APPLIED TENSOR ANALYSIS

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1. Newtonian Physics

1.1. **A Bit of History.** Newton placed his theory based on universal space and universal time; *i.e.*, space and time are independent on any external influences and on each other; In Newton's theory the position of a particle is represented by one vector x with three components each of which depends on time; *i.e.*, $x[t]$.

Einstein, on the other hand, placed time and space on equal footing with his Theory of special relativity by noting that ct, where c is the speed of light, has the dimensions of length. Minkowski, finally, united space and time in one single entity, spacetime, both conceptually and mathematically. In spacetime, the position of a particle is represented by one vector with four components; *i.e.*, the partner of x, y and z is not t, but rather ct.

1.2. **The Geometric Principle.** Physics and geometry are deeply related. So physical objects and physical processes can be modeled using geometric objects and geometric transformations. On the other hand, geometric algebra provides an efficient language to deal with geometry via algebra.

Geometric Principle:

> The laws of physics must all be expressible as geometric (coordinate-independent and reference-frame-independent) relationships – geometric transformations – between geometric objects, which represent physical entities.

There are three different conceptual frameworks for the classical laws of physics and, thus, three different geometric arenas for the laws:

(1) *General Relativity* formulates the laws as geometric relationships between geometric objects in he arena of *curved 4-dimensional spacetime*.
(2) *Special Relativity* is the limit of general relativity in the complete absence of gravity; its arena is *flat, 4-dimensional spacetime*, *aka* Minkowski spacetime.
(3) *Newtonian Physics* is the limit of general relativity when (i) gravity is weak but not necessarily absent, (ii) relative speeds of particles and materials are small compared to the speed of light c and (iii) all stresses (pressures) are small compared to the total density of mass-energy; its arena is *3-dimensional Euclidean space* with time separated off and made universal – by contrast with relativity's reference-frame-dependent time.

The aim is thus to express all physical quantities and laws in a *geometric form*: a form that is independent of any coordinate system or basis vectors.

We shall insist that the Newtonian laws of physics all obey a *Geometric Principle*: they are all geometric relationships between geometric objects, expressible without the aid of any coordinates or bases. An example is the Lorentz force law: $m\ddot{x} = q(E + v \times B)$ – a (coordinate-free) relationship between the geometric (coordinate-independent) vectors v, E and B and the scalars (the particle's) mass m and charge q; no coordinates are needed for this law of physics, nor is any description of the geometric objects as matrix-like entities. Components are secondary; they only exist after one has chosen a set of basis vectors. Components are an impediment to a clear and deep understanding of the laws of physics. The coordinate-free, component-free description is deeper and – once one becomes accustomed to it – much more clear and understandable.

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Besides, coordinate independence and basis independence strongly constrain the laws of physics. This suggests that

> Nature's physical laws *are* geometric and have nothing whatsoever to do with coordinates or components or vector bases.

1.3. **Foundational Concepts.** To lay the geometric foundations for the Newtonian laws of physics an flat, Euclidean space, consider some foundational geometric objects: points, scalars, vectors, (geometric product of vectors), inner product of vectors, distance between points.

The arena for the Newtonian laws is a spacetime composed of the 3-dimensional Euclidean space $\mathcal{E}^3$, called *3-space*, and a *universal time* t. Denote *points* (locations) in 3-space by capital script letters, such as $\mathcal{P}$ and $\mathcal{Q}$. These points and the 3-space where they live require no coordinates for their definition.

A *scalar* is a single number associated with a point, say $\mathcal{P}$, in 3-space. We are interested in scalars that represent physical quantities; *e.g.*, temperature T. When a scalar is a function of location $\mathcal{P}$ in space – *e.g.*, $T[\mathcal{P}]$, call it a *scalar field*.

A *vector* in 3-space can be thought of as a straight arrow reaching from one point, $\mathcal{P}$, to another, $\mathcal{Q}$; *i.e.*, Δx. Equivalently, Δx can be thought of as a direction at $\mathcal{P}$ and a number, the *vector's length*. Sometimes, one point $\mathcal{O}$ is selected in 3-space as an "origin" and other points, $\mathcal{P}$ and $\mathcal{Q}$, are identified by their vectorial separations $x_{\mathcal{P}}$ and $x_{\mathcal{Q}}$ from that origin.

The *Euclidean distance* Δs between two points $\mathcal{P}$ and $\mathcal{Q}$ is a scalar that requires no coordinate system for its definition. This distance Δs is also the magnitude (length) $|\Delta x|$ of the vector Δx that reaches from $\mathcal{P}$ to $\mathcal{Q}$ and the square of that length is denoted by

$$|\Delta x|^2 = \Delta x \Delta x = \Delta x^2 \coloneqq \Delta s^2 \,.$$

Of particular importance is the case when $\mathcal{P}$ and $\mathcal{Q}$ are neighboring points and Δx is a *differential quantity* $\mathrm{d}x$. By traveling along a sequence of such $\{\mathrm{d}x\}$, laying them down tail-at-tip, one after another, we can map out a *curve* to which these $\{\mathrm{d}x\}$ are tangent. The curve is $\mathcal{P}[\lambda]$, with λ a *parameter along the curve*, and the vectors that map it out are

$$\mathrm{d}x = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\lambda}\mathrm{d}\lambda \,.$$

The product of a scalar with a vector is still a vector; so if we take the change of location $\mathrm{d}x$ of a particular element of a fluid during a (universal) *time interval* $\mathrm{d}t$ and multiply it by $1/\mathrm{d}t$, then we obtain a new vector, the fluid element's *velocity* $v = \dot{x}$, at the fluid element's location $\mathcal{P}$. Performing this operation at every point $\mathcal{P}$ in the fluid defines the *velocity field* $v[\mathcal{P}]$. Similarly, the sum (or difference) of two vectors is also a vector and so taking the difference of two velocity measurements at times separated by $\mathrm{d}t$ and multiplying by $1/\mathrm{d}t$ generates the *acceleration* $a = \dot{v}$. Multiplying by the fluid element's (scalar) *mass* m gives the force $f = ma$ that produced the acceleration; dividing by an electrically produced force by the fluid element's charge q gives another vector, the electric field $E = f/q$ and so on. We can define *inner product of pairs of vectors* (*e.g.*, force and displacement) to obtain a new scalar (*e.g.*, work) and *cross products of vectors* to obtain a new vector (*e.g.*, torque). By examining how a differentiable scalar field changes from point to point, we can define its *gradient*. Thus, we can construct all of the standard scalars and vectors of Newtonian physics. What is important is that

> these physical quantities require *no* coordinate system for their definition.

They are geometric (coordinate-independent) objects residing in Euclidean 3-space at a particular time.

We can summarize this by stating that

> the Newtonian physical laws are *all* expressible as geometric relationships between these types of geometric objects and these relationships do *not* depend upon any coordinate system or orientation of axes, nor on any reference frame (on any purported velocity of the Euclidean space in which the measurements are made).

This principle is called the *Geometric Principle* for the laws of physics. It is the Newtonian analog of Einstein's Principle of Relativity.

2. Tensor Algebra

2.1. **Tensor Algebra without a Coordinate System.** We introduce, in a coordinate-free way, some fundamental concepts of differential geometry: tensors, the inner product, the metric tensor, the tensor product and contraction of tensors.

Because our space is flat, there is a unique way to transport one vector from one location to another while *keeping its length and direction unchanged*. Thus, vectors are unaffected by such a transport. Therefore, a vector is completely determined by its length and direction.

A *rank-n tensor* T is, by definition, a real-valued, linear function of n vectors. We can pictorially see a tensor as a machine that has n slots on top, into which n vectors are inserted, and one slot in its end, out of which a scalar is returned: the value that the tensor T has when evaluated as a function of the n inserted vectors. Notationally, denote tensors by T

$$T[-,-,-,-] \qquad [n \text{ slots in which to put the vectors}]\,.$$

If T is a rank-3 tensor (has 3 slots), this its value on the vectors a, b, c will be denoted by $T[a,b,c]$. Linearity of this function can be expressed as

$$T[ea+fb,c,d] = eT[a,c,d] + fT[b,c,d]\ ,$$

where e and f are real numbers.

Given two vectors a and b, the *inner product of a and B*, denoted $a \cdot b$, is defined in terms of the squared magnitude by

$$a \cdot b \coloneqq \frac{1}{4}[(a+b)^2 - (a-b)^2]\,.$$

In Euclidean space, this is the standard inner product.

Because the inner product is a linear function of each of its vectors, regard it as a tensor of rank 2. Then, the inner product is denoted $g-,-$ and is called the *metric tensor*. In other words, the metric tensor g is that linear function of two vectors whose value is given by

$$g[a,b] \coloneqq a \cdot b\,.$$

Because the inner product is symmetric, then the metric tensor is *symmetric* in its two slots [1]; *i.e.*,

$$g[a,b] = g[b,a]\ .$$

With the aid of the inner product, regard any vector a as a tensor of rank one: the real number that is produced when an arbitrary vector c is inserted into a's slot is

$$a[c] = a \cdot c\,.$$

Second-rank tensors appear frequently in the laws of physics – often in roles where one sticks a single vector into the second slot and leaves the first slot empty, thereby producing a single-slotted entity – a *vector*. An example is a rigid body's (Newtonian) moment-of-inertia tensor $I[-,-]$. Insert the body's angular velocity vector ω into the second slot and you get the body's angular momentum vector $J[-] = I[-,\omega]$. (In slot-naming notation, every slot could be regarded as an index, so the angular momentum vector equation can be written as $J_j = I_{jk}\omega^k$.)

From three (or any number of) vectors a, b, c we can construct a tensor, their *tensor product* defined by

$$a \otimes b \otimes c[e,f,g] \coloneqq a[e]\,b[f]\,c[g] = (a \cdot e)(b \cdot f)(c \cdot g)\,.$$

Here, the first expression is the notation for the value of the new tensor, $a \otimes b \otimes c$ *evaluated* on the three vectors e, f, g; the middle expression is the ordinary product of three real numbers; the value of a on e, the value of b on f and the value of c on g and the third expression is that same product with the htree numbers rewritten as scalar products.

[1] In slot-naming notation, to be developed afterwards, the components of g (in a given frame $\{\gamma_k\}$) are g_{kl}; this means that g is "waiting" to accept two vectors – this explains the notation $g[-,-]$. When a vector, say a, is plugged in, then $g[-,a]$ and thus $g_{kl}a^l = a_k$; *i.e.*, the result is the reciprocal component of a. When two vectors, say a and b, are plugged in, then $g[a,b] = a^k g_{kl} b^l = a^k b_k$. In other words, the g "disapears", leaving behind the inner product of a and b.

Similar definitions can be given for the tensor product of any two or more tensors of any rank; *e.g.*, if T has rank 2 (2 slots) and S has rank 3 (3 slots), then

$$T \otimes S[e, f, g, h, j] \coloneqq T[e, f]\, S[g, h, j] \ .$$

One last geometric (frame-independent) concept we need is *contraction* introduced by example. From two vectors a and b we can construct the tensor product $a \otimes b$ (a second rank tensor) and we can construct the scalar product $a \cdot b$ (a real number, *aka* a scalar, *aka* a rank-0 tensor). The process of *contraction is the construction of* $a \cdot b$ *from* $a \otimes b$:

$$\mathrm{cont}(a \otimes b) \coloneqq a \cdot b\,.$$

It can be shown that any second-rank tensor T can be expressed as a sum of tensor products of vectors; *viz.*,

$$T = a \otimes b + c \otimes d + \cdots \ ;$$

correspondingly, define the contraction of T to be

$$\mathrm{cont}\, T = a \cdot b + c \cdot d + \cdots \ .$$

Note that this contraction lowers the rank of the tensor by two, from 2 to 0. Similarly, for a tenor of rank n, one can construct a tensor of rank $n-2$ by contraction, but in this case, one must specify which slots are to be contracted. For instance, if T is a third rank tensor, expressible as $T = a \otimes b \otimes c + e \otimes f \otimes g + \cdots$, then the contraction of T on its first and third slots is the rank-1 tensor (vector)

$$\mathrm{cont}_{1,3}(a \otimes b \otimes c + e \otimes f \otimes g + \cdots) = (a \cdot c)b + (e \cdot g)f + \cdots \ .$$

Note that the inner product and tensor contraction lower the rank of tensors, while the tensor product increases it.

All the concepts developed in this section (vectors, tensors, metric tensor, inner product, tensor product and contraction of a tensor) can be carried over, with no change whatsoever, into *any* vector space over the real numbers that is endowed with a concept of squared length, even if the squared length is *not* Euclidean; *e.g.*, the four-dimensional spacetime of special relativity.

2.2. **Component Representation of Tensor Algebra.** In space $\mathcal{E}^3$ of Newtonian physics, there is a unique *orthonormal frame* whose vectors $\{\gamma_x, \gamma_y, \gamma_z\} = \{\gamma^1, \gamma^2, \gamma^3\}$ associated with any *Cartesian coordinate system* $\{x, y, z\} = \{x^1, x^2, x^3\} = \{x_1, x_2, x_3\}$ (In Cartesian coordinates in Euclidean space [2], place indices up or down. It doesn't matter. By definition, in Cartesian coordinates a quantity is the same whether its index is down or up). The frame element γ_k has unit length and points along the x_k coordinate direction, which is orthogonal to all the other coordinate directions, so this could be summarized by

$$\gamma_k \cdot \gamma_l = \delta_{kl}\ .$$

Any vector $a \in \mathcal{E}^3$ can be expanded in terms of these frames as

$$a = a^k \gamma_k\,.$$

Adopt Einstein summation convention: repeated indices (in this case k) are to be summed (in this $\mathcal{E}^3$ case over $j = 1, 2, 3$).

Any tensor, say the third-rank tensor $T[-,-,-]$, can be expanded in terms of tensor products of the frame elements:

$$T = T^{ijk} \gamma_i \otimes \gamma_j \otimes \gamma_k\ .$$

The components T_{ijk} of T can be computed from T and the frame elements by

$$T_{ijk} = T[\gamma_i, \gamma_j, \gamma_k]\ .$$

As an important example, the components of the metric are $g_{jk} = g[\gamma_j, \gamma_k] = \gamma_j \cdot \gamma_k = \delta_{jk}$:

$$g_{jk} = \delta_{jk} \qquad \text{in any orthonormal frame in space } \mathcal{E}^3.$$

[2] Cartesian coordinates means rectangular coordinates; *i.e.*, the frame elements all measure lengths and are orthonormal to each other. Euclidean space means that the metric is the Euclidean metric; *i.e.*, $g = \delta$.

The components of a tensor product, *e.g.*, $T[-,-,-]\otimes S[-,-]$, are deduced by inserting the frame elements into the slots; they are $T[\gamma_i,\gamma_j,\gamma_k]\otimes S[\gamma_l,\gamma_m]=T_{ijk}S_{lm}$. In words,

> the components of a tensor product are equal to the ordinary arithmetic product of the components of the individual tensors.

In components notation, the inner product of two vectors and the value of a tensor when vectors are inserted into its slots are given by

$$a\cdot b = a^k b_k \qquad \text{and} \qquad T[a,b,c]=T_{ijk}a^i b^j c^k\ ,$$

as one can show using the previous equations. Finally, the contraction of a tensor, say the fourth rank tensor $R[-,-,-,-]$, on two of its slots, say the first and third, has components that are computed from the tensor's own components:

$$\mathrm{comp}(\mathrm{cont}_{1,3}\,R)=R^{ijik}\ .$$

Note that R^{ijik} is summed on the index i, so ti has only two free indices j and k and, thus, is the components of a second rank tensor, as it must be if it is to represent the contraction of a fourth-rank tensor.

2.3. **Slot-Naming Notation.** Consider the rank-2 tensor $F[-,-]$. Define a new tensor $G[-,-]$ to be the same as F, but with the slots interchanged; *i.e.*, for two vectors a and b it is true that $G[a,b]=F[b,a]$. To simply indicate that F and G are equal but with the slots interchanged is by the notation $G[-_a,-_b]=F[-_b,-_a]$ or, in index notation, by $G_{ab}=F_{ba}$. This relationship is valid in a particular frame if and only if $G=F$ with slot interchanged is true. Then, look at any "index equation", such as $G_{ab}=F_{ba}$, momentarily as a relationship between components of tensors in a specific frame and then do a quick mind-flip and regard it quite differently: as *a relationship between geometric, frame-independent tensors with the indices playing the roles of names of slots.*

As an example of the power of this *slot-naming notation*, consider the contraction of the first and third slots of a third-rank tensor T. In any frame, the components of $\mathrm{cont}_{1,3}\,T$ are T^{aba}. Correspondingly, in slot-naming index notation, denote $\mathrm{cont}_{1,3}\,T=T^{aba}$. Say that the first and the third slots are "strangling each other", leaving free only the second slot (named b) and therefore producing a rank-1 tensor (a vector).

2.4. **Linear Maps.** A *linear map*, *aka linear transformation*, is a function between two *vector spaces* that preserves the operations of vector addition and scalar multiplication. Thus, it *always* maps linear subspaces to linear subspaces – like straight lines to straight lines or to a single point. When a linear map maps from a vector space to itself, it is called *endomorphism* or "linear operator".
Definition. Consider two vector spaces $\mathcal{V}$ and $\mathcal{W}$ over the reals $\mathcal{R}$, consider two vectors $x,y\in\mathcal{V}$ and a scalar $\alpha\in\mathcal{R}$. Then, define a linear map to be a function $f:\mathcal{V}\mapsto\mathcal{W}$ if it satisfies:

$$f[x+y]=f[x]+f[y]\ , \qquad \textit{[additivity]}$$
$$f[\alpha x]=\alpha f[x]\ . \qquad \textit{[homogeneity of degree 1]}$$

Equivalently, f is a linear map if it satisfies additivity and homogeneity for any *linear combination of vectors*; *i.e.*, consider a set of vectors $\{\gamma_1,\ldots,\gamma_m\}$ in $\mathcal{V}$ and scalars $\left\{a^1,\ldots,a^m\right\}$ in $\mathcal{R}$. Then, if f is a linear map, it satisfies

$$f\left[a^k\gamma_k\right]=\sum_{k=1}^{m}a^k f[\gamma_k]\ , \tag{2.1}$$

where Einstein summation convention and sigma notation were used [3].

[3] In traditional notation: a linear map f satisfies $f\left[a^1\gamma_1+\cdots+a^m\gamma_m\right]=a^1 f[\gamma_1]+\cdots+a^m f[\gamma_m]$. for the set $\{\gamma_k\}$ of vectors and the set $\left\{a^k\right\}$ of scalars.

2.5. **Linear Functionals.** A *linear functional*, *aka* linear form, one-form or covector, is a linear map from a vector space to its field of scalars. In $\mathcal{R}^n$, if *vectors* are represented as *column vectors*, then *linear functionals* are represented as *row vectors* and their action on vectors is given by the *dot product*, or the matrix product with the row vector on the left and the column vector on the right. In general, if $\mathcal{V}$ is a vector space over a field $\mathcal{K}$, then a linear functional f is a function from $\mathcal{V}$ to $\mathcal{K}$, which is linear.

More formally, we have the definition:

Definition. Define a linear functional a function $f : \mathcal{V} \mapsto \mathcal{R}$ if it is a linear map; i.e., if it satisfies eq. (2.1).

3. Tensor Calculus

3.1. **Directional Derivatives, Gradients, Levi-Civita Tensor, Cross Product and Curl.** Consider a tensor field $t[\mathcal{P}]$ in $\mathcal{E}^3$ and a vector a. Then, define the *directional derivative of t along a* by

$$\nabla_a t \coloneqq \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(t[x_{\mathcal{P}} + \epsilon a] - t[x_{\mathcal{P}}] \right)$$

and similarly for the directional derivative of a vector field $v[\mathcal{P}]$ and a scalar field $\phi[\mathcal{P}]$. In this definition, the quantity in parenthesis is the difference between two linear functions of vectors and we have denoted points, say $\mathcal{P}$, by the vector $x_{\mathcal{P}}$ that reaches from some arbitrary origin to the point.

The directional derivative of any tensor field t is linear in the vector a along which one differentiates. Correspondingly, if t has rank n (n slots), then there is another tensor field, denoted ∇t, with rank $n + 1$, such that

$$\nabla_a t = \nabla t[-, -, -, a] \; .$$

Here on the right hand side, the first n slots (three in the case shown) are left empty and a is put into the last slot (the "differentiation slot"). The quantity ∇t is called the *gradient of t*. In slot-naming index notation, it is conventional to denote this gradient by $t_{abc;d}$, where in general the number of indices preceding the semicolon is the rank of t. Using this notation, the directional derivative of t along a reads $t_{abc;j} a^j$.

In any Cartesian coordinate system, the *components of the gradient* are nothing but the partial derivatives of the components of the original tensor,

$$t_{abc;j} = \frac{\partial t_{abc}}{\partial x^j} \coloneqq t_{abc,j} \; .$$

In a non-Cartesian basis (*e.g.*, spherical or cylindrical bases), the components of the gradient are *not* obtained by simple partial differentiation, because of turning or length changes of the basis vectors as we go from one location to another. Later, we shall learn how to deal with this by using objects called *connection coefficients*. Thus, when dealing with Cartesian coordinates, subscripts semicolons and subscripts commas can be used interchangeably.

Because of their definition, the gradient and the directional derivative obey the standard Leibniz rule for differentiating products:

$$\nabla_A (s \otimes t) = (\nabla_A s) \otimes t + s \otimes \nabla_A t \, ; \, i.e.,$$
$$(s_{ab} t_{cde})_{;j} A^j = (s_{ab;j} A^j) t_{cde} + s_{ab} (t_{cde;j} A^j) \, ;$$

and

$$\nabla_A (f t) = (\nabla_A f) T + f \nabla_A t \, ; \, i.e.,$$
$$(f t_{abc})_{;j} A^j = (f_{;j} A^j) t_{abc} + f t_{abc;j} A^j \, ,$$

where f is a scalar-valued function, s and t tensors and A a vector.

The components g_{ab} of the metric tensor are constant in any Cartesian system, thus it is guaranteed that $g_{ab;j} = 0$; *i.e.*, the metric has vanishing gradient

$$\nabla_= g 0, \qquad i.e., \quad g_{ab;j} = 0$$

From the gradient of any vector or tensor, we can construct several other important derivatives by contracting on slots:

(1) since the gradient ∇A of a vector field A has two slots, $\nabla A[-, -]$, we can contract its slots on each other to obtain a scalar field. That scalar field is the *divergence of A* and is denoted

$$\nabla \cdot A = (\text{cont} \, \nabla A) = A_{a;a} \; .$$

(2) Similarly, if t is a tensor field of rank three, then $t_{abc;c}$ is its divergence on its third slot and $t_{abc;b}$ is its divergence on its second slot.

(3) By taking the double gradient and then contracting on the two gradient slots we obtain, from any tensor field t, a new tensor field with the same rank

$$\nabla^2 t := (\nabla \cdot \nabla) t\,, \quad \text{or, in index notation, } t_{abc;jj}\,.$$

Here and henceforth, all indices following a semicolon or comma represent gradients (or partial derivatives):

$$t_{abc;jj} = t_{abc;j;j} = t_{abc,jk} = \frac{\partial^2 t_{abc}}{\partial x^j \partial x^k}\,.$$

The operator ∇^2 is called the *Laplacian*.

The metric tensor is a fundamental property of the space in which it lives; it embodies the inner product and thence the space's notion of distance or interval and thus the space's geometry. In addition to the metric, there is one (and only one) other fundamental tensor that embodies a piece of Euclidean (or Minkowski) geometry: the *Levi-Civita tensor* ϵ.

The Levi-Civita tensor has a number of slots equal to the dimensionality of the space in which it lives: three slots in the Euclidean 3-space and four slots in the Minkowski 4-spacetime. It is anti-symmetric in each and every pair of its slots. These properties determine ϵ up to multiplicative constant. That constant is fixed by a compatibility relation between ϵ and the metric g plus the concept of "headedness": if $\{\gamma_j\}$ is a Cartesian basis and if this basis is right-handed in the usual sense, then

$$\epsilon[\gamma_1, \gamma_2, \gamma_3] = +1\,.$$

The last equation and the antisymmetry of ϵ imply that in an orthonormal, right-handed basis, the only nonzero components of ϵ are

$$\begin{cases} \epsilon_{123} & = +1\,, \\ \epsilon_{abc} & = +1\,, \quad \text{if } a,b,c \text{ is an } \textit{even} \text{ permutation of } 1,2,3\,, \\ \epsilon_{abc} & = -1\,, \quad \text{if } a,b,c \text{ is an } \textit{odd} \text{ permutation of } 1,2,3\,, \\ \epsilon_{abc} & = 0\,, \quad \text{if } a,b,c \text{ are not all different.} \end{cases}$$

The Levi-Civita tensor is used to define the cross product and the curl:

- $A \times B = \epsilon[-, A, B]$; *i.e.*, in slot-naming index notation, $\epsilon^i_{jk} A^j B^k$;
- $\nabla \times A =$ the vector field whose slotnaming index form is $\epsilon^i_{jk} A^{k;j}$.

Note. The last equation is an example of an expression that is complicated if written in index-free notation; it says that $\nabla \times A$ is the double contraction of the rank-5 tensor $\epsilon \otimes \nabla A$ on its second and fifth slots and on its third and fourth slots.

The Levi-Civita tensor in $\mathcal{E}^3$ has the following property:

$$\epsilon_{ijm}\epsilon_{klm} = \delta^{ij}_{kl} := \delta^i_k \delta^j_l - \delta^i_l \delta^j_k\,.$$

Here δ^i_k is the Kronecker delta. The 4-index delta function δ^{ij}_{kl} says that either the indices above and below each other must be the same ($i = k$ and $j = l$) with a $+$ sign or the diagonally related indices must be the same ($i = l$ and $j = k$) with a $-$ sign. With the last equation and the index-notation expressions for the cross product and curl, one can quickly and easily derive a wide variety of useful vector identities.

4. Examples

4.1. Tensors.

4.1.1. *Particle Kinetics and Lorentz Force in Geometric Language.* In Newtonian physics, a classical particle of mass m moves through space $\mathcal{E}^3$ as universal time t passes. At time t, it is located at some point $x[t]$ (its *position*). The function $x[t]$ represents a curve in space, the particle's *trajectory*. The particle's *velocity* $v[t]$ is the derivative of its position, its *momentum* $p[t]$ is the product of its mass and velocity, its *acceleration* $a[t]$ is the time derivative of its velocity and its *energy* is half its mass times velocity squared:

$$x[t]\ ,\quad v[t] = \dot{x}[t]\ ,\quad p[t] = mv[t]\ ,\quad a[t] = \dot{v}[t] = \ddot{x}[t] \quad \text{and} \quad E[t] = \tfrac{1}{2}mv^2[t]\ .$$

Since points in space $\mathcal{E}^3$ are geometric objects (defined independently of any coordinate system), so are the trajectory, velocity, momentum, acceleration and energy. (Physically, velocity has an ambiguity: it depends on one's standard of rest.)

Newton's second law of motion states that the particle's momentum can change only if a force f acts on the particle and such a change is given by

$$\dot{p} = ma = f\ .$$

If the force is produced by an electric field E and magnetic field B, then this law of motion takes the familiar Lorentz-force form:

$$\dot{p} = q(E + v \times B)\ .$$

Note that these laws of motion are geometric relationships between geometric objects, thus independent of any coordinate system.

4.1.2. *Particle Kinetics in Index Notation.* As an example of slot-naming index notation, we can rewrite the equations of particle kinetics as follows:

$$v^i = \dot{x}^i\ ,\quad \dot{p}^i = mv^i\ ,\quad a^i = \dot{v}^i = \ddot{x}^i\ ,\quad E = \frac{1}{2}mv^j v_j\ ,\quad \dot{p}^i = q(E^i + \epsilon^i_{jk}v^j B^k)\ .$$

In the last equation, ϵ^i_{jk} is the so-called *Levi-Civita tensor*, which is used to produce the cross product.

These equations could be viewed in either of two ways: (i) as the frame-independent geometric laws $v = \dot{x}$, $p = mv$, $a = \dot{v} = \ddot{x}$, $2E = mv^2$ and $\dot{p} = q(E + v \times B)$ written in slot-naming notation or ii) as equations for the components of $\{v, p, a, E, B\}$ in some particular Cartesian coordinate system.

4.1.3. *Meaning of Slot-Naming Index Notation. Example.* Find the components of the tensor $S[a, b, -]$, where a and b are vectors.

Solution. In slot-naming notation, S is rank-3, so it has 3 slots, or indices, say i, j and k. Then, S_{ijk}. The vectors a and b occupy the first and the second slots of S, respectively; so $a \to a^i$ and $b \to b^j$.

Next, since there is only one empty slot in S, then its result, say c, must be a rank-1 tensor – a vector, with the index equal to the remaining slot of S; *viz.*, k. Finally, write the product in slot-index notation as

$$c^k = S^{ijk}a^i b^j\ .$$

In the last expression, since the indices are written using Latin letters, then they all run from 1 to 3. The index k is a free index, whereas i and j are repeated, *aka* dummy, indices; *i.e.*, the (one) tensor equation represents three equations with nine terms.

Example. Find the components of the tensor $d = S[a, -, b]$.

Solution. Assign the indices $\{i, j, k\}$ to S. Then, S_{ijk}. The vectors a and b will take the first and third slots of S, respectively, while leaving the second slot to e:

$$S_{ijk}a^i b^k = d^j\ .$$

Again, the indices are Latin letters, so they all run from 1 to 3. The index j is a free index and i and k are repeated indices.

Example. Find the components of the tensor $e = T[-, -, a]$.

Solution. Assign the indices $\{i, j, k\}$ to T. Then, T_{ijk}. The vector a will take the last slot of T, while leaving the two first slots to the second-rank tensor e:

$$T_{ijk}a^k = e_{ij}\ .$$

The indices are Latin letters, so they all run from 1 to 3. The index k is a repeated index and i and j are both free indices.

Example. Convert the expression $a^i b^{ij}$ to geometric, index-free notation.

Solution. The product $a^i b^{ij}$ has two variables and three indices, so it comes from the product of a rank-1 tensor, a vector, a, and a rank-2 tensor, b; *i.e.*, $a \otimes b[-,-]$. But, by definition $a \otimes b[-,-] = b[a,-]$, which gives the result.

Example. Convert the expression $a^i b^{ij}$ to geometric, index-free notation.

Solution. The term b^{ij} is a rank-2 tensor, so $b[-,-]$ or, in slot naming notation, $b[-_i,-_j]$. Then, $a^i b^{ij}$ represents a taking the first slot, i, of b; *i.e.*, $a^i b^{ij} = b^{ij} a^i = b[a,-]$.

Example. Convert the equation $s^{ijk} = s^{kji}$ to geometric, index-free notation.

Solution. This equation represents the same tensor, but with slots interchanged. Then,

$$s^{ijk} = s^{kji} \implies s[-_i,-_j,-_k] = s[-_k,-_j,-_i] \ . \qquad \square$$

Example. Convert the equation $a^i b_i = a^i b^j g_{ij}$ to geometric, index-free notation.

Solution. This equation is the definition of the inner product for two vectors, a^i and b^j, via the metric tensor $g[-,-]$; *i.e.*, $a^i b^j g_{ij} = g[a,b]$.

4.1.4. *Numerics of Component Manipulations. Example.* The third rank tensor $S[-,-,-]$ and vectors a and b have as their only nonzero components $S^{123} = S^{231} = S^{312} = +1$, $a^1 = 3$, $b^1 = 4$, $b^2 = 5$. What are the components of the vector $c = S[a,b,-]$, the vector $d = S[a,-,b]$ and the tensor $W = a \otimes b$?

Solution. In component notation, $c^k = T^{ijk} a^i b^j$

4.1.5. *Product of Scalars and Tensors. Example.* Consider $\mathcal{E}^3$ and consider a scalar $s \in \mathcal{R}$. Then, compute the product sg in slot-naming notation.

Solution. Denote by g the metric of $\mathcal{E}^3$ and consider a standard frame $\{\gamma_k\}$ in $\mathcal{E}^3$. Then, calculate the components of the metric as $g_{ij} = \delta_{ij}$. Therefore, the product $sg = s\delta_{ij}$, in index notation.

Write the product $s\delta_{ij}$ as s_{ij}, since the diagonal components of δ_{ij} all equal s. See this in matrix notation

$$s_{ij} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} . \qquad \square$$

4.2. **Tensor Product – Again!** Consider two vectors a, b in $\mathcal{E}^n$. Consider a to be a rank-one tensor; *i.e.*, a tensor with one slot: $a[-]$. Then, the value of the tensor a when b is inserted in its slot is defined by

$$a[b] \coloneqq a \cdot b \,,$$

where the result is a real number – by the definition of tensor.

Consider the vectors $a, b, c, e, f, g \in \mathcal{E}^n$. Then, construct a tensor by using the *tensor product* of the vectors, defined by

$$a \otimes b \otimes c[e,f,g] \coloneqq a[e]\, b[f]\, c[g] \coloneqq (a \cdot e)(b \cdot f)(c \cdot g) \,.$$

Remark. Similar definitions can be given for the tensor product of any two or more tensors of any rank; for instance, if t has rank 2 and s rank 3, then

$$t \otimes s[e,f,g,h,j] \coloneqq t[e,f]\, s[g,h,j] \ ,$$

where the elements of the set $\{e,f,g,h,j\}$ are vectors in $\mathcal{E}^n$.

4.3. **Component Representation of Tensor Algebra.** Consider the geometric arena for Newtonian physics to be the 3-dimensional (Euclidean) space, $\mathcal{E}^3$, and universal time $t \in \mathcal{R}$. In this space, there is a unique *standard frame*, *aka* ordered set of *orthonormal basis vectors*, $\{\gamma_k; 1 \ldots 3\} \coloneqq \{\gamma_x, \gamma_y, \gamma_z\}$ associated with any *Cartesian coordinate system* $\{x^k\} \coloneqq \{x,y,z\}$. The frame element γ_k has unit *length* and points along the x^k *coordinate direction*, which is *orthogonal* to all the other coordinate directions. Summarize these frame elements properties by

$$\tfrac{1}{2}(\gamma_k \gamma_l + \gamma_l \gamma_k) = \gamma_k \cdot \gamma_l \coloneqq \delta_{kl} \,.$$

Expand any vector $u \in \mathcal{E}^3$ as a *linear combination* of the frame elements:

$$u = u^1\gamma_1 + u^2\gamma_2 + u^3\gamma_3 = \sum_{k=1}^{3} u^k\gamma_k \,. \tag{4.1}$$

Agree with *Einstein summation convention* to shorten eq. (4.1) as

$$u := u^k\gamma_k \,.$$

By virtue of the orthonormality of the frame, compute the components u^k of u as the *inner product*

$$u^k = u \cdot \gamma_k \,. \tag{4.2}$$

Note. Check this last definition: it is the problematic one in index manipulation! So everything fits, I think $a^k := a \cdot \gamma^k$, so "up" agrees with "up". The problem: it needs the introduction of reciprocal frames and the definition of the mixed metric δ_{jk}! However, with this, we would have:

$$a^j = (a^k\gamma_k) \cdot \gamma_j = a^k(\gamma_k \cdot \gamma_j) = a^k\delta^j_k = a^j \,. \qquad \square$$

Note. I think the correct definition would be $u_k = u \cdot \gamma_k$. In this case, analogous to the up-up case, it would be the down agrees with down. The problem: it needs the introduction of reciprocal frames and the decomposition of u onto such a frame. However, with this, we would have:

$$u_k = u^j\gamma_j \cdot \gamma_k = u^j\delta_{jk} = u_k \,.$$

One advantage over the last note is that no mixed metric is needed. Only the one already defined! Besides, another advantage is the generalization of this definition to tensors (see below). Using $u_k = u \cdot \gamma_k$ fits everything!

Expand any tensor, say the rank-3 tensor $t[-,-,-]$ as the *tensor product* of frame elements (analogous to eq. (4.1)):

$$t = t[\gamma_i, \gamma_j, \gamma_k] = t^{ijk}\,\gamma_i \otimes \gamma_j \otimes \gamma_k \,.$$

Compute the *components* t_{ijk} of t on the frame by generalizing of eq. (4.2):

$$t_{ijk} = t[\gamma_i, \gamma_j, \gamma_k] \,. \tag{4.3}$$

Proof.

Remark. Careful with index position: when the tensor t multiplies the tensor product of the frame elements, then its components go "up" t^{ijk}. However, when the tensor components have to be calculated or used, then its components go "down" t_{ijk}.

Calculate the *components of the metric* [4] tensor g as

$$g_{jk} = g[\gamma_j, \gamma_k] = \gamma_j \cdot \gamma_k = \delta_{jk} \,.$$

The last equality only holds for any standard frame, *aka* orthonormal basis, in $\mathcal{E}^3$.

Using eq. (4.3), calculate the *components of a tensor product*, *e.g.*. $t[-,-,-] \otimes s[-,-]$ by inserting the frame elements into the slots; *i.e.*,

$$t[\gamma_i, \gamma_j, \gamma_k] \otimes s[\gamma_l, \gamma_m] = t_{ijk}s_{lm} \,.$$

In words,

> the components of a tensor product equal the ordinary arithmetic product of the components of the individual tensors. The position of the indices follows that of the frame elements.

In component notation, compute the *inner product* of two vectors, say $a, b \in \mathcal{E}^3$, and the *value of a tensor* t when vectors, say three vectors $d, e, f \in \mathcal{E}^3$, are inserted into its slots by

$$a \cdot b = g_{ij}a^ib^j \qquad t[d, e, f] = t_{ijk}d^ie^jf^k \,.$$

Calculate the *components of the contraction of a tensor*, say $r[-,-,-,-]$, on two of its slots, say the first and the third, by

$$\operatorname{comp}\operatorname{cont} 1, 3r = r_{ijik} \,.$$

[4] This is the paramount example of index manipulations with tensors, because it *is* a tensor.

Note that r_{ijik} is summed on the i index, so it has only two free indices: j and k. Thus, r_{ijik} are the components of a second rank tensor, as it must be if it is to represent the contraction of a fourth-rank tensor.

5. Notation

5.1. General Commands.

- to be defined by: a defby b: $a \coloneqq b$.
- difference operator: diff a: Δa.
- text in equations: eqtxt.

5.2. Sets.

- set: set A: $\mathcal{A}$.
- elements of a set: elset(a,b,c): $\{a, b, c\}$.
- set with a property: setprop(x)(x>0): $\{x : x > 0\}$.
- Cartesian (set) product: set A sprod set B: $\mathcal{A} \otimes \mathcal{B}$.
- Cartesian power: nset An: $\mathcal{A}^n$.
- union of sets: set A union set B: $\mathcal{A} \cup \mathcal{B}$.
- intersection of sets: set A inter set B: $\mathcal{A} \cap \mathcal{B}$.
- Dim-grade space (2 is the dimension and 3 is the grade): dgspace V23: V_3^2.
- n-dim Euclidean space: espace n: $\mathcal{E}^n$.
- n-dim Minkowski space: mkspace n: $\mathcal{M}^n$.
- geometric algebra: ga: $\mathcal{G}$.
- geometric algebra on a n-dim. linear space $\mathcal{V}^n$: nga n: $\mathcal{G}^n$.
- dimension grade geometric algebra (2 is the dimension and 3 is the grade): dgga 23: $\mathcal{G}_3^2$.
- tuple: tuple(1,2,3): $[1, 2, 3]$.

5.3. Probability.

- event A: A: A.
- not event A: lnot A: $\neg A$.
- probability of event A occurring: p vat A: $p[A]$.
- probability of event A not occurring: p vat(lnot A): $p[\neg A]$.
- A and B: A land B: $A \wedge B$.
- A or B: A lor B: $A \vee B$.
- A given B (provided B): A given B: $A \mid B$.

5.4. Functions.

- function definition: fdef(f)(set A cartprod set B)(set R): $f : \mathcal{A} \otimes \mathcal{B} \to \mathcal{R}$.
- function mapping: fmap(f)(x)(x**2): $f : x \mapsto x^2$.
- maps to: x mapsto x**2: $x \mapsto x^2$.
- function class (calculus): class k: C^k.
- value at: f vat(x): $f[x]$.
- function composition: f fcomp g: $f \circ g$.
- a binary operation: a bprod b: $a * b$.
- derivative operator (on functions): fder f: $\mathrm{D}f$.
- partial derivative operator (on functions): derivative with respect to x of f: fpder xf: $\partial_x f$.

5.5. Sequences and Series.

- sequence: seq ak()(): $\{a\}_k$.
- sequence with limits: seq(a)(k)(k=1)(10): $\{a\}_k^{k=1}$ 10.
- series: serie(ak)(k=1)(n): $\sum_{a_k}^{k=1} n$.
- Fibonacci numbers: fib vat 10: fib[1] 0.

5.6. Geometric Algebra.

- point: point P: $\mathcal{P}$.
- curve: curve C: $\mathcal{C}$.
- surface: surf S: $\mathcal{S}$.
- region: region R: $\mathcal{R}$.
- bound: bound region A: $\partial\mathcal{A}$.
- vector: vec a: $\vec{a}$.
- normal (unit) vector: nvec(a): $\hat{a}$.
- omitted vector from a product: ovec(a): $\check{a}$.
- clifs or multivectors: (use capitals) A: A.
- pseudoscalar: pscl: i.
- better, less typing, use lower-case for vectors a and upper-case for other objects (bivectors, trivectors,...): A.
- use i for the pscl: i.
- orthogonal: ortho: $a \perp b$.
- parallel: parallel: $a \parallel b$.
- to be perpendicular to: perto(a)(b): $a_{\perp b}$.
- to be parallel to: parto(a)(b): $a_{\parallel b}$.
- to be orthogonal to: ortto(a)(b): $a_{\perp b}$.
- to be colinear to: colto(a)(b): $a_{\parallel b}$.
- projection of p onto q: projon pq: $p_{\parallel q}$.
- rejection of p onto q: rejon pq: $p_{\perp q}$.
- magnitude: magn(a): $|a|$.
- inverse: inv(a): a^{-1}.
- reverse: rev(a): $a^{\dagger}$.
- hodge dual: hdual(a): ${}^{*}a$.
- anticommutator: acom(a)(b): $[a, b]_+$.
- commutator: com(a)(b): $[a, b]_-$.
- expanded anticommutator: xacom(a)(b): $ab + ba$.

- expanded commutator: xcom(a)(b): $ab - ba$.
- step: step(A1): $\langle A \rangle_1$.
- scalar step: sstep(A): $\langle A \rangle_0$.
- grade operator: Grade A: $\operatorname{grade} A$.
- grade: grade A2: $\langle A \rangle_2$.
- scalar grade: sgrade A: $\langle A \rangle_0$.
- clif with step: sclif Ak: $A_{\bar{k}}$.
- even part: even(A): A_+.
- odd part: odd(A): A_-.
- gorm (geometric norm?): gorm A: $\operatorname{gorm} A$.
- expanded gorm: xgorm A: $\langle A^\dagger A \rangle_0$.
- metric: metric: g.
- Kronecker delta: kron: δ.
- signature: diag a: $\operatorname{diag} a$.
- signature: sign a: $\operatorname{sig} a$.
- inner product: iprod: $a \cdot b$.
- outer product: oprod: $a \wedge b$.
- cross product: cprod: $a \times b$.
- cannonical decomposition of the geometric product: cgprod ab: $a \cdot b + a \wedge b$.

5.7. Geometric Calculus.

- ordinary one-dim. derivative: dx x: $\mathrm{d}x$.
- ordinary time derivative (dot derivative): dt x: $\dot{x}$.
- ordinary second time derivative (dot-dot derivative): ddt x: $\ddot{x}$.
- expanded ordinary derivative: xod Hq: $\dfrac{\mathrm{d}H}{\mathrm{d}q}$.
- expanded partial derivative: xpd Hq: $\dfrac{\partial H}{\partial q}$.
- expanded material derivative: xmd phi t: $\dfrac{\partial \phi}{\partial t}$.
- expanded n-order ordinary derivative: nxod 3xt: $\dfrac{\mathrm{d}^3 x}{\mathrm{d}t^3}$.
- expanded n-order partial derivative: nxpd 3xt: $\dfrac{\partial^3 x}{\partial t^3}$.
- comma derivative: cder phi k: $\phi_{,k}$.
- semi-colon: covariant derivative: coder(cntens Aa)(k): $A^a{}_{;k}$.
- material derivative: mder phi t: $\mathrm{D}_t \phi$.
- absolute time derivative: abstder a: $\dot{\bar{a}}$.
- Christoffel symbol: chris abc: Γ^a_{bc}.
- geometric derivative: gder(a): ∇a.
- directional derivative: dder(F)(a): $\nabla_a F$.
- Laplace derivative: lder(a): $\nabla^2 a$.
- Laplace operator: lap a: $\operatorname{lap} a$.
- D'Alembert operator: dalder(phi): $\Box\phi$.
- gradient: grad(phi): $\operatorname{grad}\phi$.
- divergence: div(phi): $\operatorname{div}\phi$.
- curl: curl(phi): $\operatorname{curl}\phi$.
- rotational (curl): rot(phi): $\operatorname{rot}\phi$.

5.8. Tensors.

- spatial coordinates: scord k: x^k.
- spatial coordinates time derivative: dtscoord k: $\dot{x}^k$.
- tensor: tens T: T.
- (empty) slot: tuple(slot, a, slot): $[-, a, -]$.
- tensor product: a tprod b: $a \otimes b$.
- tensor contraction: tcont(a tprod b): $\operatorname{cont}(a \otimes b)$.
- indexed tensor contraction: itcont(1,2)(a tprod b tprod c): $\operatorname{cont}_{1,2}(a \otimes b \otimes c)$.
- tensor components: tcomp T: $\operatorname{comp} T$.
- covariant tensor components: cotens T(ij): T_{ij}.
- contravariant tensor components: cntens T(ij): T^{ij}.
- Levi-Civita tensor: lct: ϵ.
- covariant tensor time derivative: dtcotens ak: $\dot{a}_k$.
- contravariant tensor time derivative: dtcntens ak: $\dot{a}^k$.

5.9. Index Notation.

- frame element, vector: fvec: γ.
- frame: frm(k): $\{\gamma_k\}$.
- indexed frame: ifrm(k)(0)(n): $\{\gamma_k; 0 \ldots n\}$.
- reciprocal frame: rfrm k: $\left\{\gamma^k\right\}$.
- indexed frame vector: ifvec k: γ_k.
- indexed reciprocal frame vector: rfvec k: γ^k.
- components of vector in frame: comp vk: v^k.
- components of vector in reciprocal frame: rcomp vk: v_k.
- metric coefficients in frame: imet kl: g_{kl}.
- metric coefficients in reciprocal frame: rmet kl: g^{kl}.
- mixed metric coefficients: mmet kl: g^k_l.
- kronecker delta coefficients in frame: ikron kl: δ_{kl}.
- kronecker delta coefficients in reciprocal frame: rkron kl: δ^{kl}.
- mixed kronecker coefficients: mkron kl: δ^k_l.

- indexed geometric derivative (in reciprocal frame): igder k: ∂_k.
- indexed geometric derivative (in frame): rgder k: ∂^k.

5.10. Dimensional Analysis.

- dimension: dim k: $\dim k$.
- dimension and system of units (use underscore as lim): sdim(FLT)k: $\dim_{FLT} p = \left[F/L^2\right]$.
- unit: unit k: $\operatorname{unit} k$.
- physical dimension: phdim k: $[k]$.
- dimensionless quantity: kdim: Π.
- characteristic physical quantity: chpq a: a_c.
- scaled physical quantity: scpq x: $\bar{x}$.
- reynolds number: rey: Π_{re}.
- biot number: biot: Π_{bi}.

5.11. Mechanics.

- position vector: pvec vat t: $x[t]$.
- value at time and position vector (for functions): vattpvec: $[t, x[t]]$.
- separation vector between points: svec: s.
- linear momentum: lmom: p.
- kinetic energy: ken vat t: $k[t]$.
- potential energy: pen vat t: $v[t]$.
- Action functional: action: A.
- Lagrange function: lag: L.
- Hamilton function: ham: H.
- Hamilton Kinetic energy: hken: H_{kin}.
- Hamilton potential energy: hpen: H_{pot}.
- Euler-Lagrange Equation: eleqn(q)(i): $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}$.
- contra-variant vector: cnvec pi: p^i.
- (contra-variant) indexed vector: ivec pi: p^i.
- covariant vector (covector): covec pi: p_i.
- basis vector: bvec: γ.
- natural basis vector: nbvec i: γ_i.
- dual basis vector: dbvec i: γ^i.
- generalized position vector: gpvec: q.
- indexed generalized position: gpos i: q^i.
- indexed generalized velocity: gvel i: $\dot{q}^i$.
- indexed generalized momentum: gmom i: p_i.
- indexed generalized force: gfor i: f_i.

5.12. Transport Phenomena.

- thermodynamic temperature: temp: θ.
- substance: subs A: A.
- flux: flux: j.
- mass flux of substance A: mflux A: j_{A}.
- concentration of substance A: conc A: c_{A}.
- bracket concentration of substance A: bconc Aa: $[\mathrm{A}]^a$.
- chemical amount of substance A: amount A: n_{A}.
- reaction rate of substance A: rrate A: r_{A}.
- time derivative of conc.: dtconc A: $\dot{c}_{\mathrm{A}}$.
- time derivative of chem. amount: dtamount A: $\dot{n}_{\mathrm{A}}$.

5.13. Various.

- Iverson brackets: iverson(k=l): $[k = l]_{\mathrm{iv}}$.
- Poisson brackets: poisson(f,g): $[f, g]_{\mathrm{pb}}$.
- matrix representation: mtrx metric: $[g]$.
- Taylor series generated by f at the point a: tseries(f)(x)(a): $T_\infty f[x\,;a]$.
- Taylor polynomial of degree n generated by f at the point a: ntpol(n)(f)(x)(a): $T_n f[x\,;a]$.
- Fourier series generated by f at the point x: fseries fx: $F_\infty f[x]$.
- partial sums of the Fourier series generated by f at the point x: nfsum(f)(n)(x): $F_n f[x]$.
- Legendre transform of a function: ltrans f: $f_\star$.
- conjugate variable (under Legendre transf.): cvar: $x_\star$.
- average quantity: avg a: $\langle a \rangle$.

5.14. Constants.

- Boltzmann constant: boltz: k_{b}.
- speed of ligth in vacuum: ligth: c.
- Avogadro's number: avog: n_{a}.

5.15. Alphabet.

- Latin minuscules:

$$abcdefghijklmnopqrstuvwxyz\,.$$

- Latin majuscules:

$$ABCDEFGHIJKLMNOPQRSTUVWXYZ\,.$$

- Greek:

$$\alpha\beta\gamma\delta\epsilon\varepsilon\zeta\eta\theta\vartheta\iota\kappa\lambda\mu\nu\xi\pi\varpi$$

- Greek:

$$\rho\varrho\sigma\varsigma\tau\upsilon\phi\varphi\chi\psi\omega\Gamma\Delta\Theta\Lambda\Xi\Pi\Sigma\Upsilon\Phi\Psi\Omega$$