Chap 9 der 15
a)
The transformation is

$$Q = q^\alpha \cos \beta p$$
$$P = q^\alpha \sin \beta p.$$

In order to be canonical, it must satisfy

$$\begin{aligned}
[Q, P] &= 1 \\
\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} &= 1 \\
\alpha\beta q^{2\alpha-1}\cos^2 \beta p + \alpha\beta q^{2\alpha-1}\sin^2 \beta p &= 1 \\
\alpha\beta q^{2\alpha-1} &= 1 \\
\frac{1}{\alpha\beta} &= q^{2\alpha-1}.
\end{aligned}$$

This equation will be satisfied for all q only when $2\alpha - 1 = 0$ so, we must have $\alpha = 1/2$ $\beta = 2$.
b)
If this could be an extended canonical transformation, then we would have

$$\begin{aligned}
[Q, P] &= \lambda \\
\alpha\beta q^{2\alpha-1}\cos^2 \beta p + \alpha\beta q^{2\alpha-1}\sin^2 \beta p &= \lambda \\
\frac{\lambda}{\alpha\beta} &= q^{2\alpha-1}.
\end{aligned}$$

Again $\alpha = 1/2$, but this time $\beta = 2\lambda$. The transformation function of the third kind, $F_3(Q, p)$ for this transformation must satisfy

$$\begin{aligned}
-\frac{\partial F_3}{\partial p} &= \lambda q \\
-\frac{\partial F_3}{\partial Q} &= P
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial F_3}{\partial p} &= \lambda \frac{Q^2}{\cos^2(\beta p)} \\
-\frac{\partial F_3}{\partial Q} &= Q \tan \beta p.
\end{aligned}$$

The transformation function satisfying these conditions is given by

$$F_3(Q, p) = -\frac{1}{2\beta} Q^2 \tan(\beta p).$$

c)
Well, the factor of λ appears because when we take a derivative with respect to p, we get a factor of $\beta = 2\lambda$. If the transformation function was instead

$$F_3(Q, p) = -\frac{1}{2\beta} Q^2 \tan(\beta p),$$

then when we take the derivative the β in the denominator would cancel the β we get from differentiating. So, the new equations would be

$$\begin{aligned} -\frac{\partial F_3}{\partial p} &= q \\ -\frac{\partial F_3}{\partial Q} &= P \end{aligned}$$

$$\begin{aligned} -\frac{\partial F_3}{\partial p} &= \frac{Q^2}{\cos^2(\beta p)} \\ -\frac{\partial F_3}{\partial Q} &= Q\frac{1}{\beta}\tan \beta p. \end{aligned}$$

Consequently, the modified transformation equations are

$$\begin{aligned} Q &= q^\alpha \cos(\beta p) \\ P &= q^\alpha \sin(\beta p)\frac{1}{\beta}. \end{aligned}$$

Chap 9 Ex 28
a)
First, we need to express the velocities in terms of the conjugate momentum and coordinate. Assuming this charged particle is not relativistic, the Lagrangian for a charged particle moving in a magnetic field is

$$\mathcal{L} = \frac{1}{2} m\dot{q}_i\dot{q}_i + e\dot{q}_i A_i$$

So, the conjugate momentum is given by

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}_i} \\ &= m\dot{q}_i + eA_i. \end{aligned}$$

In the case of a uniform magnetic field of magnitude B pointing in the z, or third dimension, this becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} m\dot{q}_i\dot{q}_i - eB\dot{q}_1 q_2 + eB\dot{q}_2 q_1 \\ p_1 &= m\dot{q}_1 - eBq_2 \\ p_2 &= m\dot{q}_2 + eBq_1 \\ p_3 &= m\dot{q}_3. \end{aligned}$$

So, the velocities in terms of q and p are

$$\begin{aligned} v_1 &= \frac{p_1 + eBq_2}{2m} \\ v_2 &= \frac{p_2 - eBq_1}{2m} \\ v_3 &= \frac{p_3}{2m}. \end{aligned}$$

Thus, the Poisson brackets are

$$\begin{aligned} [v_i, v_j] &= \frac{\partial v_i}{\partial q_k}\frac{\partial v_j}{\partial p_k} - \frac{\partial v_j}{\partial q_k}\frac{\partial v_i}{\partial p_k} \\ [v_1, v_2] &= \frac{eB + eB}{2m} = \frac{eB}{m} \\ [v_1, v_3] &= 0 \\ [v_2, v_3] &= 0. \end{aligned}$$

b)
Since, we already have v_i in terms of p and q, evaluating these Poisson brackets is straight forward

$$[q_i, v_j] = \frac{\partial v_j}{\partial p_i} = \frac{1}{2m}\delta_{ij}$$

$$\begin{aligned} [p_i, v_j] &= -\frac{\partial v_j}{\partial q_i} \\ [p_2, v_1] &= \frac{eB}{2m} \\ [p_1, v_2] &= -\frac{eB}{2m} \end{aligned}$$

$$\begin{aligned} [q_i, \dot{p_j}] &= -[v_i, p_j] = [p_j, v_i] \\ [q_2, \dot{p_1}] &= -\frac{eB}{2m} \\ [q_1, \dot{p_2}] &= \frac{eB}{2m} \end{aligned}$$

$$\begin{aligned} [p_i, \dot{p_j}] &= [p_i, \frac{\mathcal{L}}{q_j}] \\ [p_1, \dot{p_1}] &= eB[p_1, v_2] = -\frac{e^2B^2}{2m} \\ [p_1, \dot{p_2}] &= -eB[p_2, v_1] = -\frac{e^2B^2}{2m}. \end{aligned}$$

Poisson brackets equal to 0 have not been written out explicitly. **Chap 9 Ex 31**
The Hamiltonian for the 1D harmonic oscillator is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

If $u(q,p,t) = ln(p + im\omega q) - i\omega t$ is a constant of motion, then we must have

$$\begin{aligned}
[u, \mathcal{H}] &= -\frac{\partial u}{\partial t} \\
\frac{ip\omega - m\omega^2 q}{p + im\omega q} &= i\omega \\
ip\omega - m\omega^2 q &= i\omega p - m\omega^2 q.
\end{aligned}$$

Thus, it is a constant of motion. The physical significance probably has something to do with the symmetry of the harmonic oscillator. **Chap 9 Ex 32**
The Hamiltonian we are given is

$$\mathcal{H} = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2.$$

First, lets check if $F_1 = \frac{p_1 - a q_1}{q_2}$ is a constant of motion as in the last problem

$$\begin{aligned}
[F_1, \mathcal{H}] &= -\frac{\partial F_1}{\partial t} \\
\frac{-a}{q_2}(q_1) + \frac{p_1 - a q_1}{q_2} - \left(\frac{p_1 - 2a q_1}{q_2}\right) &= 0 \\
\frac{-2a q_1}{q_2} + \frac{2a q_1}{q_2} + \frac{p_1}{q_2} - \frac{p_1}{q_2} &= 0 \\
0 &= 0
\end{aligned}$$

Thus, F_1 is a constant of motion. Now, for $F_2 = q_1 q_2$

$$\begin{aligned}
[F_2, \mathcal{H}] &= -\frac{\partial F_2}{\partial t} \\
q_1 q_2 - q_1 q_2 &= 0
\end{aligned}$$

So, F_2 is also a constant of motion. Assuming a third constant of motion F_3, can somehow be generated using Jacobi's Identity, lets see what happens when we simplify it

$$\begin{aligned}
[F_1, [F_2, \mathcal{H}]] + [\mathcal{H}, [F_1, F_2]] + [F_2, [\mathcal{H}, F_1]] &= 0 \\
[\mathcal{H}, [F_1, F_2]] &= 0 \\
[\mathcal{H}, F_3] &= 0 \\
[F_3, \mathcal{H}] &= 0
\end{aligned}$$

So, $F_3 = [F_1, F_2]$.

Chap 9 Ex 36
a)
Using the theorem concerning Poisson brackets of vector functions and components of the angular momentum, show that if **F** and **G** are two vector functions of the coordinates and momenta only, then

$$[F \cdot L, G \cdot L] = L \cdot (G \times F) + L_i L_j [F_i, G_j]$$

So, lets start with the left hand side and show that it becomes the right hand side

$$\begin{aligned}
[F \cdot L, G \cdot L] &= [F_i L_i, G_j L_j] \\
&= F_i [L_i, G_j L_j] + L_i [F_i, G_j L_j] \\
&= F_i L_j [L_i, G_j] + F_i G_j [L_i, L_j] + L_i G_j [F_i, L_j] + L_i L_j [F_i, G_j] \\
&= L_i L_j [F_i, G_j] + F_i L_j [L_i, G_j] + F_i G_j [L_i, L_j] + L_i G_j [F_i, L_j] \\
&= L_i L_j [F_i, G_j] - F_i L_j [G_j, L_i] + F_i G_j [L_i, L_j] + L_i G_j [F_i, L_j] \\
&= L_i L_j [F_i, G_j] - F_i L_j \epsilon_{jik} G_k + F_i G_j \epsilon ijk L_k + L_i G_j \epsilon_{ijk} F_k \\
&= L_i L_j [F_i, G_j] - L \cdot (F \times G) + L \cdot (F \times G) + L_i (G \times F) \\
&= L_i L_j [F_i, G_j] + L_i (G \times F) \\
&= L_i (G \times F) + L_i L_j [F_i, G_j]
\end{aligned}$$

b)
Let F be the unit vector in the μ direction, G be the unit vector in the ν direction, and λ by the third direction (so if $\mu = 1$ and $\nu = 2$ then λ will equal 3. Ok, now if we substitute the value of F and G into the equation from part a we get

$$\begin{aligned}
[F \cdot L, G \cdot L] &= L \cdot (G \times F) + L_i L_j [F_i, G_j] \\
[F_i L_i, G_j L_j] &= L_i \epsilon_{ijk} G_j F_k + L_i L_j [F_i, G_j].
\end{aligned}$$

Now, we know that the only non-zero terms of F_k and G_j are F_μ and G_ν so the equation above becomes

$$\begin{aligned}
[F_i L_i, G_j L_j] &= L_i \epsilon_{ijk} G_j F_k + L_i L_j [F_i, G_j] \\
[F_\mu L_\mu, G_\nu L_\nu] &= L_i \epsilon_{i\nu\mu} G_\nu F_\mu + L_\mu L_\nu [F_\mu, G_\nu]
\end{aligned}$$

remember that $F_\mu = G_\nu = 1$ since these are unit vectors, so the expression above becomes further simplified

$$\begin{aligned}
[F_\mu L_\mu, G_\nu L_\nu] &= L_i \epsilon_{i\nu\mu} G_\nu F_\mu + L_\mu L_\nu [F_\mu, G_\nu] \\
[L_\mu, L_\nu] &= L_i \epsilon_{i\nu\mu} + L_\mu L_\nu [1, 1] \\
[L_\mu, L_\nu] &= L_i \epsilon_{i\nu\mu}.
\end{aligned}$$

Lets call the direction perpendicular to μ and ν as λ, so the equation above becomes

$$[L_\mu, L_\nu] = L_\lambda \epsilon_{\lambda\nu\mu}. [L_\mu, L_\nu] \qquad = I_\lambda \omega_\lambda \epsilon_{\lambda\nu\mu}.$$

Therefore all of the sums reduce to simply one expression, where λ is just equal to whatever μ and ν is not equal. In the case where $\mu = \nu$ it is unclear what to pick for λ, but in this case the epsilon is equal to zero since $\mu = \nu$ so it doesn't matter. **c)**
So, the equation of motion for L_i is

$$\begin{aligned} \frac{dL_i}{dt} &= [L_i, H] + \frac{\partial L_i}{\partial t} \\ \frac{dL_i}{dt} - [L_i, H] &= \frac{\partial L_i}{\partial t} \\ \frac{dL_i}{dt} - \epsilon_{ikj}\omega_j\omega_k I_k &= N_i \\ \frac{dL_i}{dt} + \epsilon_{ijk}\omega_j\omega_k I_k &= N_i. \end{aligned}$$