

Quantum Mechanics II Cheat Sheet

Ritz Theorem

$R[\psi] = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$. Step 1: Expand $|\psi\rangle$ in energy eigenbasis. $|\psi\rangle = \sum_n c_n |n\rangle$, $c_n = \langle n | \psi \rangle$. Then $\langle \psi | \psi \rangle = \sum_n |c_n|^2$. Using $H|n\rangle = E_n |n\rangle$, $\langle \psi | H | \psi \rangle = \sum_{n,m} c_m^* c_n \langle m | H | n \rangle = \sum_n |c_n|^2 E_n$. So $R[\psi] = \frac{\sum_n |c_n|^2 E_n}{\sum_n |c_n|^2}$. Step 2: Prove $R[\psi] \geq E_0$. Since $E_n \geq E_0$, $\sum_n |c_n|^2 E_n \geq E_0 \sum_n |c_n|^2$. $R[\psi] \geq E_0$.

Thm 3.3.2 Schur's Lemma for Hermitian Ops

Let T be an IRREP of a finite group G on V . If Hermitian operator $C : V \rightarrow V$ implies $\forall g \in G, T(g)C = CT(g)$, then $C \propto I$, i.e., $C = \lambda I, \lambda \in \mathbb{C}$. Proof: C Hermitian $\implies \exists$ eigenstate $v \in V$ s.t. $Cv = \lambda v$. Subspace $\text{span}\{T(g)v\}$ is invariant. Since T is IRREP, must be V . $\forall w \in V, Cw = C \sum a_g T(g)v = \sum a_g T(g)Cv = \lambda w$. Thus $C = \lambda I$.

Thm 4.2.1 Spectrum of Hamiltonian

$H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{a}{r^s}$ with $a > 0, s > 2$ is unbounded from below. Proof: Scaled state $\Psi(\mathbf{r}) = Ne^{-r^2/r_0^2}$. Expectation values: $\langle T \rangle \sim \frac{\hbar^2}{2mr_0^2}$, $\langle V \rangle \sim -\frac{a}{r_0^s}$. $E(r_0) \approx \frac{a}{r_0^2} - \frac{\beta}{r_0^s}$. For $s > 2$, as $r_0 \rightarrow 0$, $E \rightarrow -\infty$.

Thm 3.1.1 Operators equal iff expectations equal

$\langle a | A | a \rangle = \langle a | B | a \rangle \forall |a\rangle \iff A = B$. Proof: Use $|\psi\rangle = |a\rangle + |b\rangle$ and $|\psi\rangle = |a\rangle + i|b\rangle$ to show $\langle a | A | b \rangle = \langle a | B | b \rangle$.

Thm 4.1.1 Ritz Theorem

$E = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \geq E_1$. Equality iff Ψ is ground state.

Thm 4.1.2 Generalized Ritz

Expectation value of H is stationary in neighborhood of eigenvalues. $\delta E(\Psi) = 0 \iff H\Psi = E\Psi$.

Thm 4.1.3 Variance Theorem

$\sigma^2 = \frac{\langle \Psi | (H-E)^2 | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \langle H^2 \rangle - E^2$. There is at least one eigenval in $[E - \sigma, E + \sigma]$.

Thm 5.4.1 Upper bound on j_0

Given Hilbert space with basis $\{|ab\rangle\}$ of J^2, J_0 with eigenvalues a, b . If $a \geq b^2$. Proof: $J^2 = J_0^2 + \frac{1}{2}(J_- J_+ + J_+ J_-) \implies a - b^2 \geq 0$.

Thm 3.1.2 Ops equal within phase

$A = e^{i\theta} B \iff |\langle a | A | b \rangle| = |\langle a | B | b \rangle|$.

Thm 3.1.3 Scalar Product Preserving

If $T : \mathcal{V} \rightarrow \mathcal{V}$ preserves scalar product magnitude $|\langle \phi | \psi \rangle| = |\langle T\phi | T\psi \rangle|$, then T is unitary or anti-unitary. (Wigner's Theorem).

Thm 4.2.2 For $s < 2$ spectrum of H

$H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{a}{r^s}$ ($a > 0$) contains infinite bound states.

Proof: Trial $\Psi(r) = Ne^{-(r-r_0)^2/\beta^2 r_0^2}$. For large r_0 , $E < 0$ is possible.

Thm 8.2.2 Triangular Rule

Admissible j are $|j_1 - j_2| \leq j \leq j_1 + j_2$.

Thm 12.3.1 Ops S and A

Operators S and A satisfy: (a) $S^\dagger = S, A^\dagger = A$. (b) Commute with P_g for all $g \in S_N$. $P_g S = S P_g = S$, $P_g A = A P_g = \text{sign}(g)A$. (c) Orthogonal projectors of \mathcal{H}_{so} . $S^2 = S, A^2 = A, S A = A S = 0$. Proof: $S = \frac{1}{N!} \sum_g P_g$, $A = \frac{1}{N!} \sum_g \text{sign}(g) P_g$. $P_g A = \text{sign}(g) A$.

Landau Levels Derivation (Handwritten)

$H = \frac{1}{2m} (\mathbf{p} - q\mathbf{A})^2$. $B = B\hat{z}$. Gauge $\mathbf{A} = (-By, 0, 0)$. $H = \frac{1}{2m} ((p_x + qBy)^2 + p_y^2 + p_z^2)$. $[p_x, H] = 0 \implies p_x \rightarrow \hbar k$. $H = \frac{p_y^2}{2m} + \frac{1}{2} m \omega_c^2 (y - y_0)^2 + \frac{\hbar^2 k_z^2}{2m}$. Harmonic oscillator centered at $y_0 = -\frac{\hbar k}{qB}$. $\omega_c = \frac{qB}{m}$. $E = \hbar \omega_c (n + \frac{1}{2}) + \frac{\hbar^2 k_z^2}{2m}$.

Thm 13.9.1 Optical Theorem

$\sigma_{tot}(k) = \frac{4\pi}{k} \text{Im} f_k(0)$. Proof: $f_k(\theta) = f(k, k') = -\frac{m}{2\pi\hbar^2} (2\pi\hbar)^3 \langle \mathbf{k} | T | \mathbf{k}' \rangle$. Use Lippmann-Schwinger: $\text{Im} \langle \mathbf{k} | T | \mathbf{k}' \rangle = \text{Im} \langle \mathbf{k} | V | \Psi_{\mathbf{k}}^{in} \rangle$. Principal value integral contour (semicircle over pole). $\frac{1}{E - H_0 + i\epsilon} = \text{Pr} \frac{1}{E - H_0} - i\pi \delta(E - H_0)$. Result: $\text{Im} f_k(0) = \frac{k}{4\pi} \int d\Omega' |f_k(\theta')|^2 = \frac{k}{4\pi} \sigma_{tot}$.

Thm 8.5.2 Eigenstates of J^2, J_0

If $\{|\Phi_{ajm}\rangle\}$ and $\{|\Psi_{\beta jm}\rangle\}$ are basis states of subspace V (eigenstates of J^2, J_0), then transformation matrix elements only depend on a, β (independent of j, m). $|\Psi_{\beta jm}\rangle = \sum_a \langle \Phi_{ajm} | \Psi_{\beta jm} \rangle | \Phi_{ajm} \rangle$. Elements $\langle \Phi_{ajm} | \Psi_{\beta jm} \rangle = \delta_{jj'} \delta_{mm'} \langle \Phi_{aj} | | \Psi_{\beta j} \rangle$. (Reduced matrix element).

CG Coefficients Example (Handwritten)

Addition of angular momentum $J = J_1 + J_2$. Max $m = j_1 + j_2$ is unique: $|j_1 j_2; j_1 + j_2, j_1 + j_2\rangle = |j_1, j_1\rangle |j_2, j_2\rangle$. Apply $J_- = J_{1-} + J_{2-}$ to find lower m states. Example calculation: $|j_1 j_2; j, m\rangle = \alpha | \dots \rangle + \beta | \dots \rangle$. Orthogonality used to find other states (like $j = j_1 + j_2 - 1$). Condition $\alpha^2 + \beta^2 = 1$. Values like $\alpha = \sqrt{\frac{j_1}{j_1 + j_2}}$, $\beta = -\sqrt{\frac{j_2}{j_1 + j_2}}$.

Hydrogen-like Half-Space

$V(z) = -Ze^2/z (z > 0), \infty (z < 0)$. Solving SE gives Rydberg states for odd parity (wavefunc must vanish at $z = 0$). $E_n = -\frac{Z^2 e^4 m}{2\hbar^2 n^2}$ (Same as H-atom).

Spin-Orbit Coupling

$H = \frac{\alpha}{2m} \mathbf{L} \cdot \mathbf{S} = \frac{\alpha}{2\hbar^2} (J^2 - L^2 - S^2)$. Energy shift: $\Delta E = \frac{\alpha}{2} (j(j+1) - l(l+1) - s(s+1))$. Splitting between states $j = l \pm 1/2$.

Angular Momentum Operators

$L_z |l, m\rangle = \hbar m |l, m\rangle$, $L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$. $L_{\pm} |l, m\rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} |l, m \pm 1\rangle$. $L_x = \frac{1}{2}(L_+ + L_-)$, $L_y = \frac{1}{2i}(L_+ - L_-)$.

Infinite Spherical Well

$V(r) = 0 (r < R), \infty (r > R)$. Radial eq becomes spherical Bessel eq. Sol: $R_{nl}(r) = A_{jl}(kr_n)$. Boundary $j_l(k_n R) = 0$. Energy: $E_{nl} = \frac{\hbar^2 k_{nl}^2}{2m}$ where $k_{nl} = x_{nl}/R$ (x_{nl} is n -th zero of j_l).

Permutation Group Representations (S_3)

Matrices for basis vectors (possibly defining specific rep):
 $T(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $T(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Finding eigenvalues/eigenvectors for these ops.

Spin in Magnetic Field (Rabi)

$H = -\gamma \mathbf{S} \cdot \mathbf{B}$. Let $\mathbf{B} = B_0 \hat{z} + B_1 (\cos \omega t \hat{x} + \sin \omega t \hat{y})$. Rotating frame approx. Transition probability (Rabi formula): $P(t) = \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2 \left(\frac{\sqrt{\Omega^2 + \Delta^2} t}{2} \right)$. Where $\Delta = \omega - \omega_0$ (detuning), $\Omega = \gamma B_1$ (Rabi freq). Spin precession: $\frac{d\langle \mathbf{S} \rangle}{dt} = \gamma \langle \mathbf{S} \rangle \times \mathbf{B}$.

Tensor Operators Example

Consider $V \propto (x^2 - y^2)$. Commutators: $[L_z, z^2] = 0$. Rewrite x, y using L_{\pm} ? No, using spherical harmonics. $x \pm iy \propto Y_{1,\pm 1}$. $(x \pm iy)^2 \propto Y_{2,\pm 2}$. Selection rules for matrix elements $\langle l', m' | V | l, m \rangle$: If V transforms like T_k^q (here $k = 2, q = \pm 2$), then $m' = m + q$. So $\Delta m = \pm 2$.

Time-Dependent Perturbation (HO)

$V(t) = F_0 x (0 < t < T)$. First order amp: $c_n^{(1)}(t) = -\frac{i}{\hbar} \int_0^t dt' \langle n | V(t') | i \rangle e^{i\omega_n t'}$. For HO $x \propto (a + a^\dagger)$. Selection: $\langle n | x | m \rangle \neq 0$ only if $n = m \pm 1$. Transition $0 \rightarrow 1$: $c_1^{(1)}(t) = -\frac{i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega}} \int_0^t dt' e^{i\omega t'}$. Result involves $\frac{e^{i\omega t} - 1}{i\omega}$.

Identical Particles

Two particles in 1D box. $E = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2 + n_2^2)$. Wavefunc $\Psi(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) \pm \psi_{n_1}(x_2)\psi_{n_2}(x_1)]\chi_{spin}$. Total Ψ must be: - Symmetric for Bosons. - Antisymmetric for Fermions. Spin States: - Triplet ($S = 1$): Symmetric. Requires Antisym spatial (Fermions) or Sym spatial (Bosons). - Singlet ($S = 0$): Antisymmetric. Requires Sym spatial (Fermions) or Antisym spatial (Bosons).

Slater Determinant

$$\text{N-fermion state: } \Psi(1, \dots, N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(1) & \dots & \psi_1(N) \\ \vdots & \ddots & \vdots \\ \psi_N(1) & \dots & \psi_N(N) \end{vmatrix}.$$

Zero if two particles in same state (Pauli exclusion).