Derivation 1

For a single particle, with constant mass, the equation of motion is

$$\vec{F} = m\vec{a}$$

If we differentiate the kinetic energy with respect to time, we get

$$\begin{aligned}\frac{dT}{dt} &= \frac{d}{dt}\frac{1}{2}m\vec{v}\cdot\vec{v} \\ &= m\vec{v}\cdot\vec{a} + \frac{1}{2}\dot{m}\vec{v}\cdot\vec{v}\end{aligned}$$

Since, m is constant, the second term is 0, and from the equation of motion, we know that $\vec{F} = m\vec{a}$, so the equation above becomes

$$\frac{dT}{dt} = \vec{F}\cdot\vec{v}.$$

If, m is not constant, then the equation of motion is

$$\vec{F} = m\vec{a} + \dot{m}\vec{v},$$

and differentiating mT with respect to time gives

$$\begin{aligned}\frac{dmT}{dt} &= \frac{d}{dt}\frac{1}{2}m^2\vec{v}\cdot\vec{v} \\ &= m^2\vec{v}\cdot\vec{a} + m\dot{m}\vec{v}\cdot\vec{v} \\ &= m\vec{v}\cdot(m\vec{a} + \dot{m}\vec{v}).\end{aligned}$$

From the equation of motion, it is clear that the term in parenthesis is just the force, so the equation becomes

$$\frac{dmT}{dt} = \vec{F}\cdot(m\vec{v}) = \vec{F}\cdot\vec{p}.$$

Exercise 12

First, we need to determine the work needed to move an object from the surface of the earth to infinitely far away from it (i.e. escape the earth's gravitational field). This is given by the following integral

$$\int_{r=R}^{\infty} -\frac{GmM}{r^2}dr = \frac{GmM}{R}.$$

So, the potential energy of a particle of mass m on the surface of the earth is $-\frac{GmM}{R}$. Thus, a particle must have a kinetic energy of at least $\frac{GmM}{R}$ in order to escape from the earth's gravity. This corresponds to a velocity of $\sqrt{\frac{2GM}{R}} \approx 11.187km/s$.

Exercise 13
Newton's second law states

$$F_{ext} = \frac{dP}{dt}$$

So, in order to apply newton's second law to the rocket, we need to determine the momentum P of the rocket-fuel system at an arbitrary time t. This momentum will be the sum of the momentum of the rocket plus the unspent fuel and the momentum of the ejected fuel. The momentum of the rocket is mv, where $m(t)$ is the mass of the rocket and $v(t)$ is the velocity of the rocket relative to the earth. To get the momentum of the ejected fuel, we need to integrate $(v - v')\dot{m}$ where v' is the velocity of the fuel relative to the rocket (so $v - v'$ is the velocity of the fuel relative to the earth) and $\dot{m}$ is assumed to be constant. So, the momentum of the rocket-fuel system at time t is

$$\begin{aligned} P(t) &= m(t)v(t) + \int_0^t (v - v')\dot{m}dt \\ &= mv + \dot{m}(x - v't). \end{aligned}$$

Now, all we need is to determine the external force. The problem says to assume a uniform gravitational field, so then the external force is $-mg$. Substituting these expressions into Newton's second law gives

$$\begin{aligned} F_{ext} &= \frac{dP}{dt} \\ -mg &= \dot{m}v + m\dot{v} + \dot{m}(v - v') \\ -mg &= m\dot{v} - \dot{m}v' \\ m\dot{v} &= -mg - \dot{m}v' \\ \dot{v} &= -g - \frac{\dot{m}}{m}v', \end{aligned}$$

which is the expected result. Now, we integrate both sides from time 0 to time T,

$$\begin{aligned} \int_0^T \dot{v}dt &= \int_0^T (-g - \frac{\dot{m}}{m}v')dt \\ v_f - v_i &= -gT - v'ln\left(\frac{m_e}{m_i}\right). \end{aligned} \tag{1}$$

where m_i is the initial mass, m_e is the mass of the empty rocket, and lets say m_f is the mass of the fuel. Notice that $T = m_f/\dot{m}$ assuming the rocket expends all of its fuel. Now let $r = m_f/m_e$, and then we can rewrite m_e/m_i as $1/(1+r)$, and m_f/m_e as $r/(1+r)$. Now, the problem asks to determine the fuel to rocket mass ratio (r) needed for a rocket to accelerate from rest to 11.2 km/s with an exhaust speed of 2.1 km/s and a mass loss rate of $1/60m_i$. Substituting these values into eq. 1 gives

$$11.2\text{km/s} = \left(-60g - 2.1 \times \ln\left(\frac{1}{1+r}\right)\right)\text{km/s}$$

With $60g = 60 \times .0098\text{km/s}$, this equation can be solved numerically to get $r = 273.056$.

Exercise 14

This is a two dimensional problem, and the generalized coordinates θ and ϕ are given in fig. 1 The position of the center of the rod in terms of the generalized

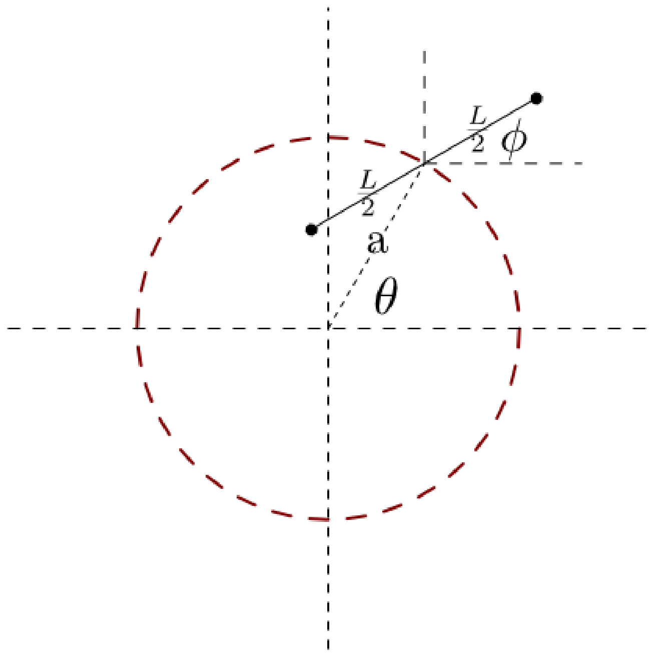

Figure 1: A graph showing the generalized coordinates θ, and ϕ. The two balls of mass m are held together by a rod of length L, the center of which is fixed to a circle of radius a.

coordinate θ is

$$\vec{rod} = a(\cos(\theta), \sin(\theta))$$

and the positions of the two masses are

$$\vec{r_1} = \vec{rod} + \frac{L}{2}(\cos\phi, \sin\phi)$$
$$\vec{r_2} = \vec{rod} - \frac{L}{2}(\cos\phi, \sin\phi)$$

so, the velocities are

$$\vec{v_1} = \frac{\partial}{\partial t}\vec{r_1} = a\dot{\theta}(-\sin(\theta), \cos(\theta)) + \frac{L}{2}\dot{\phi}(-\sin(\phi), \cos(\phi))$$
$$\vec{v_2} = \frac{\partial}{\partial t}\vec{r_2} = a\dot{\theta}(-\sin(\theta), \cos(\theta)) + \frac{L}{2}\dot{\phi}(\sin(\phi), -\cos(\phi)).$$

And, the total kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}m\left(\vec{v_1}\cdot\vec{v_1} + \vec{v_2}\cdot\vec{v_2}\right) \\
&= \frac{1}{2}m\left(\left(a\dot{\theta}\sin\theta + \frac{L}{2}\dot{\phi}\sin\phi\right)^2 + \left(a\dot{\theta}\cos\theta + \frac{L}{2}\dot{\phi}\cos\phi\right)^2 + \right. \\
&\qquad \left. \left(a\dot{\theta}\sin\theta - \frac{L}{2}\dot{\phi}\sin\phi\right)^2 + \left(a\dot{\theta}\cos\theta - \frac{L}{2}\dot{\phi}\cos\phi\right)^2\right) \\
&= \frac{1}{2}m\left(2a^2\dot{\theta}^2\sin^2\theta + \frac{2L^2}{4}\dot{\phi}\sin^2\phi + 2a^2\dot{\theta}^2\cos^2\theta + \frac{2L^2}{4}\dot{\phi}\cos^2\phi\right) \\
&= m(a^2\dot{\theta}^2 + \frac{L^2}{4}\dot{\phi}^2).
\end{aligned}$$

Exercise 18
The equations of motion for the Lagrangian,

$$L' = \frac{m}{2}\left(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2\right) - \frac{K}{2}\left(ax^2 + 2bxy + cy^2\right),$$

are

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} - \frac{\partial L'}{\partial x} &= 0 \\
ma\ddot{x} + mb\ddot{y} + Kax + Kby &= 0 \\
\frac{d}{dt}\frac{\partial L'}{\partial \dot{y}} - \frac{\partial L'}{\partial y} &= 0 \\
mc\ddot{y} + mb\ddot{x} + Kcy + Kbx &= 0.
\end{aligned}$$

If we let $\vec{a} = (\ddot{x}, \ddot{y})$ and $\vec{r} = (x, y)$, then the equation of motion becomes

$$\begin{aligned}
m\vec{a}\cdot(a, b) &= -K\vec{r}\cdot(a, b) \\
m\vec{a}\cdot(b, c) &= -K\vec{r}\cdot(b, c)
\end{aligned}$$

Remember that we can always multiply or divide an equation by a non-zero number without invalidating it. So, if we were to divide the first equation by $\sqrt{a^2+b^2}$, and the second by $\sqrt{b^2+c^2}$, we would get

$$\begin{aligned}
m\vec{a}\cdot\widehat{(a, b)} &= -K\vec{r}\cdot\widehat{(a, b)} \\
m\vec{a}\cdot\widehat{(b, c)} &= -K\vec{r}\cdot\widehat{(b, c)}
\end{aligned}$$

Where the hat signifies that the vector is a unit vector pointing in the same direction. Now, it is clear that the first equation relates the acceleration along the (a, b) direction to the position along the same direction, and the second equation does the same in the (b, c) direction. So, if we define new coordinates through a point transformation

$$\tilde{x} = \frac{ax + by}{\sqrt{a^2 + b^2}} = r \cdot \widehat{(a, b)}$$
$$\tilde{y} = \frac{bx + cy}{\sqrt{b^2 + c^2}} = r \cdot \widehat{(b, c)}$$

$$x = \frac{\tilde{y} b \sqrt{b^2 + c^2} - \tilde{x} c \sqrt{a^2 + b^2}}{\sqrt{b^2 - ac}}$$
$$y = \frac{\tilde{x} b \sqrt{a^2 + b^2} - \tilde{y} a \sqrt{b^2 + c^2}}{\sqrt{b^2 - ac}}$$

This point transformation basically exchanges the x and y axis for new axis pointing in the (b, c) and (a, b) directions. If we apply this transformation to the equations of motion, we get

$$m\vec{a} \cdot \widehat{(a, b)} = -K\vec{r} \cdot \widehat{(a, b)}$$
$$m\vec{a} \cdot \widehat{(b, c)} = -K\vec{r} \cdot \widehat{(b, c)}$$

$$m\ddot{\tilde{x}} = -k\tilde{x}$$
$$m\ddot{\tilde{y}} = -k\tilde{y},$$

and it becomes clear that this Lagrangian describes a spring-like central force system, where $\vec{F} = -K\vec{r}$. The usual Lagrangian for this system is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{K}{2}\left(x^2 + y^2\right),$$

and it is related to the given Lagrangian by the point transformation discussed above. The two cases that the problem asks you to look at: $a = 0 = c$, and $b = 0, a = -c$ correspond to no change in the axis, since the new axis are $(0, 1), (1, 0)$ and $(1, 0), (0, 1)$ respectively. Also, now the condition on $b^2 - ac$ can be reinterpreted as the condition that the new axes (a, b), and (b, c), must not be parallel, since then the transformation would not be one to one.

$$(a\hat{,}b) \neq (b\hat{,}c)$$
$$\sqrt{b^2+c^2}(a,b) \neq \sqrt{a^2+b^2}(b,c)$$

$$\sqrt{b^2+c^2}a \neq \sqrt{a^2+b^2}b$$
$$\sqrt{b^2+c^2}b \neq \sqrt{a^2+b^2}c$$

$$(b^2+c^2)a \neq (a^2+b^2)b$$
$$(b^2+c^2)b \neq (a^2+b^2)c$$

$$\frac{a}{b} \neq \frac{b}{c}$$
$$b^2 \neq ac$$
$$b^2 - ac \neq 0.$$

In fact, it is even apparent from the form of the point transformation that $b^2 - ac \neq 0$ since this term appears in the denominator.