Chap 9 der 2
We are given the transformation

$$Q = q\cos\alpha - p\sin\alpha$$
$$P = q\sin\alpha + p\cos\alpha.$$

Lets see if this transformation satisfies the symplectic condition. The matrix M is given by

$$\begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

If, we evaluate the product MJM^{T}, we get

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

thus this transformation satisfies the symplectic condition. Now, we need to find a generating function for this transformation. I have decided to find a function of the first kind. First, I will need to solve for p

$$Q = q\cos\alpha - p\sin\alpha$$
$$P = q\sin\alpha + p\cos\alpha$$

$$p = \frac{q\cos\alpha - Q}{\sin\alpha}$$
$$p = \frac{P - q\sin\alpha}{\cos\alpha}.$$

Notice that I can solve for p in terms of q and Q, or in terms of q and P. The first solution is valid only if $\sin\alpha \neq 0$, and the second is valid only if $\cos\alpha \neq 0$. Since I have decided to find a generating function of the first kind, I will need to use the solution for p in terms of q and Q. This means I must restrict myself to the case where $\sin\alpha \neq 0$, or equivalently $\alpha \neq n\pi$ $n \in \mathbb{Z}$. For these points, the transformation simplifies to the identity transformation, whose generating function is well known. Now, we can solve for the generating function

$$-\frac{\partial F_1}{\partial Q} = P$$
$$-\frac{\partial F_1}{\partial Q} = q\sin\alpha + p\cos\alpha$$
$$-\frac{\partial F_1}{\partial Q} = q\sin\alpha + \frac{q\cos\alpha - Q}{\sin\alpha}\cos\alpha$$
$$F_1 = -Qq\sin\alpha + \left(\frac{1}{2}Q^2 - Qq\cos\alpha\right)\cot\alpha + g(q).$$

However, we also require that

$$\begin{aligned}
\frac{\partial F_1}{\partial q} &= p \\
-Q\sin\alpha + (-Q\cos\alpha)\cot\alpha + g'(q) &= \frac{q\cos\alpha - Q}{\sin\alpha} \\
-Q\sin^2\alpha + -Q\cos^2\alpha + \sin\alpha g'(q) &= q\cos\alpha - Q \\
\sin\alpha g'(q) &= q\cos\alpha \\
g(q) &= \frac{1}{2}q^2\cot\alpha.
\end{aligned}$$

So, the generating function is

$$F_1 = -Qq\sin\alpha + \left(\frac{1}{2}Q^2 + \frac{1}{2}q^2 - Qq\cos\alpha\right)\cot\alpha + g(q).$$

The physical significance of the transformation at $\alpha = 0$, is the identity transformation, and the physical significance of the transformation at $\alpha = \frac{\pi}{2}$ is that coordinate is exchanged with the negative of momentum, and momentum is exchanged with coordinate. In the first case, the generating function fails, this is because a generating function of the first kind assumes that q and Q are independent, which is not the case if $q = Q$.

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The transformation is given by

$$\begin{aligned}
Q &= \frac{\alpha p}{x} \\
P &= \beta x^2.
\end{aligned}$$

In order for this transformation to be canonical, we must have

$$\begin{aligned}
[Q, P]_{qp} &= 1 \\
\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} &= 1 \\
-2\alpha\beta &= 1 \\
\beta &= -\frac{1}{2\alpha}.
\end{aligned}$$

Thus, the transformation is only canonical if $\beta = -\frac{1}{2\alpha}$. A generating function of the first kind for this transformation is

$$F_1 = \frac{Qx^2}{2\alpha}.$$

This is easy to verify

$$\begin{aligned}
\frac{\partial F_1}{\partial x} &= \frac{Qx}{\alpha} = p \\
-\frac{\partial F_1}{\partial Q} &= -\frac{x^2}{\alpha} = P.
\end{aligned}$$

The Hamiltonian in the new variables is

$$H = \frac{p^2}{2m} + \frac{kx^2}{2} = -\frac{Q^2 P}{m\alpha} - Pk\alpha,$$

and the equations of motion are

$$\begin{aligned}\dot{P} &= -\frac{\partial H}{\partial Q} \\ \dot{Q} &= \frac{\partial H}{\partial P} \\ \dot{P} &= \frac{2QP}{m\alpha} \\ \dot{Q} &= -\frac{Q^2}{m\alpha} - k\alpha.\end{aligned}$$

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In terms of Poisson brackets, the canonical condition is the following system of equations

$$\begin{aligned}\frac{\partial P_1}{\partial p_1} + \frac{\partial P_1}{\partial p_2} &= 0 \\ 2q_1 \frac{\partial P_2}{\partial p_1} &= 0 \\ 2q_1 \frac{\partial P_1}{\partial p_1} &= 1 \\ q_2 \frac{\partial P_2}{\partial p_1} + q_1 \frac{\partial P_2}{\partial p_2} &= 1.\end{aligned}$$

One of the poisson brackets is zero for any choice of P, and the last poisson bracket I will deal with later. Here is the most general form for P_1 and P_2

$$\begin{aligned}P_1 &= \frac{p_1 - p_2}{2q_1} + f_1(q_1, q_2) \\ P_2 &= p_2 + f_2(q_1, q_2).\end{aligned}$$

Now, a good choice for P_1 and P_2 is

$$\begin{aligned}P_1 &= \frac{p_1 - p_2}{2q_1} \\ P_2 &= p_2 + (q_1 + q_2)^2.\end{aligned}$$

The hamiltonian can be written entirely in terms of P_1 and P_2, but lets see if this transformation is canonical. The last poisson bracket is

$$\begin{aligned}\frac{\partial P_1}{\partial p_1}\frac{\partial P_2}{\partial q_1} + \frac{\partial P_1}{\partial p_2}\frac{\partial P_2}{\partial q_2} - \frac{\partial P_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} &= 0 \\ \frac{partial f_1}{\partial q_1} - \left(\frac{\partial f_2}{\partial q_1}\frac{1}{2q_1} - \frac{\partial f_2}{\partial q_2}\frac{1}{2q_1}\right) &= 0.\end{aligned}$$

So, if $f_1 = 0$ and $f_2 = (q_1 + q_2)^2$, then this condition will be satisfied so the transformation will be canonical. The new hamiltonian is

$$\begin{aligned} H &= \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2 \\ &= (P_1)^2 + P_2. \end{aligned}$$

So, the equations of motion are

$$\begin{aligned} P_1 &= \alpha \\ P_2 &= \beta \\ Q_1 &= 2\alpha t + c_1 \\ Q_2 &= t + c_2. \end{aligned}$$

With a little bit more algebra, these equations can be put in terms of q_1, q_2, and p_1, p_2 using the inverse of the original canonical transformation. In this for the solution will probably not be linear, so by doing a transformation we essentially solved the problem without having to solve the original Hamilton's equations.

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a)

I will show that this transformation is canonical using Poisson brackets

$$\begin{aligned} [Q, P]_{qp} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ [Q, P]_{qp} &= \frac{i\alpha}{2i\alpha} + \frac{1}{2} = 1. \end{aligned}$$

Thus, the transformation is canonical. A generating function of the first kind is given by

$$F_1 = -\frac{Q^2}{4i\alpha} + qQ - \frac{i\alpha}{2}q^2.$$

This can be easily verified

$$\begin{aligned} -\frac{\partial F_1}{\partial Q} &= \frac{Q}{2i\alpha} - q \\ &= \frac{Q - 2qi\alpha}{2i\alpha} = \frac{p - qi\alpha}{2i\alpha} = P \end{aligned}$$

$$\frac{\partial F_1}{\partial q} = Q - i\alpha q = p.$$

b)

In terms of there new variables, the solution to the harmonic oscillator is given by

$$\dot{Q} = -kq + i\alpha\frac{p}{m} = \frac{Q}{m} - P\frac{(1+k)2i\alpha}{(1-i\alpha)m}$$
$$\dot{P} = \frac{-kq}{2i\alpha} - \frac{p}{2m} = \frac{-Q}{2mi\alpha} - P\frac{(k-1)}{m(1-i\alpha)}$$

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a)
The equations of motion are given by:

$$\dot{p} = -\frac{\partial H}{\partial q}$$
$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = \frac{1}{2q} + 2p^2q^3$$
$$\dot{q} = pq^4.$$

b)
The following transformation:

$$q = \frac{1}{Q\sqrt{k}}$$
$$p = \frac{PQ^2k}{\sqrt{m}},$$

will make the Hamiltonian in part a reduce to a harmonic oscillator. However, this transformation is not canonical, since $[Q, P] = \sqrt{\frac{k}{m}}$. The only canonical transformation I could find was

$$q = \frac{\sqrt{m}}{P}$$
$$p = \frac{QP^2}{\sqrt{m}},$$

but this reduces to the special harmonic oscillator with $k = m$. For this transformation, the equation of motion from part a is

$$\dot{q} = pq^4$$
$$-\dot{P}\frac{\sqrt{m}}{P^2} = \frac{m^2}{P^2}\frac{Q}{\sqrt{m}}$$
$$\dot{P} = -mQ,$$

which is just the equation of motion for the harmonic oscillator with $k = m$. So, the equation from part a is satisfied.