Chap 8 der 3

If we wish to use $\dot{p}_i$ and $\dot{q}_i$ as our independent variables, then We will need the following Hamiltonian.

$$G(\dot{p}_i, \dot{q}_i, t) = q_i\dot{p}_i - L.$$

Its differential is

$$\begin{aligned}dG &= \dot{p}_i dq_i + q_i d\dot{p}_i - \dot{p}_i dq_i - p_i d\dot{q}_i - \frac{\partial L}{\partial t}dt\\ dG &= q_i d\dot{p}_i - p_i d\dot{q}_i - \frac{\partial L}{\partial t}dt.\end{aligned}$$

Of course we can also find the differential of G by taking partial derivatives with respect to $\dot{p}_i$ and $\dot{q}_i$. So, we have the following equations

$$\begin{aligned}q_i &= \frac{\partial G}{\partial \dot{p}_i}\\ -p_i &= \frac{\partial G}{\partial \dot{q}_i}\\ -\frac{\partial L}{\partial t} &= \frac{\partial H}{\partial t}.\end{aligned}$$

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So, the location of the mass is given by θ and x, see fig. 1 for a picture.

$$\begin{aligned}x_{\text{particle}} &= x + l\cos\theta\\ z &= ax^2 - l\cos\theta.\end{aligned}$$

So, the potential and kinetic energies are

$$\begin{aligned}T &= \frac{1}{2}m(\dot{x}^2_{\text{particle}} + \dot{z}^2)\\ &= \frac{1}{2}m\left(\left(\dot{x} - l\dot{\theta}\sin\theta\right)^2 + \left(2ax\dot{x} + l\dot{\theta}\sin\theta\right)^2\right)\end{aligned}$$

$$\begin{aligned}V &= mgz\\ &= mg\left(ax^2 - l\cos\theta\right).\end{aligned}$$

So, the canonical momenta are

$$p_x = m\left(\left(\dot{x} - l\dot{\theta}\sin\theta\right) + 2ax\left(2ax\dot{x} + l\dot{\theta}\sin\theta\right)\right)$$

$$p_\theta = ml\sin\theta\left(-\left(\dot{x} - l\dot{\theta}\sin\theta\right) + \left(2ax\dot{x} + l\dot{\theta}\sin\theta\right)\right)$$

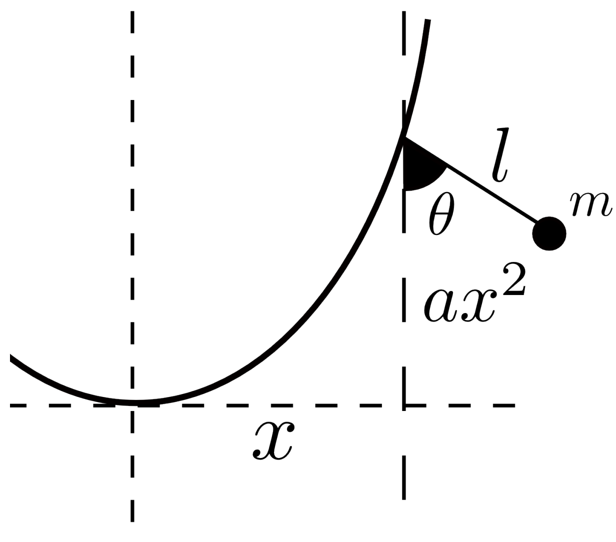

Figure 1: This figure shows the generalized coordinates for the pendulum problem.

Thus, the Hamiltonian is

$$
\begin{aligned}
H &= \dot{x}p_x + \dot{\theta}p_\theta - L \\
&= \dot{x}p_x + \dot{\theta}p_\theta - T + V \\
&= \dot{x}p_x + \dot{\theta}p_\theta - \frac{1}{2}\left(\dot{x}p_x + \dot{\theta}d\theta\right) + mg\left(ax^2 - l\cos\theta\right) \\
&= \frac{1}{2}\left(\dot{x}p_x + \dot{\theta}d\theta\right) + mg\left(ax^2 - l\cos\theta\right) \\
&= \frac{1}{2ml^2\sin^2\theta(1+2ax)^2}\left((p_\theta + lp_x\sin\theta)^2 + (2axp_\theta - lp_x\sin\theta)^2\right) + mg\left(ax^2 - l\cos\theta\right) \\
&= \frac{\mathcal{A}}{2}\left((p_\theta + lp_x\sin\theta)^2 + (2axp_\theta - lp_x\sin\theta)^2\right) + mg\left(ax^2 - l\cos\theta\right).
\end{aligned}
$$

So, essentially we replaced $\dot{\theta}$ with p_x and $\dot{x}$ with p_θ, and had to change the sign of the p_x terms. Also notice that as expected the Hamiltonian was just the total energy. The Hamiltonian equations of motion are

$$
\begin{aligned}
\dot{x} &= \mathcal{A}l\sin\theta\left((p_\theta + lp_x\sin\theta) - (2axp_\theta - lp_x\sin\theta)\right) \\
\dot{\theta} &= \mathcal{A}\left((p_\theta + lp_x\sin\theta) + 2ax\left(2axp_\theta - lp_x\sin\theta\right)\right) \\
-\dot{p}_x &= \mathcal{A}\left(2ap_\theta\left(2axp_\theta + lp_x\sin\theta\right)\right) + 2axmg \\
-\dot{p}_\theta &= \mathcal{A}lp_x\cos\theta\left((p_\theta + lp_x\sin\theta) - (2axp_\theta - lp_x\sin\theta)\right) + mg\left(ax^2 - l\cos\theta\right)
\end{aligned}
$$

A quick check of the units shows that a must have units of inverse length, but that otherwise the units match. **Chap 8 Ex 23**
a)
Notice that the curl of A as defined in the book will give a uniform magnetic field of strength B pointing in the z direction. Also, A can be defined as $A = (B \times (a_1 x\hat{\imath} + a_2 y\hat{\jmath}))$, with the condition that $a_1 + a_2 = 1$, so the book chose $a_1 = a_2 = \frac{1}{2}$, but one could also choose $a_1 = 0$ and $a_2 = 1$ and that could simplify the problem.

The kinetic and potential energies are

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ V &= V(r) - qA\dot{v} \\ &= V(r) - \frac{qB}{2}(x\dot{y} - \dot{x}y). \end{aligned}$$

So, the canonical momentum are

$$\begin{aligned} p_x &= m\dot{x} - \frac{qB}{2}y \\ p_y &= m\dot{y} + \frac{qB}{2}x. \end{aligned}$$

So, $\dot{x}$, and $\dot{y}$ are

$$\begin{aligned} m\dot{x} &= p_x + \frac{qB}{2}y \\ m\dot{y} &= p_y - \frac{qB}{2}x. \end{aligned}$$

Hamiltonian should be equal to the total energy again, but lets use the general formula one more time.

$$\begin{aligned} H &= p_i q_i - L \\ H &= \frac{1}{2m}\left[p_x^2 + p_y^2 - qB(yp_x - xp_y) - \frac{q^2B^2}{4}(x^2 + y^2)\right]. \end{aligned}$$

All of the algebra has been omitted because it was done on a computer, and again as expected the Hamiltonian is the total energy.
b)
We can derived ω, where ω is the angular velocity of the rotating body frame. We know that there must be a centripetal force in order for the particle to move in a circle, the magnetic field provides this force, but it can also be written in

terms of ω. Thus, we have

$$\begin{aligned} F_c &= -m\frac{d}{dt}(\omega r) \\ e\dot{r}B &= -m\frac{d}{dt}(\omega r) \\ erB &= -m(\omega r) \\ \omega &= -\frac{eB}{m}. \end{aligned}$$

Assuming that m, e, and B are fixed. This is where the book gets the ω value. So, it turns out that the magnetic force the charged particle experiences will cause it to rotate with angular velocity $-\frac{eB}{m}$ that is independent of its position or velocity, which means if we use the non-inertial reference frame the book suggests, the fictitious centrifugal forces should cancel with the magnetic force. We will define one general coordinate: r. Notice that there are three forces besides the central potential acting on the particle. They are the centrifugal force, the Coriolis force, and the magnetic force. Their magnitudes are

$$\begin{aligned} F_{\text{magnetic}} &= 2eB\dot{r}\hat{\theta} - eBr\omega\hat{r} \\ F_{\text{centrefugal}} &= m\omega^2 r\hat{r} \\ F_{\text{coriolis}} &= -2m\omega\dot{r}\hat{\theta}. \end{aligned}$$

Notice that all these forces perfectly cancel each other. So, our kinetic and potential energies are given by

$$\begin{aligned} T &= \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) \\ V &= V(r). \end{aligned}$$

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This problem involves a two dimensional motion because the mass can move along the track, and the cylinder can spin. Let generalized coordinates be l, the distance the mass has moved along the track, and θ the angle that the cylinder has rotated through. The kinetic and potential energies are given by

$$\begin{aligned} T &= \frac{1}{2}\left(M\frac{a^2}{2}\omega^2 + mf'(l)^2\dot{l}^2 + m(g'(l)\dot{l} + \omega a)^2\right) \\ &= \frac{1}{2}\left(M\frac{a^2}{2}\omega^2 + mf'(l)^2\dot{l}^2 + gm'(l)\dot{l}^2 + 2mg'(l)\omega a\dot{l} + m\omega^2 a^2\right) \\ &= \frac{1}{2}\left(M\frac{a^2}{2}\omega^2 + m\dot{l}^2 + 2mg'(l)\omega a\dot{l} + m\omega^2 a^2\right) \\ V &= -mgf(l), \end{aligned}$$

where M is the mass of the cylinder, and f is a monotonically increasing function that computes the vertical distance the mass has travelled given the distance it has moved along the track, and g is a function that computes the angular distance the mass has rotated. These functions depends on the shape of the track. So, the canonical momenta are

$$\begin{aligned} p_l &= m\dot{l} + g'(l)m\omega a \\ p_\theta &= \left(M\frac{a^2}{2}\omega + ma^2\omega + mg'(l)\dot{l}a\right) \end{aligned}$$

The Hamiltonian will be the total energy since the Lagrangian does not depend on time, and all forces are derivable from a conservative potential. Also keep in mind that since f and g return a distance, their derivative with respect to l will be unitless.

$$\begin{aligned} H &= T + V \\ &= \frac{1}{2}\left(M\frac{a^2}{2}\omega^2 + m\dot{l}^2 + 2mg'(l)\omega a\dot{l} + m\omega^2a^2\right) - mgf(l) \\ &= \frac{-1}{a^2m((-1+g^2)m - M)}\left(M\frac{a^2}{2}p_l^2 + mp_\theta^2 - 2mg'(l)p_lap_\theta + mp_l^2a^2\right) - mgf(l) \\ &= \mathcal{A}\left(M\frac{a^2}{2}p_l^2 + mp_\theta^2 - 2mg'(l)p_lap_\theta + mp_l^2a^2\right) - mgf(l) \end{aligned}$$

So, the Hamiltonian equations of motion are

$$\begin{aligned} \dot{l} &= \mathcal{A}\left(Ma^2p_l - 2mg'(l)ap_\theta + 2mp_la^2\right) \\ \dot{\theta} &= \mathcal{A}\left(2mp_\theta - 2mg'(l)p_la\right) \\ -\dot{p}_l &= -mgf'(l) - \mathcal{A}2mg''(l)p_lap_\theta \\ -\dot{p}_\theta &= 0. \end{aligned}$$

I cannot find a solution without knowing g and f. Lets assume that $f'(l) = c$ and $g'(l) = b$, so $g''(l) = 0$ and $c^2 + b^2 = 1$. This means that the track is shaped in such a way that the ratio of tangential to vertical motion remains constant. If this is the case, our equations of motion become.

$$\begin{aligned} \dot{l} &= \mathcal{A}\left(Ma^2p_l - 2mbap_\theta + 2mp_la^2\right) \\ \dot{\theta} &= \mathcal{A}\left(2mp_\theta - 2mbp_la\right) \\ -\dot{p}_l &= -mgc \\ -\dot{p}_\theta &= 0. \end{aligned}$$

So, the canonical momentum is given by

$$\begin{aligned} p_\theta &= \text{Constant} \\ p_l &= mgct. \end{aligned}$$

From here, the canonical momentum can by substituted into the equations for $\dot{l}$ and $\dot{\theta}$, and then we will have $\dot{l}$ and $\dot{\theta}$ as function of time. If we integrate once, we will even have l and θ as functions of time, but this algebra is unnecessary if you want a qualitative understanding of the motion. Basically, as time goes on, the mass picks up speed along the track, and the cylinder-mass system starts spinning in the counter clockwise direction the rate at which the system gains angular momentum depends on cb, and if $cb = 0$ (as in a flat track or a track going straight down), then there is no change in angular momentum. **Chap 8 Ex 26**

a)

Lets take the generalized coordinate to be x, the distance the mass is from the center of the system with values of x to the right of the center being positive. The kinetic and potential energies are

$$T = \frac{1}{2}m\dot{x}^2$$
$$U = \frac{1}{2}k_1\left(x - \frac{a}{2}\right)^2 + \frac{1}{2}k_2\left(x + \frac{a}{2}\right)^2$$

The Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k_1\left(x - \frac{a}{2}\right)^2 - \frac{1}{2}k_2\left(x + \frac{a}{2}\right)^2. \end{aligned}$$

The canonical momentum is $p_x = m\dot{x}$, and the Hamiltonian is

$$\begin{aligned} H &= T + U \\ &= \frac{p_x^2}{2m} + \frac{1}{2}k_1\left(x - \frac{a}{2}\right)^2 + \frac{1}{2}k_2\left(x + \frac{a}{2}\right)^2. \end{aligned}$$

The energy is conserved, since the Hamiltonian is just the total energy, and does not depend on time.

b)

Assuming that this coordinate is 0 at the center of the system, the kinetic and potential energies are

$$T = \frac{1}{2}m\left(\dot{Q}\right)^2$$
$$T = \frac{1}{2}m\left(\dot{q} - \omega b\cos\omega t\right)^2$$
$$U = \frac{1}{2}k_1\left(Q - \frac{a}{2}\right)^2 + \frac{1}{2}k_2\left(Q + \frac{a}{2}\right)^2$$

The Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m\left(\dot{Q}\right)^2 - \frac{1}{2}k_1\left(Q - \frac{a}{2}\right)^2 - \frac{1}{2}k_2\left(Q + \frac{a}{2}\right)^2. \end{aligned}$$

The canonical momentum is $p_Q = m\dot{Q}$. The Hamiltonian is

$$
\begin{aligned}
H &= \dot{Q}p_Q - L \\
&= \dot{Q}p_Q - \frac{1}{2}m\left(\dot{Q}\right)^2 + \frac{1}{2}k_1\left(Q - \frac{a}{2}\right)^2 + \frac{1}{2}k_2\left(Q + \frac{a}{2}\right)^2 \\
&= \frac{1}{2}m\left(\dot{Q}\right)^2 + \frac{1}{2}k_1\left(Q - \frac{a}{2}\right)^2 + \frac{1}{2}k_2\left(Q + \frac{a}{2}\right)^2.
\end{aligned}
$$

The Hamiltonian is not conserved, since it is an explicit function of time, the energy will also not be conserved.