A BRIEF INTRODUCTION TO INDEX NOTATION

DIEGO HERRERA

Abstract. This is the abstract.

Contents

1. Remembering Basic Concepts

Let's quickly review how we note, work and operate with vectors in engineering using traditional notation.

1.1. The Cartesian Coordinate System.

In engineering, we are used to work with vectors using the *Cartesian coordinate system* in three-dimensional space. Let's look at how it is constructed, some of its properties and the notation that we have inherited from it.

The Cartesian coordinate system is a system for denoting points in space. It uniquely assigns a triplet of real numbers, called *coordinates*, to each point. This system can be constructed by

(1) choosing an ordered triplet of lines, called *coordinate axes* or simply *axes*, any two of them being perpendicular;
(2) labeling the point where the three lines meet as the *origin*, noted $\mathcal{O}$;
(3) assigning a *single unit of length* for all of the axes;
(4) assigning an orientation for each axes – orientation sets the direction in which an axis increases or decreases; and,
(5) finally, labeling the axes as X, Y and Z.

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With the system thereby constructed, the coordinates of any point, say $\mathcal{P}$, are obtained by drawing a perpendicular line through $\mathcal{P}$ to each coordinate axis, by reading the points where these perpendiculars meet the axes and by assigning these three numbers, joined as (x, y, z), to $\mathcal{P}$, where x is the reading from $\mathcal{P}$ to the X-axis, y to the Y-axis and z to the Z-axis. Notice that these readings are signed readings; *i.e.*, they may well be negative (decreasing orientation).

A key feature of the Cartesian coordinate system is that, by construction, its coordinates measure lengths. Thus any triplet of Cartesian coordinates has physical dimensions of [L] in all of its components. This feature distinguishes this system from other systems, such as cylindrical or spherical, some of whose components measure lengths and some others angles.

1.2. **Unit Vectors And Basis.** [close the gap between coordinates and vectors: up to here only coordinates have been defined, from here down below, vectors are used, but never defined :( ]

[arguments: so far, we know how to reach any point $\mathcal{X}$ from $\mathcal{O}$, by means of measuring the distances from the coordinate axes to the point itself, the result would be a triplet of the form (x, y, z). Using this representation, we could define equality of triplets, ie, equality of points, addition of triplets, scaling of triplets (multiplication with scalars), and so on. For instance, let's define addition: let $\mathcal{P} = (a, b, c)$ and let $Q = (d, e, f)$, then $P + Q = (a + d, b + e, c + f)$. The problem arises in how to define distances, lengths and angles or in how to define transformations, such as translations, rotations, calculus [1] using this representation. The problem seems to increase when we think about calculus. How to define derivatives and integrals to calculate slopes, speeds, accelerations, areas, volumes and so forth?

Let's us look for another representation of points using Cartesian coordinates. Perhaps, we could find a representation similar to polynomials. Polynomials have a structure similar to our triplets: their "powers of x". Polynomials are great for computations for they provide, by its "exponents of x", a natural and efficient way of doing calculations. For instance, let $p = a + bx + cx^2$ and let $q = d + ex + fx^2$, then $p + q = (a + d) + (b + e)x + (c + f)x^2$. benefits of the + between components, we know how to graph them, we know how to do calculus with them...]

Complementarily to the coordinate representation of points as triplets of numbers, points in a Cartesian coordinate system may also be represented by a position vector, $\vec{r}$, which can be thought of as an arrow pointing from the origin of the coordinate system to the point.

A step forward in the development of the 3D Cartesian system is the introduction of *unit vectors*. Unit vectors are vectors that have the coordinates [2] $(1, 0, 0) \equiv e_x$, $(0, 1, 0) \equiv e_y$ and $(0, 0, 1) \equiv e_z$. Thus, e_x points to the direction to which the values of the X-axis increase, e_y to which the values of the Y-axis increase and e_z to which the values of the Z-axis increase. These basis vectors have two important properties:

(1) by construction, e_x, e_y and e_z have unit length – *normalization*; and
(2) they form an angle of 90° with one another – *orthogonality*.

Now, to ease computation with vectors and their coordinates, it will be useful to determine a way to decompose *any* vector into the axes by means of unit vectors, because unit vectors provide a scale and a direction. These two properties are essential for engineering applications. Such a decomposition is done thanks to the fact that e_x, e_y and e_z, thereby defined, form a *basis* for 3D Cartesian systems.

A basis, denoted by B, is a set of vectors that satisfies two conditions:

[1] This geometric objects and geometric transformations are essential to develop geometric theories, such as Euclidean geometry, and essential to develop physical theories.

[2] $\equiv$ means *is defined as*.

(1) linear independence: for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$, if $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ for $v_1, v_2, \ldots, v_n \in B$, then necessarily $a_1 = a_2 = \cdots = a_n = 0$; and
(2) spanning: for every x in $\mathbb{R}^3$ it is possible to choose $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$.

It can be shown, although not here, that e_x, e_y and e_z do, in fact, form a basis. Moreover, they form an orthonormal (orthogonal and normalized) basis for the 3D Cartesian coordinate system.

This property is important, for it eases computation with vectors and their components by allowing the decomposition of *any* vector in 3D to the form

$$v = v_x e_x + v_y e_y + v_z e_z. \tag{1}$$

Notice that each term in Eq. 1 is composed of a real number – v_x, v_y and v_z – multiplied by a vector – e_x, e_y and e_z. v_x indicates the *magnitude of* $\boldsymbol{v}$ in the x-direction, v_y in the y-direction and v_z in the z-direction.

To summarize this section, it can be said that e_x, e_y and e_z form an orthonormal basis for the Cartesian coordinate system; *i.e.*, we can now express 3D vectors in a suitable form that eases computations: .

1.3. **The Engineering Vector.** In traditional notation, a 3-D vector $\boldsymbol{v}$ is thought of as a sum of three terms looking like [3]

$$\boldsymbol{v} = v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}} + v_z \hat{\boldsymbol{k}}, \tag{2}$$

where v_x, v_y and v_z are real numbers called the *components of* $\boldsymbol{v}$ onto the basis vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. The three basis vectors are called *the elements of the orthonormal basis of the 3-D space*.

Notice that each term has two different parts: a real number and a vector. v_x indicates the *magnitude of* $\boldsymbol{v}$ in the x-direction, v_y in the y-direction and v_z in the z-direction.

Orthonormal basis means that the vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ form an angle of 90° with one another [4] (*orthogonality*), have unit length (*normalization*), and *every* vector of $\mathbb{R}^3$ can be *uniquely* decomposed as shown in Eq. 2. Such a descomposition is also called *the representation of* $\vec{v}$ *in* $\mathbb{R}^3$. The basis set of $\mathbb{R}^3$, *aka the 3-D basis*, is the set that has $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ as elements.

1.4. **Vector Operations.** In basic curses, we learn that, with vectors, we can do addition, subtraction, multiplication, but not division, at least not between vectors. Addition and subtraction are presented *component wise*; *e.g.*, for two vectors $\vec{u}$ and $\vec{v}$, we have

$$\vec{u} + \vec{v} = (u_x + v_x)\hat{\boldsymbol{\imath}} + (u_y + v_y)\hat{\boldsymbol{\jmath}} + (u_z + v_z)\hat{\boldsymbol{k}},$$
$$\vec{u} - \vec{v} = (u_x - v_x)\hat{\boldsymbol{\imath}} + (u_y - v_y)\hat{\boldsymbol{\jmath}} + (u_z - v_z)\hat{\boldsymbol{k}}.$$

Addition and subtraction of vectors with scalars is not defined. So, a thing such as $a + \vec{v}$, where $a \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^3$, is not even mentioned.

So far, so good. What is left is to review the interesting operation of multiplication. Interesting, because it comes in three flavors:

(1) Multiplication with scalars: if $a \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^3$, then $a\vec{v}$ is another vector defined as

$$a(v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}} + v_z \hat{\boldsymbol{k}}) = a v_x \hat{\boldsymbol{\imath}} + a v_y \hat{\boldsymbol{\jmath}} + a v_z \hat{\boldsymbol{k}}.$$

[3] I could have used $\vec{v}$, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ instead of $\boldsymbol{v}$, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$, respectively, but I wanted to stress how heavy traditional notation looks like.

[4] In Cartesian coordinate system, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ point in every increasing direction of Cartesian axes. $\hat{\boldsymbol{\imath}}$ points in the direction of increasing x's, $\hat{\boldsymbol{\jmath}}$ in the direction of increasing y's and $\hat{\boldsymbol{k}}$ in the direction of increasing z's and they are placed one after another using the *right-hand rule*.

(2) Dot product: let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$. Then the *dot product of* $\vec{u}$ *and* $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z.$$

The dot product is also called *scalar product*, since its result is a scalar.

(3) Cross product: let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$. Then the *cross product of* $\vec{u}$ *and* $\vec{v}$ is given by

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\hat{\boldsymbol{\imath}} + (u_3 v_1 - u_1 v_3)\hat{\boldsymbol{\jmath}} + (u_1 v_2 - u_2 v_1)\hat{\boldsymbol{k}}.$$

The cross product is also called *vector product*, since it results in another vector.

Of the three definitions, that of the cross product is the weirdest, because it seems arbitrary [5]. Although cumbersome, did you notice the symmetry in the scalar components of $\vec{u} \times \vec{v}$, $(u_2 v_3 - u_3 v_2)$, $(u_3 v_1 - u_1 v_3)$, and $(u_1 v_2 - u_2 v_1)$?

One last thing about the cross product: it is only defined for vectors in 3D, no more, no less. This seems not to be a problem, since in engineering we do not work in more than 3D; but it severely and unnecessarily limits theory.

2. Important note

Each term in $\gamma_i v^i$ is composed of a scalar, v^i, and a vector, γ_i. By virtue of our definition of multiplication of vectors with scalars, $\gamma_i v^i$ can be rearranged to $v^i \gamma_i$ without changing the result [6]. Additionally, see that v^i indicate which component; while γ_i, which direction.

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[5] Under its limits of validity, the cross product is nevertheless well behaved, mathematically consistent, and thus useful. Besides, it has a proper geometrical interpretation that we will see later.

[6] We will be forced to change this rule once we meet vectorial calculus.

3. Notation, Notation, Notation!

Working with vectors as currently notated in engineering is exhausting: engineering notation is too verbose and inconsistent. Inconsistent, because, while the spatial axes are named x, y and z and while the components of $\vec{v}$ onto such axes are logically called v_x, v_y, and v_z (following the axis names), the unit vectors – supposed to represent the axes themselves – are called $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. Besides, the notation does not extend well to higher dimensions. What would happen with notation if we added another dimension to our studies? Let us say we wanted to form a 4-D vector, how would we note the fourth basis vector? $\hat{\boldsymbol{h}}$, $\tilde{l}$, $\check{m}$, $\bar{n}$? What if we needed to work with vectors in 10-D? We need notation that eases working with vectors, extends nicely vector treatment to higher dimensions and saves some typing in the meantime.

3.1. Undecorated Vars.

Notation 1 (Undecorated variables). *From now on, we will generally* not *decorate variables;* e.g.*, a vector v will be denoted as v, not as $\vec{v}$, $\boldsymbol{v}$, $\hat{v}$, or alike. We will denote mathematical objects with italic font types. Scalars will be denoted with the first letters of the alphabet,* e.g.*, a, b, c; vectors with the last letters of the alphabet,* e.g.*, u, v, w; unit vectors with* one *Greek letter: γ. When needed, superscripts and subscripts will be added to variables. They will help to enforce this policy.*

Decorations look "cute", but are troublesome. How would we tell the difference among scalars, vectors, unit vectors, *etc.*, in equations then? We found our rationale in [1]

> [...] a name is not the same as an explanation. Do not expect the structure of a name or symbol to tell you everything you need to know. Most of what you need to know belongs in the legend. The name or symbol should allow you to look up the explanation in the legend.

Let us explain now what we mean by indexing variables. In Eq. 2, for instance, we noted the components of the vector v using subindices that indicate *space variables*; v_x, v_y and v_z. This is the piece of notation that does not extend nicely to higher dimensions. To overcome this limitation, let us agree that

3.2. Index Notation.

Notation 2 (Index notation). *Numeric indices will replace space variable symbols in the following way:*

- *Basis vectors will be denoted with the Greek lowercase letter* gamma [7]*: γ. Thus, $\hat{\boldsymbol{\imath}} \to \gamma$, $\hat{\boldsymbol{\jmath}} \to \gamma$ and $\hat{\boldsymbol{k}} \to \gamma$.*
- *The* three spatial coordinates *will be replaced with* one indexed spatial coordinate *in the following way: $x \to x^1$, $y \to x^2$ and $z \to x^3$. For a spatial vector, say $x \in \mathbb{R}^3$, this replacement means*
$$x = \gamma_1 x^1 + \gamma_2 x^2 + \gamma_3 x^3.$$
- *For* non-spatial vectors*, we will index them as though they were spatial vectors. For a vector v in $\mathbb{R}^3$, this means $v_x \to v^1$, $v_y \to v^2$ and $v_z \to v^3$ or, more fully,*
$$v = \gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3.$$

It is to be noted that

(1) the number of indices represents the dimension of the variable: a scalar will have no indices, a vector will have one, a matrix will have two, and so on.

[7] This choice goes against the notation used in vector algebra and linear algebra, which use e instead. But, in geometric algebra, our ultimate goal, the usage of γ is the common choice.

(2) the range of the indexing variables may be understood from context – a vector $v \in \mathbb{R}^3$ will have one index ranging from 1 to 3;
(3) superscripts are indices rather than a exponents – "x^2" is therefore to be read "second component of v", instead of "x to the power of 2".
(4) the basis vectors, γ_1, γ_2, γ_3, follow the numbering of their respective vector component indices, but are *sub*scripts rather than *super*scripts. We will explain what it means *geometrically* in the following sections.

One immediate advantage of using these notational conventions is that vectors can be written in a rather compact way, for instance, by using the *sigma notation*. For our vector v in $\mathbb{R}^3$ defined in Eq. 2, we have

$$\begin{aligned}\vec{v} &= v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}} \rightarrow \\ v &= v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}, \\ &= v_x\gamma_x + v_y\gamma_y + v_z\gamma_z, \\ &= v^1\gamma_1 + v^2\gamma_2 + v^3\gamma_3, \\ &= \sum_{i=1}^{3}\gamma_i v^i,\end{aligned} \tag{3}$$

or, *iff* [8] the range of i is understood,

$$v = \sum \gamma_i v^i.$$

Sweet! But, we could do even better. The problem is that, if you work quite a bit with vector components or summations, you could end up with many $\sum$'s. So many that they could amount the hundreds [9]. We would like to become proficient on vectors, not on the Greek alphabet. To overcome this, Einstein [10] came up with an idea that will cut down verbosity, while providing an elegant syntax:

3.3. Einstein Summation Convention.

NOTATION 3 (Einstein summation convention). *When an index variable appears* twice *in a single term, the convention implies summation of that term over all the values of the index. This means simply to eliminate the summation symbol, $\sum$, and the summation bounds, while keeping the summand. For instance, in Eq. 3, Einstein convention means*

$$v = \sum_{i=1}^{3}\gamma_i v^i \qquad \rightarrow \qquad v = \gamma_i v^i. \tag{4}$$

An index that is summed over is called a summation index*, in this case i. It is also called a* dummy index *since any symbol can replace i without changing the meaning of the expression, provided that it does not collide with index symbols in the same term. A free index appears* once and only once *within each additive term in an expression.*

An index that is not summed over is a free index *and should be found in each term of the equation or formula if it appears in any term. Compare dummy indices and free indices with free variables and bound variables.*

Free indices only appear as either super- or sub-script, never as both, and they must occur exactly once in every term.

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[8] *iff* is the math lingo for "if and only if", not a typo.
[9] No kidding! Don Knuth claimed to have used around 1000 $\sum$'s [2]!
[10] Yep, *that* Einstein.

Dummy indices occur once as super- and once as sub-script, so summation over these indices is implied. Dummy indices must never have the same label as a free index.

NOTE 1 (Summary of Einstein summation convention). *A brief summary on how index notation works in the form of "rules"[4]:*

Rule 1: *Repeated, doubled indices in quantities multiplied together are implicitly summed.*
Rule 2: *Indices that are not summed over (free indices) are allowed to take all possible values unless stated otherwise.*
Rule 3: *It is illegal to use the same dummy index more than twice in a term unless its meaning is made explicit.*

NOTE 2 (Warning! Bad notation). *When establishing index notation or Einstein summation convention – like Eq. 4, some authors go from $\gamma_i v^i$ to simply v^i and say that "v^i are the components of v". They just drop the basis* [11]*. This notation is useless, so it will* not *be used here. It is useless, because the notation $\gamma_i v^i$ says* explicitly *where v is being* decomposed *onto. It gives the basis γ_i. Only when knowing the basis, phrases like "the components of v" make sense. Dropping the basis, like v^i, gives no information onto where the vector was decomposed.*

NOTE 3 (Warning! Source of errors). *We will see that when multiplying two vectors, say u and v, in index notation, we can be tempted to violate.*

3.4. **Summary.** To express eng. vectors in index notation the general process is

(1) drop decorations (no bold faces, no arrows, no accents, *etc.*);
(2) choose a basis onto which the vector will be *decomposed*;
(3) for every vector, ;
(4) ...

Wow! Quite a journey from Eq. 2 to Eq. 4.

Sec. 4 is devoted to ... proficiency in using index notation. If you feel you need practice, check it out.

[11] For instance, in [3], the author even says that bases are not needed.

4. Exercises

Since the notations that we have just introduced are a bit too much to digest at once, let us dedicate this section to practice to which we have agreed. Note, while checking the examples, how the conventions ease working with vectors and how "good looking" [12] the resulting equations become.

Example 4.1. Express the engineering vector $\vec{v}$, Eq. 2, in index notation, using sigma notation and Einstein summation convention.

Solution. Let's us begin by writing the engineering vector again, just as a remainder:

$$\vec{v} = v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}.$$

Now, let's work towards the solution.

- Change the typography; *i.e.*, no decorations: $\vec{v} = v$.
- Things

Let v be an element of the $\mathbb{R}^n$ space and let $\{\gamma_i\,;\ i : 1, \ldots, n\}$ denote the space basis set. Then, v can be decomposed onto such a basis as

$$v = \gamma_1 v^1 + \gamma_2 v^2 + \cdots + \gamma_n v^n,$$

or, using sigma notation,

$$v = \sum_{i=1}^{3} \gamma_i v^i,$$

or, using index notation and Einstein summation convention,

$$v = \gamma_i v^i,$$

The engineering vector is found by setting n to 3; that is, $v = \gamma_i v^i$ where i runs from 1 to 3. Verbosely,

$$v = \gamma_i v^i = \gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3.$$

□

Example 4.2. Explicitly expand the 3-D vector $\gamma_i v^i$.

Solution.

- The first step is to notice any free indices. In this case, there are none.
- Then find any dummy indices. There's one, i: $\gamma_i v^i$.
- Establish the upper limit of each index. In this case, $v \in \mathbb{R}^3$, so i runs from 1 to 3.
- Find the terms for each dummy index one at time until you run out of values and then join them with a + (*summation* convention, remember?):

$$\begin{aligned} \gamma_i v^i &= \gamma_1 v^1 + \cdots && [i = 1] \\ &= \gamma_1 v^1 + \gamma_2 v^2 + \cdots && [i = 2] \\ &= \gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3. && [i = 3] \end{aligned}$$

The final answer is found when you run out of values; *i.e.*,

$$v = \gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3.$$

□

[12] Good looking equations form a part of what mathematicians call *mathematical beauty*.

EXAMPLE 4.3. Express the engineering vector $\vec{v}$, Eq. 2, in index notation, using sigma notation and Einstein summation convention.

Solution. Instead of doing that for 3-D, we will generalize the notation for vectors in n-D and get the solution as a special case [13].

- Change the typography; *i.e.*, no decorations: $\vec{v} = v$.

Let v be an element of the $\mathbb{R}^n$ space and let $\{\gamma_i\,;\ i : 1, \ldots, n\}$ denote the space basis set. Then, v can be decomposed onto such a basis as

$$v = \gamma_1 v^1 + \gamma_2 v^2 + \cdots + \gamma_n v^n,$$

or, using sigma notation,

$$v = \sum_{i=1}^{3} \gamma_i v^i,$$

or, using index notation and Einstein summation convention,

$$v = \gamma_i v^i,$$

The engineering vector is found by setting n to 3; that is, $v = \gamma_i v^i$ where i runs from 1 to 3. Verbosely,

$$v = \gamma_i v^i = \gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3.$$

□

EXAMPLE 4.4. Redefine vector addition.

Solution. Let u and v be two elements of $\mathbb{R}^n$. Then $w = u + v$, called *the addition of* u *and* v, is another vector in $\mathbb{R}^n$ defined as

$$\gamma_i w^i \equiv \gamma_i u^i + \gamma_i v^i.$$

□

Discussion. If u and v were in $\mathbb{R}^1$, *aka*, $\mathbb{R}$, the index i will be 1 – *i.e.*, we would be working in 1-D; if u and v were in $\mathbb{R}^2$, the index i will run from 1 to 2 – *i.e.*, we would be working in 2-D; if u and v were in $\mathbb{R}^3$, the index i will run from 1 to 3 – *i.e.*, we would be working in 3-D; and so on. For all of such cases, we get the correct results with the "improved" definition of vector addition. Nice! Although more abstract, our redefinition works nicely in *any* dimension. All that achieved with little typing and few explaining, at least not in the definition itself.

EXAMPLE 4.5. Redefine multiplication of vectors with scalars.

Solution. Consider $a \in \mathbb{R}$ and $v \in \mathbb{R}^3$. Then $w = av$, called *the product of* a *and* v, is another vector in $\mathbb{R}^n$ defined [14] as

$$w \equiv a\gamma_i v^i = \gamma_i v^i a.$$

□

So far we have been going from vector notation to index notation. What about going the other way around. Let's show it with a classic example. Do you remember that systems of linear equations can be expressed by means of vectors? Let us do some practicing with that.

EXAMPLE 4.6. Expand the following equations accordingly:

$$a_i = b_i + c_i,$$

where i runs from 1 to 3.

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[13] Phrases like this one are the origin of so many jokes about mathematicians.
[14] The symbol $\equiv$ is being used to denote definition. You probably have already figured it out.

Solution.

$$a_1 = b_1 + c_1,$$
$$a_2 = b_2 + c_2,$$
$$a_3 = b_3 + c_3.$$

□

Wow, that was easy, right? We simply replaced the values of i one at a time. The key was to notice that i is a free index: a free index appears only once in every term and always means distinct equations, in this case three. Also notice that there is not repeated index, so no summation is implied.

Let's practice with double indices.

EXAMPLE 4.7. Expand a_{ii} in 3D.

Solution.

$$a_{ii} = a_{11} + a_{22} + a_{33}.$$

□

In this case, i is a dummy index. Dummy indices are the ones that imply summation.

What about matrices? If vectors can be expressed with indices, and matrices are composed out of vectors, then, maybe, matrices can be written using index notation. Yes, they can. But we need a bit more work to do. Since matrices have both rows and columns, we need to agree which index will represent what.

NOTATION 4 (Matrices in index notation). *A matrix M with n rows and m columns,* i.e.*, a $n \times m$ matrix, can be expressed in index notation as*

$$M = M_{ij}.$$

where i runs from 1 to n and j from 1 to m.

Our first matrix to represent in index notation will be a classical one.

EXAMPLE 4.8. Represent the Kronecker delta in index notation.

Solution. Let δ be a $n \times m$ matrix whose elements are as follows

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where i runs from 1 to n and j from 1 to m. The matrix δ is also called *the Kronecker delta.* □

EXAMPLE 4.9. Represent the dot product in index notation and Einstein convention.

Solution. Let u and v be two elements of $\mathbb{R}^n$. Then $w = u \cdot v$, called the *dot product of u and v*, is defined as

$$w \equiv u^i v^i. \tag{5}$$

□

NOTE 4. *The last equation, Eq. 5, is sometimes taken as the "definition" of the dot product. It works, but it hides some things. Can you spot where the hidden bits lie and why?*

To see this, let us write the dot product as a bilinear, *i.e.*, distributive, operation between vectors rather than components in the following way:

- Accept the two vectors u and v as elements of $\mathbb{R}^n$.
- Express them using index notation: $\gamma_i u^i$ and $\gamma_j v^j$, with i and j both running from 1 to n.

- Calculate the dot product:
$$u \cdot v = \gamma_i u^i \cdot \gamma_j v^j = u^i v^j \gamma_i \cdot \gamma_j = u^i v^i.$$

Can you spot the bits now? Here is a list:

- Why i for u and j for v if both are 3-D vectors, and thus their indices run through the same values?
- What happened with the dot product of the basis vectors, $\gamma_i \cdot \gamma_j$? Was the dot product of the basis vectors set to one? To zero? To both?
- Why at the end the index of v was switched to i?

The answer for the first question is the Einstein summation convention rule 3, see Note 1. Thus, u and v need different indices for their expansion. In fact, every time that two or more vector expansions are joint together, every expansion needs a different index.

The answer for the rest of the questions are more elaborate and will be answered, but not now, because they require a bit more concepts to be developed. What we could say is that the dot product of basis "substitutes indices"; *e.g.*, in $u^i v^j \gamma_i \cdot \gamma_j$, the dot product takes and changes the index j of vi to i, leaving thus $u^i v^i$ as a result.

References

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