INTRODUCTION TO GEOMETRIC ALGEBRA

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1. Introduction

1.1. **Engineering Vectors.** In engineering, three dimensional (3D) vectors are commonly presented under the form

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}\,,$$

where v_x, v_y and v_z are real numbers called the *components of the vector* and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ unit length vectors, *aka unit vectors*, pointing in every direction of space.

Then, two basic operations, *vector addition* and *multiplication by scalars*, are introduced:

$$\begin{aligned}\boldsymbol{v} + \boldsymbol{w} &= (v_x + w_x)\boldsymbol{i} + (v_y + w_y)\boldsymbol{j} + (v_z + w_z)\boldsymbol{k}\\ \alpha\boldsymbol{v} &= \alpha v_x\boldsymbol{i} + \alpha v_y\boldsymbol{j} + \alpha v_z\boldsymbol{k}\,,\end{aligned}$$

where α is a real number and $\boldsymbol{v}$ and $\boldsymbol{w}$ vectors. Both of these basic operations result in a vector.

Additionally to the basic operations, two other ones are introduced: *cross product* of vectors, which results in a vector and the *inner product* of vectors, which results in a real number, instead of in a vector.

1.2. **Inner Product.** Given two 3-D vectors, say $\boldsymbol{v}$ and $\boldsymbol{w}$, the inner product between them, $\boldsymbol{v}\cdot\boldsymbol{w}$, is defined by the formula

$$\boldsymbol{v}\cdot\boldsymbol{w} = v_xw_x + v_yw_y + v_zw_z\,.$$

2. Conventions

basis: a basis is a set of linearly independent vectors that, in a linear combination, can represent every vector in a given vector space or free module, or, more simply put, which define a "coordinate system" (as long as the basis is given a definite order). In more general terms, a basis is a linearly independent spanning set.

frame: An ordered basis.

In order to reduce cluttering and optimize vector manipulation, agree with the following

- ⋄ no decoration: drop any decoration for vectors; *i.e.*, use $\mathbf{u}$ to denote an element of $\mathcal{E}^n$, instead of $\boldsymbol{u}$, $\vec{u}$, or alike;
- ⋄ index notation: relabel the Cartesian axes according to $x \to x_1$, $y \to x_2$ and $z \to x_3$. The frame elements thus become γ_1, γ_2 and γ_3. Write then $\mathbf{u} \in \mathcal{E}^3$ as a linear combination of frame elements as $\mathbf{u} = \gamma_1u^1 + \gamma_2u^2 + \gamma_3u^3$ or more compactly $\mathbf{u} = \sum_{k=1}^3 \gamma_ku^k$;
- ⋄ Einstein summation convention: drop the summation symbol $\sum$; *i.e.*, $\mathbf{u} = \gamma_ku^k$, where the (repeated) index k is assumed to run from 1 to 3.

Example 2.1. *With these conventions, rewrite the definitions of vector addition, multiplication by scalars, inner and outer product of vectors.*

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Solution. Let $\alpha \in \mathcal{R}$, $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$ and the frame $\{\gamma_k;\, k : 1 \ldots n\}$. Then

$$\begin{aligned}
\alpha\mathbf{u} &\doteq \alpha\gamma_k u^k = \alpha u^k \gamma_k\,, && \text{[multiplication by scalars]}\\
\mathbf{u}\mathbf{v} &\doteq \gamma_k u^k + \gamma_k v^k = \gamma_k(u^k + v^k) = (u^k + v^k)\gamma_k\,, && \text{[vector addition]}\\
\mathbf{u}\cdot\mathbf{v} &\doteq \gamma_k u^k \cdot \gamma_l v^l = u^k(\gamma_k \cdot \gamma_l)v^l = u^k g_{kl} v^l\,, && \text{[inner product]}\\
\mathbf{u}\wedge\mathbf{v} &\doteq \gamma_k u^k \wedge \gamma_l v^l = u^k(\gamma_k \wedge \gamma_l)v^l = u^k v^l(\gamma_k \wedge \gamma_l)\,. && \text{[outer product]}
\end{aligned}$$

□

3. Exploratory Maths

Given two vectors, $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$, what would happen if we now multiply $\mathbf{u}, \mathbf{v}$ term wise?

Note term-wise multiplication of $\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{u} \otimes \mathbf{v}$.

To find the result, consider $\mathbf{u}, \mathbf{v} \in \mathcal{E}^2$ and a frame $\{\gamma_1, \gamma_2\}$ of orthonormal vectors. Term-wise multiply $\mathbf{u}$ and $\mathbf{v}$:

$$\begin{aligned}
\mathbf{u} \otimes \mathbf{v} &= \gamma_k u^k \otimes \mathbf{v} && [\mathbf{u} \text{ onto frame}]\\
&= \gamma_1 u^1 \otimes \mathbf{v} + \gamma_2 u^2 \otimes \mathbf{v} && [k \text{ from 1 to 2}]\\
&= \gamma_1 u^1 \otimes \gamma_l v^l + \gamma_2 u^2 \otimes \gamma_l v^l && [\mathbf{v} \text{ onto frame}]\\
&= \gamma_1 u^1 \otimes \gamma_1 v^1 + \gamma_1 u^1 \otimes \gamma_2 v^2 + \gamma_2 u^2 \otimes \gamma_1 v^1 + \gamma_2 u^2 \otimes \gamma_2 v^2 && [l \text{ from 1 to 2}]\\
&= u^1 v^1 \gamma_1 \otimes \gamma_1 + u^1 v^2 \gamma_1 \otimes \gamma_2 + u^2 v^1 \gamma_2 \otimes \gamma_1 + u^2 v^2 \gamma_2 \otimes \gamma_2 && [\text{rearranging scalars}]\\
&= (u^1 v^1 \gamma_1 \otimes \gamma_1 + u^2 v^2 \gamma_2 \otimes \gamma_2) + (u^1 v^2 \gamma_1 \otimes \gamma_2 + u^2 v^1 \gamma_2 \otimes \gamma_1) && [\text{agrouping terms}]
\end{aligned}$$

To recover the inner product and the outer product from $\mathbf{u} \otimes \mathbf{v}$, the frame elements must satisfy

$$\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_1 = 1 \qquad \text{and} \qquad \gamma_1 \otimes \gamma_2 = \gamma_k \gamma_l = -\gamma_l \gamma_k\,,$$

or, more abstractly,

4. Geometric Algebra

> [From the middle of the nineteenth century] the primary focus [of mathematicians] was no longer on performing a calculation or computing an answer, but formulating and understanding abstract concepts and relationships. [...] Mathematical objects were no longer thought of as given primarily by formulas, but rather as carriers of conceptual properties.
>
> — keith devlin, Introduction to Mathematical Thinking (Fall 2012) – Background Reading

4.1. **Geometric Product.** Let $\mathcal{L}$ be a linear space over $\mathcal{R}$ and let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{L}$. Then, assume a *geometric product of* $\mathbf{u}$ *and* $\mathbf{v}$, denoted $\mathbf{uv}$, that satisfies

(1) associativity: $(\mathbf{uv})\mathbf{w} = \mathbf{u}(\mathbf{vw})\,;$

(2) left distributivity: $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{uv} + \mathbf{uw}\,;$

(3) right distributivity: $(\mathbf{v} + \mathbf{w})\mathbf{u} = \mathbf{vu} + \mathbf{wu}\,;$

(4) contraction: $|\mathbf{u}|^2 \doteq \mathbf{u}^2 \doteq \mathbf{uu}$, where $|\mathbf{u}| \in \mathcal{R}$; *i.e.*, a scalar.

Call *vector space over* $\mathcal{R}$ a linear space $\mathcal{V}$ where the geometric product is defined and call *vectors* the elements of $\mathcal{V}$.

Note that, since $\mathcal{R}^n$ is a linear space, then, if the geometric product is also defined in $\mathcal{R}^n$, it turns $\mathcal{R}^n$ into a vector space. Call this vector space the *n-dimensional Euclidean space*, denoted $\mathcal{E}^n$. In other words, $\mathcal{E}^n$ is $\mathcal{R}^n$ equipped with a geometric product.

4.2. **Magnitude of Vectors.** Consider $\mathbf{u} \in \mathcal{E}^n$. Then, call the *magnitude of* $\mathbf{u}$, denoted $|\mathbf{u}|$, the quantity

$$|\mathbf{u}|^2 \doteq \mathbf{u}^2 \doteq \mathbf{u}\mathbf{u}\,.$$

Also, call $|\mathbf{u}|$ the *length of* $\mathbf{u}$. The expression $|\mathbf{u}| = 0$ implies $\mathbf{u} = \mathbf{0}$.

Equivalently, since, by axiom, $\mathbf{u}\mathbf{u} \in \mathcal{R}$, calculate the magnitude of $\mathbf{u}$ as

$$|\mathbf{u}| = \sqrt{\mathbf{u}^2} = \sqrt{\mathbf{u}\mathbf{u}}\,.$$

4.3. **Inverse of Vectors.** Let $\mathbf{u} \in \mathcal{E}^n$ and assume $\mathbf{u} \neq \mathbf{0}$. Then, define the *inverse of* $\mathbf{u}$, denoted $\mathbf{u}^{-1}$, by

$$\mathbf{u}^{-1} \doteq \frac{\mathbf{u}}{\mathbf{u}\mathbf{u}} = \frac{\mathbf{u}}{\mathbf{u}^2} = \frac{\mathbf{u}}{|\mathbf{u}|^2}\,.$$

Motivation. Contract the vector $\mathbf{u}$: $\mathbf{u}\mathbf{u} = \mathbf{u}^2$. Since, by assumption, $\mathbf{u} \neq 0$ and, by axiom, $\mathbf{u}^2 \in \mathcal{R}$, divide then both sides of $\mathbf{u}\mathbf{u} = \mathbf{u}^2$ by $\mathbf{u}^2$ to find

$$\mathbf{u}\frac{\mathbf{u}}{\mathbf{u}^2} = 1\,,$$

which motivates the definition. □

4.4. **Commutator and Anticommutator.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$ and $\mathbf{u}\mathbf{v}$ the geometric product of $\mathbf{u}$ and $\mathbf{v}$. Then, define the *commutator of* $\mathbf{u}$ *and* $\mathbf{v}$, denoted $[\mathbf{u}, \mathbf{v}]_-$, by

$$[\mathbf{u}, \mathbf{v}]_- \doteq \mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u}$$

and define the *anticommutator of* $\mathbf{u}$ *and* $\mathbf{v}$, denoted $[\mathbf{u}, \mathbf{v}]_+$, by

$$[\mathbf{u}, \mathbf{v}]_+ \doteq \mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}\,.$$

4.5. **Inner Product.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Define the *inner product of* $\mathbf{u}$ *and* $\mathbf{v}$, denoted $\mathbf{u}\cdot\mathbf{v}$, by

$$\mathbf{u}\cdot\mathbf{v} \doteq \tfrac{1}{2}\,[\mathbf{u}, \mathbf{v}]_+ = \tfrac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})\,.$$

By construction, the inner product is *symmetric*; *i.e.*, $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.

The inner product $\mathbf{u}\cdot\mathbf{v}$ results in a scalar.

Proof. Contract (square) the vector $(\mathbf{u} + \mathbf{v})$ and multiply terms:

$$(\mathbf{u} + \mathbf{v})^2 = (\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v}) = \mathbf{u}^2 + \mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} + \mathbf{v}^2\,,$$

Isolate $(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})$ to find

$$\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} = (\mathbf{u} + \mathbf{v})^2 - \mathbf{u}^2 - \mathbf{v}^2\,.$$

Note that, by definition, $\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u} = 2(\mathbf{u}\cdot\mathbf{v})$, thus

$$2(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{u} + \mathbf{v})^2 - \mathbf{u}^2 - \mathbf{v}^2\,.$$

Since, by axiom, contracting a vector produces a scalar, then $(\mathbf{u} + \mathbf{v})^2$, $\mathbf{u}^2$ and $\mathbf{v}^2$ are scalars. Therefore, $(\mathbf{u}\cdot\mathbf{v})$ is also a scalar, which was needed be proved. □

4.6. **Outer Product.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Define the *outer product of* $\mathbf{u}$ *and* $\mathbf{v}$, denoted $\mathbf{u} \wedge \mathbf{v}$, by

$$\mathbf{u} \wedge \mathbf{v} \doteq \tfrac{1}{2}\left[\mathbf{u}, \mathbf{v}\right]_{-} = \tfrac{1}{2}(\mathbf{uv} - \mathbf{vu})\,.$$

Call $\mathbf{u} \wedge \mathbf{v}$ a *bivector*. By construction, the outer product is *anti*-symmetric; *i.e.*, $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.

Note that $\mathbf{u} \wedge \mathbf{u} = 0$, where $0 \in \mathcal{R}$.

Proof. Apply the definition of the outer product to $\mathbf{u} \wedge \mathbf{u}$

$$\mathbf{u} \wedge \mathbf{u} = \tfrac{1}{2}\left[\mathbf{u}, \mathbf{u}\right]_{-} = \tfrac{1}{2}(\mathbf{uu} - \mathbf{uu}) = \tfrac{1}{2}(\mathbf{u}^2 - \mathbf{u}^2)\,.$$

Since $\mathbf{u}^2 \in \mathcal{R}$, then $\mathbf{u}^2 - \mathbf{u}^2 = 0$, which yields the result. □

4.7. **Fundamental Decomposition of the Geometric Product.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Write the geometric product as a function of its symmetric and anti-symmetric parts; *i.e.*, as a function of the inner and outer products,

$$\mathbf{uv} = \tfrac{1}{2}\left[\mathbf{u}, \mathbf{v}\right]_{+} + \tfrac{1}{2}\left[\mathbf{u}, \mathbf{v}\right]_{-} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}\,.$$

Call the last equation the *fundamental decomposition of the geometric product*.

4.8. **Angle between Vectors.** Consider $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Then, define the angle θ between $\mathbf{u}$ and $\mathbf{v}$ by

$$2|\mathbf{u}||\mathbf{v}|\cos\theta \doteq \left[u, v\right]_{+} = \mathbf{uv} + \mathbf{vu}\,.$$

Using the definition of the inner product, write the last equation in alternative form

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\,.$$

4.9. **Magnitude of the Outer Product.** Consider $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Then, the following identity holds

$$\mathbf{uv} = 2(\mathbf{u} \cdot \mathbf{v}) - \mathbf{vu}\,.$$

Proof. Express $\mathbf{uv}$ and $\mathbf{vu}$ using the fundamental decomposition of the geometric product, then, in the decomposition of $\mathbf{vu}$ use the symmetric and anti-symmetric properties of the inner and outer products, respectively, to find

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}\,,$$
$$\mathbf{vu} = \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \wedge \mathbf{u} = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \wedge \mathbf{v}\,.$$

Add the two equations to find

$$\mathbf{uv} + \mathbf{vu} = 2(\mathbf{u} \cdot \mathbf{v})\,.$$

Finally, isolate the product $\mathbf{uv}$ to find the desired result. □

Next, consider θ to be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then, the magnitude of $\mathbf{u} \wedge \mathbf{v}$ is given by

$$|\mathbf{u} \wedge \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2|\sin\theta|^2\,.$$

Proof. Express $\mathbf{uv}$ using the fundamental decomposition of the geometric product and isolate $\mathbf{u} \wedge \mathbf{v}$:

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}\,,$$
$$\implies \mathbf{u} \wedge \mathbf{v} = \mathbf{uv} - \mathbf{u} \cdot \mathbf{v}\,.$$

Then, contract both sides of the last equation and multiply termwise the right-hand side of it

$$(\mathbf{u} \wedge \mathbf{v})^2 = (\mathbf{uv} - \mathbf{u} \cdot \mathbf{v})(\mathbf{uv} - \mathbf{u} \cdot \mathbf{v}) = \mathbf{uvuv} - \mathbf{uv}(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})\mathbf{uv} + (\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{v})\,.$$

Since $(\mathbf{u} \cdot \mathbf{v})$ results in a scalar, it commutes with the other members of the algebra. Thus, $\mathbf{uv}(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v})\mathbf{uv}$. Additionally, since $(\mathbf{u} \cdot \mathbf{v})$ results in a scalar, then $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v})^2$. Therefore, the contracted outer product becomes

$$(\mathbf{u} \wedge \mathbf{v})^2 = \mathbf{uvuv} - 2\mathbf{uv}(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{v})^2\,.$$

Replace next the product $\mathbf{uv}$ using the identity $\mathbf{uv} = 2(\mathbf{u}\cdot\mathbf{v}) - \mathbf{vu}$:

$$(\mathbf{u}\wedge\mathbf{v})^2 = \mathbf{uv}(2(\mathbf{u}\cdot\mathbf{v}) - \mathbf{vu}) - 2\mathbf{uv}(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})^2 .$$

Work on the right-hand side of the last equation to find

$$(\mathbf{u}\wedge\mathbf{v})^2 = 2\mathbf{uv}(\mathbf{u}\cdot\mathbf{v}) - \mathbf{uvvu} - 2\mathbf{uv}(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{v})^2 .$$

Simplify terms in the last equation and use the identity $\mathbf{uvvu} = \mathbf{u}(\mathbf{vv})\mathbf{u} = |\mathbf{u}|^2|\mathbf{v}|^2$ to have

$$(\mathbf{u}\wedge\mathbf{v})^2 = (\mathbf{u}\cdot\mathbf{v})^2 - |\mathbf{u}|^2|\mathbf{v}|^2 .$$

Using the definition of the angle between $\mathbf{u}$ and $\mathbf{v}$, denoted θ, the last equations turns into

$$(\mathbf{u}\wedge\mathbf{v})^2 = (|\mathbf{u}||\mathbf{v}|\cos\theta)^2 - |\mathbf{u}|^2|\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta - |\mathbf{u}|^2|\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2(\cos^2\theta - 1) .$$

Use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ to find

$$(\mathbf{u}\wedge\mathbf{v})^2 = -|\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta .$$

Finally, take the magnitude (absolute value) in both sides of the last equation to find

$$|\mathbf{u}\wedge\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2|\sin\theta|^2 ,$$

which yields the desired result. □

4.10. **Law of Cosines.** Let $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathcal{E}^n$. Consider $\mathbf{a}$ and $\mathbf{b}$ separated by an angle θ and consider $\mathbf{c} = \mathbf{a} - \mathbf{b}$. Then, the following identity holds

$$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta .$$

Call the last equation the *law of cosines*.

Proof. Square both sides of the equation $\mathbf{c} = \mathbf{a} - \mathbf{b}$; that is, $\mathbf{c}^2 = (\mathbf{a}-\mathbf{b})^2 = (\mathbf{a}-\mathbf{b})(\mathbf{a}-\mathbf{b})$. Multiply term by term the last equality to find $\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 - (\mathbf{ab} + \mathbf{ba})$.

By definitions, $\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ and $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. Thus, $(\mathbf{ab} + \mathbf{ba}) = 2|\mathbf{a}||\mathbf{b}|\cos\theta$. Therefore,

$$\mathbf{c}^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta ,$$

which yields the desired result. □

4.11. **Orthogonality and Collinearity.** Consider $\mathbf{u},\mathbf{v} \in \mathcal{E}^n$. Call two vectors *orthogonal* or *perpendicular* if their inner product is zero; *i.e.*, $\mathbf{u} \perp \mathbf{v} \iff \mathbf{u}\cdot\mathbf{v} = 0$.

Call two vectors *collinear* or *parallel* if their outer product is zero; *i.e.*, $\mathbf{u} \parallel \mathbf{v} \iff \mathbf{u}\wedge\mathbf{v} = 0$.

Consider $\mathbf{u},\mathbf{v} \in \mathcal{E}^n$ to be two orthogonal vectors. Then,

$$\mathbf{u}\cdot\mathbf{v} = 0 \iff \mathbf{uv} = -\mathbf{vu} .$$

On the other hand, consider $\mathbf{u},\mathbf{v} \in \mathcal{E}^n$ to be two collinear vectors. Then,

$$\mathbf{u}\wedge\mathbf{v} = 0 \iff \mathbf{uv} = \mathbf{vu} .$$

Thus, the geometric product $\mathbf{uv}$ provides a measure of the relative direction of the vectors: commutativity means that the vectors are collinear, whereas anticommutativity means that they are orthogonal.

4.12. **Magnitude of Vectors – Review.** Consider $\mathbf{u} \in \mathcal{E}^n$. Since

$$|\mathbf{u}|^2 = \mathbf{uu} = [\mathbf{u},\mathbf{u}]_+ + [\mathbf{u},\mathbf{u}]_- = \mathbf{u}\cdot\mathbf{u} ,$$

then, also calculate the magnitude (or length) of $\mathbf{u}$ by

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} .$$

4.13. **Normality.** Let $\mathbf{u} \in \mathcal{E}^n$. Call $\mathbf{u}$ a *normal vector* (or *unit vector*) if its magnitude (or length) equals unity; *i.e.*, if it satisfies

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = 1 \,.$$

4.14. **Euclidean Metric.** Consider $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Then, define a function $d : \mathcal{E}^n \otimes \mathcal{E}^n \to \mathcal{R}$ whose formula is given by

$$d(\mathbf{u}, \mathbf{v}) = |\mathbf{u} - \mathbf{v}| \,.$$

Then, since d is defined in n-dimensional *Euclidean* space and since d satisfies the definition of metric, call thus the function d *Euclidean metric* or *Euclidean distance function*.

Moreover, let $\mathbf{a} \in \mathcal{E}^n$. Then,

$$\begin{aligned} d(\mathbf{u} + \mathbf{a}, \mathbf{v} + \mathbf{a}) &= |\mathbf{u} + \mathbf{a} - (\mathbf{v} + \mathbf{a})| \,, \\ &= |\mathbf{u} - \mathbf{v}| \,, \\ &= d(\mathbf{u}, \mathbf{v}) \ ; \end{aligned}$$

that is, Euclidean metric is a *translation invariant metric*.

Use the definition of magnitude of vectors to find

$$d(\mathbf{u}, \mathbf{v}) = |\mathbf{u} - \mathbf{v}| = \sqrt{(\mathbf{u} - \mathbf{v})(\mathbf{u} - \mathbf{v})} = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \,.$$

4.15. **Multivectors.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Then, the geometric product $\mathbf{uv}$ is the addition of a scalar and a bivector:

$$\mathbf{uv} = \underbrace{\mathbf{u} \cdot \mathbf{v}}_{\text{scalar}} + \underbrace{\mathbf{u} \wedge \mathbf{v}}_{\text{bivector}} \ .$$

In other words, geometric multiplying two vectors does not result in a vector; *i.e.*, *the algebra does not close.*

To overcome this, define a *multivector* to be the addition of scalars, vectors, bivectors and so on.

4.16. **Reciprocal Frames.** Consider $\mathcal{V}$ to be a n-dimensional vector space, a frame $\{\gamma_k\} \in \mathcal{V}$ whose elements need not be orthogonal and i to be the unit pseudoscalar. Then, there is a reciprocal frame $\left\{\gamma^k\right\}$ whose elements are given by

$$\gamma^k = (-1)^{(k-1)} \gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \check{\gamma}_k \wedge \cdots \wedge \gamma_k i^{-1} \,,$$

where $\check{\gamma}_k$ means that γ_k is to be omitted from the product and i^{-1} means the inverse of i.

4.17. **Construction of the Reciprocal Frame.** Example of construction of the reciprocal frame elements for the three-dimensional Euclidean space, $\mathcal{E}^3$.

Consider in $\mathcal{E}^3$ an orthonormal frame $\{\gamma_k\}$. Then, construct the *reciprocal frame elements* $\left\{\gamma^k\right\}$ applying the following procedure:

(1) Since the frame elements are orthonormal, then $\gamma_k \gamma_l + \gamma_l \gamma_k = g_{kl}$ where $g_{kl} = \delta_{kl}$; thus, $\gamma_k \gamma_l = \gamma_k \wedge \gamma_l$.

(2) Define the shorthand notation $\gamma_{kl} = \gamma_k \gamma_l$.

(3) Find $i = \gamma_1 \gamma_2 \gamma_3 = \gamma_{123}$ and thus $ii = \gamma_{123123} = -1$.

(4) Find $i^{-1} = 1/i = i/ii = -i$.

(5) Apply the equation

$$\gamma^k = (-1)^{(k-1)} \gamma_1 \wedge \gamma_2 \wedge \gamma_3 (-i)$$

for each k.

For $k = 1$,

$$\gamma^1 = (-1)^{(1-1)} \wedge \gamma_2 \wedge \gamma_3 (-i) = (1) \gamma_2 \wedge \gamma_3 (-\gamma_{123}) = \gamma_1 \,.$$

Similarly, for $k = 2$, then γ^2 and, for $k = 3$, then $\gamma^3 = \gamma_3$.

Note: As a conclusion of the previous construction. For $\mathcal{E}^3$, the reciprocal frame elements equal the frame elements.

4.18. **Alternate Construction of the Reciprocal Frame.** Example of construction of the reciprocal frame elements for the fourth-dimensional Minkowski space, $\mathcal{M}^4$.

Consider in $\mathcal{M}^4$ an orthonormal frame $\{\gamma_\mu\}$. Then, define the *reciprocal frame elements* $\{\gamma^\mu\}$ by $\{\gamma^\mu = {\gamma_\mu}^{-1}\}$.

Using this definition, construct the reciprocal frame elements by applying:

(1) Since the frame elements are orthonormal in $\mathcal{M}^4$, then $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \eta_{\mu\nu}$ where the signature $\operatorname{sig}\eta = (-+++)$.
(2) Define the shorthand notation $\gamma_{\mu\nu} = \gamma_\mu\gamma_\nu$.
(3) For the timelike reciprocal frame element, γ^0, find $\gamma_0\gamma_0 = \eta_{00} = -1$, then $\gamma^0 = {\gamma_0}^{-1} = \gamma_0/\gamma_0\gamma_0 = -\gamma_0$.
(4) For the spacelike reciprocal frame elements, γ^k, find $\gamma^k\gamma^k = \eta_{kk} = 1$, then $\gamma^k = {\gamma_k}^{-1} = \gamma_k/\gamma_k\gamma_k = \gamma_k$.

Note: As a conclusion of the previous construction. For $\mathcal{M}^4$, the reciprocal frame spacelike elements equal the frame elements ($\gamma^k = \gamma_k$) – just as in $\mathcal{E}^3$ – whereas the reciprocal frame timelike element equals the negative of the frame timelike element ($\gamma^0 = -\gamma_0$).

4.19. **Representation of Vectors in Reciprocal Frames.**

Exercise 4.1. *Consider $\mathcal{E}^n$ and an orthonormal frame $\{\gamma_k;\, k : 1 \dots n\}$ in $\mathcal{E}^n$. Consider a reciprocal frame $\left\{\gamma^k\right\}$ defined by $\gamma^k \cdot \gamma_l = g_{kl}$. Then, given a vector $v \in \mathcal{E}^n$, how does it transform when we change to the reciprocal frame?*

Solution. The frame is orthonormal, so, by definition, its elements satisfy $\gamma_k \cdot \gamma_l = g_{kl}$.

On the other hand, the vector v is a geometric object. Thus, it is independent of the coordinate system used to represent it; *i.e.*, it must be the same in both frames:

$$x_k\gamma^k = x^k\gamma_k\,.$$

Inner multiply both sides with γ_l to find

$$x_k\gamma^k \cdot \gamma_l = x^k\gamma_k \cdot \gamma_l$$

or, equivalently,

$$x_l = x^k g_{kl}\,,$$

where the identity $x_k\gamma^k \cdot \gamma_l = x_k g_{kl} = x^l$ was used. This yields the desired result. □

4.20. **Derivatives in Spacetime.** Consider $\mathcal{M}^4$ and a frame $\{\gamma_k;\, k : 0 \dots 4\}$. Then, expand the geometric derivative ∇ onto this frame:

$$\nabla = \nabla^\mu\gamma_\mu\,,$$

where $\nabla^1 = (\partial/\partial x^1)$ and similarly for x^2 and x^3, but $\nabla^0 = -(\partial/\partial x^0)$; *i.e.*, $\nabla = -\nabla^0\gamma_0 + \nabla^k\gamma_k$.

Note: As a mnemonic, to know where the minus sign goes: Imagine a dimensionless scalar field $\phi(x)$; that is, a scalar-valued function of the position vector. Positions are measured in units of length. However, the length of the gradient vector $\nabla\phi$ is *not* measured in the same units as the position, but in *reciprocal* units of length. So, using this fact, write

$$\nabla = \frac{1}{\gamma_0}\frac{\partial}{\partial x^0} + \frac{1}{\gamma_1}\frac{\partial}{\partial x^1} + \frac{1}{\gamma_2}\frac{\partial}{\partial x^2} + \frac{1}{\gamma_3}\frac{\partial}{\partial x^3}\,.$$

Then, evaluate the reciprocals of $\{\gamma_k\}$ according to the results of the last section to find

$$\nabla = -\gamma_0\frac{\partial}{\partial x^0} + \gamma_1\frac{\partial}{\partial x^1} + \gamma_2\frac{\partial}{\partial x^2} + \gamma_3\frac{\partial}{\partial x^3} + = -\gamma_0\frac{\partial}{\partial x^0} + \gamma_k\frac{\partial}{\partial x^k}\,.$$

This equation shows explicitly the crucial minus sign in front of the first term (timelike derivative) and has the basis vectors in the numerators where they normally belong.

4.21. **Product of a Vector and a Bivector.** Let $\mathcal{V}$ be an n-dimensional vector space and $\mathcal{G} = \mathcal{G}(\mathcal{V})$ be a geometric algebra on $\mathcal{V}$. Let $u \in \mathcal{G}$ be a vector and $B \in \mathcal{G}$ be a bivector. Then, define the *inner product of* u *and* B by

$$\langle uB \rangle_1 = u \cdot B = \frac{1}{2}(uB - Bu) = -B \cdot u$$

and define the *outer product of* u *and* B by

$$\langle uB \rangle_3 = u \wedge B = \frac{1}{2}(uB + Bu) = B \wedge u\,.$$

In general, if you multiply an object of grade r by an object of grade s, then the geometric product is liable to contain terms of all grades from $|r - s|$ to $|r + s|$, counting by twos, as we see in the following:

Example 4.1. *Let $\{\gamma_k;\, k : 1 \ldots 4\}$ be an orthonormal frame satisfying $[\gamma_k, \gamma_l]_+ = 2\,[k = l]_{Iv}$. Then, calculate the geometric, inner and outer products of $A = \gamma_1 \wedge \gamma_2$ and $B = (\gamma_2 + \gamma_3) \wedge (\gamma_4 + \gamma_1)$.*

Solution. Because the frame satisfies $[\gamma_k, \gamma_l]_+ = 2\,[k = l]_{\mathrm{Iv}}$, then $\gamma_k \gamma_l = \gamma_k \wedge \gamma_l$. Therefore, introduce a shorthand notation:

$$\gamma_{kl} = \gamma_k \gamma_l = \gamma_k \wedge \gamma_l\,.$$

With this shorthand, A becomes $A = \gamma_{12}$.

Term-wise multiply the terms of B

$$B = (\gamma_2 + \gamma_3) \wedge (\gamma_4 + \gamma_1) = \gamma_{24} + \gamma_{21} + \gamma_{34} + \gamma_{31}\,.$$

Both A and B are homogeneous of grade 2 – they're 2-blades. So, we expect AB to contain terms of grades from $|2 - 2| = 0$ to $|2 + 2| = 4$, counting by two; *i.e.*, terms of grade 0, 2 and 4.

Then, calculate AB taking into account the orthonormality properties of the frame:

$$\begin{aligned} AB &= (\gamma_{12})(\gamma_{24} + \gamma_{21} + \gamma_{34} + \gamma_{31})\,, \\ &= \gamma_{1224} + \gamma_{1221} + \gamma_{1234} + \gamma_{1231}\,, \\ &= 1 + \gamma_{14} + \gamma_{23} + \gamma_{1234}\,, \\ &= 1 + \gamma_{14} + \gamma_{23} + i\,, \end{aligned}$$

where, in the last equation, $i = \gamma_{1234}$ is the unit pseudoscalar.

Finally, from the definition of inner and outer products, note that

$$\begin{aligned} A \cdot B &= \langle AB \rangle_0 = 1\,, \\ &= \langle AB \rangle_2 = \gamma_{14} + \gamma_{23}\,, \\ A \wedge B &= \langle AB \rangle_4 = i\,, \end{aligned}$$

which yields the desired result. □

5. Geometric Interpretations of Multiplications

5.1. **Geometric Int. of the Inner Product.** Consider $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Then, define the angle θ between $\mathbf{u}$ and $\mathbf{v}$ by

$$\cos\theta \doteq \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\,.$$

Call two vectors *orthogonal* or *perpendicular* if their inner product is zero; *i.e.*, $\mathbf{u} \perp \mathbf{v} \iff \mathbf{u} \cdot \mathbf{v} = 0$.

5.2. **Geometric Int. of the Outer Product.** Since by construction the outer product is antisymmetric, then $\mathbf{u} \wedge \mathbf{u} = 0$.

Geometrically, the outer product $\mathbf{u} \wedge \mathbf{v}$ can be interpreted as an oriented plane element whose magnitude is equal to the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Call two vectors *collinear* or *parallel* if their outer product is zero [1]; *i.e.*, $\mathbf{u} \parallel \mathbf{v} \iff \mathbf{u} \wedge \mathbf{v} = 0$.

5.3. **Geometric Int. of the Geometric Product.** Consider $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$ to be two orthogonal vectors. Then,

$$\mathbf{u} \cdot \mathbf{v} = 0 \iff \mathbf{uv} = -\mathbf{vu}\,.$$

On the other hand, consider $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$ to be two collinear vectors. Then,

$$\mathbf{u} \wedge \mathbf{v} = 0 \iff \mathbf{uv} = \mathbf{vu}\,.$$

Thus, the geometric product $\mathbf{uv}$ provides a measure of the relative direction of the vectors: commutativity means that the vectors are collinear, whereas anticommutativity means that they are orthogonal.

6. Geometric Algebra of the Plane

Hereafter, for the sake of convenience, define the *tau number* τ

$$\tau \doteq 2\pi \approx 6.283185307179586\ldots\,,$$

where π is the *circle constant*; *i.e.*, the ratio of a circle's circumference to its diameter.

6.1. **Algebra of the Plane.** Let $\mathcal{F} = \{\gamma_1, \gamma_2\}$ be a frame of orthonormal vectors in $\mathcal{E}^2$. Then, the frame elements satisfy

$$[\gamma_k, \gamma_l]_+ = 2\,[k = l]_{\mathrm{Iv}}\;;$$

that is, $\gamma_1^2 = \gamma_2^2 = 1$, $\gamma_1 \cdot \gamma_2 = 0$ and $\gamma_1 \wedge \gamma_2 = \gamma_1\gamma_2 = -\gamma_2 \wedge \gamma_1$.

Define the shorthand γ_{12}

$$\gamma_{12} = \gamma_1\gamma_2 = \gamma_1 \wedge \gamma_2$$

and, with this, define the *unit pseudoscalar* i by

$$i \doteq \gamma_{12} = \gamma_1\gamma_2 = \gamma_1 \wedge \gamma_2\,.$$

Note that i squares to -1:

$$i^2 = \gamma_{12}\gamma_{12} = \gamma_{1212} = -1\,.$$

The unit pseudoscalar *anti-commutes* with vectors in $\mathcal{E}^2$:

$$\begin{aligned}
i\gamma_1 &= \gamma_{121} = -\gamma_{112} = -\gamma_1 i\,,\\
\gamma_1 i &= \gamma_{112} = -\gamma_{121} = -i\gamma_1\,,\\
i\gamma_2 &= \gamma_{122} = -\gamma_{212} = -\gamma_2 i\,,\\
\gamma_2 i &= \gamma_{212} = -\gamma_{122} = -i\gamma_2\,.
\end{aligned}$$

The basis set

$$1, \qquad \{\gamma_1, \gamma_2\} \qquad \text{and} \qquad i\,,$$

spans the full algebra. Denote this algebra by $\mathcal{G}_2$.

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[1] Collinearity implies that two vectors determine a parallelogram with vanishing area.

6.2. **Geometry of the Plane.** Left- and right-multiply γ_1 by i

$$\begin{aligned} i\gamma_1 &= \gamma_{121} = -\gamma_{112} = -\gamma_2\,, \\ \gamma_1 i &= \gamma_{112} = \gamma_2\,. \end{aligned}$$

Then, left- and right-multiply γ_2 by i

$$\begin{aligned} i\gamma_2 &= \gamma_{122} = \gamma_1\,, \\ \gamma_2 i &= \gamma_{212} = -\gamma_{122} = -\gamma_1\,. \end{aligned}$$

Thus, left-multiplication by i rotates vectors $\tau/4$ (90°) clockwise (*i.e.*, negative sense), while right-multiplication by i rotates vectors $\tau/4$ counterclockwise (*i.e.*, positive sense) [2].

From the last results, it follows that two successive left (right) multiplications of a vector by i rotates the vector through $\tau/2$ (180°). This is equivalent to multiply the vector by -1 or by i^2, for $i^2 = -1$.

Appendix A. Functions

Denote by $\mathcal{N}$ the set of natural numbers; *i.e.*, $\mathcal{N} = \{1, 2, 3, \ldots\}$, and by $\mathcal{R}$ the set of real numbers (*aka* reals or *scalars*).

A.1. **Cartesian Product.** Consider two sets $\mathcal{A}$ and $\mathcal{B}$. Let $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then, call *Cartesian product of $\mathcal{A}$ and $\mathcal{B}$*, denoted by $\mathcal{A} \otimes \mathcal{B}$, the set consisting of all ordered pairs (a, b). Call a and b the *components of the ordered pair* (a, b).

A.2. **Function.** Consider two sets $\mathcal{A}$ and $\mathcal{B}$. Let f be a subset of $\mathcal{A} \otimes \mathcal{B}$. Then, call f a *function from $\mathcal{A}$ to $\mathcal{B}$* if every element of $\mathcal{A}$ is the first component of *one and only one* ordered pair in the subset.

Call $\mathcal{A}$ the *domain of f* and $\mathcal{B}$ the *codomain of f*.

A.3. **Function Notation.** Denote a function f with domain $\mathcal{A}$ and codomain $\mathcal{B}$ by $f : \mathcal{A} \to \mathcal{B}$. Call the elements of $\mathcal{A}$ *arguments of f* and call, for each argument a, the corresponding unique b in the codomain the *function value at a* or the *image of a under f*, written as $f(a)$.

Additionally, say that f *associates b with a* or *maps a to b*. Abbreviate this by $b = f(a)$.

To specify a function, use the $\mapsto$ notation; *e.g.*, $f : \mathcal{R} \to \mathcal{R}$ defined by $x \mapsto x + 1$; read this: »f is a function from $\mathcal{R}$ (the set of real numbers) to $\mathcal{R}$, where x maps to $x + 1$«.

A.4. **Metric.** Consider a set $\mathcal{X}$ and $x, y, z \in \mathcal{X}$. Consider a function $d : \mathcal{X} \otimes \mathcal{X} \to \mathcal{R}$. Call d a *metric* if it satisfies

(1) non-negativity (separation axiom): $d(x, y) \geq 0$;

(2) identity of indiscernibles (coincidence axiom): $d(x, y) = 0$ if and only if $x = y$;

(3) symmetry: $d(x, y) = d(y, x)$;

(4) subadditivity (triangle inequality): $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a metric, it is also called a *distance function* or, simply, *distance*.

Consider a set $\mathcal{Y}$ and $a, x, y \in \mathcal{Y}$. Assume an addition $+ : \mathcal{Y} \otimes \mathcal{Y} \to \mathcal{Y}$ defined in $\mathcal{Y}$. Call d a *translation invariant metric* if it satisfies

$$d(x, y) = d(x + a, y + a)\,.$$

[2] Mnemonic: left-multiplication by i rotates a vector in the sense of closing the *left* hand, while right-multiplication by i in the sense of closing the *right* hand.

Appendix B. Linear Spaces

B.1. **Linear Space.** Let $\mathcal{L}$ be a non-empty set of objects called the *elements of* $\mathcal{L}$. Consider $a, b \in \mathcal{R}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{L}$. Then, call $\mathcal{L}$ a *linear space over* $\mathcal{R}$ if it satisfies:

(1) closure over addition: there is a unique element $(\mathbf{u}+\mathbf{v}) \in \mathcal{V}$ called the *sum of* $\mathbf{u}$ *and* $\mathbf{v}$;

(2) closure by multiplication of scalars: there is a unique element $a\mathbf{u} \in \mathcal{L}$ called the *product of* a *and* $\mathbf{u}$;

(3) commutative law for addition: $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$;

(4) associative law for addition: $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$;

(5) existence of zero element: there is an element in $\mathcal{L}$, denoted by $\mathbf{0}$, such that $\mathbf{u}+\mathbf{0} = \mathbf{u}$, for all $\mathbf{u} \in \mathcal{L}$;

(6) existence of negatives: the element $(-1)\mathbf{u}$ has the property $\mathbf{u}+(-1)\mathbf{u} = \mathbf{0}$;

(7) associative law for multiplication of scalars: $a(b\mathbf{u}) = (ab)\mathbf{u}$;

(8) distributive law for addition in $\mathcal{V}$: $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$;

(9) distributive law for addition of scalars: $(a+b)\mathbf{u} = a\mathbf{u}+b\mathbf{u}$;

(10) existence of identity: $1\mathbf{u} = \mathbf{u}$.

B.2. **Basis.** Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a finite subset of a linear space $\mathcal{V}$ over $\mathcal{R}$ and let $a^1, a^2, \ldots, a^n \in \mathcal{R}$. Call $\mathcal{B}$ a *basis* if it satisfies:

(1) the linear independence property: if $a^1\mathbf{v}_1 + a^2\mathbf{v}_2 + \cdots + a^n\mathbf{v}_n = 0$, then necessarily $a^1 = a^2 = \cdots = a^n = 0$; and

(2) the spanning property: for every $\mathbf{x} \in \mathcal{V}$, it is possible to choose $a^1, a^2, \ldots, a^n \in \mathcal{R}$ such that $\mathbf{x} = a^1\mathbf{v}_1 + a^2\mathbf{v}_2 + \cdots + a^n\mathbf{v}_n$.

Call the numbers a^i the *components of* $\mathbf{x}$ *with respect to* $\mathcal{B}$. By the first property, they are uniquely determined.

B.3. **Real Linear Space.** Consider $n \in \mathcal{N}$. Then, define the *n-Cartesian power of* $\mathcal{R}$, denoted $\mathcal{R}^n$, by

$$\mathcal{R}^n = \left\{ \left(x^1, x^2, \ldots, x^n\right) \, : \, x^i \in \mathcal{R} \text{ for all } 1 \leq i \leq n \right\}.$$

Write an element of $\mathcal{R}^n$, say $\mathbf{x}$, as

$$\mathbf{x} = \left(x^1, x^2, \ldots, x^n\right) \, ,$$

where the is in the x^is are *indices* (or placeholders) rather than exponents. Call the x^is the *components of* $\mathbf{x}$.

Sometimes, the x^is are called the *coordinates of* $\mathbf{x}$ and, for this reason, $\mathcal{R}^n$ is also called *real coordinate space*.

Let $a \in \mathcal{R}$ and $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$. Then, define the operations

$$\begin{aligned} \mathbf{x}+\mathbf{y} &\doteq \left(x^1+y^1, x^2+y^2, \ldots, x^n+y^n\right) \, , \\ a\mathbf{x} &\doteq \left(ax^1, ax^2, \ldots, ax^n\right) \, . \end{aligned}$$

Since $\mathcal{R}^n$ and the operations therein defined satisfy the linear space axioms, then $\mathcal{R}^n$ forms a linear space over $\mathcal{R}$. Call $\mathcal{R}^n$ the *n-dimensional real linear space* (sometimes shortened to *n-D space*; *e.g.*, 2-D space, 3-D space and so on).

Appendix C. Index Notation

C.1. **Einstein Summation Convention.** Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis, $\mathbf{x} \in \mathcal{V}$ be a vector and $a^1, a^2, \ldots, a^n \in \mathcal{R}$ be the components of $\mathbf{x}$ with respect to $\mathcal{B}$. Then, by means of the spanning property of $\mathcal{B}$, the components of $\mathbf{x}$ with respect to $\mathcal{B}$ are

$$\mathbf{x} = a^1\mathbf{v}_1 + a^2\mathbf{v}_2 + \cdots + a^n\mathbf{v}_n \,.$$

Shorten the sum by using the *sigma notation*; *i.e.*,

$$\mathbf{x} = \sum_{k=1}^{n} \mathbf{v}_k a^k \,.$$

Call $\sum$ the *summation symbol* and k the *summation index*.

Adopt *Einstein summation convention*: drop the summation symbol and assume the summation index k runs from 1 to n to rewrite $\mathbf{x}$ as

$$\mathbf{x} = \mathbf{v}_k a^k \,.$$

C.2. **Frames.** Consider a set $\mathcal{F} \subset \mathcal{E}^n$ of vectors $\{\gamma_k;\, k : 1 \ldots n\}$. Then, call $\mathcal{F}$ a *frame* and its elements *frame vectors* if the γ_ks satisfy

$$[\gamma_k, \gamma_l]_+ = 2\,[k = l]_{\mathrm{Iv}} \ ,$$

where $[k = l]_{\mathrm{Iv}}$ are *Iverson brackets*.

Note that, if $k = l$, then $\gamma_k\gamma_k + \gamma_k\gamma_k = 2$ (no summation implied). Thus, $\gamma_k\gamma_k = \gamma_k^2 = 1$; *i.e.*, $|\gamma_k| = 1$. So, the frame vectors are normal vectors.

On the other hand, if $k \neq l$, then $\gamma_k\gamma_l + \gamma_l\gamma_k = 2(\gamma_k \cdot \gamma_l) = 0$, then $\gamma_k \cdot \gamma_l = 0$. Thus, by the definition of orthogonality, γ_ks are mutually orthogonal.

Therefore, since the elements of $\mathcal{F}$ are orthogonal and normal, call them *orthonormal*. (For this reason, $\mathcal{F}$ is called an *orthonormal frame*.)

C.3. **Components of Vectors.** Consider $\mathcal{F} \subset \mathcal{E}^n$ and $\mathcal{F} = \{\gamma_k;\, k : 1 \ldots n\}$. Let $\mathbf{u} \in \mathcal{E}^n$. Then, find the the *components (or coordinates) of* $\mathbf{x}$ *with respect to* $\mathcal{F}$, denoted as u^ks, by

$$u^k = \mathbf{u} \cdot \gamma_k \,.$$

Thus, $\mathbf{u}$ can be written as a function of its components onto $\mathcal{F}$ by

$$\mathbf{u} = \gamma_k u^k \,.$$

In particular, consider a set $\mathcal{F} \subset \mathcal{E}^n$ of vectors $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ defined by

$$\gamma_1 \doteq (1, 0, \ldots, 0) \ , \qquad \gamma_2 \doteq (0, 1, \ldots, 0) \ , \qquad \cdots , \qquad \gamma_n \doteq (0, 0, \ldots, n) \ ;$$

then, since $\mathcal{F}$ satisfies the definition of basis, write thus any vector $\mathbf{u} \in \mathcal{E}^n$ in the form

$$\mathbf{u} = \gamma_k u^k \,.$$

Call $\mathcal{F}$ a *frame*, the vectors $\{\gamma_k;\, k : 1 \ldots n\}$ *basis vectors* and the u^ks the *components (or coordinates) of* $\mathbf{x}$ *with respect to* $\mathcal{F}$.

C.4. **Operations with Basis Vectors.** Let $\mathcal{F} = \{\gamma_k;\, k : 1 \dots n\}$ be a frame in $\mathcal{E}^n$. Consider the basis vectors to satisfy the algebra

$$[\gamma_k, \gamma_l]_+ = 2\,[k = l]_{\mathrm{Iv}} = 2g_{kl}\,,$$

where $[k = l]_{\mathrm{Iv}}$ are *Iverson brackets* and g_{kl} is *Kronecker delta*.

Expand the expression $[\gamma_k, \gamma_l]_+ = 2\,[k = l]_{\mathrm{Iv}}$. On the one hand, if $k = l$,

- ⋄ $\gamma_k\gamma_k + \gamma_k\gamma_k = 2$, thus $\gamma_k\gamma_k = 1$;
- ⋄ $\gamma_k \cdot \gamma_k = \frac{1}{2}(\gamma_k\gamma_k + \gamma_k\gamma_k) = 1$; and
- ⋄ $\gamma_k \wedge \gamma_k = \frac{1}{2}(\gamma_k\gamma_k - \gamma_k\gamma_k) = 0$.

On the other hand, if $k \neq l$,

- ⋄ $\gamma_k\gamma_l + \gamma_l\gamma_k = 0$, thus $\gamma_k\gamma_l = -\gamma_l\gamma_k$;
- ⋄ $\gamma_k \cdot \gamma_l = \frac{1}{2}(\gamma_k\gamma_l + \gamma_l\gamma_k) = 0$; and
- ⋄ $\gamma_k \wedge \gamma_l = \frac{1}{2}(\gamma_k\gamma_l - \gamma_l\gamma_k) = \gamma_k\gamma_l = -\gamma_l\gamma_k$.

Use the equality in the last item to define a shorthand

$$\gamma_{kl} \doteq \gamma_k\gamma_l = \gamma_k \wedge \gamma_l\,.$$

Denote by u^k the components of $\mathbf{u}$ onto $\mathcal{F}$. Determine each u^k by

$$u^k = \mathbf{u} \cdot \gamma_k\,;$$

that is, write any vector $\mathbf{u}$ as

$$\mathbf{u} = \gamma_k u^k\,.$$

Write the inner product $\mathbf{u} \cdot \mathbf{v}$ in index notation

$$\mathbf{u} \cdot \mathbf{v} = \gamma_k u^k \cdot \gamma_l v^l = u^k v^l \gamma_k \cdot \gamma_l = u^k v^l g_{kl}\,;$$

the outer product $\mathbf{u} \wedge \mathbf{v}$

$$\mathbf{u} \wedge \mathbf{v} = \gamma_k u^k \wedge \gamma_l v^l = u^k v^l \gamma_k \wedge \gamma_l\,;$$

and finally the geometric product $\mathbf{uv}$

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = u^k g_{kl} v^l + u^k v^l \gamma_k \wedge \gamma_l\,.$$

C.5. **Index Formula for the Inner Product.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$ and let $\{\gamma_k;\, k : 1 \dots n\}$ be a frame in $\mathcal{E}^n$. Then, $\mathbf{u} \cdot \mathbf{v}$ becomes

$$\mathbf{u} \cdot \mathbf{v} = \gamma_k u^k \cdot \gamma_l v^l = u^k v^l \gamma_k \cdot \gamma_l = u^k v^l g_{kl}\,.$$

Expand the indices k and l, apply $g_{kl} = 1$ for $k = l$ and $g_{kl} = 0$ for $k \neq l$ to find

$$\mathbf{u} \cdot \mathbf{v} = u^1 v^1 + u^2 v^2 + \cdots + u^n v^n = \sum_{m=1}^{n} u^m v^m\,.$$

Call the last equation the *index formula for the inner product*.

C.6. **Index Formula for the Magnitude of Vectors.** Let $\mathbf{u} \in \mathcal{E}^n$. Then, find $|\mathbf{u}|$ by

$$|\mathbf{u}|^2 = \mathbf{u}\mathbf{u}\,,$$
$$\Longrightarrow\ |\mathbf{u}| = \sqrt{\mathbf{u}\mathbf{u}} = \sqrt{\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\wedge\mathbf{u}} = \sqrt{\mathbf{u}\cdot\mathbf{u}}\,.$$

Let $\{\gamma_k;\, k : 1 \dots n\}$ be a frame of orthonormal vectors in $\mathcal{E}^n$. Then, use the index for the inner product to find $\mathbf{u}\cdot\mathbf{u}$

$$\mathbf{u}\cdot\mathbf{u} = \sum_{m=1}^{n} (u^m)^2\,.$$

Therefore,

$$|\mathbf{u}| = \sqrt{\sum_{m=1}^{n} (u^m)^2}\,.$$

Call the last equation the *index formula for the magnitude of vectors*.

C.7. **Pseudoscalar.** Let $\{\gamma_k;\, k : 1 \dots n\}$ be a frame of orthonormal vectors in $\mathcal{E}^n$. Then, the *unit pseudoscalar* is defined by

$$i \doteq \gamma_1\gamma_2 \cdots \gamma_n = \gamma_{12\cdots n}\,.$$

Appendix D. General Relativity

[Consider Newtonian theory.] Consider a particle moving through a gravitational field of intensity g. Note by t time, by $x(t)$ the particle's position while it moves through space and by $\dot{x}(t)$ the particle's velocity. Then, calculate the particle momentum by

$$p(t) = m_i\dot{x}(t)\ ,$$

where m_i is the particle's *inertial mass*.

On the other hand, the field interaction with the particle produces a force f given by

$$f = m_g g\,,$$

where m_g is the particle's *gravitational mass*.

Then, according with Newton's second law, we have

$$f = dp(t)\,/dt = d(m_i\dot{x}(t))/dt = m_i \implies m_i\ddot{x}(t) = m_g g\,.$$