Chap 6 Ex 4
The potential energy is given by

$$V = -m_1 gl\cos\theta_1 - m_2 gl(\cos\theta_1 + \cos\theta_2)$$
$$V = -lg\left((m_1+m_2)\cos\theta_1 + m_2\cos\theta_2\right),$$

where m_1 is the mass of the upper weight, m_2 the mass of the lower weight θ_1 is the angle that the upper bar makes with the vertical axis, and θ_2 the angle the lower bar makes. The position of the lower weight is

$$\vec{r_2} = l(\cos\theta_1 + \cos\theta_1)\hat{j} + l(\sin\theta_1 + \sin\theta_1)\hat{\imath}.$$

So, the kinetic energy is

$$T = \frac{l^2}{2}\left(\dot{\theta}_1^2(m_1+m_2) + \dot{\theta}_2^2 m_2 + 2\dot{\theta}_1\dot{\theta}_2 m_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)\right)$$
$$T = \frac{l^2}{2}\left(\dot{\theta}_1^2(m_1+m_2) + \dot{\theta}_2^2 m_2 + 2\dot{\theta}_1\dot{\theta}_2 m_2\cos(\theta_1-\theta_2)\right).$$

If we use the small angle approximation, then the kinetic and potential energies become.

$$V = -\frac{lg}{2}\left((m_1+m_2)(2-\theta_1^2) + m_2(2-\theta_2^2)\right)$$
$$T = \frac{l^2}{2}\left(\dot{\theta}_1^2(m_1+m_2) + \dot{\theta}_2^2 m_2 + 2\dot{\theta}_1\dot{\theta}_2 m_2\right).$$

So, the T and V tensors are

$$V = lg\begin{bmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{bmatrix}$$
$$T = l^2\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}$$

So, we need to solve $det(V-\omega^2 T) = 0$, let $\lambda = \omega^2$

$$\begin{vmatrix} (g-l\lambda)(m_1+m_2) & -l\lambda m_2 \\ -l\lambda m_2 & (g-l\lambda)m_2 \end{vmatrix} = 0$$

This has the following solutions

$$\lambda = \frac{2g}{l}\left(1-\sqrt{\frac{m_2}{m_1}}\right), \frac{2g}{l}\left(1+\frac{2m_2}{m_1}+\sqrt{\frac{m_2}{m_1}}\right).$$

Notice that the units of ω^2 are Hz^2, as expected. Also if $m_1 >> m_2$, then we get that

$$\frac{m_2}{m_1} \approx 0$$

So, if $m_1 >> m_2$ the two frequencies are approximately equal to $\frac{2g}{l}$. The normal modes are

$$\left(\sqrt{\frac{m_2}{m_1+m_2}},1\right) \text{ and } \left(-\sqrt{\frac{\mathrm{m}_2}{\mathrm{m}_1+\mathrm{m}_2}},1\right)$$

where the first one corresponds to the eigenvalue $\frac{2g}{l}\left(1-\sqrt{\frac{m_2}{m_1}}\right)$. Notice that the eigenvectors are approximately equal if $m_1 >> m_2$. Let ω_1, and ω_2 denote the two eigenvalues, and $\vec{c_1}$, and $\vec{c_2}$ denote the two eigenvectors. The most general motion this system can have is given by

$$\vec{r}(t) = f_1 c_1 \cos(\omega_1 t + \delta_1) + f_2 \cos(\omega_2 t + \delta_2)$$

If we have the initial condition

$$\vec{r}(0) = (\theta_0, 0)$$
$$\dot{\vec{r}}(0) = (0, 0)$$

(i.e. the upper mass is slightly pulled away from the vertical, and the lower mass is allowed to hang free), then we have

$$\vec{r}(0) = (\theta_0, 0)$$
$$f_1 c_1 \cos(\delta_1) + f_2 c_2 \cos(\delta_2) = (\theta_0, 0)$$

$$\dot{\vec{r}}(0) = (0, 0)$$
$$f_1 c_1 \omega_1 \sin(\delta_1) + f_2 c_2 \omega_2 \sin(\delta_2) = (0, 0)$$

$$f_1 \cos(\delta_1) - f_2 \cos(\delta_2) = \theta_0 \sqrt{\frac{m_1+m_2}{m_2}}$$
$$f_1 \cos(\delta_1) + f_2 \cos(\delta_2) = 0$$

$$f_1 \omega_1 \sin(\delta_1) - f_2 \omega_2 \sin(\delta_2) = 0$$
$$f_1 \omega_1 \sin(\delta_1) + f_2 \omega_2 \sin(\delta_2) = 0$$

This equation can be solved for the amplitudes and phase shifts, but even without solving it, it can be seen that this equation will give rise to beats, since when $\cos(\omega_1 t+\delta_1) = 1$, and $\cos(\omega_2 t+\delta_2) = -1$ the first pendulum will have maximum amplitude, and $\sin(\omega_1 t + \delta_1) = \sin(\omega_2 t + \delta_2) = 0$, so the second pendulum will be at rest.

Chap 6 Ex 5
For the tri-atomic molecule, the eigenvectors and eigenvalues are

$$\omega_1 = 0$$
$$\vec{c}_1 = \frac{1}{\sqrt{2m+M}}(1,1,1)$$
$$\omega_2 = \sqrt{\frac{k}{m}}$$
$$\vec{c}_2 = \frac{1}{\sqrt{2m}}(1,0,-1)$$
$$\omega_3 = \sqrt{\frac{k}{m}\left(1+\frac{2m}{M}\right)}$$
$$\vec{c}_3 = \left(\frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}}, \frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}}, \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}}\right)$$

Assuming that there is no translational motion (f_1=0), then the general solution is

$$x_1(t) = \frac{1}{\sqrt{2m}} f_2 \cos(\omega_2 t + \delta_2) + \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} f_3 \cos(\omega_3 t + \delta_3)$$
$$x_2(t) = \frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} f_3 \cos(\omega_3 t + \delta_3)$$
$$x_3(t) = -\frac{1}{\sqrt{2m}} f_2 \cos(\omega_2 t + \delta_2) + \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} f_3 \cos(\omega_3 t + \delta_3)$$

a)
The initial conditions are that everyone starts at rest, and $x_2(0) = a_0$, with the

other masses being at equilibrium. The equations for x_2 gives

$$x_2(0) = a_0$$
$$\dot{x}_2(0) = 0$$

$$\frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} f_3 \cos(\delta_3) = a_0$$
$$\frac{2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} \omega_3 f_3 \sin(\delta_3) = 0$$

$$f_3 = \frac{a_0\sqrt{2M\left(2+\frac{M}{m}\right)}}{-2}$$
$$d_3 = 0.$$

From the other equations, we get

$$f_2 = \frac{-a_0}{2}\sqrt{2M\left(\frac{2+\frac{M}{m}}{1+\frac{2m}{M}}\right)}$$
$$d_2 = 0.$$

b)

If the middle mass had an initial velocity of v_0, then

$$x_2(0) = a_0$$
$$\dot{x}_2(0) = v_0$$

$$\frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} f_3 \cos(\delta_3) = a_0$$
$$\frac{2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} \omega_3 f_3 \sin(\delta_3) = v_0$$

for simplicity of notation, let $a = \frac{a_0\sqrt{2M\left(2+\frac{M}{m}\right)}}{-2}$ and $b = \frac{v_0\sqrt{2M\left(2+\frac{M}{m}\right)}}{2\omega_3}$, so the equations become

$$f_3 \cos(\delta_3) = a$$
$$f_3 \sin(\delta_3) = b$$

$$f_3 = \sqrt{a^2 + b^2}$$
$$d_3 = \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right).$$

Similarly, f_2 and d_2 can be solved for using the other initial condition equations

$$f_2 = \frac{1}{\sqrt{\left(1+\frac{2m}{M}\right)}} a$$
$$d_2 = \cos^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right).$$

Chap 6 Ex 8

The potential energy is given by

$$V = \frac{k}{2}\left((x_1-x_2)^2 + (x_3-x_2)^2 + (x_3-x_1)^2 + (y_1-y_2)^2 + (y_3-y_2)^2 + (y_3-y_1)^2\right).$$

So, the potential and kinetic energy tensors are given by

$$V = k\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

$$T = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix}$$

where the first three columns are the coordinates x_1, x_2, x_3, and the last three are y_1, y_2, y_3. So, the eigenvalues and eigenvectors are

$$\text{Translation in } x$$
$$\omega_1 = 0$$
$$\vec{c}_1 = (1, 1, 1, 0, 0, 0)$$

$$\text{Translation in } y$$
$$\omega_2 = 0$$
$$\vec{c}_2 = (0, 0, 0, 1, 1, 1)$$

$$\text{Oscillation between mass 1 and 2 in } y$$
$$\omega_3 = \sqrt{\frac{3k}{m}}$$
$$\vec{c}_2 = (0, 0, 0, 1, -1, 0)$$

$$\text{Oscillation between mass 1 and 3 in } y$$
$$\omega_3 = \sqrt{\frac{3k}{m}}$$
$$\vec{c}_2 = (0, 0, 0, 1, 0, -1)$$

$$\text{Oscillation between mass 1 and 2 in } x$$
$$\omega_3 = \sqrt{\frac{3k}{m}}$$
$$\vec{c}_2 = (1, -1, 0, 0, 0, 0)$$

$$\text{Oscillation between mass 1 and 3 in } x$$
$$\omega_3 = \sqrt{\frac{3k}{m}}$$
$$\vec{c}_2 = (1, 0, -1, 0, 0, 0)).$$

So, I found a double root instead of a triple root. This must mean that either there is a mistake in my expression for the kinetic or potential energy, or that I did not set up the system properly. **Chap 6 Ex 9**
The most general solution to the equation of motion of the last problem is

$$x_j = f_i c_{ij} \cos(\omega_i t + \delta_i).$$

Where there is a sum over i, and ω_i are the normal frequencies, and c_i are the corresponding eigenvectors. So, any solution that can be written in this way will satisfy the equations of motion. Clearly translation in x and y is a solution since those are the first two eigenvectors. I believe that rotation about the z should probably be the third eigenvector, but either there is a mistake in my expression for the kinetic or potential energy, or my system is not set up correctly.

Chap 6 Ex 14

Chap 6 Ex 18

Rectangular plate problem

The kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{z}^2 + \frac{1}{6}w^2\dot{\phi}^2 + \frac{1}{6}h^2\dot{\theta}^2\right).$$

The potential energy is given by

$$V = \frac{1}{2}k\left(4z^2 + w^2\sin^2\phi + h^2\sin^2\theta\right)$$
$$V = \frac{1}{2}k\left(4z^2 + w^2\phi^2 + h^2\theta^2\right)$$

So, the kinetic and potential energy tensors are given by

$$V = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$T = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

where the coordinates are z, $w\phi$, $h\theta$ in that order. The coordinates have been chosen so that the frequency has units of Hz. The normal modes and frequencies are

$$\text{Oscillation of plate up and down}$$
$$\omega_1 = \sqrt{\frac{4k}{m}}$$
$$\vec{c}_1 = (1, 0, 0, 0))$$
$$\text{Oscillation of plate right and left}$$
$$\omega_2 = \sqrt{\frac{6k}{m}}$$
$$\vec{c}_2 = (0, 1, 0))$$
$$\text{Oscillation of plate forward and backward}$$
$$\omega_3 = \sqrt{\frac{6k}{m}}$$
$$\vec{c}_3 = (0, 0, 1))$$

remember that the second two coordinates are $w\phi$ and $h\theta$ not ϕ and θ.