MATHEMATICAL MODELING FOR PHYSICAL SCIENCES

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Abstract. Abstract goes here :) In the following, the introduction to each section contains the action plan, while the subsections are the plan applied to an example: the motion of a gravitational pendulum.

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1. Guide

This section provides with guidelines to approach the mathematical modeling of physical phenomena.

1.1. Problem background.

A mathematical model of a physical phenomenon begins with the *problem background*, a description or introduction to the object. It should contain

- a *description* of the essential features of the physical process and
- an identification of the *objectives*, the key questions requiring answers.

Answering what, who, where, how and why questions guides to write down the description. Additionally, including graphical illustrations aids not only in the description, but in the definition of physical quantities and the establishment of hypotheses, as well.

1.2. Problem formulation.

The *problem formulation* aims to:

- identify key physical processes;
- interpret these processes mathematically;
- establish a mathematical model – governing equations and suitable initial and boundary conditions;
- state clearly the assumptions.

The formulation must be based on sound physical principles, experimental facts or laws expressed in mathematical terms. As a guide, then, define the physical framework (geometry, kinematics, dynamics, thermal transfer and so on), state a dimensional set and then define the physical quantities, constants, parameters, coefficients and provide their dimensions in the chosen set.

A more formal approach is to begin with educated guessing, followed by dimensional analysis, order of magnitude analysis [1], analysis of extreme cases, simplifications and ends with a restricted model. The end result may be less accurate to fit experimental data, but less complex and thus more understandable. If fitting is not satisfactory, one can relax simplifications, one at a time, until a desired, or required, agreement is found.

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[1] Order of magnitude analysis is preceded by dimensional analysis, since *only* the comparison of *dimensionless* quantities is meaningful!

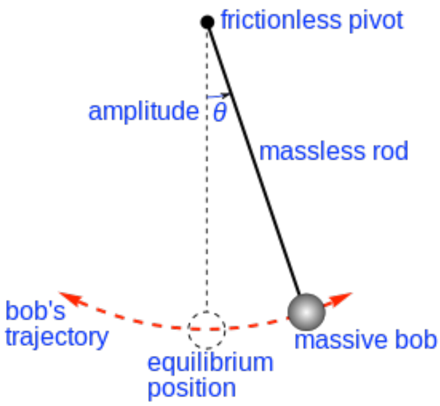

FIGURE 1. Schema of a simple gravitational pendulum

1.3. **Analysis.** Once the physical and mathematical models and their assumptions have been proposed, one regularly faces a set of equations, probably differential equations together with initial and boundary conditions. The next step is then to analyze the set by

- non-dimensionalize the equations, included initial and boundary conditions;
- making analogies with other related problems or phenomena, as the case of mass, energy and momentum transport and
- relying on analytic and numeric methods, obtaining solutions (results).

In the case of obtaining analytic solutions to differential equations, it is necessary to verify that they satisfy the differential equations object to the initial and boundary conditions.

As a final step, uncertainty analysis should be performed, to obtained ranges of validity, instead of punctual solutions.

1.4. **Results.** The final step of modeling physical phenomena is to present results, give conclusions and discussion them. Specifically, one should

- interpret results with respect to the original physical process and objectives;
- verify assumptions by confronting the results with reference values or experimental data and
- identify the solution limitations and possible extensions.

2. DESCRIPTION

[In this section, we set the description of a physical phenomenon: the motion of a gravitational pendulum. We focus on answer the questions: what is a pendulum? what is a gravitational pendulum? How the pendulum is set into motion? What keeps the pendulum moving? What forces act on the pendulum that damp its motion?]

A pendulum is a mechanical system consisting of a bob hanging by a rod attached, in turn, to a pivot, see fig. 1. A gravitational pendulum is a pendulum object only to gravitational interactions.

There are basically three ways of setting a gravitational pendulum into motion:

(1) by moving the bob from its equilibrium position to an initial angle, θ_0, at time $t = 0$;
(2) by applying a force that imprints an angular velocity to the bob, $\dot{\theta}_0$, at time $t = 0$;
(3) or by both at the same time $t = 0$.

Once the pendulum is swinging, an interplay between kinetic energy and gravitational potential energy keeps the system in motion. Kinetic energy initiates motion by the initial displacement or initial forces that perturbed the system from its equilibrium position, while the gravitational potential tries to restore the bob to its equilibrium position. This

Quantity	Symbol	Dimension
pendulum amplitude	θ	1
initial amplitude	θ_0	1
time	t	T
free fall acceleration	g	L/T^2
rod length	l	L
bob mass	m_{bob}	M
bob density	ρ_{bob}	M/L^3
bob diameter	d_{bob}	L
fluid density	ρ_{fl}	ML^3
fluid dynamic viscosity	μ_{fl}	M/LT

Table 1. Physical quantities involved in the motion of a gravitational pendulum

interplay will continue indefinitely, unless damping forces, eventually, stop the pendulum from moving. Friction on the pivot or drag – if the pendulum is partially or totally submerged in a viscous fluid – are examples of damping forces.

3. Objective

To seek for a mathematical relation, perhaps in a closed form, to predict the pendulum amplitude variation with time; *i.e.*, deduce a function of the form $\theta[t]$.

4. Physical processes

A swinging gravitational pendulum is an instance of a dynamics process, since a description of motion, based on its causes, is the final aim.

There are two main cases to consider when studying the motion of a gravitational pendulum: undamped motion, where no frictional forces and no drag are taken into account, and damped motion, where frictional forces or drag are considered. In both cases, however, the interplay between kinetic energy and gravitational potential must be regarded, since it drives motion.

To uncover the relationships between the different physical quantities that affect the pendulum motion, one firstly has to propose them; then, join them as dimensionless quantities and, finally, use physical considerations and order of magnitude analysis to restrain the physical model to thus decrease complexity. The last step will pave the path to a precise mathematical model.

4.1. **Dimensional and order of magnitude analyses.** First, one can hypothesize that the pendulum bob hangs by a massless, frictionless and inflexible rod. This implies that the system center of gravity will coincide with the bob center of gravity (massless rod), that no damping due to friction will happen and that the bob will trace a circular orbit of radius equal to the length of the rod (inflexible rod).

Since the problem belongs to dynamics, we choose the dimensional set $\{\mathsf{L}, \mathsf{M}, \mathsf{T}\}$, with a cardinality of three. Next, let us list the possible physical quantities that may influence the pendulum motion together with their symbols and dimensions, see table 1.

According to our first approach, table 1, the pendulum amplitude may be model by a function f of the form

$$\theta = f[\theta_0, t, g, l, m_{\text{bob}}, \rho_{\text{bob}}, d_{\text{bob}}, \rho_{\text{fl}}, \mu_{\text{fl}}] \ .$$

To begin with, since ρ_{bob}, m_{bob} and d_{bob} are related, we can keep two of the three quantities. We keep diameter and density. Then, note that there are 6 physical quantities and 3 independent dimensions. Thus, according to the Pi-theorem, the number of dimensionless quantities that can be formed is $9 - 3 = 6$. These quantities are

$$\Pi_1 = \theta\,, \quad \Pi_2 = \theta_0\,, \quad \Pi_3 = \frac{d}{l}\,, \quad \Pi_4 = \frac{\rho_{\text{fl}}}{\rho_{\text{bob}}}\,, \quad \Pi_5 = t\sqrt{\frac{g}{l}}\,, \quad \Pi_6 = \frac{\rho_{\text{fl}} d\sqrt{lg}}{\mu}\ .$$

Again, according to the Pi-theorem, the mathematical function we seek is of the *form*:

$$\theta = \psi\left[\theta_0, \frac{d}{l}, \frac{\rho_{\mathrm{fl}}}{\rho_{\mathrm{bob}}}, t\sqrt{\frac{g}{l}}, \frac{\rho_{\mathrm{fl}} d\sqrt{lg}}{\mu}\right] .$$

Now, we use order of magnitude analysis to further restrain the physical model so to simplify the mathematical model. We assume:

- $l \gg d$, then $d/l \to 0$. The validity of this assumption depends on one's tolerance; *e.g.*, if the bob diameter is, say, 5 cm and the rod length 1 m, then the ratio $d/l = 0.05$, which might be discarded for some purposes.
- $\rho_{\mathrm{bob}} \gg \rho_{\mathrm{fl}}$, then $\rho_{\mathrm{fl}}/\rho_{\mathrm{bob}} \to 0$. Say, for instance, that the bob is made of steel and swings through air at 15 °C and at sea level, then

$$\rho_{\mathrm{air}}/\rho_{\mathrm{steel}} = 1.225\,\mathrm{kg/m^3}/7750\,\mathrm{kg/m^3} \sim 0.158 \times 10^{-4} ,$$

  which can be ignored. However, if the steel bob swings through oil with $\rho_{\mathrm{oil}} = 850\,\mathrm{kg/m^3}$, then

$$\rho_{\mathrm{oil}}/\rho_{\mathrm{steel}} = 850\,\mathrm{kg/m^3}/7750\,\mathrm{kg/m^3} \sim 0.110 ,$$

  which may be important for some applications.
- an inviscid fluid. An inviscid fluid is a fluid with no viscosity, then

$$\rho_{\mathrm{fl}} d\sqrt{lg}/\mu = 0 .$$

  The validity of this assumption will also depend on the circumstances.

Then, after having restrained the physical model, we seek for a mathematical function of the form

$$\theta = \psi\left[\theta_0, t\sqrt{\frac{g}{l}}\right] .$$

The functional form of ψ must be found by a more refined theoretical analysis or by experimentation. However, dimensional analysis reduced the complex model by passing from ten to six dimensionless quantities. Moreover, order of magnitude analysis further limited the model to two dimensionless quantities. The physical model is more restricted, but based on sensible assumptions. At the end, only confrontation of the model with experimental data will provide the accuracy of the reductions made.

4.2. **Mathematical interpretation.** The mathematical model wishes to find a function to predict the pendulum amplitude variation with time: $\theta = \theta[t]$.

Now, based on the previous section, we can combine all the previous assumptions by defining a *simple gravitational pendulum* as a pendulum composed of a massive bob of mass m hanging by a massless, frictionless and inflexible rod of length l attached to a frictionless pivot. Under the influence of gravitational interactions, the pendulum swings through an inviscid fluid of negligible density.

The last definition leads to a very restricted physical model. However, this will allow us to finally find a closed form for the pendulum amplitude.

4.3. **Mathematical model.** Consider a simple gravitational pendulum composed of a bob of mass m and a rod of length l. Let θ be the amplitude of the pendulum for any time t. Let the initial angle be θ_0 and the initial velocity be $\dot{\theta}_0 = 0$. Finally, let g be the free fall acceleration. Then, find the equation of motion for the pendulum.

Using θ as the generalized position and $\dot{\theta}$ as the generalized velocity, write the Lagrangian for the system:

$$e_{\mathrm{lag}} = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos[\theta] .$$

Find next the generalized momentum and its temporal change

$$\partial_{\dot{\theta}} e_{\mathrm{lag}} = m l^2 \dot{\theta} \implies \mathrm{d}_t \partial_{\dot{\theta}} e_{\mathrm{lag}} = m l^2 \ddot{\theta} .$$

Calculate the generalized force:

$$\partial_\theta e_{\mathrm{lag}} = -mgl\sin[\theta] .$$

Replace the generalized force and the temporal change of the generalized momentum in the Euler-Lagrange equation to find:

$$\Box_{\theta\dot{\theta}} e_{\text{lag}} = ml^2\ddot{\theta} + mgl\sin[\theta] = 0\,.$$

Since $ml > 0$, divide the last equation through ml^2 to have

$$\ddot{\theta} + \frac{g}{l}\sin[\theta] = 0\,,$$

object to the initial conditions $\theta[0] = \theta_0$ and $\dot{\theta}[0] = 0$.

Note that we deduced the equation by means of Lagrangian mechanics. Writing down the Lagrangian was possible due to the fact that the system was assumed to be conservative; *i.e.*, neither friction nor drag were considered.

4.4. **Assumptions.** The equation of motion for the pendulum was found using the following assumptions: