A BRIEF INTRODUCTION TO GEOMETRIC ALGEBRA

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Definition 2.1 (Linear Space): *Let V denote a non-empty set of objects, called elements. Let x, y and z be arbitrary elements in V and let a and b be arbitrary real numbers. The set V is called a linear space if it satisfies the following ten axioms listed in three groups.*

(1) Closure Axioms

Axiom 2.1 (Closure under addition): *There is a unique element in V called the sum of x and y, denoted by $x + y$.*

Axiom 2.2 (Closure by multiplication of real numbers): *There is a unique element in V called the product of a and x, denoted by ax.*

(2) Axioms For Addition

Axiom 2.3 (Commutative law): $x + y = y + x$.

Axiom 2.4 (Associative law): $(x + y) + z = x + (y + z)$.

Axiom 2.5 (Existence of zero element): *There is an element in V, denoted by 0, such that $x + 0 = x$.*

Axiom 2.6 (Existence of negatives): *The vector $(-1)x$ has the property $x + (-1)x = 0$, where $(-1) \in \mathbb{R}$.*

(3) Axioms For Multiplication By Numbers

Axiom 2.7 (Associative law): $a(bx) = (ab)x$.

Axiom 2.8 (Distributive law for addition in V): $a(x + y) = ax + ay$.

Axiom 2.9 (Distributive law for addition of numbers): $(a + b)x = ax + bx$.

Axiom 2.10 (Existence of identity): $1x = x$, *where* $1 \in \mathbb{R}$.

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Herein, we will be concerned with vectors as depicted in geometry; *i.e.*, line segments from one point (tail) to another point (tip), where an arrow is placed into the vector tip. In such a context, if V is a set whose elements are *vectors* and if V satisfies the aforementioned axioms, then V will be called a *vector space*, instead of linear space. Additionally, the arbitrary real numbers will be referred to as *scalars*. Moreover, because we have used real numbers as scalars, a set V that satisfies the axioms of a linear space is called a *vector space over the real numbers*.

NOTE 1. *In the definition of a vector space, coordinates are nowhere to be found! Vectors and vector spaces are treated as pure mathematical entities on their own right; i.e., they are independent of any coordinate system, any choice of basis, any* $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$*, and so on.*

NOTE 2. *The axioms do not tell us how to do computations, they tell us that the operations therein defined exist. For instance, let* x *and* y *in* V*. The axioms tell us that* $x + y$ *is defined, is a vector and is in* V*, but they do not tell us how to calculate its value(s).*

NOTATION 1. *For any two vectors* x *and* y*, the negative of* x*,* $[(-1)x]$*, is denoted by* $-x$ *and the difference* $y - x$ *is defined to be the sum* $y + (-x)$*.*

The definition of linear spaces is perhaps daunting, because it is abstract. However, instead of thinking of the axioms as a list of statements about abstract elements and abstract operations, it is better to regard the axioms as a *license* to do algebra with its elements.

DEFINITION 2.2 (Tuple): *Let* X *be a non-empty set whose* n *elements are* $x_1, x_2, \ldots, x_n$*. A* n*-tuple is an ordered list of* X *written as*

$$(x_1, x_2, \ldots, x_n) \, .$$

DEFINITION 2.3 (Ordered Pair): *Let* X *and* Y *be two sets. An ordered pair, denoted by* (x, y)*, is defined as*

$$(x, y) = \{\{x\}, \{x\} : x \in X \text{ and } y \in Y\}.$$

DEFINITION 2.4 (Cartesian Product): *The Cartesian product of two sets* X *and* Y*, denoted* $X \times Y$*, is the set of all possible ordered pairs whose first component is a member of* X *and whose second component is a member of* Y*; i.e.,*

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

DEFINITION 2.5 (Cartesian Power): *The Cartesian power of a set* X *is defined as*

$$X^n = \underbrace{X \times X \cdots \times X}_{n} = \{(x_1, x_2, \ldots, x_n) : x_i \in X \text{ for all } 1 \leq i \leq n\}.$$

DEFINITION 2.6 (Function): *A function* f *from* X *to* Y *is a subset of the Cartesian product* $X \times Y$ *such that every element of* X *is the first component of one and only one ordered pair in the subset; i.e., for every* x *in* X *there is exactly one element* y *such that the ordered pair* (x, y) *is contained in the subset defining* f*. The set* X *is called the domain of* f*, the set* Y *is called the codomain of* f *and the set of ordered pairs is called the graph of* f*. A function* f *from* X *to* Y *is denoted as*

$$f : X \to Y.$$

The elements of X are called arguments of f. For each argument x, the corresponding unique y in the codomain is called the function value at x or the image of x under f. It is written as $f(x)$. One says that f associates y to x or maps x to y. This is abbreviated by $y = f(x)$.

In order to specify a concrete function, the notation $\mapsto$ is used.

Example 2.1. A function is specified as $f : \mathbb{N} \to \mathbb{Z}$, such that $x \mapsto 4 - x$. That is, f is a function from $\mathbb{N}$ (the set of natural numbers) to $\mathbb{Z}$ (the set of integers) where x maps to $4 - x$.

Definition 2.7 (Inner Product): *Consider a vector space V over a field F. A inner product in V is a map*

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

that satisfies the following three axioms for all vectors x, y, z in V and all scalars a in F:

(1) *Symmetry: $\langle x, y \rangle = \langle y, x \rangle$.*
(2) *Linearity in the first argument:*

$$\langle ax, y \rangle = a \langle x, y \rangle,$$
$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle.$$

(3) *Positive-definiteness: $\langle x, x \rangle \geq 0$ with equality holding for $x = 0$.*

3. Familiar Vector Products

Definition 3.1 (Basis Vectors): *A basis B of a vector space V over a field F is a linearly independent subset of V that spans V. That is, in more detail, suppose that $B = \{b_1, b_2, \ldots, b_n\}$ is a finite subset of V over F. Then B is a basis if it satisfies the following conditions:*

(1) *the* linear independence *property: for all $a_1, a_2, \ldots, a_n \in F$, if $a_i v_i = 0$, then necessarily $a_1 = \cdots = a_n = 0$; and*
(2) *the* spanning *property: for every x in V it is possible to choose $a_1, a_2, \ldots, a_n \in F$, such that $x = a_i v_i$.*

The vectors b_i are called the basis vectors of the basis B. The numbers a_i are called the coordinates of the vector x with respect to the basis B. By the first property, the coordinates of x with respect to B are uniquely determined.

An ordered basis is also called a frame (wiki: basis vectors).

Example 3.1. Let the field K be the set $\mathbb{R}$ of real numbers, and let the vector space V be the (Euclidean) space $\mathbb{R}^3$. Consider the vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Then any vector in $\mathbb{R}^3$ is a linear combination of e_1, e_2 and e_3.

Definition 3.2 (Cartesian Coordinate System): *A Cartesian coordinate system specifies each point uniquely in a plane by a pair of numerical* coordinates, *which are the signed distances from the point to two fixed perpendicular directed lines, measured in the same unit of length. Each reference line is called a* coordinate axis *or just* axis *of the system, and the point where they meet is its* origin, *usually at ordered pair $(0, 0)$. The coordinates can also be defined as the positions of the perpendicular projections of the point onto the two axes, expressed as signed distances from the origin.*

One can use the same principle to specify the position of any point in three-dimensional space by three Cartesian coordinates, its signed distances to three mutually perpendicular planes (or, equivalently, by its perpendicular projection onto three mutually perpendicular lines). In general, one can specify a point in a space of any dimension n by use of n Cartesian coordinates, the signed distances from n mutually perpendicular hyperplanes.

The invention of Cartesian coordinates in the 17th century by René Descartes (Latinized name: Cartesius) revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra. Using the Cartesian coordinate system, geometric shapes (such as curves) can be described by Cartesian equations: algebraic equations involving the coordinates of the points lying on the shape. For example, a circle of radius 2 may be described as the set of all points whose coordinates x and y satisfy the equation $x^2 + y^2 = 4$.

Definition 3.3 (Standard Basis): *In mathematics, the standard basis (also called* natural basis *or* canonical basis*) for a Euclidean space consists of one unit vector pointing in the direction of each axis of the Cartesian coordinate system. For example, the standard basis for three-dimensional space are the vectors*

$$e_x = (1, 0, 0), \qquad e_y = (0, 1, 0), \text{ and } e_z = (0, 0, 1).$$

Here the vector e_x points in the x direction, the vector e_y points in the y direction, and the vector e_z points in the z direction. There are several common notations for these vectors, including $\{e_x, e_y, e_z\}$, $\{e_1, e_2, e_3\}$, $\{i, j, k\}$, and $\{x, y, z\}$. These vectors are sometimes written with a hat to emphasize their status as unit vectors; e.g., $\{\hat{\imath}, \hat{\jmath}, \hat{z}\}$.

In n-dimensional Euclidean space, the standard basis consists of n distinct vectors

$$\{e_i : 1 \leq i \leq n\},$$

where e_i denotes the vector with a 1 in the i-th coordinate and 0's elsewhere.

These are a basis in the sense that any other vector can be expressed uniquely as a linear combination of these. For example, every vector v in three-dimensional space can be written uniquely as

$$v = v_x e_x + v_y e_y + v_z e_z.$$

The scalars v_x, v_y, v_z being the scalar components of the vector v.

Or, using summation convention, with $x \to x_1$, $y \to x_2$, $z \to x_3$, $v_x \to v_1$, $v_y \to v_2$, $v_z \to v_3$, then

$$v = v_i e_i.$$

Definition 3.4 (Real Coordinate Space): *Let $\mathbb{R}$ denote the field of real numbers. For any positive integer n, the set of all n-tuples of real numbers forms an n-dimensional vector space over $\mathbb{R}$, which is denoted $\mathbb{R}^n$ and sometimes called* real coordinate space*. An element of $\mathbb{R}^n$ is written*

$$x = (x_1, x_2, \ldots, x_n),$$

where each x_i is a real number. The vector space operations on $\mathbb{R}^n$ are defined by (for $\alpha \in \mathbb{R}$)

$$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \alpha x &= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n). \end{aligned}$$

The vector space $\mathbb{R}^n$ comes with a standard basis:

$$\begin{aligned} e_1 &= (1, 0, \ldots, 0), \\ e_2 &= (0, 1, \ldots, 0), \\ &\vdots \\ e_n &= (0, 0, \ldots, n). \end{aligned}$$

An arbitrary vector in $\mathbb{R}^n$ can then be written in the form

$$v = x_i e_i,$$

with i ranging from 1 to n.

$\mathbb{R}^n$ is the prototypical example of a real n-dimensional vector space.

Definition 3.5 (Coordinate Vector): *Let V be a vector space of dimension n over a field F and let*

$$B = \{b_1, b_2, \ldots, b_n\} = \{b_i\,;\ i : 1 \ldots n\}$$

be an ordered basis for V. Then, for every v in V there is a unique linear combination of the basis vectors that equals v:

$$v = \alpha_i b_i. \qquad \text{(Summation convention implied!)}$$

The linear independence of vectors in the basis ensures that the αs are determined uniquely by v and B. Now, we define the coordinate vector of v relative to B to be the following sequence of coordinates:

$$[v]_B = (\alpha_1, \alpha_2, \ldots, \alpha_n)\,.$$

This is also called the representation of v with respect of B, or the B representation of v. The αs are called the coordinates of v. The order of the basis becomes important here, since it determines the order in which the coefficients are listed in the coordinate vector.

Note: Coordinate vectors of finite dimensional vector spaces are herein represented as elements of column *vectors; e.g., for v in the last definition*

$$[v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

Definition 3.6 (Dot Product): *Let a and b be two vectors in a vector space V whose representation is given by*

$$a = (a_1, a_2, \ldots, a_n)\,,$$
$$b = (b_1, b_2, \ldots, b_n)\,.$$

Then, the dot product between a and b is defined as

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = \sum_{1 \le i \le n} a_i b_i = \sum\nolimits_i a_i b_i \ \ [1 \le i \le n] = a_i b_i;$$

where $\sum$ denotes summation notation and n is the dimension of V.

Notice that the inputs for the dot product are two vectors, but the result of it is a scalar. For this reason, the dot product is also called *scalar product*.

Note: dot prod is not generally equal to the inner prod (check if it satisfies the inner product axioms).

4. Geometric Algebra

4.1. **Geometric Product.**

Definition 4.1 (Geometric Product): *Let V be a vector space over the field of real numbers, $\mathbb{R}$. Let a, b and c be elements of V. The* geometric product *of a and b, denoted as ab, is an operation in V that follows the rules:*

(1) *associative: $(ab)c = a(bc)$,*
(2) *left distributive: $a(b + c) = ab + ac$,*

object	visualized as	geometric extent	grade
scalar	point	no geometric extent	0
vector	line segment	extent in 1 direction	1
bivector	patch of surface	extent in 2 directions	2
trivector	piece of space	extent in 3 directions	3
etc.			

TABLE 1. GA grading system

(3) *right distributive:* $(b+c)a = ba + ca$,
(4) *contraction:* $a^2 = |a|^2$.

4.2. **Visualization.** A scalar can be visualized as an ideal point in space, which has no geometric extent. A vector can be visualized as line segment, which has length and orientation. A bivector can be visualized as a patch of flat surface, which has area and orientation. Continuing down this road, a trivector can be visualized as a piece of three-dimensional space, which has a volume and an orientation.

Each such object has a *grade*, according to how many dimensions are involved in its geometric extent. Therefore we say GA is a graded algebra. The situation is summarized in **??**.

For any vector V you can visualize $2V$ as being twice as much length, and for any bivector B you can visualize $2B$ as having twice as much area. Alas this system of geometric visualization breaks down for scalars; geometrically all scalars "look" equally pointlike. Perhaps for a scalar s you can visualize $2s$ as being twice as hot, or something like that.

Presumably you are familiar with the idea of adding scalars to scalars, and adding vectors to vectors (tip to tail). We now introduce the idea that any element of the GA can be added to any other. This includes adding scalars to vectors, adding vectors to bivectors, and every other combination. So it would not be unusual to find an element C such that

$$C = s + V + B;$$

where s is a scalar, V is a vector, and B is a bivector.

Given any clif C, we can talk about the grade-0 piece of it, the grade-1 piece of it, *etc.* Notation: The grade-N piece of C is denoted $\langle C \rangle_N$. We will often be particularly interested in the scalar piece, $\langle C \rangle_0$.

geometric product of two vectors:

$$ab = a \cdot b + a \wedge b.$$

parallel vectors: $a \parallel b$.

perpendicular vectors: $a \perp b$.

reverse: $\tilde{C}$.

The gorm of any clif C is formed by multiplying C by the reverse of $\tilde{C}$, and keeping the scalar part of the product. That is:

$$\operatorname{gorm} C \equiv \langle \tilde{C} C \rangle_0.$$

basis vectors: γ.

natural coordinate frame (reference frame, std basis, canonical frame, std coord system): $\{\gamma_i\,;\ i : 1 \ldots 3\}$.

5. Appendix

5.1. **Einstein Notation.** From previous studies, we known that any vector can be uniquely expressed as a linear combination of basis vectors. For instance, in traditional engineering notation, a vector $v \in \mathbb{R}^3$ can be decompose in a sum of its components, $\{v_x, v_y, v_z\}$, onto the (Cartesian) unit basis vectors, $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, that is,

$$v = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}\,.$$

Arguably, the result of the decomposition into axis looks verbose, and inelegant.

Einstein [1] *notation* (*aka*, Einstein summation convention, index notation, tensorial notation) provides an elegant and compact way to cope with such verbosity. The process is:

- replace the space coordinates: $x \to x^1$, $y \to x^2$ and $z \to x^3$;
- replace the vector components: $v_x \to v^1$, $v_y \to v^2$ and $v_z \to v^3$;
- replace the three eng. unit vectors with a single indexed variable: $\hat{\imath} \to e_1$, $\hat{\jmath} \to e_2$ and $\hat{k} \to e_3$;
- express the sum using the standard *sigma convention*:

$$v = \sum_{i=1}^{3} v^i e_i;$$

- agree that, when an index variable appears twice in a single term, the convention implies summation of that term over all the values of the index; *i.e.*,

$$v = v^i e_i.$$

The upper indices are *not* exponents, but rather *indices of coordinates*, *coefficients* or *basis vectors*. For example, x^2 should be read as "x-two", *not* "x squared", and typically (x^1, x^2, x^3) would be equivalent to the traditional (x, y, z).

In general, indices can range over any indexing set, including an infinite set.

An index that is summed over is a *summation index*, in this case i. It is also called a *dummy index* since any symbol can replace i without changing the meaning of the expression, provided that it does not collide with index symbols in the same term.

An index that is not summed over is a *free index* and should be found in each term of the equation or formula if it appears in any term. Compare dummy indices and free indices with free variables and bound variables.

6. Origin Of The Geometric Product

Consider two vectors, x and y, elements of the real 2-dimensional space. We can decompose x and y using the $\mathbb{R}^2$ standard basis, $\{e_i\,;\; i : 1 \ldots 2\}$:

$$\begin{aligned} x &= x^i e_i = x^1 e_1 + x^2 e_2,\\ y &= y^i e_i = y^1 e_1 + y^2 e_2. \end{aligned}$$

Now let us multiply x and y component wise, as though we were doing algebra with real numbers,

$$\begin{aligned} xy &= x^1y^1e_1e_1 + x^1y^2e_1e_2 + x^2y^1e_2e_1 + x^2y^2e_2e_2,\\ &= x^1y^1e_1e_1 + x^2y^2e_2e_2 + x^1y^2e_1e_2 + x^2y^1e_2e_1. \end{aligned} \tag{1}$$

[1] Yep, *that* Einstein!

At this point we have to remember that the $x^i y^j$ are scalars, so we can rearrange them in any way; *e.g.*, $x^2y^1 = y^1x^2$. However, the terms e_ie_j are vectors. We do not know if rearrangement is possible. We do not know if, for instance, $e_1e_2 \stackrel{?}{=} e_2e_1$. More considerations are needed.

In order to open more avenues of research, let us take a look at other vector products, dot product and vector product. Let us begin with the dot product of x and y:

$$v \cdot y = x^i y^i = x^1y^1 + x^2y^2. \tag{2}$$

Now look at the coefficients of e_1e_1 and e_2e_2 in **??** and compare them with the correspondent ones in **??**. It seems that we can recover the result of the dot product if we allow $e_1e_1 = e_2e_2 = 1$.

Maybe something similar happens with the cross product. But wait, not that fast! We face a problem: the cross product is not defined in 2D; it is defined only in 3D. So, in order to do our component wise multiplication, we must rewrite the 2D vectors in 3D in the following way:

$$x = x^ie_i = x^1e_1 + x^2e_2 + 0e_3,$$
$$y = y^ie_i = y^1e_1 + y^2e_2 + 0e_3.$$

With this little trick [2], our component wise product, **??**, does not change at all.

Let us calculate the cross product then

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x^1 & x^2 & 0 \\ y^1 & y^2 & 0 \end{vmatrix} = e_3(x^1y^2 - x^2y^1). \tag{3}$$

Look at the coefficients of e_ie_j in **??** and compare them with the coefficient [3] of e_3 in **??**. The coefficient of e_1e_2 seems to be OK, but the coefficient of e_2e_1 has a different signature: it's negative! However, we can recover the cross product if we do $e_1e_2 = -e_2e_1$.

Let us recalculate the component wise product of x and y,

$$\begin{aligned} xy &= x^1y^1e_1e_1 + x^1y^2e_1e_2 + x^2y^1e_2e_1 + x^2y^2e_2e_2, \\ &= x^1y^1e_1e_1 + x^2y^2e_2e_2 + x^1y^2e_1e_2 + x^2y^1e_2e_1, \\ &= (x^1y^1 + x^2y^2) + (x^1y^2 - x^2y^1)e_1e_2. \end{aligned} \tag{4}$$

where, in the last step, we have applied our findings: $e_1e_1 = e_2e_2 = 1$ and $e_1e_2 = -e_2e_1$.

Notice some oddities in **??**.

- No e_3. It means that if we do component wise product of vectors, we do not have to change space. This is great news, because it opens the possibility to study *local* geometry instead of *embedded* geometry.
- The term $(x^1y^2 - x^2y^1)e_1e_2$ in **??** is *neither* a scalar (if it were, there would be no vectors multiplying it), *nor* a vector (it has two vectors multiplying it: e_1e_2, instead of only one). What is that then? It's a new algebraic object, yet to be explained. If we remember that Descartes' analytic geometry was so successful, because it provided a bridge between algebra and geometry, maybe we would need to explain what e_1e_2 is in geometrical terms. We will, in the next sections. But there's something we know: it is *neither* a scalar *nor* is it a vector.
- Finally, the greatest oddity of them all! If we remember that x^iy^i represents a scalar (the result of the dot product of x and y) and if we have just found that e_1e_2 is not a scalar, **??** is telling us that xy yields an entity with two different parts, one scalar and one not

[2] We will soon correct the cross product, so logic and methodology prevail over "math trickery".

[3] Yep. When you cross multiply two-2D vectors, you have to do it in 3D, because the vector you get is perpendicular to the multiplicands! That's odd.

scalar? **??** is telling us to add two different mathematical objects. It's like adding scalars to vectors! Yep.

7. Quick Review

Let's review what we already know about vectors and their algebra.

In engineering applications, a vector, say $\vec{v}$, is though of as mathematical object having a form

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{z}, \tag{5}$$

where v_x, v_y and v_z are the so called *components of* v and the vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{z}$ are the so called *unit vectors* of the *2-D space*.

For our further treatment, it is better to make some changes in notation and in terminology.

Notation 2. *The following replacements will allow us to generalize our ideas to more than two dimensions, preserving mathematical rigor, while adding consistency. Additionally, they will save some typing.*

- *Replace the the three spacial coordinates by one indexed spacial coordinate: $x \to x^1$, $y \to x^2$ and $z \to x^3$, where the superscripts are indices rather than a exponents. So x^2 is to be read second component of v, rather than "x to the power of 2".*
- *Replace the components of vectors with numeric indices ranging from 1 to the dimension where the vector "lives"; e.g., for a vector v in $\mathbb{R}^3$, $v_x \to v^1$, $v_y \to v^2$ and $v_z \to v^3$.*
- *Replace the unit vectors: $\hat{\imath} \to e_1$, $\hat{\jmath}$ and $\hat{k}$.*
- *Drop the arrow in vector typography: $\vec{v} \to v$; i.e., no decorations for vectors.*

One immediate advantage of using these notational conventions is that vectors can be written in a rather compact (and elegant [4]) way, using the *sigma notation*. For instance, for our vector v in $\mathbb{R}^3$, we have

$$v = \sum_{i=1}^{3} e_i v^i. \tag{6}$$

Consider an orthonormal basis, $\{e_1, e_2\}$, of the real coordinate space $\mathbb{R}^2$. Two arbitrary vectors, say u and v in $\mathbb{R}^2$, can be decomposed as a linear combination of the basis elements;

$$\begin{aligned} u &= u^1e_1 + u^2e_2 = u^ie_i, \\ v &= v^1e_1 + v^2e_2 = v^ie_i \end{aligned}$$

[4] Compare **??** with **??**.