1 APPLIED MODEL THEORY

> The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work.
>
> — VON NEUMANN, [1]

To illustrate various problem solving techniques, we will analyze the motion of a charged particle using Newtonian Physics. We will do so by showing various math and physics methods in different levels of sophistication: guessing, dimensional analysis, approximations and analytic techniques. Finally, we present a final wrapped-up solution.

Since I like to keep things informal, if I say « we » in the text, I really mean you and I. [2, p. 2]

one has to assume, derive and test the model. The issue is strengthened in design.

1.1 PROBLEM STATEMENT

In professional work, seldom does one find a « well-posed problem », where all the problem data, the information regarding... one has to work towards that goal: to well pose a problem (well posed problems have more chances to find a solution).

However, herein we will not be concerned by posing a problem, we will rather take one from a book (the reference exercise) and illustrate the ideas on how to solve problems by applying different techniques to solve such an exercise.

He who seeks for methods without having a definite problem in mind seeks in the most part in vain. [3]

1.1.1 *Reference exercise*

As a *reference exercise*, I chose a nice one presented in the *Particle Kinetics and Lorentz Force in Geometric Language* section in [4, chap. 1, p. 8]. There, geometric ideas, *via* the *geometric principle*, [1] are applied to Newtonian physics.

[1] The laws of physics must all be expressible as geometric [...] relationships between geometric objects [...], which represent physical entities. [5, part I, p. iii]

Now, to save your time for finding it, I quote the exercise verbatim:

ENERGY CHANGE FOR CHARGED PARTICLE *Without introducing any coordinates or basis vectors, show that, when a particle with charge q interacts with electric and magnetic fields, its energy changes at a rate*

$$dE/dt = \mathbf{v} \cdot \mathbf{E} \,. \tag{1}$$

In eq. (1), E represents the particle's kinetic energy, t (Newton's) universal time, $\mathbf{v}$ the particle's velocity and $\mathbf{E}$ an electric field.

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1.1.2 *Reference exercise analysis*

In the reference exercise, we are asked to derive (match) a given formula. As a healthy advice, always check if a formula, specially one to match, is correct (in this case, derivable). But, how to know if a formula is correct without a formal derivation? Catch-22! Well, not really. We have a simple (but powerful) method to analyze formula correctness without the need of long computations: *dimensional analysis* – in a correct equation, all of its terms have the same dimensions. Let's see if eq. (1) passes this test.

Upon seeing any equation, first check its dimensions [...]. If all terms do not have identical dimensions, the equation is not worth solving – a great savings of effort. [6, p. 42]

Since we are dealing with electrodynamics, we choose the dimensions of force F, length L, electric charge Q and time T as base dimensions for the analysis. Then, for eq. (1), we have that

$$\begin{aligned} \frac{\dim dE}{\dim dt} &= \frac{\mathsf{FL}}{\mathsf{T}} && [\text{LHS of eq. (1)}] \\ &\neq \\ \dim \mathbf{v} \cdot \dim \mathbf{E} &= \frac{\mathsf{FL}}{\mathsf{QT}}\,. && [\text{RHS of eq. (1)}] \end{aligned}$$

Note the additional Q in the RHS of eq. (1) (or the lack thereof in the LHS). Dimensions do not match, thus the formula is false! Then, we could well stop here and move on. But, we are a bit curious: what went wrong? Misprint, mistype or bad derivation are possible causes. But, as an error pointer, we realize that in the exercise statement the particle's electric charge q is given and a magnetic field is mentioned; however, neither appear in eq. (1). Let's use this observation to guess that

$$dE/dt = q\mathbf{v} \cdot \mathbf{E}\,. \tag{2}$$

The last equation is dimensionally homogeneous and, thus, plausible.

1.1.3 *Reference exercise reformulation*

In the last section, we briefly analyzed the reference exercise, found out its conclusion is incorrect and guessed a plausible formula. But, we are far from the end: a plausible equation is not our final answer.

The next step is to reform the exercise statement itself. I think the reference exercise aim was not to asked us to « show » a wrong conclusion. We replace then the *show* bit for *find*. In this way, we do not have to worry about any formula to match. We will have to derive one. Let's try:

ENERGY CHANGE FOR CHARGED PARTICLE *Without introducing any coordinates or basis vectors, find the energy change rate dE/dt of a particle with charge q when it interacts with an electric field $\mathbf{E}$ and a magnetic field.*

1.1.4 *Working exercise*

The reference exercise was analyzed and reformulated to avoid its wrong conclusion. We could work with the reformulated reference exercise. However, there are some additional changes I would like to make before having a *working exercise*:

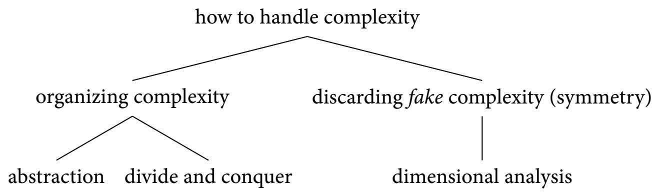

Figure 1 Dimensional analysis and abstraction as methods to handle complexity when solving physics or math problems, adapted from [8, p. 2]

- I will generalize the statement by relaxing hypotheses and removing data. Specifically, I will remove the recommendation of not using coordinates nor basis vectors; [2] will relax the « particle » model by hypothesizing a « body », $\mathfrak{B}$, instead; will replace the data « charge q », « electric field $\mathbf{E}$ » and « magnetic field » with « electrically charged ». These replacements will correct the lack of q and the magnetic field in eq. (1) – perhaps they are not needed.

- Notice that in the reference exercise the particle interacts with fields. However, it is not mentioned the *agent* that creates the fields. This is a serious omission that perpetuates the *impetus believe*: « that a force can be imparted to an object and act on it independently of any agent » [7]. It is particularly notorious in the reference exercise: we are told, in electromagnetic theories, that any charged particle creates electric and magnetic fields. So, is the $\mathbf{E}$ in the reference exercise due to the moving particle or to another one? We *interpret* the statement as « there is a particle, different from the moving one, that creates the electric and magnetic fields with which the moving particle interacts ». We correct this lack of agent by adding a second electrically charged body, $\mathfrak{B}'$.

- Finally, I like how mathematicians present propositions. They explicitly write the *premises* (or hypotheses) and the *conclusion* (or question) in a way that nothing is left to interpretation. In such a fashion, the problem becomes *self-contained*. Self-contained statements tend to sound a bit pedantic; but, the result pays off in understanding.

[2] although this is an important reminder of working with geometric objects rather than with coordinates, it limits generalization.

[On Newton's laws] the laws fail to explicitly state that *every force has an agent*, that every force is a binary function describing the action of an agent on an object. [7]

With these changes in mind, we present a working version of the exercise:

CHARGED BODY ENERGY CHANGE *Consider a massive, electrically charged body $\mathfrak{B}$ moving toward an electrically charged body $\mathfrak{B}'$. Then, find $\mathfrak{B}$ temporal change of energy.*

1.2 WORKING EXERCISE SOLUTION

Now we are ready to work on the exercise. But, wait! Without you noticing it, we already stared working on it. We started by applying two of my favorite methods: *dimensional analysis* and *abstraction*. These methods are ideal to handle complexity, *vide* fig. 1. While dimensional analysis discards « fake » complexity by compressing information, abstraction organizes it.

Dimensional analysis was briefly treated in section 1.1.2 and will be shown more fully in section 1.2.5. In the next section, I will explain abstraction.

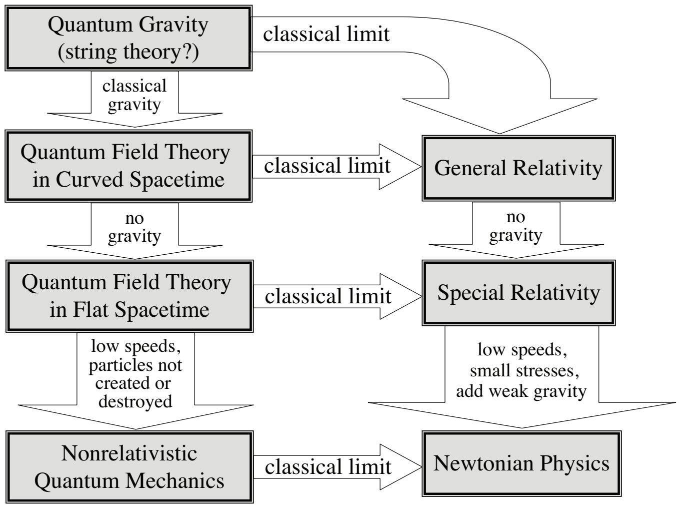

Figure 2 The relationship of the three frameworks for classical physics (on right) to four frameworks for quantum physics (on left). Each arrow indicates an approximation. All other frameworks are approximations to the ultimate laws of quantum gravity (whatever they may be – perhaps a variant of string theory). [5, chap. 1, p. iv]

1.2.1 *Abstraction*

As seen in section 1.1.4, abstraction is the process of relaxing hypotheses and removing data to leave only the bare bones of an statement. There are some reasons to do this:

The problem statement should be very general and free of as much data as possible, as later stages in the modelling process will consider and gather what is needed. [9, p. 8]

1. a neater, crisper, easier to picture exercise statement, due to the lack of data and suggested notation – *cf.* the reference exercise statement, section 1.1.1, with the working exercise statement, section 1.1.4;

2. as the statement becomes more abstract, keywords that show governing effects begin to emerge;

3. as the governing effects appear, theories can be proposed to model those effects. Then, we have to make assumptions and filled out data to satisfy theories frameworks. These latter steps engage us in a better understanding of the physics behind the model and its limitations.

For instance, in the case of the working exercise, note the keywords *electrically charged* and *moving*. They point to a theory of electromagnetism and to a theory of motion: an electrodynamic theory. We can now choose any of the available ones. For motion, we could choose Newton's, Lagrange's, Hamilton's, Einstein's or quantum theories; for electromagnetism, classical, quantum, field theories and so on.

For the present case, in order to retain the spirit of the reference exercise, we choose Newtonian physics as the main physical framework in which to work. (The relationships among several physical theories is presented in fig. 2.)

1.2.2 *Notation*

Often, scalars, vectors and other mathematical objects are typeset with different font faces for each math type. Although this convention works fine for printed texts, it posses issues when working on paper. See, for instance, that, in the reference exercise, E is used to represent energy and $\mathbf{E}$ to represent electric field. Now, how to distinguish between E and $\mathbf{E}$ with pen on paper without both e's getting confused?

The alternatives, then, are to decorate objects, like using arrows on top of letters for vectors – $\vec{E}$ for electric field – or to use majuscules and minuscules to distinguish objects. Herein I *could* follow any of such conventions, but I am not going to. I do not like how arrows, bold typefaces or majuscules look like. Instead I will use different symbols for different quantities and minuscules to typeset variables; for instance, f would represent force, e electric field, k kinetic energy and so forth. Even though this latter convention appears error prone, it constantly reminds me to be careful when working with mixed types of math objects, for I do not rely on typographical decoration anymore.

A name is not the same as an explanation. Do not expect the structure of a name or symbol to tell you everything you need to know. Most of what you need to know belongs in the legend. The name or symbol should allow you to look up the explanation in the legend. [10]

Finally, in the writer's eyes, besides honoring math objects while quenching notation, this flat, undecorated typography seems to give equations an air of elegance and simplicity unmatchable by heavy decoration. Compare, for instance, the undecorated version of Newton's second law of motion

$$f = ma\,,$$

with $\vec{F} = m\vec{a}$ or $\mathbf{F} = m\mathbf{a}$, its decorated and bold counterparts.

Typography exists to honor content. [11, p. 17]

But in our opinion truths of this kind should be drawn from notions rather than from notations. [12]

1.2.3 *Adoption of physical framework – model theory*

via the model theory [7].

Before developing the necessary mathematics, survey the crucial physics. [13, p. 11]

THEORY Newtonian electrodynamics: (Newtonian physics)

OBJECT body $\mathfrak{B}$ modeled as a moving particle: (state variables, object variables, ...) with charge q and mass m.

AGENT body $\mathfrak{B}'$ modeled as a non-moving particle.

DYNAMIC LAWS

$$p = mv \qquad [\text{def. momentum}] \tag{3}$$

$$f = \dot{p} \qquad [\text{Newton's second law}] \tag{4}$$

$$2k = mv^2 \qquad [\text{def. kinetic energy}] \tag{5}$$

INTERACTION LAWS

$$f = q\,(e' + v \times b') \qquad [\text{Lorentz force}] \tag{6}$$

INTERPRETATION ...

QED

Notice that interpretation forms a part of the proof!

how to handle complexity

- discarding *fake* complexity (symmetry)
  - dimensional analysis
- discarding *actual* complexity
  - approximation

Figure 3 Approximation as methods to handle complexity when solving physics or math problems, adapted from [8, p. 2]

1.2.4 *Approximate solution*

Now that we have the physical framework in place, it is time to mathematically model the phenomenon using laws, definitions and theorems from the framework. However, we will not use the full-blown formulas; we will use approximations instead.

We do this because we do not want to be distracted by fancy math, detailed calculations, extra accurate results, unnecessary math factors (like τ or π) in our first contact with the model formulation, *vide* fig. 3. We want understanding first, then we polish the model little by little; *i.e.*, we will firstly focus on estimating the backbone effects influencing the phenomenon. There are some recommended estimations available [14]:

Too much mathematical rigor teaches *rigor mortis*: the fear of making an unjustified leap even when it lands on a correct result. Instead of paralysis, have courage – shoot first and ask questions later. Although unwise as public policy, it is a valuable problem-solving philosophy. [6, p. viii]

- discarding unnecessary factors;
- number guessing;
- geometry tinkering (everything is a cube or a sphere);
- usage of ratios;
- usage of conservation laws;
- dimensional analysis;
- plausibility checks.

In our case, we will mainly discard unnecessary factors and use the *secant method* for approximating derivatives [6, p. 38].

The secant method for approximating derivatives consists in replacing derivatives by divisions:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \sim \frac{f}{x}\,,$$

where f is a function whose derivative with respect to x exists and $\sim$ means *is similar to*.

Discarding factors and approximating derivatives by the secant can be illustrated by estimating the kinetic energy temporal change of a moving particle:

$$k = \frac{1}{2}mv^2 \sim mv^2 \implies \dot{k} \sim \frac{k}{t} \sim \frac{mv^2}{t}\,,$$

where we have discarded the factor of $1/2$ and approximated the kinetic energy time derivative.

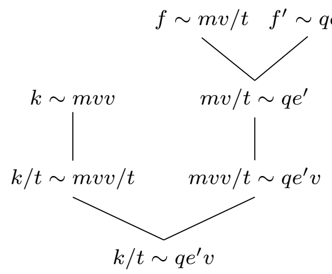

Figure 4 Effect of electric field on particle $\mathfrak{P}$

Working back on the exercise, first, we write the set of equations obtained in section 1.2.3:

$$2k = mv^2\,, \qquad \text{[kinetic energy]}$$
$$f = m\dot{p} = m\dot{v}\,. \qquad \text{[Newton's second law]}$$
$$f' = q\left(e' + v \times b'\right)\,, \qquad \text{[Lorentz force]}$$

where unprimed quantities represent $\mathfrak{P}$ (moving body – *object* – modeled as particle) properties, while primed quantities $\mathfrak{P}'$ (non-moving body – *agent* – modeled as particle) properties.

Then, we drop numeric factors, product between vectors and use the secant method to approximate derivatives to find

$$k \sim mv^2\,,$$
$$\dot{k} \sim k/t \sim mv^2/t \sim (mv/t)\,v\,,$$
$$f \sim mv/t\,,$$
$$f' \sim q\left(e' + vb'\right)\,.$$

We begin the analysis by *dividing and conquering*, *vide* fig. 1, the last set. We first analyze the electric field effect, then the magnetic field effect on the particle motion and finally we join them both.

Electric field effect

To understand the electric field effect, we use a *tree diagram* that contains all the computational work with the quantities and the formulas governing the particle's motion, *vide* fig. 4.

Atop fig. 4, we wrote the three main dynamic laws governing the particle's motion: kinetic energy definition, Newton's second law of motion and Lorentz force law. On the LHS, we begun with the kinetic energy definition and then approximated its time derivative. On the RHS, we begun with Newton's second law and Lorentz force law, then equated them and multiplied the result by v (to match the vv in the LHS). Finally, we equated the LHS to the RHS to find that

$$\dot{k} \sim k/t \sim qe'v\,, \tag{7}$$

which gives the effect of the electric field created by $\mathfrak{P}'$ on $\mathfrak{P}$ motion. This equation shows that $\dot{k}$ depends neither on forces nor on masses. A fact we will use a bit later.

$$
\begin{tikzcd}[column sep=tiny]
 & & f \sim mv/t \arrow[dr, dash] & & f' \sim qvb' \arrow[dl, dash] \\
k \sim mv^2 \arrow[d, dash] & & & mv/t \sim qvb' \arrow[d, dash] & \\
k/t \sim mv^2/t \arrow[dr, dash] & & & mv^2/t \sim qvb'v \arrow[dll, dash] & \\
 & k/t \sim qvb'v & & &
\end{tikzcd}
$$

Figure 5 Effect of magnetic field on particle $\mathfrak{P}$

Notice that we were careful when writing vector multiplications, since there are some: inner product, cross product, outer product and geometric product. However, using this very same information, we can go a bit further with eq. (7). We know that e' and v are vectors, so we need a product between them, and we also know that $\dot{k}$ is a scalar. Thus, for the two sides of eq. (7) to agree, the only product between e' and v is the inner product, since is the only one that returns a scalar. With this, eq. (7) becomes:

$$\dot{k} \sim qe' \cdot v \,,$$

which agrees nicely with eq. (2); the equation we guessed when correcting the reference exercise.

It is worth to mention that eq. (2) and eq. (7) illustrate the power of dimensional analysis and approximations. With these two, plus some insights, we were able to independently derive meaningful expressions without the need of full-blow calculations! Plausible equations with little job done.

Magnetic field effect

To understand the magnetic field effect, we repeat the same methodology used in the previous section: a tree diagram, *vide* fig. 5. The result is that

$$\dot{k} \sim k/t \sim qvb'v \,,$$

which gives the effect of the electric field created by $\mathfrak{P}'$ on $\mathfrak{P}$ motion. This equation shows again that $\dot{k}$ does not depend on f, f' or m.

In this case, note the triple product of the vectors $vb'v$. It can be traced back to the dynamic laws as

$$vb'v \sim (v \times b') \cdot v \,.$$

Again, the right hand side v must enter as an inner product, since the LHS of the equation is a scalar, $\dot{k}$, and the result of $v \times b'$ is a vector. Using vector algebra, it is possible to show that the triple product vanishes. We could show it that way, but we prefer to argue geometrically. The product $v \times b'$ returns a vector perpendicular to the plane formed by v and b'. Then, the triple product *must* vanish, for the angle formed by the vector resulting from $v \times b'$ and v is[3] $\tau/4$ or 90°. Thus,

$$\dot{k} \sim 0 \,. \tag{8}$$

[3] For convenience, define $\tau \doteq 2\pi$. [15]

$$f \sim mv/t \quad f' \sim q\,(e' + vb')$$

$$k \sim mv^2 \qquad mv/t \sim q\,(e' + vb')$$

$$k/t \sim mv^2/t \qquad mv^2/t \sim q\,(e' + vb')\,v$$

$$k/t \sim q\,(e' + vb')\,v$$

Figure 6 Effect of electromagnetic field on particle $\mathfrak{P}$

The last equation agrees with the physical interpretation of the phenomenon. Call f_{m} the force due to the magnetic field; *i.e.*, the RHS term of Lorentz force: $q\,(v \times b)$. Note that f_{m} is always perpendicular to both the v and the b that created it – mathematically expressed by the (cross) product $v \times b$. Then, when a charged particle moves through the field, it traces an helical path in which the helix axis is parallel to the field and where v remains constant. Because the magnetic force is *always* perpendicular to the motion, the b can do *no* work.

Finally, we see again that approximate methods and some physical insights yield meaningful results.

Electromagnetic field effect

Once we have understood a bit more on the physics of the phenomenon, we can join both effects, *vide* fig. 6. The result is that

$$\dot{k} \sim qv \cdot e' \,,$$

which gives the effect of the electric field created by $\mathfrak{P}'$ on $\mathfrak{P}$ motion. This equation shows that $\dot{k}$ does not depend on f, f', m or b'.

One more thing. The last equation is fine, but we can do even better. See that both sides have the same dimensions and that the inner product of e and v, two vectors, is a scalar. Then, we can present the last formula as

$$\dot{k}/qe \cdot v \sim 1 \,. \tag{9}$$

The presentation of eq. (9) has two advantages: it is *scaled* (dimensionless and of order unity) and it stresses the scalar character the solution.

1.2.5 *Dimensional analysis*

For the next solution, we will use *dimensional analysis* to determine the *functional form* of the model to the phenomenon. To find the functional form of the physical model by means of dimensional analysis, follow the steps:

- Experience has shown us that to analyze geometric problems we need only the dimension of length, L; to analyze kinematic problems we need add time T; to analyze dynamics, we can add either mass M or force F. We cheese

PHYSICAL QUANTITY	SYMBOL	DIMENSIONS
$\mathfrak{P}$ electric charge	q	Q
$\mathfrak{P}$ kinetic energy	k	FL
$\mathfrak{P}$ velocity	v	$\mathsf{L/T}$
$\mathfrak{P}'$ electric field	e'	$\mathsf{F/Q}$
$\mathfrak{P}'$ magnetic field	b'	$\mathsf{FT/LQ}$
Time	t	T

Table 1 Physical model for an electrically charged particle $\mathfrak{P}$ moving towards an electrically charged particle $\mathfrak{P}'$

- Physical model: a list of the relevant variables: *vide* table 1, [18, p. 4]. [4]
- In the chosen set, the dimensions of the *six* physical quantities that model the phenomenon are $\dim k = \mathsf{FL}$, $\dim t = \mathsf{T}$, $\dim q = \mathsf{Q}$, $\dim e = \mathsf{F/Q}$, $\dim v = \mathsf{L/T}$ and $\dim b = \mathsf{FT/LQ}$. [5]
- According to the Buckinham's theorem, there are $6-4 = 2$ dimensionless quantities Π. The first one is $\Pi_1 = k/tevq$ and the second $\Pi_2 = bv/e$.
- Finally, the model should have the form:

$$\Xi'[\Pi_1, \Pi_2] = \Xi'\left[\frac{k}{tevq}, \frac{bv}{e}\right] = 0 \implies \frac{k}{t} = evq\,\Xi\left[\frac{bv}{e}\right] ,$$

where Ξ is an *unknown* dimensionless functions of dimensionless arguments.

In the last equation, the precise form of the function Ξ must be determined by experimentation or by analytic means. However, dimensional analysis confirms our suspicion: $\dot{k} \sim k/t \sim qev$; *i.e.*, the product qev « lives upstairs » in the equation.

Finally, the function Ξ should be equal to a dimensionless parameter Π if our guess is to be correct. We will keep Ξ, nevertheless, for it may be that our guess is incorrect.

[4] We use this set because it doesn't include mass as we guessed so in the previous section.
[5] Again, mass doesn't appear in the physical model.

2 THINGS TO ORGANIZE

2.0.6 *Interpretation of the solution*

[Apply the case to the electron-proton. Only e-field interaction needed :)]

2.0.7 *Proof scratch work*

Suppose we didn't... Two column style: left-hand side for calculations, right-hand side for explanations. We use the approx. method solution style.

2.1 WORDY DERIVATION

We solve the problem now by presenting a « wordy-version » of the analytic solution: we describe the math derivation in detail.

The particle kinetic energy is $2k = mv^2$. This can be rewritten as

$$2k = mv \cdot v \,,$$

since v is colinear to itself; *i.e.*, its outer product is zero; *viz.*, $v^2 = vv = v \cdot v + v \wedge v = v \cdot v$.

Then, calculate the kinetic energy change rate with time by

$$2k = mv{\cdot}v \implies 2\dot{k} = m\,(\dot{v} \cdot v + v \cdot \dot{v}) = m\,(\dot{v} \cdot v + \dot{v} \cdot v) = 2m\dot{v}{\cdot}v \,,$$

where the product rule for the differentiation of the inner product, the commutativity property of the inner product and the dot notation for derivatives were used.

Next, one cancels out the numerical factor 2 in both sides of the equality to find that

$$k = m\dot{v} \cdot v \,.$$

$$\dfrac{\dfrac{k \sim mv^2}{k/t \sim mv^2/t} \qquad \dfrac{\dfrac{f \sim mv/t \qquad f' \sim q\,(e' + vb')}{mv/t \sim q\,(e' + vb')}}{mv^2/t \sim q\,(e' + vb')\,v}}{k/t \sim q\,(e' + vb')\,v}$$

Figure 1 Natural deduction proof tree (Gentzen style) of the charged particles case. At the top of the tree, there are the main dynamic laws: definition of kinetic energy, Newton's second law of motion and definition of Lorentz force law. From them and using, throughout, the secant approximation for derivatives, it is possible to derive an approximate formula for the change of kinetic energy of an electrically charged particle toward another electrically charged particle in the Newtonian framework. (ref. here)

		Object model and state quantities	
[A]	(1)	$\mathfrak{B}$ is a particle	
[A]	(2)	$\mathfrak{B}$ has mass m	
[A]	(3)	$\mathfrak{B}$ has electric charge q	
[A]	(4)	$\mathfrak{B}$ moves with velocity v	
		Agent model and state quantities	
[A]	(5)	$\mathfrak{B}'$ is a particle	
[A]	(6)	$\mathfrak{B}'$ has mass m'	
[A]	(7)	$\mathfrak{B}'$ is static	
[A]	(8)	$\mathfrak{B}'$ has electric charge q'	
[A]	(9)	$\mathfrak{B}'$ has electric field e'	
[A]	(10)	$\mathfrak{B}'$ has magnetic field b'	
[A]	(11)	t is universal	
		Dynamic laws	
[1,2,4]	(12)	$k \sim mv^2$	$\mathfrak{B}$ kin. energy
[1,2,4,11]	(13)	$f \sim mv/t$	$\mathfrak{B}$ motion: Newton's second
		Interaction laws	
[1,3,4,5,7,8,9,10]	(14)	$f' \sim q\,(e' + vb')$	$\mathfrak{B}, \mathfrak{B}'$ interact: Lorentz force
		Model derivation	
[12]	(15)	$k/t \sim mv^2/t$	time derivative
[13.14]	(16)	$mv/t \sim q\,(e' + vb')$	=
[16]	(17)	$mv^2/t \sim q\,(e' + vb')\,v$	times v
[17]	(18)	$k/t \sim q\,(e' + vb')\,v$	=

Table 1 Sketch work to model the electrically charged bodies interaction. The left column contains the statements (formulas) being used, the right column their justifications [16, p. 3]. Note the usage of approximations throughout the derivation – specially the secant approximation for derivatives [6, p. 38] –, of apostrophes to differentiate $\mathfrak{P}$ quantities from $\mathfrak{P}'$ quantities and of geometric algebra [17]. [A] means assumption.

On the other hand, the particle's motion can be modeled by equating Newton's second law of motion with Lorentz force, since the particle interacts with an electromagnetic field. Thus, we find that

$$\dot{p} = q\left(e + v \times b\right) \,,$$

where p is the particle's linear momentum. By definition, $p = mv$, so $\dot{p} = \dot{m}v + m\dot{v} = m\dot{v}$, because mass is constant, $\dot{m} = 0$, then we have that

$$m\dot{v} = q\left(e + v \times b\right) \,.$$

Plug in the last equation (equation of motion) into the $\dot{k}$ expression:

$$\dot{k} = qe \cdot v + q\left(v \times b\right) \cdot v \,.$$

Since the triple product vanishes, one finally finds

$$\dot{k} = qe \cdot v \,,$$

the rate at which the particle's kinetic energy changes with respect to time.

This (analytic) solution confirms our guessed model and the approximate solutions. Then, it creates confidence, not only on our intuition, but also on the efficacy of approximate methods.

2.2 FORMAL SOLUTION

Finally, we present a terser solution.

Agree on the given hypotheses and on the symbols and notation previously established.

First, model the movement of the particle (equation of motion) by equating Newton's second law to Lorentz force law:

$$m\dot{v} = q\left(e + v \times b\right) \,. \tag{1}$$

Write next the particle's kinetic energy as $2k = mv \cdot v$ and then calculate its temporal change $\dot{k}$:

$$\dot{k} = m\dot{v} \cdot v \,. \tag{2}$$

Plug eq. (1) into eq. (2) to find: $\dot{k}/q = e \cdot v + v \times b \cdot v$. Since the scalar triple product vanishes, the model is then

$$\dot{k} = qe \cdot v \,,$$

which can be scaled to

$$\frac{\dot{k}}{qe \cdot v} = 1 \,.$$

The last formula models the temporal change of kinetic energy of a charged particle moving through a constant electromagnetic field.

The formal solution was obtained from the derivation of the wordy solution. They only differ in presentation. In the formal solution,

- the presentation is brief, concise, straight to the point, but not incomplete. It only leaves « obvious details » to be filled in; *e.g.*, nowhere it is written that $\dot{p} = \dot{m}v + m\dot{v} = m\dot{v}$, because under hypotheses, m is constant, so it is « well-known » that $\dot{p} = m\dot{v}$ in such a case;

- equations are referred to by proper, technical names: Newton's second law of motion, scalar triple product and so on;
- only « important » equations, derivations and results are displayed, whereas small equations, non-trivial, but small, derivations and partial results are presented in-line – with the running text;
- verbs changed to the imperative to avoid the use of personal grammar forms – we, us, one and so on – and of the passive voice.

2.3 MATH PROOF

... [19, chap. 1] ... [20].

Consider two electrically charged bodies $\mathfrak{B}$ and $\mathfrak{B}'$. Consider $\mathfrak{B}$ mass to be constant and consider $\mathfrak{B}$ moving towards $\mathfrak{B}'$. Let q and v represent $\mathfrak{B}$ electric charge and velocity and let e represent $\mathfrak{B}'$ electric field. Then, the formula

$$\dot{k}/qe \cdot v = 1$$

models $\mathfrak{B}$ temporal change of kinetic energy $\dot{k}$.

2.3.1 *Lamport's proof style*

2.3.2 *Traditional proof style*

2.3.3 *Wrong*

the answer to the problem was wrong in [4] and then corrected in [5].

quote: nuilluis in verba :).

2.4 FINAL REMARKS

The method herein presented is far from being perfect. But it has worked nicely for me, not only when solving textbook exercises, but also in personal research and professional work. In textbooks, authors can write shorter, open statements, because the context given by the surrounding text allows them to do so. In real research, however, one never finds a textbook problem with a back-of-the-book solution.

The most important aspects on solving exercises are, according to my experience:

- having a problem that interests me;
- working hard on having a good description of the problem;
- making assumptions.

If all of the previous premises are satisfied, I ....

My working methodology is heavily influenced by John Denker, David Hestenes and Sanjoy Mahajan's ideas.

David Hestenes and Kip Thorne ideas on working with the geometric principle...

Math proofs from Houston.

the writing style comes from AIP Style Manual, Denker, Linder, Mahajan and mainly Fourier (theory of heat!).

I like to write in first person (singular and plural)

The old taboo against using the first person in formal prose has long been deplored by the best authorities and ignored by some of the best writers. [...] A single author should also use « we » in the common construction that includes the reader. [21, p. 14]

Dynamic and interaction laws			
[A]	(1)	$2k = mv \cdot v$	kinetic energy
[A]	(2)	$\dot{p} = m\dot{v}$	Newton's second
[A]	(3)	$f' = q\,(e' + v \times b')$	Lorentz force
Model derivation			
[1]	(4)	$\dot{k} = m\dot{v} \cdot v$	d_t
[2,3]	(5)	$m\dot{v} = q\,(e' + v \times b')$	$=$
[5]	(6)	$m\dot{v} \cdot v = q\,(e' + v \times b') \cdot v$	$\cdot\, v$
[6]	(7)	$m\dot{v} \cdot v = qe' \cdot v$	$v \times b' \cdot v = 0$
[4,7]	(8)	$\dot{k} = qe' \cdot v$	$=$
[8]	(9)	$\dot{k}/qe' \cdot v = 1$	$1/qe' \cdot v$

Table 2 [Ga] means geometric algebra identity.

2.5 FORMAL PROOF

Natural deduction proof (tree proof): Gentzen style.

For this section consider the assumptions made in ... and geometric algebra.

We adapt Lamport's math proof style [16, 22] to physics,

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>
> — NEWTON, [23]

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DOCUMENT REVISION HISTORY

The following table describes the changes to « Applied model theory ».

VERSION	DATE	NOTES
0.0.1	20/08/2014	First release
0.0.2	21/08/2014	Changes in text organization. Typo corrections
0.0.3	22/08/2014	Title and subtitle changed
0.0.4	29/08/2014	Current document compilation