2m} + \langle V\rangle \sim \frac{\hbar^2}{2m\beta^2 r_0^2} - \frac{\alpha}{r_0^s}\,. \tag{4.20}$$

If $s < 2$ we can make $\langle H\rangle$ negative by fine-tuning r_0 to sufficiently large values. Hence, it follows that there are stationary states of negative energy, in which the particle may be found, with a fair probability, at large distances from the origin. This means that there are levels of arbitrary small negative energy. It must be recalled that the wave functions of bound states tend to zero exponentially fast in the region of space where classical motion is forbidden, while here $\langle H\rangle$ decays according to power law. ■

Theorem 3.1.1 Two linear operators A and B are equal iff

$$\langle\alpha|A|\alpha\rangle = \langle\alpha|B|\alpha\rangle \quad \text{for any} \quad |\alpha\rangle.$$

Proof. One direction is trivial, if $A = B$ then clearly $\langle\alpha|A|\alpha\rangle = \langle\alpha|B|\alpha\rangle$. To prove the other direction we need to show that $\langle\alpha|A|\beta\rangle = \langle\alpha|B|\beta\rangle$ for any vectors $|\alpha\rangle, |\beta\rangle$. To this end we note that by assumption the equality of the expectation values must

also hold for the states $|\varphi\rangle = |\alpha\rangle + |\beta\rangle$, and $|\psi\rangle = |\alpha\rangle + i|\beta\rangle$, i.e.

$$\begin{aligned} (\langle\alpha| + \langle\beta|)\,A\,(|\alpha\rangle + |\beta\rangle) &= (\langle\alpha| + \langle\beta|)\,B\,(|\alpha\rangle + |\beta\rangle) \\ (\langle\alpha| - i\langle\beta|)\,A\,(|\alpha\rangle + i|\beta\rangle) &= (\langle\alpha| - i\langle\beta|)\,B\,(|\alpha\rangle + i|\beta\rangle)\,. \end{aligned} \tag{3.7}$$

Hence, utilizing the equality of the expectation values, we get

$$\begin{aligned} \langle\alpha|A|\beta\rangle + \langle\beta|A|\alpha\rangle &= \langle\alpha|B|\beta\rangle + \langle\beta|B|\alpha\rangle \\ \langle\alpha|A|\beta\rangle - \langle\beta|A|\alpha\rangle &= \langle\alpha|B|\beta\rangle - \langle\beta|B|\alpha\rangle \end{aligned} \tag{3.8}$$

By adding and subtracting these two equations we see that $\langle\alpha|A|\beta\rangle = \langle\alpha|B|\beta\rangle$ for any $|\alpha\rangle, |\beta\rangle$. ■

Theorem 4.1.1 — Ritz Theorem. For an arbitrary function Ψ in our Hilbert space, the expectation value of H fulfills the relation

$$E \equiv \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} \geq E_1. \tag{4.2}$$

The equality holds if and only if Ψ is the ground state wave-function of H.

Proof. As the eigenstates of H form a complete basis of the Hilbert space, Ψ may be expanded as

$$\Psi = \sum_k c_k\Phi_k. \tag{4.3}$$

It follows that

$$\frac{\langle\Psi|H - E_1|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\sum_{k=1}^\infty (E_k - E_1)|c_k|^2}{\sum_{k=1}^\infty |c_k|^2} \geq 0\,. \tag{4.4}$$

The last relation is due to the ordering of the energies, proving Ritz's theorem. ■

Theorem 4.1.2 — Generalized Ritz Theorem*. The expectation value of the Hamiltonian H is stationary in the neighborhood of its eigenvalues.

Proof. To prove this theorem we induce a small variation in the wave function Ψ, thus if $E\langle\Psi|\Psi\rangle = \langle\Psi|H|\Psi\rangle$ then

$$\begin{aligned} \delta E\langle\Psi|\Psi\rangle &= \delta(\langle\Psi|H|\Psi\rangle) - E\delta(\langle\Psi|\Psi\rangle) \\ &= \langle\delta\Psi|H - E|\Psi\rangle + \langle\Psi|H - E|\delta\Psi\rangle \end{aligned} \tag{4.5}$$

If we choose $\delta\Psi = \epsilon(H - E)\Psi$, then on the right hand side we get the norm of the function $\epsilon(H - E)\Psi$. It follows that E is stationary iff $(H - E)\Psi = 0$, i.e. Ψ is an exact eigenstate of H. ■

Let us assume that $\Psi = \Phi_n + \delta\Phi$, with $\langle\Phi_n|\delta\Phi\rangle = 0$. In this case

$$E = \frac{\langle\Phi_n + \delta\Phi|H|\Phi_n + \delta\Phi\rangle}{\langle\Phi_n + \delta\Phi|\Phi_n + \delta\Phi\rangle} = \frac{E_n\langle\Phi_n|\Phi_n\rangle + \langle\delta\Phi|H|\delta\Phi\rangle}{\langle\Phi_n|\Phi_n\rangle + \langle\delta\Phi|\delta\Phi\rangle} = E_n + O(\delta\Phi^2)\,. \tag{4.6}$$

Thus, we conclude that the energy converge as $O(\delta\Phi^2)$. We note that all other observables converge as $O(\delta\Phi)$.

Theorem 4.1.3 — The variance theorem*. Given the variance of the energy expectation value

$$\sigma^2 \equiv \frac{\langle\Psi|(H - E)^2|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\langle\Psi|H^2|\Psi\rangle}{\langle\Psi|\Psi\rangle} - E^2, \tag{4.7}$$

there is at least one exact eigenvalue of H in the interval $[E - \sigma, E + \sigma]$.

Proof. Expanding the wave-function Ψ using the normalized eigenstates of H we can rewrite the variance as

$$\sigma^2 = \sum_{i=1}^\infty (E_i - E)^2|c_i|^2. \tag{4.8}$$

Now let E_K be the eigenstates closest to E, then

$$\sum_{i=1}^\infty (E_i - E)^2|c_i|^2 \geq (E_K - E)^2\sum_{i=1}^\infty |c_i|^2 = (E_K - E)^2. \tag{4.9}$$

Thus $\sigma^2 \geq (E_K - E)^2$, QED. ■

Theorem 5.4.1 — Upper bound on the eigenvalues of J_0. Given a Hilbert space with an orthonormal basis $\{|ab\rangle\}$ such that the states $|ab\rangle$ are eigenstates of J^2, J_0. If a, b are the eigenvalues of J^2, J_0

$$J^2|ab\rangle = a|ab\rangle \quad \text{and} \quad J_0|ab\rangle = b|ab\rangle \tag{5.60}$$

then $a \geq b^2$.

Proof. To prove this statement we notice that

$$J^2 = J_0^2 + \frac{1}{2}(J_-J_+ + J_+J_-) \tag{5.61}$$

and therefore

$$J^2 - J_0^2 = \frac{1}{2}\left(J_+^\dagger J_+ + J_+J_+^\dagger\right) \tag{5.62}$$

It follows that

$$\langle ab|J^2 - J_0^2|ab\rangle = \langle ab|\frac{1}{2}\left(J_+^\dagger J_+ + J_+J_+^\dagger\right)|ab\rangle \geq 0. \tag{5.63}$$

Therefore

$$\langle ab|J^2 - J_0^2|ab\rangle = a - b^2 \geq 0 \;\Rightarrow\; a \geq b^2. \tag{5.64}$$

■

Theorem 3.1.2 Two linear operators A and B are equal to within a phase

$$A = e^{i\alpha}B$$

iff

$$|\langle\alpha|A|\beta\rangle| = |\langle\alpha|B|\beta\rangle| \quad \text{for any} \quad |\alpha\rangle, |\beta\rangle.$$

"⇒"

If $A = e^{i\alpha}B$ then

$$\langle a|A|b\rangle = e^{i\alpha}\langle a|B|b\rangle \quad\Rightarrow\quad |\langle a|A|b\rangle| = |\langle a|B|b\rangle|.$$

Done.

"⇐"

$$A|b_j\rangle = e^{i\theta_j}B|b_j\rangle, \qquad \theta_j := \theta(b_j).$$

Apply the same relation to $|b_1\rangle + |b_2\rangle$:

$$A(|b_1\rangle + |b_2\rangle) = e^{i\theta_{12}}B(|b_1\rangle + |b_2\rangle)$$

for some θ_{12}. But linearity gives

$$e^{i\theta_1}B|b_1\rangle + e^{i\theta_2}B|b_2\rangle = e^{i\theta_{12}}\big(B|b_1\rangle + B|b_2\rangle\big).$$

Rearrange:

$$\big(e^{i\theta_1} - e^{i\theta_{12}}\big)B|b_1\rangle + \big(e^{i\theta_2} - e^{i\theta_{12}}\big)B|b_2\rangle = 0.$$

Take inner product with $B|b_1\rangle$ to get

$$\big(e^{i\theta_1} - e^{i\theta_{12}}\big)\|B|b_1\rangle\|^2 + \big(e^{i\theta_2} - e^{i\theta_{12}}\big)\langle Bb_1|Bb_2\rangle = 0.$$

Similarly inner product with $B|b_2\rangle$. Because $\langle Bb_1|Bb_2\rangle \neq 0$, the only way both equations hold is

$$e^{i\theta_1} = e^{i\theta_2} = e^{i\theta_{12}}.$$

So the phase does **not** depend on b (on the non-kernel part). Call it $e^{i\alpha}$. Then for all $|b\rangle$,

$$A|b\rangle = e^{i\alpha}B|b\rangle,$$

hence $A = e^{i\alpha}B$. ■

Theorem 3.1.3 Given a linear subspace $\mathcal{V}$. If T is a one-to-one transformation $T : \mathcal{V} \to \mathcal{V}$ which conserves the scalar product on $\mathcal{V}$ and is defined up to an arbitrary phase, then the phases can be always chosen so as to make T either linear and unitary or anti-linear and anti-unitary.

Consider any two basis vectors $|1\rangle, |2\rangle$ and a normalized superposition

$$|\psi\rangle = \frac{|1\rangle + z|2\rangle}{\sqrt{1 + |z|^2}} \qquad (z \in \mathbb{C}).$$

Its transition probabilities with $|1\rangle$ and $|2\rangle$ are

$$|\langle 1|\psi\rangle|^2 = \frac{1}{1 + |z|^2}, \qquad |\langle 2|\psi\rangle|^2 = \frac{|z|^2}{1 + |z|^2}.$$

Because T preserves these, $T|\psi\rangle$ must have the same squared overlaps with $|1'\rangle$ and $|2'\rangle$. Hence (up to an overall phase depending on ψ) it must be of the form

$$T|\psi\rangle \sim \frac{|1'\rangle + w(z)\,|2'\rangle}{\sqrt{1 + |w(z)|^2}}$$

with $|w(z)| = |z|$. Therefore

$$w(z) = e^{i\phi(z)}\,z \quad \text{or} \quad w(z) = e^{i\phi(z)}\,z^*,$$

for some real $\phi(z)$.

Now use **one more** preserved transition probability: compare $|\psi(z)\rangle$ and $|\psi(z')\rangle$:

$$|\langle\psi(z)|\psi(z')\rangle| = \frac{|1 + z^*z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}.$$

On the primed side, this becomes

$$\frac{|1 + w(z)^*w(z')|}{\sqrt{(1 + |w(z)|^2)(1 + |w(z')|^2)}}.$$

Because $|w(z)| = |z|$, denominators match, so we need

$$|1 + z^*z'| = |1 + w(z)^*w(z')| \quad \forall z, z'.$$

A standard functional-analysis argument shows this forces **one of two consistent choices** on the whole 2D subspace:

- either $w(z) = z$ for all z (after absorbing a constant phase into $|2'\rangle$), or
- $w(z) = z^*$ for all z (again after a constant phase adjustment).

In other words: on the span of $|1\rangle, |2\rangle$, after choosing phases consistently,

$$T(a|1\rangle + b|2\rangle) \;\sim\; a|1'\rangle + b|2'\rangle \quad \text{(linear case)}$$

or

$$T(a|1\rangle + b|2\rangle) \;\sim\; a^*|1'\rangle + b^*|2'\rangle \quad \text{(anti-linear case)}.$$

Step 3: Show the same choice holds for the whole $\mathcal{V}$

Repeat the 2D analysis for every pair $(|n\rangle, |m\rangle)$. If one pair forced "linear" and another forced "anti-linear", you get a contradiction by considering a 3D superposition (e.g. involving $|1\rangle, |2\rangle, |3\rangle$) and comparing preserved transition probabilities in two [illegible]erent ways. So the choice (linear vs conjugate-linear) must be **global** on $\mathcal{V}$. ■

Theorem 8.2.2 — The triangular rule. The admissible values of j are limited by the triangular rule

$$|j_1 - j_2| \leq j \leq j_1 + j_2 \tag{8.27}$$

Proof. The maximum value of m is $j_1 + j_2$. There is only one state with this value of m, the state $|j_1 j_2 j = (j_1 + j_2), m = (j_1 + j_2)\rangle$. For $m = j_1 + j_2 - 1$ there are two possible states $m_1 = j_1, m_2 = j_2 - 1$ and $m_1 = j_1 - 1, m_2 = j_2$. One of these states corresponds to the IRREP $j = j_1 + j_2$, the other to $j = j_1 + j_2 - 1$. We can go on lowering the value of m adding a new IRREP j until $m_1 = -j_1$ or $m_2 = -j_2$. Beyond this point we won't get any new j states. ■

Theorem 12.3.1 The operators $\mathcal{S}$ and $\mathcal{A}$ satisfy the following properties:

(a) $\mathcal{S}$ and $\mathcal{A}$ are Hermitian operators: $\mathcal{S}^\dagger = \mathcal{S}$, $\mathcal{A}^\dagger = \mathcal{A}$.

(b) $\mathcal{S}$ and $\mathcal{A}$ commute with P_g for all $g \in S_N$,

$$P_g\mathcal{S} = \mathcal{S}P_g = \mathcal{S}\,,$$
$$P_g\mathcal{A} = \mathcal{A}P_g = \text{sign}(g)\mathcal{A}\,. \tag{12.24}$$

(c) $\mathcal{S}$ and $\mathcal{A}$ are projectors to orthogonal subspaces of $\mathcal{H}_w$, i.e.

$$\mathcal{S}^2 = \mathcal{S}\,, \quad \mathcal{A}^2 = \mathcal{A}\,, \quad \mathcal{S}\mathcal{A} = \mathcal{A}\mathcal{S} = 0\,. \tag{12.25}$$

Proof. **(a)** By definition

$$\mathcal{S}^\dagger = \frac{1}{N!}\sum_{g\in S_N} P_g^\dagger = \frac{1}{N!}\sum_{g\in S_N} P_g^{-1} = \frac{1}{N!}\sum_{g\in S_N} P_{g^{-1}} = \mathcal{S}\,, \tag{12.26}$$

where 2nd equality follows from unitarity of representation $P_g^\dagger = P_g^{-1}$, the 3rd equality rests on $P_g^{-1} = P_{g^{-1}}$ which follows from (12.22). Finally, the last equality is true because inverse operation is a bijection from S_N to itself. The proof of $\mathcal{A}^\dagger = \mathcal{A}$ is almost identical. We only need to note that since $g^{-1}g = I$ and $\text{sign}(I) = 1$ we must have $\text{sign}(g^{-1}) = \text{sign}(g)$.

(b) We have

$$P_g\mathcal{A} = \frac{1}{N!}\sum_{g'\in S_N}\text{sign}(g')\,P_gP_{g'} = \frac{1}{N!}\sum_{g'\in S_N}\text{sign}(g)^2\text{sign}(g')\,P_{gg'}$$
$$= \frac{\text{sign}(g)}{N!}\sum_{g'\in S_N}\text{sign}(gg')\,P_{gg'} = \text{sign}(g)\mathcal{A}\,, \tag{12.27}$$

where in the 2nd equality we used (12.22) and $\text{sign}(g)^2 = 1$, $\forall g \in S_N$, whereas in the 1st equality of the second line we substituted $\text{sign}(g)\text{sign}(g') = \text{sign}(gg')$. Finally, last equality rests on the bijective nature of the map $g' \to gg'$. The proof for $\mathcal{S}$ follow suit.

(c) Using the definition of $\mathcal{S}$ and **(b)**, yields

$$\mathcal{S}^2 = \frac{1}{N!}\sum_{g\in S_N}P_g\mathcal{S} = \frac{1}{N!}\sum_{g\in S_N}\mathcal{S} = \mathcal{S}\,. \tag{12.28}$$

Similarly,

$$\mathcal{A}^2 = \frac{1}{N!}\sum_{g\in S_N}\text{sign}(g)P_g\mathcal{A} = \frac{1}{N!}\sum_{g\in S_N}\text{sign}(g)^2\mathcal{A} = \mathcal{A}\,. \tag{12.29}$$

Now for $\mathcal{S}\mathcal{A}$ we see that

$$\mathcal{S}\mathcal{A} = \mathcal{S}(P_g^{-1}P_g)\mathcal{A} = (\mathcal{S}P_g^{-1})(P_g\mathcal{A}) = \mathcal{S}\,\text{sign}(g)\mathcal{A}, \tag{12.30}$$

and choosing an odd parity g we much have $\mathcal{S}\mathcal{A} = 0$. ■

מציאת k עבור שדה מגנטי וחשמלי:

$$H = \frac{1}{2m}(P - q\vec{A})^2 - q\Phi(x)$$

$$B = B\hat{z}\,,\quad E = \mathcal{E}\hat{x}$$

$$B = \nabla\times A = \partial_x A_y - \partial_y A_x \longrightarrow A = Bx\hat{y}$$:נבחר

$$E = \nabla\Phi = \partial_x\Phi \longrightarrow \Phi = \mathcal{E}x$$:נבחר

$$H = \frac{1}{2m}(-i\hbar\vec\nabla - qBx\hat{y})^2 - q\mathcal{E}x$$

$$= \frac{1}{2m}\begin{pmatrix}-i\hbar\partial_x\\ -i\hbar\partial_y - qBx\\ -i\hbar\partial_z\end{pmatrix}\begin{pmatrix}-i\hbar\partial_x\\ -i\hbar\partial_y - qBx\\ -i\hbar\partial_z\end{pmatrix} - q\mathcal{E}x$$

$$= \frac{1}{2m}\left(-\hbar^2\partial_x^2 + (-i\hbar\partial_y - qBx)^2 - \hbar^2\partial_z^2\right) - q\mathcal{E}x$$

$$H\psi = E\psi$$

$$\left(\frac{1}{2m}\left(-\hbar^2\partial_x^2 + (-i\hbar\partial_y - qBx)^2 - \hbar^2\partial_z^2\right) - q\mathcal{E}x\right)\psi = E\psi$$

ההמילטוניאן עם פוטנציאל שתלוי רק ב-x לכן בשאר הכיוונים יהיה לנו חלקיק חופשי וננחש פתרון:

$$\psi = e^{ik_zz + ik_yy}\psi(x)$$

$$\left(\frac{1}{2m}(\hbar k_y + qBx)^2 - q\mathcal{E}x - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\hbar^2k_z^2}{2m}\right)e^{ik_zz+ik_yy}\psi(x) = Ee^{ik_zz+ik_yy}\psi(x)$$

$$\left(\frac{1}{2m}(\hbar k_y + qBx)^2 - q\mathcal{E}x - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(\frac{1}{2m}(\hbar^2k_y^2 + 2\hbar k_yqBx + q^2B^2x^2) - q\mathcal{E}x - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(\frac{1}{2m}\left(\hbar^2k_y^2 + 2q(k_y\hbar B - \mathcal{E}m)\,x + q^2B^2x^2\right) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(\frac{q^2B^2}{2m}\left(x^2 + 2\left(\frac{\hbar k_yB - m\mathcal{E}}{qB^2}\right)x + \left(\frac{\hbar k_y}{qB}\right)^2\right) - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(\frac{q^2B^2}{2m}\left(x^2 + 2\left(\frac{\hbar k_yB - m\mathcal{E}}{qB^2}\right)x + \left(\frac{\hbar k_y}{qB} - \frac{m\mathcal{E}}{qB^2}\right)^2 - \left(\frac{\hbar k_y}{qB} - \frac{m\mathcal{E}}{qB^2}\right)^2 + \left(\frac{\hbar k_y}{qB}\right)^2\right)\right.$$
$$\left.- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(\frac{q^2B^2}{2m}\left(\left(x + \frac{\hbar k_y}{qB} - \frac{m\mathcal{E}}{qB^2}\right)^2 - \left(\cancel{\left(\frac{\hbar k_y}{qB}\right)^2} - 2\left(\frac{\hbar k_ym\mathcal{E}}{q^2B^3}\right) + \left(\frac{m\mathcal{E}}{qB^2}\right)^2\right) + \cancel{\left(\frac{\hbar k_y}{qB}\right)^2}\right.$$
$$\left.- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(\frac{q^2B^2}{2m}\left(x + \frac{\hbar k_y}{qB} - \frac{m\mathcal{E}}{qB^2}\right)^2 + \frac{q^2B^2}{2m}\left(\frac{2\hbar k_ym\mathcal{E}}{q^2B^3} - \frac{m^2\mathcal{E}^2}{q^2B^4}\right)\right.$$
$$\left.- \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(\frac{q^2B^2}{2m}\left(x + \frac{\hbar k_y}{qB} - \frac{m\mathcal{E}}{qB^2}\right)^2 + \frac{\hbar k_y\mathcal{E}}{B} - \frac{m\mathcal{E}^2}{2B^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$w = \frac{qB}{m} \qquad x_0 = \frac{m\mathcal{E}}{qB^2} - \frac{\hbar k_y}{qB}$$

$$\left(\frac{m}{2}w^2(x - x_0)^2 + \frac{\hbar k_y\mathcal{E}}{B} - \frac{m\mathcal{E}^2}{2B^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\psi(x) = \left(E - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}w^2(x - x_0)^2\right)\psi(x) = \left(E + \frac{m\mathcal{E}^2}{2B^2} - \frac{\hbar k_y\mathcal{E}}{B} - \frac{\hbar^2k_z^2}{2m}\right)\psi(x)$$

אוסילאטור הרמוני מוזז.

$$\left(E + \frac{m\mathcal{E}^2}{2B^2} - \frac{\hbar k_y\mathcal{E}}{B} - \frac{\hbar^2k_z^2}{2m}\right) = \hbar w\left(n + \tfrac{1}{2}\right)$$

$$E = \hbar w\left(n + \tfrac{1}{2}\right) - \frac{m\mathcal{E}^2}{2B^2} + \frac{\hbar k_y\mathcal{E}}{B} + \frac{\hbar^2k_z^2}{2m}$$

ראשית לינארית תלויות ב- $\frac{\mathcal{E}}{B}$.

Theorem 13.9.1 — The Optical theorem. The optical theorem states that

$$\sigma_{\text{tot}}(k) = \frac{4\pi}{k}\text{Im}\,f_k(0)\,. \tag{13.59}$$

Proof. To prove the optical theorem we use the T-matrix

$$f_k(0) = f(\boldsymbol{k},\boldsymbol{k}) = -\frac{m}{2\pi\hbar^2}(2\pi\hbar)^3\langle\boldsymbol{k}|T|\boldsymbol{k}\rangle. \tag{13.60}$$

Now using the Lippmann-Schwinger equation (13.24) and (13.43) we have

$$\text{Im}\,\langle k|T|k\rangle = \text{Im}\,\langle k|V|\Psi_k^{in}\rangle$$
$$= \text{Im}\left[\left(\langle\Psi_k^{in}| - \langle\Psi_k^{in}|V\frac{1}{E - H_0 + i\epsilon}\right)V|\Psi_k^{in}\rangle\right]. \tag{13.61}$$

To proceed we recall the definition of the principle value integral

$$\text{Pr}\int_{-\infty}^{\infty}dx\frac{f(x)}{x - x_0} = \lim_{\delta\to 0}\left[\int_{-\infty}^{-\delta}dx\frac{f(x)}{x - x_0} + \int_{\delta}^{\infty}dx\frac{f(x)}{x - x_0}\right], \tag{13.62}$$

and note that

$$\int_{-\infty}^{\infty}dE\frac{f(E)}{E - E_0 + i\epsilon} = \lim_{\delta\to 0}\left[\int_{-\infty}^{-\delta}dE\frac{f(E)}{E - E_0 + i\epsilon} + \int_c dE\frac{f(E)}{E - E_0 + i\epsilon} + \int_{\delta}^{\infty}dE\frac{f(E)}{E - E_0 + i\epsilon}\right]$$
$$= \text{Pr}\int_{-\infty}^{\infty}dE\frac{f(E)}{E - E_0 + i\epsilon} + i\pi f(E_0), \tag{13.63}$$

where c stands for integration over a semicircle from $-\delta$ to δ, see Fig. 13.2.

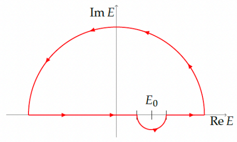

Figure 13.2: A principle value integral.

It follows that

$$\frac{1}{E - H_0 + i\epsilon} = \text{Pr}\left(\frac{1}{E - H_0}\right) + i\pi\delta(E - H_0). \tag{13.64}$$

Utilizing this relation we see that

$$\text{Im}\,\langle k|T|k\rangle = \text{Im}\left[\langle\Psi_k^{in}|V|\Psi_k^{in}\rangle - \langle\Psi_k^{in}|V\,\text{Pr}\left(\frac{1}{E - H_0}\right)V|\Psi_k^{in}\rangle\right.$$
$$\left.- i\pi\langle\Psi_k^{in}|V\delta(E - H_0)V|\Psi_k^{in}\rangle\right] \tag{13.65}$$

Due to the hermiticity of V and H_0 the first two term are real, hence

$$\text{Im}\,\langle k|T|k\rangle = -\pi\langle\Psi_k^{in}|V\delta(E - H_0)V|\Psi_k^{in}\rangle. \tag{13.66}$$

Utilizing once more the relation (13.43) between V and T we get

$$\text{Im}\,\langle k|T|k\rangle = -\pi\langle k|T\delta(E - H_0)T|k\rangle$$
$$= -\pi\int\hbar^3dk'\langle k|T|k'\rangle\langle k'|T|k\rangle\delta\left(E - \frac{\hbar^2k'^2}{2m}\right)$$
$$= -\pi\int\hbar^3d\hat{k}'\frac{mk}{\hbar^2}|\langle k'|T|k\rangle|^2, \tag{13.67}$$

where in the last step we have used $dk' = k'^2dE_k'(dk'/dE_k')d\hat{k}'$. With a little help form (13.46) we can rewrite this result in terms of the total cross-section,

$$\text{Im}\,\langle k|T|k\rangle = -\pi\left(\frac{2\pi\hbar^2}{m(2\pi\hbar)^3}\right)^2\int d\hat{k}'\hbar mk|f_k(k')|^2,$$
$$= -\pi\left(\frac{2\pi\hbar^2}{m(2\pi\hbar)^3}\right)^2\hbar mk\sigma_{\text{tot}}. \tag{13.68}$$

Now

$$\text{Im}\,f_k(0) = -\frac{m}{2\pi\hbar^2}(2\pi\hbar)^3(-\pi)\left(\frac{2\pi\hbar^2}{m(2\pi\hbar)^3}\right)^2\hbar mk\sigma_{\text{tot}}$$
$$= \pi\left(\frac{2\pi\hbar^2}{m(2\pi\hbar)^3}\right)\hbar mk\sigma_{\text{tot}} = \frac{k}{4\pi}\sigma_{\text{tot}}, \tag{13.69}$$

which concludes our proof. ■

גוף קשיח:

$$H = \frac{K_x^2}{2I_x} + \frac{K_y^2}{2I_y} + \frac{K_z^2}{2I_z}$$

$$= \left(\frac{1}{4I_x} + \frac{1}{4I_y}\right)(K_x^2 + K_y^2) + \left(\frac{1}{4I_x} - \frac{1}{4I_y}\right)(K_x^2 - K_y^2) + \frac{K_z^2}{2I_z}$$

$$= \left(\frac{1}{4I_x} + \frac{1}{4I_y}\right)(K^2 - K_z^2) + \left(\frac{1}{4I_x} - \frac{1}{4I_y}\right)(K_x^2 - K_y^2) + \frac{K_z^2}{2I_z}$$

$$= \left(\frac{1}{4I_x} + \frac{1}{4I_y}\right)K^2 + \left(\frac{1}{2I_z} - \frac{1}{4I_x} - \frac{1}{4I_y}\right)K_z^2 + \left(\frac{1}{4I_x} - \frac{1}{4I_y}\right)(K_x^2 - K_y^2)$$

$$= AK^2 + BK_z^2 + C(K_x^2 - K_y^2)$$

$$K_\pm = K_x \pm iK_y \quad \Rightarrow \quad K_x = \frac{1}{2}(K_+ + K_-)$$
$$K_y = \frac{1}{2i}(K_+ - K_-)$$

$$K_x^2 = \frac{1}{4}(K_+^2 + K_+K_- + K_-K_+ + K_-^2)$$
$$K_y^2 = -\frac{1}{4}(K_+^2 - K_+K_- - K_-K_+ + K_-^2) \quad \Rightarrow \quad K_x^2 - K_y^2 = \frac{1}{2}(K_+^2 + K_-^2)$$

$$H = AK^2 + BK_z^2 + \frac{C}{2}(K_+^2 + K_-^2)$$

$$H|LMK\rangle = A\hbar^2L(L+1)|LMK\rangle + B\hbar^2K^2|LMK\rangle + \frac{C}{2}\hbar^2\sqrt{L(L+1) - K(K+1)}\sqrt{L(L+1) - (K+1)(K+2)}$$
$$|LMK+2\rangle + \frac{C}{2}\hbar^2\sqrt{(L+K)(L-K+1)}\sqrt{(L+K-1)(L-K+2)}\;|LMK-2\rangle$$

<u>L=1:</u>

$$\langle LMK'|H|LMK\rangle = \begin{array}{c|ccc} & -1 & 0 & +1 \\ \hline -1 & 2A\hbar^2 + B\hbar^2 & 0 & \hbar^2C \\ 0 & 0 & 2A\hbar^2 & 0 \\ +1 & \hbar^2C & 0 & 2A\hbar^2 + B\hbar^2 \end{array}$$

Theorem 8.5.2 If $\{|\Phi_{\alpha_j m_a}\rangle\}$ and $\{|\Psi_{\beta_j m_b}\rangle\}$ are two sets of basis states spanning subspace $\mathcal{V}$, and if both sets are eigenstates of J^2 and J_0 characterized by the j, m quantum numbers, then the transformation matrix elements $\langle\Phi_{\alpha_j a m_a}|\Psi_{\beta_j b m_b}\rangle$ from one basis to the other are diagonal in the j, m quantum numbers and independent of m, i.e.

$$\langle\Phi_{\alpha j_a m_a}|\Psi_{\beta j_b m_b}\rangle = \delta_{j_aj_b}\delta_{m_am_b}\langle\Phi_{\alpha j_a}||\Psi_{\beta j_b}\rangle\,, \tag{8.74}$$

where $\langle\Phi_{\alpha j_a}||\Psi_{\beta j_b}\rangle$ are the transformation coefficients.

Proof. The proof of this statement is rather simple. Since both J^2 and J_0 are Hermitian, it is evident that the matrix-elements $\langle\Phi_{\alpha j_a m_a}|\Psi_{\beta j_b m_b}\rangle$ are orthogonal in both j, and m. Setting, therefore, $j_a = j_b = j$ (and the same for m_a, m_b) we can write

$$|\Psi_{\beta jm}\rangle = \sum_\alpha\langle\Phi_{\alpha jm}|\Psi_{\beta jm}\rangle|\Phi_{\alpha jm}\rangle. \tag{8.75}$$

It follows that for any $m < j$

$$|\Psi_{\beta jm+1}\rangle = \sum_\alpha\langle\Phi_{\alpha jm+1}|\Psi_{\beta jm+1}\rangle|\Phi_{\alpha jm+1}\rangle, \tag{8.76}$$

but since

$$|\Psi_{\beta jm+1}\rangle = \frac{J_+}{\sqrt{(j-m)(j+m+1)}}|\Psi_{\beta jm}\rangle$$
$$= \sum_\alpha\langle\Phi_{\alpha jm}|\Psi_{\beta jm}\rangle|\Phi_{\alpha jm+1}\rangle, \tag{8.77}$$

it follows that

$$\langle\Phi_{\alpha jm+1}|\Psi_{\beta jm+1}\rangle = \langle\Phi_{\alpha jm}|\Psi_{\beta jm}\rangle. \tag{8.78}$$

■

* מציאת מקדמי קלבש גורדן:

$$m = m_1 + m_2$$

$$|j_1 - j_2| \le j \le j_1 + j_2$$

$$J_+|jm\rangle = \hbar\sqrt{(j-m)(j+m+1)}\;|j, m+1\rangle$$

$$J_-|jm\rangle = \hbar\sqrt{(j+m)(j-m+1)}\;|j, m-1\rangle$$

נפעל עם J_- על $J_{max} = m_1 + m_2$:

$$|m_1m_2;\, j = j_{max},\, m = m_1 + m_2\rangle = |m_1m_2;\, m_1 = m_1,\, m_2 = m_2\rangle$$

נפעל על שני האגפים: $J_- = J_{1-} + J_{2-}$

$$\hbar\sqrt{(j+m)(j-m+1)}\;|m_1m_2;\, j = j_{max},\, m = m_1 + m_2 - 1\rangle =$$

$$\hbar\sqrt{(m_1+m_1)(m_1-m_1+1)}\;|m_1m_2;\, m_1 = m_1 - 1,\, m_2 = m_2\rangle +$$

$$\hbar\sqrt{(m_2+m_2)(m_2-m_2+1)}\;|m_1m_2;\, m_1 = m_1,\, m_2 = m_2 - 1\rangle$$

נפעל שוב עד ש: $-j = m$ ... על ערכים מקדמים

אחר כך נמצא את j ... עד $j = |j_1 - j_2|$.

הפעם יש כמה אפשרויות לערכי m בהתחלה:

$J_{max} = m_1 + m_2$ מתוך הדרישה וההורדה של j ל: $j = j - 1$ $\longrightarrow$ $m_1 = m_1\,,\ m_2 = m_2 - 1$

$m_1 = m_1 - 1\,,\ m_2 = m_2 - 1$

* דוגמה עם $m_1 = 1\,,\ m_2 = \frac{1}{2}$ אחרי הורדה של j ל- $j - 1$:

$$|j_1j_2;\, j = \tfrac{1}{2},\, m = \tfrac{1}{2}\rangle = \alpha|j_1j_2;\, m_1 = 0,\, m_2 = \tfrac{1}{2}\rangle + \beta|j_1j_2;\, m_1 = 1,\, m_2 = -\tfrac{1}{2}\rangle$$

$$0 = \langle j_1j_2;\, j = \tfrac{3}{2},\, m = \tfrac{1}{2}|j_1j_2;\, j = \tfrac{1}{2},\, m = \tfrac{1}{2}\rangle = \alpha\sqrt{\frac{j_1}{j_1 + j_2}} + \beta\sqrt{\frac{j_2}{j_1 + j_2}}$$

$$= \alpha\sqrt{\frac{2}{3}} + \beta\sqrt{\frac{1}{3}}$$

דרישת נרמול: $\alpha^2 + \beta^2 = 1$

$$\alpha = \sqrt{\frac{j_2}{j_1 + j_2}} = \sqrt{\frac{1}{3}} \qquad \beta = -\sqrt{\frac{j_1}{j_1 + j_2}} = -\sqrt{\frac{2}{3}}$$

$$|j_1j_2;\, j = \tfrac{1}{2},\, m = \tfrac{1}{2}\rangle = \sqrt{\frac{1}{3}}\,|j_1j_2;\, m_1 = 0,\, m_2 = \tfrac{1}{2}\rangle - \sqrt{\frac{2}{3}}\,|j_1j_2;\, m_1 = 1,\, m_2 = -\tfrac{1}{2}\rangle$$

רמות אנרגיה של אלקטרון בפוטנציאל:

$$V=\begin{cases}-\frac{Z_{He}e^2}{z} & z>0\\ \infty & z\le 0\end{cases}$$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}+V(z)\right)\psi(z)=E\psi(z)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi(z)-\frac{Z_{He}e^2}{z}\psi(z)=E\psi(z)$$

$$\frac{\partial^2}{\partial z^2}\psi(z)=\left(-\frac{2m}{\hbar^2}\frac{Z_{He}e^2}{z}-\frac{2m}{\hbar^2}E\right)\psi(z)$$

$$z\to\infty$$

$$\frac{\partial^2}{\partial z^2}\psi(z)=-\frac{2m}{\hbar^2}E\psi(z)\longrightarrow a=\frac{2m}{\hbar^2}E$$

$$\psi=e^{-\sqrt{a}z}H(z)$$

$$H''(z)-2\sqrt{a}H'(z)-\underbrace{\frac{2m}{\hbar^2}\frac{Z_{He}e^2}{z}}_{=b}H(z)=0$$

$$H(z)=\sum a_n z^n$$

$$\sum_{n=0}a_n n(n-1)z^{n-2}-2\sqrt{a}\sum_{n=0}a_n n z^{n-1}+\frac{b}{z}\sum_{n=0}a_n z^n=0$$

$$\sum_{n=2}a_n n(n-1)z^{n-2}-2\sqrt{a}\sum_{n=0}a_n n z^{n-1}+b\sum_{n=0}a_n z^{n-1}=0$$

$$k=n-1$$

$$\sum_{n=2}a_n n(n-1)z^{n-2}=\sum_{k=1}a_{k+1}k(k+1)z^{k-1}$$

$$\sum_{n=1}\left(a_{n+1}n(n+1)-(2\sqrt{a}n-b)a_n\right)z^{n-1}=0$$

$$a_{n+1}=\frac{2\sqrt{a}n-b}{n(n+1)}a_n$$

$$2\sqrt{a}n-b=0\quad\Rightarrow\quad n=\frac{b}{2\sqrt{a}}=-\frac{mZ_{He}e^2}{\hbar^2\sqrt{\frac{2mE}{\hbar^2}}}$$

$$E_z=-\frac{Z_{He}^2e^4m}{2\hbar^2n^2}$$

בכיוון x, y יש חלקיק חופשי לכן האנרגיות והפונקציות העצמיות יהיו של חלקיק חופשי.

$$E_{x,y,z}=\frac{\hbar^2}{2m}(k_x+k_y)-\frac{Z_{He}^2e^4m}{2\hbar^2n^2}$$

$$a_{n+1}=\frac{2\sqrt{a}n-b}{n(n+1)}a_n$$

$$n=1\qquad a_2=\frac{2\sqrt{a}-b}{2}a_1$$

$$n=2\qquad a_3=\frac{4\sqrt{a}-b}{6}a_2=\frac{(2\sqrt{a}-\frac{b}{2})(2\sqrt{a}-b)}{6}$$

$$n=3\qquad a_4=\frac{6\sqrt{a}-b}{12}a_3=\frac{(2\sqrt{a}-\frac{b}{3})(2\sqrt{a}-\frac{b}{2})(2\sqrt{a}-b)}{24}$$

$$a_n=\frac{1}{n!}\prod_{k}^{n-1}\left(2\sqrt{a}-\frac{b}{k}\right)$$

$$\psi_{(x,y,z)}=Ne^{i(k_x x+k_y y)}\cdot e^{-\frac{mZ_{He}e^2}{\hbar^2 n}z}\sum_{n=1}\frac{1}{n!}\left(\frac{2mZ_{He}e^2}{\hbar^2}\right)^{n-1}\prod_{k}^{n-1}\left(\frac{1}{n}-\frac{1}{k}\right)z^n$$

מציאת אנרגיות אפשריות להמילטוניאן:

$$H=\frac{\alpha}{h}L\cdot S$$

$$L\cdot S=\frac{1}{2}(J^2-L^2-S^2)$$

$$H=\frac{\alpha}{2\hbar}(J^2-S^2-L^2)$$

נמצא את J האפשריים מכללי ברירה:

$$J=\frac{3}{2},\ \frac{1}{2}$$

$$E=\frac{\alpha\hbar}{2}\left(j(j+1)-l(l+1)-s(s+1)\right)$$

$$E_{\frac{3}{2}}=\frac{\alpha}{2\hbar}\left(\frac{15}{4}-2-\frac{3}{4}\right)=\frac{\alpha\hbar}{2}\ \longrightarrow$$ ניוון $2j+1=4$

$$E_{\frac{1}{2}}=\frac{\alpha}{2\hbar}\left(\frac{3}{4}-2-\frac{3}{4}\right)=-\alpha\hbar\ \longrightarrow$$ ניוון $2j+1=2$

הצגה מטריציונית של L:

$L_z=L_0$:

$$\sum_{mm'}\langle l,m'|L_0|l,m\rangle=\hbar m\,\delta_{mm'}$$

L^2:

$$\sum_{mm'}\langle l,m'|L^2|l,m\rangle=\hbar^2 l(l+1)\,\delta_{mm'}$$

$L_x=\frac{1}{2}(L_++L_-)$:

$$\frac{1}{2}\sum_{m,m}\langle l,m'|L_+|l,m\rangle+\langle l,m'|L_-|l,m\rangle=\frac{\hbar}{2}\left(\sqrt{l(l+1)-m(m+1)}\right)\delta_{m'(m+1)}+\left(\sqrt{l(l+1)-m(m-1)}\right)\delta_{m'(m-1)}\frac{\hbar}{2}$$

$L_y=\frac{1}{2}(L_+-L_-)$:

$$\frac{1}{2}\sum_{m,m}\langle l,m'|L_+|l,m\rangle-\langle l,m'|L_-|l,m\rangle=\frac{\hbar}{2}\left(\sqrt{l(l+1)-m(m+1)}\right)\delta_{m'(m+1)}-\sqrt{l(l+1)-m(m-1)}\,\delta_{m'(m-1)}\frac{\hbar}{2}$$

L_z:

$$\begin{array}{c|ccc} & -1 & 0 & 1\\ \hline -1 & -\hbar & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & \hbar\end{array}$$

L^2:

$$\begin{pmatrix}2\hbar^2 & 0 & 0\\ 0 & 2\hbar^2 & 0\\ 0 & 0 & 2\hbar^2\end{pmatrix}$$

L_x:

$$\begin{array}{c|ccc} & -1 & 0 & 1\\ \hline -1 & 0 & \frac{\hbar}{2}\sqrt{2} & 0\\ 0 & \frac{\hbar}{2}\sqrt{2} & 0 & \frac{\hbar}{2}\sqrt{2}\\ 1 & 0 & \frac{\hbar}{2}\sqrt{2} & 0\end{array}$$

L_y:

$$\begin{array}{c|ccc} & -1 & 0 & 1\\ \hline -1 & 0 & -\frac{\hbar}{2}\sqrt{2} & 0\\ 0 & \frac{\hbar}{2}\sqrt{2} & 0 & -\frac{\hbar}{2}\sqrt{2}\\ 1 & 0 & \frac{\hbar}{2}\sqrt{2} & 0\end{array}$$

* אם נתון המילטוניאן מהצורה $H=AL_z^2+L_x-L_y$ נציב מטריצות את המטריצות ונלכסן כדי למצוא את הפונקציות והערכים העצמיים.

תנועת חלקיק עם ספין תחת פוטנציאל:

$$H=\frac{p^2}{2m}+\hat{n}\cdot S\qquad V=\begin{cases}n\cdot S & r\le R\\ \infty & r>R\end{cases}$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2+S_n\right)\psi=E\psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi=(E-m_s)\psi$$

נפתור את החלק הרדיאלי ע"י הפרדת משתנים:

$$-\frac{\hbar}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)\right)=(E-m_s)\psi$$

$$\psi=\frac{u}{r}$$

$$-\frac{\hbar}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{u'}{r}-\frac{u}{r^2}\right)\right)\right)=(E-m_s)\psi$$

$$-\frac{\hbar}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru'-u\right)\right)=(E-m_s)u$$

$$-\frac{\hbar}{2m}\left(\frac{1}{r}\left(u'+ru''-u'\right)\right)=(E-m_s)u$$

$$u''=-\frac{2m}{\hbar}(E-m_s)u$$

$$u=A\sin(kr)+B\cos(kr)\qquad k=\sqrt{-\frac{2m}{\hbar}(E-m_s)}$$

$$u(r=0)=0\qquad\Rightarrow\qquad u=\sin(kr)$$
$$u(r=R)=0$$

$$0=A\sin(kR)\quad\Rightarrow\quad k=\frac{\pi n}{R}\quad\Rightarrow\quad E=-\frac{\pi^2n^2\hbar^2}{R^22m}+m_s$$

$$1=\int_0^R A^2\sin(kr)\sin(kr)\,dr$$

$$1=A^2\frac{R}{2}$$

$$A=\sqrt{\frac{2}{R}}$$

$$u=\sqrt{\frac{2}{R}}\sin(kr)$$

$$\begin{cases}\psi=\frac{u}{r}\cdot\varphi_{0,0}\cdot X_m & \leftarrow\ X_m=|\uparrow\rangle,|\downarrow\rangle\\ E=-\frac{\pi^2n^2\hbar^2}{R^22m}+m_s & \leftarrow\ m_s=\pm\frac{\hbar}{2}\end{cases}$$

* סימטריה להחלפת חלקיקים: (מקדמי קלפש ג'ורדן)
עבור $J=J_{max}$ יש סימטריה זוגית להחלפת חלקיקים
ובכל פעם כשיורדים ב-J הסימטריה מתחלפת.

מציאת הצגה בלתי פריקה של חבורה:

נבחר וקטורים ונראה שמייצגים את הפעולה:

$$V_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\quad V_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\quad V_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\quad V_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

$$T_{(123)}=\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\qquad T_{(12)(34)}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

אלה הצגות פריקות של החבורה.
כדי למצוא הצגה לא פריקה:

$$V_0=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\qquad\underbrace{V_1=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\quad V_2=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\quad V_3=\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}}_{\text{נמצא 3 וקטורים מאונכים אליו}}$$

נבחר וקטור שלא משתנה תחת הפעולה

$$P=\begin{pmatrix}1&1&0&0\\1&-1&1&0\\1&0&-1&1\\1&0&0&-1\end{pmatrix}$$

$$P^{-1}TP=\begin{pmatrix}1&0\\0&T_s(g)\end{pmatrix}\qquad \dim(T_s(g))=3,\quad \dim(1)=1$$

שתי הצגות בלתי פריקות ממימד 1, 3

$$P^{-1}T_{(123)}P=\begin{pmatrix}1&0&0&0\\0&0&-1&1\\0&1&-1&1\\0&0&0&1\end{pmatrix}$$

$$T_{s(123)}=\begin{pmatrix}0&-1&1\\1&-1&1\\0&0&1\end{pmatrix}$$

* מציאת מצב עצמי של ספין מסובב:

$$\hat{n}=\frac{B_0\hat{z}+B_1\hat{x}}{\sqrt{B_0^2+B_1^2}}$$

$$\cos\theta=\frac{\hat{n}\cdot\hat{z}}{\|\hat{n}\|\|\hat{z}\|}=\frac{B_0}{\sqrt{B_0^2+B_1^2}}$$

$$\theta=\cos^{-1}\left(\frac{B_0}{\sqrt{B_0^2+B_1^2}}\right)=\tan^{-1}\left(\frac{B_1}{B_0}\right)$$

$$\hat{n}=\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}$$

$$\vec{\sigma}\cdot\hat{n}=\sin\theta\cos\varphi\begin{pmatrix}0&1\\1&0\end{pmatrix}+\sin\theta\cos\varphi\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\cos\theta\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$
$$=\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}$$

$$\det(\vec{\sigma}\hat{n}-\lambda)=\lambda^2-\cos^2\theta-\sin^2\theta=\lambda^2-1=0$$

$$\boxed{\lambda=\pm1}$$

$\lambda=1$:

$$\begin{pmatrix}\cos\theta-1 & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow|\uparrow_n\rangle=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix}=\cos\frac{\theta}{2}|\uparrow\rangle+\sin\frac{\theta}{2}e^{i\varphi}|\downarrow\rangle$$

$\lambda=-1$:

$$\begin{pmatrix}\cos\theta+1 & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta+1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow|\downarrow_n\rangle=\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{-i\varphi}\end{pmatrix}=\sin\frac{\theta}{2}|\uparrow\rangle-\cos\frac{\theta}{2}e^{-i\varphi}|\downarrow\rangle$$

* מציאת ערכים עצמיים למצב ספין בשדה מגנטי מסתובב:

$$H=-\gamma S\cdot B$$

כאשר פועל שדה קבוע:

$$B_0=B_0\hat{z}$$

וגם שדה מסתובב:

$$B_1=B_1\cos(wt)\hat{x}+B_1\sin(wt)\hat{y}$$

$$H=w_0S_z+w_1(\cos(wt)S_x+\sin(wt)S_y)\qquad w_i=-\gamma B_i,\quad S_i=\frac{\hbar}{2}\sigma_i$$

נסובב את ההמילטוניאן:

$$H'=e^{\frac{i}{\hbar}wtS_z}He^{-\frac{i}{\hbar}wtS_z}-wS_z$$

כאשר:

$$e^{\frac{i}{\hbar}wtS_z}S_xe^{-\frac{i}{\hbar}wtS_z}=S_x\sin(wt)-S_y\sin(wt)$$
$$e^{\frac{i}{\hbar}wtS_z}S_ye^{-\frac{i}{\hbar}wtS_z}=S_x\sin(wt)+S_y\cos(wt)$$
$$e^{\frac{i}{\hbar}wtS_z}S_ze^{-\frac{i}{\hbar}wtS_z}=S_z$$

ונקבל:

$$H'=\frac{\hbar}{2}\begin{pmatrix}w_0-w & w_1\\ w_1 & w_0-w\end{pmatrix}$$

נלכסן בעזרת:

$$A=a_0I+a\cdot\sigma$$

$$\lambda_\pm=a_0\pm|a|$$

* קידום בזמן של מצב מסובב:

$$|\uparrow\rangle_z=\cos\frac{\theta}{2}|\uparrow\rangle-\sin\frac{\theta}{2}|\downarrow\rangle$$

$$|\uparrow(t)\rangle_z=e^{-\frac{i}{\hbar}Ht}|\uparrow\rangle_z=e^{-\frac{i}{\hbar}\gamma\beta}\cos\frac{\theta}{2}|\uparrow\rangle-e^{-\frac{i}{\hbar}\gamma\beta}\sin\frac{\theta}{2}|\downarrow\rangle$$

כדי למצוא הסתברות בזמן מסוים נציב את המצבים $|\uparrow\rangle_z$, $|\downarrow\rangle_z$ שוב במקום $|\uparrow\rangle$, $|\downarrow\rangle$.

תורת הפרעות מנוונת: $V = \lambda(x^2-y^2)$

נחליף את כל האופרטורים באופרטורים ספריים:

$$[L_0, A_\mu^{(\lambda)}] = \hbar\mu A_\mu^{(\lambda)}$$

$$[L_\pm, A_\mu^{(\lambda)}] = \hbar\sqrt{(\lambda\mp\mu)(\lambda\pm\mu+1)}\,A_{\mu\pm1}^{(\lambda)}$$

$$\boxed{\begin{array}{l}[\partial_z, z] = 1 \\ [\partial_{x,y}, z] = 0\end{array}}$$

$$z = \Theta_\mu^\lambda = \Theta_0^1$$

$$L_x = -i(y\partial_z - z\partial_y) \quad , \quad L_y = -i(z\partial_x - x\partial_z) \quad , \quad L_\pm = L_x \pm iL_y$$

$$[L_z, z^2] = 0 \Rightarrow \mu = 0$$

נשתמש באופרטור סולם כדי לדעת מתי מתאפסים ולפי כך את λ:

$$[L_+, z^2] = [-i(y\partial_z - z\partial_y) + (z\partial_x - x\partial_z), z^2] = -i2yz - 2xz$$

$$[L_+, -i2yz - 2xz] = [-i(y\partial_z - z\partial_y) + (z\partial_x - x\partial_z), -i2yz - 2xz] = (x+iy)^2$$

$$[L_+, (x+iy)^2] = [-i(y\partial_z - z\partial_y) + (z\partial_x - x\partial_z), (x+iy)^2] = 0 \Rightarrow \boxed{\lambda = 2}$$

$$[L_+, z^2] = \sqrt{\lambda(\lambda+1) + \mu(\mu+1)}\;\Theta_1^2 = -i2yz - 2xz$$

$$[L_+^2, z^2] = \sqrt{\lambda(\lambda+1) - \mu(\mu+1)}\;\Theta_2^2 = 2\,\Theta_2^2 = (x+iy)^2$$

$$[L_-^2, z^2] = \sqrt{\lambda(\lambda+1) + \mu(\mu-1)}\;\Theta_{-2}^2 = 2\,\Theta_{-2}^2 = (x-iy)^2$$

$$\Downarrow$$

$$\Theta_{-2}^2 = \frac{1}{2}(x-iy)^2 = \frac{1}{2}(x^2 - ixy - y^2)$$

$$\Theta_2^2 = \frac{1}{2}(x+iy)^2 = \frac{1}{2}(x^2 + ixy - y^2)$$

$$\Downarrow$$

$$V = \lambda(\Theta_2^2 + \Theta_{-2}^2)$$

כללי ברירה:

$$|\lambda\mu l'm'\rangle = |lm\rangle$$

$$m = m' + \mu \Rightarrow m = m' \pm 2 \quad \leftarrow \quad \Theta_2^2 \to 2, \;\; \Theta_{-2}^2 \to -2$$

$$|\lambda - l'| \le l \le \lambda + l' \Rightarrow |2 - l'| \le l \le 2 + l'$$

שיקולי זוגיות:

$$\langle l'm'|x^2|lm\rangle = \langle l'm'|\pi^\dagger\pi x^2 \pi^\dagger\pi|lm\rangle = (-1)^{l+l'}\langle l'm'|x|lm\rangle$$

$$\Downarrow$$

$$l + l' = \text{even}$$

$|nlm\rangle$

$$V = \lambda \cdot \left(\begin{array}{c|cccc} & 211 & 210 & 21{-1} & 200 \\ \hline |211\rangle & 0 & 0 & \alpha^* & 0 \\ |210\rangle & 0 & 0 & 0 & 0 \\ |21{-1}\rangle & 0 & 0 & 0 & 0 \\ |200\rangle & 0 & 0 & 0 & 0 \end{array} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right)$$

$$\qquad\qquad \Theta_{-2}^2 \qquad\qquad\qquad \Theta_2^2$$

$$V = \lambda \begin{pmatrix} 0 & 0 & \alpha^* & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

כלומר רק איבר אחד והצמוד אליו לא מתאפסים:
(רק $|211\rangle$, $|21{-1}\rangle$ מקבלים תיקון לאנרגיה).

$$\alpha = \langle 21{-1}|x^2-y^2|211\rangle = \int \psi_{21-1}^*(x^2-y^2)\psi_{211}\,d^3r$$

נמצא את α, נציב בהצגה ב-V ונלכסן את V כדי למצוא את המצבים העצמיים והאנרגיות העצמיות.

תורת הפרעות תלויה בזמן עבור אוסילטור הרמוני:

$$V(t) = \begin{cases} F_0 x & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$x = \sqrt{\frac{\hbar}{2m\omega_0}}(a + a^\dagger) \qquad H_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$V(t) = F_0 x(t) = \underbrace{F_0\sqrt{\frac{\hbar}{2m\omega_0}}}_{\equiv g}(a + a^\dagger)$$

$$|\psi(t)\rangle = \sum c_n(t) e^{-\frac{iH_0 t}{\hbar}}|\psi\rangle = \sum c_n(t) e^{-i\omega_n t}|\psi_n\rangle$$

$$|\psi_0\rangle = |\psi_0\rangle$$

סדר שני:

תנאי התחלה:
$c_m(t_0) = 1$
$m = 0$

$$C_n^{(2)}(t) = -\frac{i}{\hbar}\sum_{mm'} c_m(t_0)\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, e^{i\omega_{nm'}t'} V_{nm'}(t')\, e^{i\omega_{m'm}t''} V_{m'm}(t'')$$

$$V_{mm'} = g\langle 0|a + a^\dagger|m'\rangle = g\langle 0|a|1\rangle = g \qquad m' = 1$$

$$C_n^{(2)}(t) = -\frac{ig}{\hbar}\int_0^t dt'\, e^{i\omega_{n1}t'} V_{n1}(t')\left(1 - e^{i\omega_{10}t'}\right)$$

$$V_{n1} = g\langle n|a + a^\dagger|1\rangle \longrightarrow g\langle 0|a|1\rangle = g \qquad n = 0$$

$$g\langle 2|a^\dagger|1\rangle = \sqrt{2}g \qquad n = 2$$

$$C_2^{(2)} = -\frac{ig^2\sqrt{2}}{\hbar}\int_0^t dt'\, e^{i\omega_{21}t'}\left(1 - e^{i\omega_{10}t'}\right) = -\frac{ig^2\sqrt{2}}{\hbar}\left(\frac{e^{i\omega_{21}t} - 1}{i\omega_{21}} - \frac{e^{i(\omega_{21}+\omega_{10})t} - 1}{i(\omega_{21}+\omega_{10})}\right)$$

$$C_0^{(2)} = -\frac{ig^2}{\hbar}\int_0^t dt'\, e^{i\omega_{01}t'}\left(1 - e^{i\omega_{10}t'}\right) = -\frac{ig^2}{\hbar}\left(\frac{e^{i\omega_{01}t} - 1}{i\omega_{01}} - \frac{e^{i(\omega_{01}+\omega_{10})t} - 1}{i(\omega_{01}+\omega_{10})}\right)$$

סדר ראשון:

תנאי התחלה:
$c_m(t_0) = 1$
$m = 0$

$$C_n^{(1)} = -\frac{i}{\hbar}\sum_m c_m(t_0)\int_0^t dt'\, e^{i\omega_{nm}t'} V_{nm}(t')$$

$$= -\frac{i}{\hbar}\int_0^t dt'\, e^{i\omega_{n0}t'} V_{n0}(t')$$

$$V_{n0}(t') = g\langle n|(a + a^\dagger)|0\rangle = g\langle 1|a^\dagger|0\rangle = g \qquad n = 1$$

לכל n-ה אחרים
חוץ מ-1

$$C_1^{(1)}(t) = -\frac{i}{\hbar}g\int_0^t e^{i\omega_{10}t'}dt' = \frac{g}{\hbar\omega_{01}}\left(1 - e^{i\omega_{10}t}\right)$$

* מצב יסוד ורמה מעוררת של שני חלקיקים לבוש חופשיים בעלי ספין 1:

חלקיק חופשי:

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n}{L}x\right)$$

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

שני חלקיקים:

$$\psi_{nm} = \psi(x_1)\cdot\psi(x_2)\cdot\chi_{spin} = \frac{2}{L}\sin\left(\frac{\pi n}{L}x_1\right)\sin\left(\frac{\pi m}{L}x_2\right)\chi_{spin}$$

$$E_{nm} = \frac{\hbar^2\pi^2 n^2}{2mL^2} + \frac{\hbar^2\pi^2 m^2}{2mL^2} + \chi_{spin}$$

במצב היסוד:

$n = 1, \; m = 1$

$$\psi_{11} = \psi(x_1)\cdot\psi(x_2)\cdot\chi_{spin} = \frac{2}{L}\sin\left(\frac{\pi}{L}x_1\right)\sin\left(\frac{\pi}{L}x_2\right)\chi_{spin}$$

$$E_{11} = \frac{\hbar^2\pi^2}{mL^2} \quad \leftarrow \quad \text{ניוון 1 למצב הבסיס}$$

ספין 1 ⇐ לכל חלקיק יכול להיות $s = -1, 0, 1$.

$$S = s_1 + s_2$$

$$\Downarrow$$

$$|s_1 - s_2| \le S \le s_1 + s_2$$

$$\Downarrow$$

$$0 \le S \le 2$$

$S = 0$	$M = 0$	ניוון 1
$S = 1$	$M = -1, 0, 1$	ניוון 3
$S = 2$	$M = -2, -1, 0, 1, 2$	ניוון 5

אבל בגלל שיש לנו בוזונים נצטרך פונקציה כוללת סימטרית ובגלל שהפונקציה המרחבית סימטרית נצטרך גם ספין כולל סימטרי:

$$P_{12}|S, M\rangle = (-1)^{2s - S}|S, M\rangle \qquad s = 1$$

$S = 2$:

$$P_{12}|S, M\rangle = |S, M\rangle \qquad \text{סימטרי}$$

$S = 1$:

$$P_{12}|S, M\rangle = -|S, M\rangle \qquad \text{אנטי סימטרי}$$

$S = 0$:

$$P_{12}|S, M\rangle = |S, M\rangle \qquad \text{סימטרי}$$

$$\Downarrow$$

$(S = 2 + S = 0)$ ניוון 6

במצב מעורר ראשון אפשר לבנות פונקציה סימטרית אם ספין סימטרי או פונקציה אנטי סימטרית אם ספין אנטי סימטרי:

$$S = 0, 2 \qquad \psi_+ = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right]\cdot\chi_{spin\text{-}sym}$$

$$S = 1 \qquad \psi_- = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)\right]\cdot\chi_{spin\text{-}A\text{-}sym}$$

סך הכל ניוון 9.

$$E_n = \frac{5\pi^2\hbar^2}{2mL^2} + \chi_{spin}$$

(בגלל אורתוגונליות כיוון שאין ניוון בפונקציה המרחבית).

* 3 חלקיקים זהים בפוטנציאל הרמוני חד ממדי בעלי ספין:

מצב יסוד של אוסילטור הרמוני חד ממדי:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$E_0 = \hbar\omega\left(n + \frac{1}{2}\right) = \frac{1}{2}\hbar\omega$$

בגלל שאין ניוון (הוא מצב 0), מצב יסוד של 3 חלקיקים:

$$\psi_0 = \psi_{1,0}\cdot\psi_{2,0}\cdot\psi_{3,0} = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{3}{2}} e^{-\frac{m\omega(x_1^2 + x_2^2 + x_3^2)}{2\hbar}}$$

$$E_n = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right) = \frac{3}{2}\hbar\omega \qquad n_{1,2,3} = 0$$

ספין $\frac{1}{2}$: כל פרמיונים לא יכולים שני חלקיקים עם אותו ספין באותו מצב. לכן מצב יסוד: $n = 0$ לשני חלקיקים ומצב מעורר $n = 1$.

$$E = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right) = \hbar\omega\left(0 + 0 + 1 + \frac{3}{2}\right) = \frac{5}{2}\hbar\omega$$

$$\psi_0 = \frac{1}{\sqrt{3!}}\det\begin{pmatrix} \psi_0(x_1)\chi_{\frac{1}{2}}(s_1) & \psi_0(x_1)\chi_{-\frac{1}{2}}(s_1) & \psi_1(x_1)\chi(s_1) \\ \psi_0(x_2)\chi_{\frac{1}{2}}(s_2) & \psi_0(x_2)\chi_{-\frac{1}{2}}(s_2) & \psi_1(x_2)\chi(s_2) \\ \psi_0(x_3)\chi_{\frac{1}{2}}(s_3) & \psi_0(x_3)\chi_{-\frac{1}{2}}(s_3) & \psi_1(x_3)\chi(s_3) \end{pmatrix}$$

$\psi_1(x_n)\chi(s_n)$ יכול להיות עם ספין $\frac{1}{2}$ או $-\frac{1}{2}$ ולכן ניוון 2.