GEOMETRIC ALGEBRA NOTATION, INDEX NOTATION AND MATRIX NOTATION

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Contents

1. Older Vector Notation

Newton's second law of motion:
With boldface typography: $\boldsymbol{F} = m\boldsymbol{a}$;
with arrows: $\vec{F} = m\vec{a}$;
with "$\boldsymbol{e}$" with hats for unit vectors: $F\hat{\boldsymbol{e}}_F = ma\hat{\boldsymbol{e}}_a$;
the "engineering version":

$$F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}} = m\left(a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}\right).$$

2. Geometric Algebra Notation

No decoration! Text, context, legends and so on give the type of the physical quantity!
Scalars: a, b, c.
Vectors: u, v, w.
Geometric product: uv.
Inner product: $u \cdot v$.
Outer product: $u \wedge v$.
Magnitude or length (scalars, vectors, ...): $\|u\|$.

3. Index Notation

Basis vector: γ.
k ranges from 1 to 3.

3.1. Natural coordinate frame.

Natural coordinate frame (natural coordinate system – tangents): $\{\gamma_k\}$.

Explicit expansion (decomposition or projection) of a vector in the frame $\{\gamma_k\}$ (v^k are the components or projections of the vector v onto the natural basis vectors):

$$v = v^1\gamma_1 + v^2\gamma_2 + v^3\gamma_3.$$

Expansion of a vector v in the frame $\{\gamma_k\}$ using summation notation:

$$v = \sum_{k=1}^{3} v^k\gamma_k.$$

Expansion of a vector v in the frame $\{\gamma_k\}$ using summation notation with interval:

$$v = \sum_{1\le k\le 3} v^k\gamma_k.$$

Expansion of a vector v in the frame $\{\gamma_k\}$ using Iverson brackets:

$$v = \sum\nolimits_k v^k\gamma_k\ [1 \le k \le 3].$$

Expansion of a vector v in the frame $\{\gamma_k\}$ using Einstein Summation Convention:

$$v = v^k\gamma_k.$$

3.2. **Dual coordinate frame.** Dual coordinate frame (dual coordinate system – normals): $\{\gamma^k\}$.

Explicit expansion (decomposition or projection) of a vector in the frame $\{\gamma^k\}$ (v_k are the components or projections of the vector v onto the dual basis vectors):

$$v = v_1\gamma^1 + v_2\gamma^2 + v_3\gamma^3.$$

Expansion of a vector v in the frame $\{\gamma^k\}$:

$$v = \sum_{k=1}^{3} v_k\gamma^k.$$

Expansion of a vector v in the frame $\{\gamma^k\}$ using summation notation in interval:

$$v = \sum_{1\le k\le 3} v_k\gamma^k.$$

Expansion of a vector v in the frame $\{\gamma^k\}$ using Iverson brackets:

$$v = \sum\nolimits_k v_k\gamma^k \ \ [1 \le k \le 3] \, .$$

Expansion of a vector v in the frame $\{\gamma^k\}$ using Einstein Summation Convention:

$$v = v_k\gamma^k.$$

3.3. **Kronecker delta.** In the natural coordinate frame $\{\gamma_k\}$, Kronecker delta is defined as, using Iverson brackets,

$$\delta_{ij} = [i = j] \, .$$

In the dual coordinate frame $\{\gamma^k\}$, it is given by

$$\delta^{ij} = [i = j] \, .$$

Mixed Kronecker delta:

$$\delta^i{}_j.$$

4. Linear Algebra (Matrix) Notation

Expansion of a vector v in the frame $\{\gamma^k\}$ (matrix vector):

$$v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} .$$

Transpose of covariant vectors (matrix vector):

$$v = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T .$$

5. Geometric Calculus

Covariant (partial) derivative of v with respect to t: $\partial_t v$.
Contravariant (partial) derivative of v with respect to t: $\partial^t v$.
In index notation, with v in $\{\gamma_k\}$, partial derivative of v with respect to t: $\partial^t v^k\gamma_k$;
With v in $\{\gamma^k\}$, partial derivative of v with respect to t: $\partial_t v_k\gamma^k$.

6. Iverson Bracket

Iverson bracket returns 1 if the logical condition in square brackets is satisfied, and 0 otherwise; that is,

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true;} \\ 0 & \text{otherwise.} \end{cases}$$

where P is a logical statement that can be true or false.

The notation is useful in expressing sums or integrals without boundary conditions. For example,

$$\sum_{1 \le i \le 10} i^2 = \sum\nolimits_i i^2 \; [1 \le i \le 10] \,,$$

In the first sum, the index i is limited to be in the range 1 to 10. The second sum is allowed to range over all integers, but where i is strictly less than 1 or strictly greater than 10, the summand is 0, contributing nothing to the sum. Such use of Iverson bracket can permit easier manipulation of these expressions.

Another example is Kronecker delta. In the natural coordinate frame $\{\gamma_k\}$, Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Or, using Iverson brackets,

$$\delta_{ij} = [i = j] \,.$$

7. Functions

The formal definition of a function relies on the notion of Cartesian products. The Cartesian product of two sets $\mathbb{X}$ and $\mathbb{Y}$ is the set consisting of ordered pairs, that is, expressions of the form

$$(x, y),$$

where x and y are arbitrary elements of $\mathbb{X}$ and $\mathbb{Y}$, respectively. Thus, an ordered pair is just the datum consisting of one element of $\mathbb{X}$ and one of $\mathbb{Y}$, and the element of $\mathbb{X}$ is always written first. In terms of set theory, the Cartesian product between $\mathbb{X}$ and $\mathbb{Y}$ is denoted as $X \times Y$.

Now, a function from $\mathbb{X}$ to $\mathbb{Y}$ is a set $\mathbb{F}$ of ordered pairs, subject to the following condition: for every $x \in \mathbb{X}$ there is a unique $y \in \mathbb{Y}$ for which the pair $(x, y) \in \mathbb{F}$. Additionally, the set of all permitted inputs, $\mathbb{X}$, to a given function is called the domain of the function, while the set of permissible outputs, $\mathbb{Y}$, is called the codomain.

A function f with domain $\mathbb{X}$ and codomain $\mathbb{Y}$ is commonly denoted by

$$f : \mathbb{X} \to \mathbb{Y};$$

or

$$\mathbb{X} \xrightarrow{f} \mathbb{Y}.$$

In this context, the elements of $\mathbb{X}$ are called arguments of f. For each argument x, the corresponding unique y in the codomain is called the function value at x or the image of x under f. It is written as $f(x)$. One says that f associates y to x or maps x to y.

In order to specify a concrete function, the notation $\mapsto$ (an arrow with a bar at its tail) is used. For example,

$$\begin{aligned} f : \mathbb{N} &\to \mathbb{Z} \\ x &\mapsto 4 - x. \end{aligned}$$

The first part, $f : N \to Z$, reads: "f is a function from $\mathbb{N}$ (the set of natural numbers) to $\mathbb{Z}$ (the set of integers)". The second part, $x \mapsto 4 - x$, reads: "x maps to $4-x$". In other words, this function has the natural numbers as domain, the integers as codomain. A function is properly defined only when the domain and codomain are specified.

The set of all y is known as the image of the function, and need not be the whole of the codomain.

8. Fundamental Physics Definitions

Inertial reference frame: an inertial reference frame is a frame that moves freely in space; ie, a frame that is no subject to any force.

Spacetime: time and space are treated in equal footing. Time has the dimension of length, $\dim \mathsf{L}$. In the SI, time is measured in m. This comes from the definition of speed and the speed of light, c:

$$x = ct.$$

It can be seen that c is just a scaling factor. It scales the time dimension to the space dimension.

9. Del Operator

Del is a symbol used in mathematics, in particular, in vector calculus, as a vector differential operator, usually represented by the nabla symbol del. When applied to a function defined on a one-dimensional domain, it denotes its standard derivative as defined in calculus. When applied to a field (a function defined on a multi-dimensional domain), del may denote the gradient (locally steepest slope) of a scalar field, the divergence of a vector field, or the curl (rotation) of a vector field, depending on the way it is applied.

In the Cartesian coordinate system with coordinates $(x^1, x^2, \ldots, x^n)$, del is written as

$$\mathrm{del} = \gamma_i \partial^i;$$

where $\{\gamma_i\}$, $1 \leq i \leq n$, is the standard basis in this space.

10. Metric Tensor

A metric tensor is a type of function defined on a manifold (such as a surface in space) which takes as input a pair of tangent vectors v and w and produces a real number (scalar) $g(v, w)$ in a way that generalizes many of the familiar properties of the inner product of vectors in Euclidean space. In the same way as an inner product, a metric tensor is used to define the length of and angle between tangent vectors, but is not required to be positive-definite. By integration, the metric tensor allows one to define and compute the length of curves on the manifold.

11. Scalar and Vector Fields

11.1. **Scalar Fields.** In mathematics and physics, a scalar field associates a scalar value to every point in a space. The scalar may either be a mathematical number, or a physical quantity. Scalar fields are required to be coordinate-independent, meaning that any two observers using the same units will agree on the value of the scalar field at the same point in space (or spacetime). Examples used in physics include the temperature distribution throughout space, the pressure distribution in a fluid, and so on. These fields are the subject of scalar field theory.

Physically, a scalar field is additionally distinguished by having units of measurement associated with it. In this context, a scalar field should also be independent of the coordinate system used to describe the physical system; that is, any two observers using the same units must agree on the numerical value of a scalar field at any given point of physical space.

In physics, scalar fields often describe the potential energy associated with a particular force. The force is a vector field, which can be obtained as the gradient of the potential energy scalar field. Examples include:

- Potential fields, such as the Newtonian gravitational potential, or the electric potential in electrostatics, are scalar fields which describe the more familiar forces.
- A temperature, humidity or pressure field, such as those used in meteorology.

11.2. **Vector Fields.** In vector calculus, a vector field is an assignment of a vector to each point in a subset of Euclidean space. A vector field in the plane, for instance, can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane. Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.

In coordinates, a vector field on a domain in n-dimensional Euclidean space can be represented as a vector-valued function that associates an n-tuple of real numbers to each point of the domain. This representation of a vector field depends on the coordinate system, and there is a well-defined transformation law in passing from one coordinate system to the other. Vector fields are often discussed on open subsets of Euclidean space, but also make sense on other subsets such as surfaces, where they associate an arrow tangent to the surface at each point (a tangent vector).

12. Grad, Div, Curl, and All That

12.1. **Grad, gradient.** The variation in space of any quantity can be represented (e.g., graphically) by a slope. The gradient represents the steepness and direction of that slope.

In vector calculus, the gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field, and whose magnitude is that rate of increase.

The gradient (or gradient vector field) of a scalar function, $f(x_1, x_2, \ldots, x_n)$, is denoted ∇f where ∇ (the nabla symbol) denotes the vector differential operator, del. The notation is also commonly used for the gradient. The gradient of f is defined as the unique vector field whose dot product with any unit vector γ at each point x is the directional derivative of f along γ.

12.2. **Div, divergence.** The divergence of a vector field, F, on Euclidean space is a function (or scalar field) defined by

$$\operatorname{div} F = \nabla \cdot F = \partial^i F^i;$$

The divergence at a point represents the degree to which a small volume around the point is a source or a sink for the vector flow, a result which is made precise by the divergence theorem.

13. How to Solve Physics Problems

Choose the framework for the physical model: Classical Physics, Quantum Mechanics. Within Classical Physics, General Relativity, Spacial Relativity or Newtonian Physics. Led by the physical model, choose the arena for the geometrical model: General Relativity –¿ curved spacetime, Special Relativity –¿ flat, Minkowski spacetime, Newtonian Physics –¿ flat, Euclidean 3-space, plus universal time. Remember: geometry, geometric objects (scalars, vectors, tensors, fields, and so on) and their relationships are meant to represent the material (physical) "reality". (*Ubi materia, ibi geometria.* — Johaness Kepler.)

Pose the problem using physics laws (as conservation laws) aided by geometry: physical analogy, symmetry, geometric objects, vector algebra and vector calculus are your friends here. The result should be a set of coordinate-free equations that relate different geometric objects based on physics principles; that is, a vector notation model. (The physics is in the arrows and contours, not in the matrix elements — John Denker.)

Use dimensional analysis when developing the mathematical and physical model. Dimensional analysis of the model is not only a requisite for the formula(e) to be correct and consistent, but it can be also used to sketch the solution plan and to test the solution itself.

Choose a suitable coordinate (reference) frame where to express the model. Mention it explicitly. Expand the geometric objects onto the chosen frame(s).

Work towards the answer using index notation.

If no calculation is required, such as in proving theorems, translate the index notation solution in terms of vector notation.

If calculation is required, express the index notation in terms of matrix notation. Use linear algebra to solve the problem.

Verify the solution backwards.

Present the results.

14. Physics Principles

14.1. **New foundations for classical mechanics — Hestenes, p. 81.** Physical points (events): vectors.

Distances: metric, intervals.

Physical space: The set of all physical points determined by physical rules.

Euclidean geometry has certain implications when interpreted as a physical theory.

Zeroth law of physics: Physical space is a 3-D Euclidean Space (valid in Newtonian arena).

We label the points of physical space by vectors regarding each vector as identical to the point it labels. In math apps, it is quite sufficient to regard each Euclidean Space as a vector space. However, for physics purposes, it is essential to distinguish between the point and the vector which labels it.

The study of lines, curves, volumes in 3-D Euclidean Space is a pure math enterprise, but the correspondence of 3-D Euclidean space warranties its relevance to the study of physical space.

14.2. **Applications of Classical Physics — Roger D. Blandford, Kip S. Thorne, 2011.** Geometric Principle: The laws of physics must all be expressible as geometric (coordinate-independent and reference-frame-independent) relationships between geometric objects, which represent physical entities.

There are three different conceptual frameworks for the classical laws of physics, and correspondingly three different geometric arenas for the laws; [see Fig. 1]. General Relativity is the most accurate classical framework; it formulates the laws as geometric relationships between geometric objects in the arena of curved 4-dimensional spacetime. Special Relativity is the limit of general relativity in the complete absence of gravity; its arena is flat, 4-dimensional Minkowski spacetime. Newtonian Physics is the limit of general relativity when (i) gravity is weak but not necessarily absent, (ii) relative speeds of particles and materials are small compared to the speed of light c, and (iii) all stresses (pressures) are small compared to the total density of mass-energy; its arena is flat, 3-dimensional Euclidean space with time separated off and made universal (by contrast with relativity's reference-frame-dependent time).

In most textbooks, physical laws are expressed in terms of quantities (locations in space, momenta of particles, etc.) that are measured in some coordinate system. For example, Newtonian vectorial quantities are expressed as triplets of numbers, e.g., $p = (p_x, p_y, p_z) = (1, 9, -4)$, representing the components of a particle's momentum on the axes of a spatial coordinate system.

We shall insist, on the other hand, that the Newtonian laws of physics all obey a Geometric Principle: they are all geometric relationships between geometric objects, expressible without the aid of any coordinates or bases. An example is the Lorentz force law $m\partial_t v = q(E + v \times B)$ – a (coordinate-free) relationship between the geometric (coordinate-independent) vectors (v, E

and B) and scalars (the particle's mass m and charge q). No coordinates or basis vectors are needed for this law of physics, nor is any description of v, E or B as a matrix-like entities with components v^i, E^i and B^i.

Components are secondary; they only exist after one has chosen a set of basis vectors. Components are an impediment to a clear and deep understanding of the laws of physics. The coordinate-free, component-free description is deeper, and – once one becomes accustomed to it – much more clear and understandable.

By adopting this geometric viewpoint, we shall gain great conceptual power, and often also computational power. For example, when we ignore experiment and simply ask what forms the laws of physics can possibly take (what forms are allowed by the requirement that the laws be geometric), we shall find that there is remarkably little freedom. Coordinate independence and basis independence strongly constrain the laws of physics. This power, together with the elegance of the geometric formulation, suggests that in some deep (ill-understood) sense, Nature's physical laws are geometric and have nothing whatsoever to do with coordinates or components or vector bases.

14.2.1. *Newtonian Physics.* The arena for the Newtonian laws is a spacetime composed of the familiar 3-dimensional Euclidean (universal) space of everyday experience (which we shall call 3-space), and a universal time t. We shall denote point (locations) in 3-space by capital script letters such as $\mathcal{P}$ and $\mathcal{Q}$. These points and the 3-space in which they live require no coordinates for their definition.

A scalar is a single number that we associate with a point, $\mathcal{P}$, in 3-space. We are interested in scalars that represent physical quantities, e.g., temperature T. When a scalar is a function of location $\mathcal{P}$ in space, e.g. $T(\mathcal{P})$, we call it a scalar field.

A vector in Euclidean 3-space can be thought of as a straight arrow that reaches from one point, $\mathcal{P}$, to another, $\mathcal{Q}$ (e.g., the arrow Δx of [Fig. 1.1a]). Equivalently, Δx can be thought of as a direction at $\mathcal{P}$ and a number, the vector's length. Sometimes we shall select one point $\mathcal{O}$ in 3-space as an "origin" and identify all other points, say $\mathcal{Q}$ and $\mathcal{P}$, by their vectorial separations $x_{\mathcal{Q}}$ and $x_{\mathcal{P}}$ from that origin.

The Euclidean distance $\Delta\sigma$ between two points $\mathcal{P}$ and $\mathcal{Q}$ in 3-space can be measured with a ruler and so, of course, requires no coordinate system for its definition. (If one does have a Cartesian coordinate system, it can be computed by the Pythagorean formula, a precursor to the "invariant interval" of flat spacetime, Sec. ??.) This distance $\Delta\sigma$ is also the length $\|\Delta x\|$ of the vector Δx that reaches from $\mathcal{P}$ to $\mathcal{Q}$, and the square of that length is denoted

$$\|\Delta x\|^2 \equiv (\Delta x)^2 \equiv \Delta\sigma^2.$$

Of particular importance is the case when $\mathcal{P}$ and $\mathcal{Q}$ are neighboring points and Δx is a differential (infinitesimal) quantity dx. By traveling along a sequence of such dx's, laying them down tail-at-tip, one after another, we can map out a curve to which these dx's are tangent (Fig. 1.1b). The curve is $\mathcal{P}(\lambda)$, with λ a parameter along the curve; and the infinitesimal vectors that map it out are $dx = (d\mathcal{P}/d\lambda)d\lambda$.

The product of a scalar with a vector is still a vector; so if we take the change of location dx of a particular element of a fluid during a (universal) time interval dt, and multiply it by $1/dt$, we obtain a new vector, the fluid element's velocity $v = dx/dt$, at the fluid element's location $\mathcal{P}$. Performing this operation at every $\mathcal{P}$ in the fluid defines the velocity field $v(\mathcal{P})$. Similarly, the sum (or difference) of two vectors is also a vector and so taking the difference of two velocity measurements at times separated by dt and multiplying by $1/dt$ generates the acceleration $a = dv/dt$. Multiplying by the fluid element's (scalar) mass m gives the force $F = ma$ that produced the acceleration; dividing an electrically produced force by the fluid element's charge q gives another vector, the electric field $E = F/q$, and so on.

We can define inner products [Eq. (1.4a) below] of pairs of vectors at a point (e.g., force and displacement) to obtain a new scalar (e.g., work), and cross products [Eq. (1.21a)] of vectors to obtain a new vector (e.g., torque). By examining how a differentiable scalar field changes from point to point, we can define its gradient [Eq. (1.15b)]. In this fashion, which should be familiar to the reader and will be elucidated and generalized below, we can construct all of the standard scalars and vectors of Newtonian physics. What is important is that

> these physical quantities require no coordinate system for their definition.

They are geometric (coordinate-independent) objects residing in Euclidean 3-space at a particular time.

Geometric objects: points, locations, distances, vectors, (scalar, vector) fields.

Geometric relations: inner products, cross products, gradients.

Physical quantities: mass, force, velocity, acceleration, work, ...

Main tenet: laws of physics can be expressed in terms of geometric relations of geometric objects. These geometric objects, in their turn, are meant to represent physical quantities.

14.2.2. *Newtonian Physics Fundamental Principle.* It is a fundamental (though often ignored) principle of physics that

> the Newtonian physical laws are all expressible as geometric relationships between these types of geometric objects, and these relationships do not depend upon any coordinate system or orientation of axes, nor on any reference frame (on any purported velocity of the Euclidean space in which the measurements are made).

We shall call this the Geometric Principle for the laws of physics, and we shall use it throughout this book. It is the Newtonian analog of Einstein's Principle of Relativity (Sec. ?? below).

14.3. **Newtonian Space-time vs. Einstein Spacetime.**

14.3.1. *Newtonian Absolute Space and Absolute Time – Principia, Newton. 17..* Absolute, true and mathematical time, of itself, and from its own nature flows equably without regard to any thing external.

Absolute Space, in its own nature, without any thing external, remains always similar and immovable.

14.3.2. *Newtonian Space-Time – Principia for the Common Reader. S. Chandrasekhar. 1995.* In current terminology, the space-time manifold that is assumed by Newton is the Cartesian product

$$t \times \text{Euclidean 3-space}$$

where t is Newton's 'equable time'.

14.3.3. *Continuity of spacetime.* Grassmann placed a correspondence between the geometry "continuum" an the real line; that is, he established a bridge between geometry and algebra (and calculus). Thus, geometry is suitable to represent the spacetime continuum.

14.3.4. *Einstein Spacetime.* Space and time are not two separated entities anymore. They are to be taken as one entity: spacetime. Minkowski give it a fully mathematical representation – via the metric:

> Henceforth space by itself, and time by itself, are doomed to fade into shadows,
> and only a union of the two will preserve and independent reality.

Newton's space and time are absolute, fixed, immovable, not affected by anything external; but, Einstein's spacetime is relative, flexible. It changes (curves, bends) under the presence of mass. That is, Einstein discovered that mass affects spacetime, telling it how to curve. In its turn, spacetime affects mass, telling it how to move:

Ubis materia ibis geometria.

15. Advice – Electromagnetism with Spacetime Algebra. John F. Lindner

Before developing the necessary mathematics, survey the crucial physics.

16. Some notation

Rank of a tensor (scalar, vectors, tensors...): for v^i; then, $\operatorname{rank} v = 3$.

Let $\mathcal{P}$ represent a point in 3-D space with coordinates $(t\,; x_i)$ and let ϕ be a scalar field in $\mathcal{P}$, that is $\phi\,(\mathcal{P}) = \phi(t, x_i)$. The temporal derivative of ϕ is defined as

$$\frac{\partial \phi}{\partial t} \equiv \partial_0 \phi.$$

The spatial derivatives of ϕ are defined as

$$\frac{\partial \phi}{\partial x_1} \equiv \partial_1 \phi;$$
$$\frac{\partial \phi}{\partial x_2} \equiv \partial_2 \phi;$$
$$\frac{\partial \phi}{\partial x_3} \equiv \partial_3 \phi.$$

These last three equations can be gather as

$$\frac{\partial \phi}{\partial x_i} \equiv \partial_i \phi.$$

In vector notation, $\partial_i \phi$ is called the gradient and it is written as $\operatorname{grad} \phi$ or $\nabla \phi$. Note that $\partial_i \phi$ is a rank-1 tensor (ie, a vector).

Some equivalent notations for $\partial_i \phi$ are

$$\partial_i \phi \equiv \partial_{x_i} \phi$$

or, occasionally,

$$\partial_i \phi \equiv \phi_{,i}\,.$$

Newtonian event: $\mathcal{P}$. Newtonian event coordinates: $(t\,; x_i)$.

17. Heat Equation

Let $T : \mathbb{R}^4 \to \mathbb{R}$ represent the temperature of a body $\mathcal{B}$ at point $\mathcal{P}$ with coordinates $(t\,; x_i)$. The temperature distribution inside $\mathcal{B}$ is thus described by the heat equation, viz.,

$$\partial_0 T = \kappa\,(\partial_i \partial_i T) = \kappa \triangle T;$$

where i runs from 1 to 3.

17.1. **Laplacian.**

Let $f : \mathbb{R}^3 \to \mathbb{R}$. The Laplacian operator, denoted by $\triangle$, is defined as

$$\triangle f \equiv \frac{\partial^2 f}{\partial x} + \frac{\partial^2 f}{\partial y} + \frac{\partial^2 f}{\partial z} \equiv \partial_i \partial_i f.$$

17.2. **Wave Equation.** The wave equation describes the motion of a vibrating string that connects two points $\mathcal{A}$ and $\mathcal{B}$. Suppose that the height of the string at distance x from $\mathcal{A}$ at time t is written $h(x, t)$. Then the wave equation says that

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 h}{\partial x^2} \Longrightarrow \frac{1}{v^2}\partial_0\partial_0 h = \partial_1\partial_1 h.$$

In three-dimensions, the wave equation reads

$$\frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} = \triangle\, h \Longrightarrow \Box^2\, h = 0;$$

where the operator $\Box^2$ is called D'Alembertian and is a shorthand for

$$\triangle\, h - \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2}.$$

or, in index notation,

$$\Box^2\, h = \triangle\, h - \frac{1}{\|v_i\|}\partial_0\partial_0 h.$$

18. Sets Test

$\mathbb{R}$. $\mathbb{R}^{\not k}$. $\mathbb{R}^2$. $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}^3$. $\mathbb{D}^n$.

19. Matrices

Matrix: A.
Element of a 2D matrix: a_{ij}
Explicit matrix elements:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Determinant: $\det A$.
Explicit determinant elements:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

20. Bases and matrices - Matrix algebra for beginners, Part II - linear transformations, eigenvectors and eigenvalues – Jeremy Gunawardena

We need one more idea to see where matrices enter the picture. It goes back to the French philosopher and mathematician René Descartes in the 17th century. Descartes showed how geometry can be turned into algebra by using a *system of coordinates*. Descartes did not know about vector spaces and linear transformations but we can readily translate his ideas into this context. We choose two vectors (for a 2 dimensional vector space) as *basis vectors*, or, in Descartes' language, as the axes of the coordinate system. Descartes took his axes to be at right angles to each other, which is what we still do whenever we draw a graph. Axes which are at right angles define a *Cartesian* coordinate system. However, there is no necessity to do this and our basis vectors can be at any angle to each other, provided only that the angle is not zero. It is not much use to have coincident axes (see the comments below on bases in higher dimensions). Let us call the basis vectors b_1 and b_2. It is important to remember that their choice can be arbitrary, provided only that $b_1 \neq \alpha b_2$. Although the order of the basis vectors is not important for them to constitute a basis, it is important for some of the things we want to do below, so it is best to think of b_1, b_2 as a different basis to b_2, b_1.

Now, if we have any vector x then we can use the parallelogram rule in reverse to project it into the two basis vectors, as shown in [Figure 2]. This gives us two component vectors lying in the direction of the basis vectors. Each component is some scalar multiple of the corresponding basis vector. In other words, there are scalars α_1 and α_2, which are often called the scalar components of x, such that

$$x = \alpha_1 b_1 + \alpha_2 b_2.$$

All we have done, in effect, is to work out the coordinates of the point at the end of x in the coordinate system defined by b_1 and b_2. Descartes would still recognize this, despite all the fancy new language.

> The name relativity theory was an unfortunate choice: The relativity of space and time is not the essential thing, which is the independence of laws of Nature from the viewpoint of the observer.
> — Arnold Sommerfeld