Ex 12

If, the Lagrangian depended on second derivatives of the coordinates, then when we vary it, we get

$$\delta I = \delta \int L(q_1, \ldots q_n, \dot{q}_1 \ldots \dot{q}_n, \ddot{q}_1 \ldots \ddot{q}_n, t) dt.$$

As usual, let the coordinates depend on t and some parameter α_i, such that $q_i(t, \alpha) = q_i(t, 0) + \alpha \eta_i(t)$, and $q_i(t, 0)$, is the solution that makes the action stationary. Then, the variation of I is given by

$$\delta I = \frac{\partial I}{\partial \alpha} = \int \sum_i \left(\frac{\partial L}{\partial q_i} \frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \alpha} + \frac{\partial L}{\partial \ddot{q}_i} \frac{\partial \ddot{q}_i}{\partial \alpha} \right) dt$$

We can integrate the second term by parts once, and the third term twice to get,

$$\delta I = \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} \right) \frac{\partial q_i}{\partial \alpha}.$$

The surface terms arising from the integration by parts go to zero, because there is no variation on the ends. This is then set to zero, in order to find the solution that minimizes the action, and using the fundamental lemma of the calculus of variations, we get

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}_i} = 0,$$

as expected. Now, if $L = -(m/2)q\ddot{q} - (k/2)q^2$, then we get

$$-\frac{m}{2}\ddot{q} - kq - \frac{d}{dt} 0 + \frac{d^2}{dt^2}' \left(\frac{m}{2} q \right) = 0,$$

and if the mass is constant, this becomes,

$$m\ddot{q} = -kq,$$

the equation for the 1D spring obeying Hooke's law.

Ex 14

Fig. 1 shows the coordinate system. In order to simplify the notation R_c is the constant radius of the cylinder, R_h is the constant radius of the hoop, and r is a generalized coordinate specifying the distance between the hoop and the cylinder. There are two constraints, one is that the hoop does not slip, which can be written as $R_2^2 \omega^2 = (r - R_2)^2 \dot{\theta}^2 + \dot{r}^2$. Since we are not asked to determine the force of constrain associated with this constraint (friction), we will not incorporate this constraint using a Lagrange multiplier, but rather we will let the choice of generalized coordinates take care of it. The second
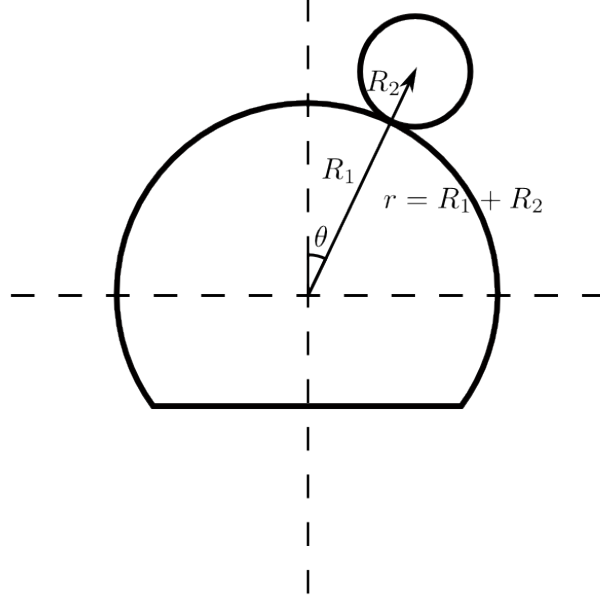
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Figure 1: A graph showing the generalized coordinates r, and θ. The uniform hoop of mass m and radius R_2 rolls without slipping on the surface of the fixed cylinder of radius R_2.

constraint, the that the hoop rolls along the cylinder, which can be written as $r = R_1 + R_2$. So, our Lagrangian is

$$L' = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 - mgr\cos\theta + \lambda(r - R_1 - R_2)$$

$$L' = \frac{1}{2}m((r - R_2)^2\dot{\theta}^2 + \dot{r}^2) + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta + \lambda(r - R_1 - R_2)$$

Now, we can find the equation of motion for r, and from that get an expression for λ

$$m(r - R_2)\dot{\theta}^2 + mr\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$m(2r - R_2)\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$m(2R_1 + R_2)\dot{\theta}^2 - mg\cos\theta + \lambda = 0, \tag{1}$$

note that all terms involving $\dot{r}$ have been dropped since $r = R_1 + R_2$. Now, we need an expression for $\dot{\theta}^2$ in terms of θ, we get this from conservation of energy, and for conciseness, the constraint $r = R_1 + R_2$ is applied to eliminate r. The initial energy of the system is $mg(R_1 + R_2)$, since initially, the system is at rest

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on top of the cylinder, thus

$$\frac{1}{2}mR_1\dot{\theta}^2 + \frac{1}{2}m\left((R_1+R_2)^2\dot{\theta}^2\right) + mg(R_1+R_2)\cos\theta = mg(R_1+R_2)$$

$$\dot{\theta}^2 = (1-\cos\theta)\frac{2g(R_1+R_2)}{(R_1+R_2)^2 + R_1^2}$$

$$\dot{\theta}^2 = (1-\cos\theta)\frac{Ag}{2R_1+R_2}.$$

Where A is $2(R_1+R_2)(2R_1+R_2)/\left((R_1+R_2)^2 + R_1^2\right)$. Now substituting this expression into eq. 1, gives

$$\lambda = mg(\cos\theta - A(1-\cos\theta))$$
$$\lambda = mg((A+1)\cos\theta - A).$$

The hoop will fall off once the constraint force becomes 0, so

$$0 = ((A+1)\cos\theta - A)$$
$$\cos\theta = \frac{A}{A+1}.$$

Notice that the hoop was instead a point mass (ie $R_2 = 0$), then we get

$$\cos\theta = \frac{2}{3},$$

which is the result that we got in class for the point mass.

Ex 16
The equations of motion for the Lagrangian

$$L = e^{yt}\left(m\frac{\dot{q}^2}{2} - \frac{kq^2}{2}\right)$$

are

$$-e^{yt}kq - \frac{d}{dt}(e^{yt}m\dot{q}) = 0,$$

and if $\dot{m} = 0$, this becomes

$$e^{yt}\left(-kq - ym\dot{q} - m\ddot{q}\right) = 0.$$

So, $m\ddot{q} = -kq - ym\dot{q}$, which means there are tow forces: the force of friction $-ym\dot{q}$, and the spring force $-kq$. So this is a damped 1D oscillator, or a 1D oscillator with friction The energy function for this Lagrangian is

$$h = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$
$$= e^{yt}\left(\frac{m\dot{q}^2}{2} + \frac{kq^2}{2}\right)$$
$$= e^{yt}\left(T + V\right),$$

where T is the kinetic, and V is the potential energy. Since this function depends explicitly on time, it is not conserved. In fact, we know that

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

$$ye^{yt}\left(T+V\right) + e^{yt}\frac{d}{dt}\left(T+V\right) = -ye^{ty}\left(T-V\right)$$

$$\frac{dE}{dt} = -2yT$$

$$\frac{dE}{dt} = -ym\dot{q}^2.$$

So, the energy is not conserved, but it dissipated by the second force (other than $-kq$).

Under the point transformation described in the book, the Lagrangian becomes

$$L = e^{-yt}\left(m\frac{(\dot{s}-ys)^2}{2} - \frac{ks^2}{2}\right),$$

and the equations of motion are

$$e^{-yt}\left(-ym(\dot{s}-ys)-ks\right) - \frac{d}{dt}\left(e^{-yt}m\dot{s}\right) = 0$$

$$e^{-yt}\left(-ym(\dot{s}-ys)-ks\right) + ye^{-yt}m\dot{s} - e^{-yt}m\ddot{s} = 0$$

$$e^{-yt}\left((y^2m-k)s - m\ddot{s}\right) = 0.$$

Ex 17

If, we are able to write the kinetic and potential energy terms as

$$T = \sum_i f_i(q_i)\dot{q}_i^2 \qquad\qquad V = \sum_i V_i(q_i)$$

Then, the equations of motion becomes

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\frac{\partial f_i(q_i)}{\partial q_i}\dot{q}_i^2 - \frac{\partial V_i(q_i)}{\partial q_i} - \frac{d}{dt}\left(2f_i(q_i)\dot{q}_i\right) = 0$$

$$-\frac{\partial f_i(q_i)}{\partial q_i}\dot{q}_i^2 - \frac{\partial V_i(q_i)}{\partial q_i} - 2f_i(q_i)\ddot{q}_i = 0$$

All other terms in the sum disappear because the other terms don't depend on q_i, so the partial differentiation yields 0. Thus, the terms separate into i ODEs.

Ex 20

Let, x, and y specify the position of the mass m, with y being vertical and x being horizontal, and the x_w be the position of the end of the wedge. Thus the constraint is $(x-x_w)\tan\alpha - y = 0$, and the Lagrangian is

$$L = \frac{1}{2}\left(M\dot{x}_w^2 + m(\dot{x}^2 + \dot{y}^2)\right) - mgy + \lambda(x - x_w - y\cot\alpha).$$

The equations of motion arising from this Lagrangian are

$$m\ddot{x} - \lambda = 0$$
$$m\ddot{y} + mg + \lambda \cot \alpha = 0$$
$$M\ddot{x}_w + \lambda = 0$$
$$\lambda = m\ddot{x} = -M\ddot{x}_w.$$

If we differentiate the constraint equation twice with respect to time, we get $\ddot{y} = (\ddot{x} - \ddot{x}_w)\tan \alpha$, which is the same as $m\ddot{y} = (\lambda + \lambda/M)\tan \alpha$. Substituting this expression into the equation of motion for y gives

$$\lambda \left(\frac{M+m}{M} \right) \tan mg + \lambda \cot \alpha = 0$$

$$\lambda = \frac{-mg}{\left(\frac{M+m}{M} \right) \tan \alpha + \cot \alpha}.$$

So, the forces of constraint are

$$Q_x = \lambda$$
$$Q_y = -\tan \alpha \lambda$$
$$Q_{x_w} = -\lambda.$$

In time t, these force do work

$$dW = -\lambda dx_w + \lambda dx - \lambda \tan \alpha dy$$

$$\int_0^t \dot{W} dt = \int_0^t \left(-\lambda \dot{x}_w + \lambda \dot{x} - \lambda \tan \alpha \dot{y} \right) dt$$

$$W = -\lambda(x_w(t) - x_w(0)) + \lambda(x(t) - x(0)) - \lambda \tan \alpha(y(t) - y(0))$$

$$W = \lambda \left(-(x_w(t) - x_w(0)) + (x(t) - x(0)) - \tan \alpha(y(t) - y(0)) \right)$$

$$W = 0,$$

where in the second to last step I used the constraint equation. So, the forces of constraint do no work. One of the constants of motion, will be the energy function, which in this case is just the total energy.

$$E = \frac{1}{2} \left(M\dot{x}_w^2 + m(\dot{x}^2 + \dot{y}^2) \right) + mgy.$$