LATEX API FOR GEOMETRIC ALGEBRA DOCUMENTS

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Contents

Date: September 18, 2012.
Key words and phrases. engineering notation index notation scalars vectors.

1. Sample Text

Sample text was taken from [3].

1.1. Geometric Product. Let u, v and w be in [1] $\mathfrak{R}^3$. Assume a *geometric product*, denoted by juxtaposition,

$$uv$$

that is *associative*

$$u(vw) = (uv)w = uvw,$$

but *not* necessarily *commutative*

$$uv \neq vu.$$

1.2. Normalized, Anti-symmetric Basis. Expand (or *project*) a vector v onto the *frame* (reference frame or basis) $\{\sigma_k\}$ with *components* v^k to get a *linear combination* of the vector components (onto the frame)

$$v = \sum_{k=1}^{3} \sigma_k v^k,$$

where the superscript k is an *index* rather than an exponent. (Save the subscript for a future index.) Adopt the *Einstein summation convention* and shorten this to

$$v = \sigma_k v^k,$$

with an *implied sum* over repeated lowered and raised (raised and lowered) indices. We agree that Latin indices will always range from 1 to 3.

Assume the *basis vectors*

$$\{\sigma_k\} = \{\sigma_1, \sigma_2, \sigma_3\}$$

[1] $\mathfrak{R}^3$ denotes the 3-D real space.

cliff (multivector)	number	name	grade
1	1	scalar	0
$\{\sigma_k\}$	3	vector	1
$\{\sigma_k\sigma_l\} = \{\mathcal{I}\sigma_m\}$	3	bivector (pseudovector)	2
$\sigma_{123} = \mathcal{I}$	1	trivector (pseudoscalar)	3

TABLE 1. Basis for geometric algebra of three-dimensional space

satisfy the *abstract algebra*

$$\sigma_k\sigma_l + \sigma_l\sigma_k = 2\delta_{kl}.$$

where the *Kronecker delta*

$$\delta_{kl} = [k = l]\,,$$

using *Iverson brackets*, is also the *flat space metric* [2]. In δ_{kl}, k indicates rows and l columns of the matrix representation of the delta; *i.e.*, $[\delta]_.$

Thus, under the geometric product, the basis vectors $\{\sigma_k\}$ are *anti-symmetric and normalized*; *e.g.*, $\sigma_1\sigma_2 = -\sigma_2\sigma_1$ and $\sigma_3\sigma_3 = 1$ and so on.

1.3. **Cliffs – Multivectors.** Unlike traditional vector algebra, geometric algebra combines scalars and vectors to form *cliffs*, *aka*, *multivectors*. This is analogous to combining real and imaginary numbers to form complex numbers. The geometric algebra of space is spanned by the $1 + 3 + 3 + 1 = 8 = 23$ cliffs of Tab. 1.

The most general such cliff (multivector)

$$C = s + v + B + T,$$

or [3]

$$C = C^0 + C^1\sigma_1 + C^2\sigma_2 + C^3\sigma_3 + C^4\sigma_{12} + C^5\sigma_{23} + C^6\sigma_{31} + C^7\sigma_{123}$$

has grade-0 *scalar*, grade-1 *vector*, grade-2 *bivector*, and grade-3 *trivector* parts:

$$\begin{aligned}
s &= \langle C\rangle_0 = C^0, \\
v &= \langle C\rangle_1 = C^1\sigma_1 + C^2\sigma_2 + C^3\sigma_3, \\
B &= \langle C\rangle_2 = C^4\sigma_{12} + C^5\sigma_{23} + C^6\sigma_{31}, \\
T &= \langle C\rangle_3 = C^7\sigma_{123}.
\end{aligned}$$

Attempt to create a grade-4 *quadvector* (or higher grade multivector) by multiplying a trivector by a vector, and it contracts to form a bivector instead [4]. Geometrically interpret scalars as directionless points, vectors as directed lines, bivectors as directed areas, and trivectors as directed volumes.

[2] Metric is generally noted as g. However, in the case of *Euclidean space*, $g = g_{kl} = \delta_{kl}$. Because, Euclidean space is flat.

[3] To find out the order of the indices, remember the *cyclic* rule: $1 \to 2 \to 3 \to 1$. This explains why σ_{31} instead of σ_{13} in $C^6\sigma_{31}$.

[4] For instance, $\sigma_{123}\sigma_3 = (\sigma_{12}\sigma_3)\sigma_3 = \sigma_{12}\sigma_3\sigma_3$, but $\sigma_3\sigma_3 = 1$, so $\sigma_{123}\sigma_3 = \sigma_{12}$; *i.e.*, a bivector.

1.4. **Pseudoscalar.** Because of the anti-symmetry and normalization of the $\{\sigma_k\}$, the trivector $I = \sigma_1\sigma_2\sigma_3$ commutes with all basis vectors [5],

$$\begin{aligned}
\mathcal{I}\sigma_1 &= \sigma_{123}\sigma_1 = \sigma_{1231} = -\sigma_{1213} = \sigma_{1123} = \sigma_1\sigma_{123} = \sigma_1\mathcal{I},\\
\mathcal{I}\sigma_2 &= \sigma_{123}\sigma_2 = \sigma_{1232} = -\sigma_{1223} = \sigma_{2123} = \sigma_2\mathcal{I},\\
\mathcal{I}\sigma_3 &= \sigma_{123}\sigma_3 = \sigma_{1233} = -\sigma_{1323} = \sigma_{3123} = \sigma_3\mathcal{I},
\end{aligned}$$

and hence commutes with all vectors $v = \sigma_k v^k$

$$\mathcal{I}v = v\mathcal{I}.$$

It also squares to negative unity [6]

$$\mathcal{I}^2 = \mathcal{I}\mathcal{I} = \sigma_{123}\sigma_{123} = \sigma_{123123} = -\sigma_{121323} = \sigma_{112323} = \sigma_{2323} = -\sigma_{2233} = -\sigma_{33} = -1.$$

Since the trivector $\mathcal{I}$ acts like an (imaginary) scalar, refer to it as a *pseudoscalar*.

1.5. **Pseudovectors.** The pseudoscalar $\mathcal{I}$ relates vectors and bivectors via the duality transformations

$$\begin{aligned}
\mathcal{I}\sigma_1 &= \sigma_{1231} = -\sigma_{1213} = +\sigma_{1123} = +\sigma_{23} = +\sigma_2\sigma_3,\\
\mathcal{I}\sigma_2 &= \sigma_{1232} = -\sigma_{1223} = +\sigma_{13} = +\sigma_{13} = +\sigma_1\sigma_3,\\
\mathcal{I}\sigma_3 &= \sigma_{1233} = +\sigma_{12} = +\sigma_{12} = +\sigma_1\sigma_2.
\end{aligned}$$

or, by further permutations,

$$\begin{aligned}
\sigma_1\mathcal{I} &= \mathcal{I}\sigma_1 = \sigma_2\sigma_3,\\
\sigma_2\mathcal{I} &= \mathcal{I}\sigma_2 = \sigma_1\sigma_3,\\
\sigma_3\mathcal{I} &= \mathcal{I}\sigma_3 = \sigma_1\sigma_3.
\end{aligned}$$

Since bivectors are *dual* to vectors in this one-to-one (or isomorphic) way, refer to them as *pseudovectors*.

1.6. **Inner and Outer Product Decomposition.** Form the geometric product of two generic vectors

$$uv = \sigma_k u^k \sigma_l v^l = \sigma_k \sigma_l u^k v^l.$$

Explicitly expand the implied sums to find

$$\begin{aligned}
uv = &+ \sigma_1\sigma_1 u^1 v^1 + \sigma_1\sigma_2 u^1 v^2 + \sigma_1\sigma_3 u^1 v^3,\\
&+ \sigma_2\sigma_1 u^2 v^1 + \sigma_2\sigma_2 u^2 v^2 + \sigma_2\sigma_3 u^2 v^3,\\
&+ \sigma_3\sigma_1 u^3 v^1 + \sigma_3\sigma_2 u^3 v^2 + \sigma_3\sigma_3 u^3 v^3.
\end{aligned}$$

Invoke the anti-commutation and normalization of the basis vectors to segregate the symmetric and antisymmetric parts and write

$$\begin{aligned}
uv = &+ \sigma_1\sigma_1 u^1 v^1 + \sigma_2\sigma_2 u^2 v^2 + \sigma_3\sigma_3 u^3 v^3\\
&+ \sigma_2\sigma_3(u^2 v^3 - u^3 v^2) + \sigma_3\sigma_1(u^3 v^1 - u^1 v^3) + \sigma_1\sigma_2(u^1 v^2 - u^2 v^1),
\end{aligned}$$

[5] Joining and splitting indices only works for *orthonormal* basis.
[6] Remember that $\sigma_{112323} = \sigma_{2323}$, because $\sigma_{11} = \sigma_1\sigma_1 = 1$.

which is a scalar [7] plus a bivector [8]

$$
\begin{aligned}
uv &= u \cdot v + u \wedge v, && (1)\\
&= \langle uv \rangle_0 + \langle uv \rangle_2.
\end{aligned}
$$

The symmetric, scalar part is the *inner product*, *aka* dot or scalar product,

$$\langle uv \rangle_0 = u \cdot v = u^k \delta_{kl} v^l = u^k v^l \, [k = l] = u^1 u^1 + u^2 u^2 + u^3 u^3. \qquad (2)$$

and the antisymmetric, bivector part is the *outer product*, *aka* wedge or bivector product

$$
\begin{aligned}
\langle uv \rangle_2 = u \wedge v &= \sigma_2\sigma_3(u^2v^3 - u^3v^2) + \sigma_3\sigma_1(u^3v^1 - u^1v^3) + \sigma_1\sigma_2(u^1v^2 - u^2v^1) && (3)\\
&= \mathcal{I}\sigma_1(u^2v^3 - u^3v^2) + \mathcal{I}\sigma_2(u^3v^1 - u^1v^3) + \mathcal{I}\sigma_3(u^1v^2 - u^2v^1) && (4)\\
&= \mathcal{I} u \times v. && (5)
\end{aligned}
$$

which is the dual of the traditional *cross product*, *aka* vector product

$$u \times v = \sigma_1(u^2v^3 - u^3v^2) + \sigma_2(u^3v^1 - u^1v^3) + \sigma_3(u^1v^2 - u^2v^1).$$

Unlike the cross product, which exists in only 3 dimensions (as only in 3 dimensions is there a unique perpendicular to a plane), the outer product generalizes to any number of dimensions, including the $3 + 1 = 4$ dimensions of spacetime.

Use these identifications to recover the Eq. 1 decomposition of a generic geometric product,

$$
\begin{aligned}
uv &= \sigma_k u^k \sigma_l v^l,\\
&= \sigma_k \sigma_l u^k v^l,\\
&= (\sigma_k \cdot \sigma_l + \sigma_k \wedge \sigma_l) u^k v^l,\\
&= \sigma_k \cdot \sigma_l u^k v^l + \sigma_k \wedge \sigma_l u^k v^l,\\
&= \delta_{kl} u^k v^l + \sigma_k \wedge \sigma_l u^k v^l,\\
&= u \cdot v + u \wedge v.
\end{aligned}
$$

where

$$
\begin{aligned}
u \cdot v &= \delta_{kl} u^k v^l = u^k \delta_{kl} v^l,\\
u \wedge v &= \sigma_k \wedge \sigma_l u^k v^l.
\end{aligned}
$$

1.7. **Symmetries.** By Eq. 2 and Eq. 2, the inner product is symmetric and the outer product is antisymmetric,

$$
\begin{aligned}
u \cdot v &= +v \cdot u,\\
u \wedge v &= -v \wedge u,
\end{aligned}
$$

where the latter implies $v \wedge v = 0$. Use these symmetries to write the fundamental decomposition of Eq. 1

$$uv = u \cdot v + u \wedge v,$$

as

$$vu = v \cdot u + v \wedge u = uv = u \cdot v - u \wedge v$$

[7] By definition, $\sigma_k \sigma_l = [k = l] = \delta_{kl}$.

[8] This is the reason for having to add cliffs (multivectors) to the algebra. Geometric multiplying two vectors result in a *non*-vector, so the algebra does *not* close. To *close* the algebra, we need a higher element that contains vectors (multiplicands) and scalars and bivectors (the results of the geometric product). Then, when product multiplying two vectors, we get a scalar and a bivector, all of them are part of the algebra, and the algebra closes.

case	product
generic	$uv = u \cdot v + u \wedge v$
$u \parallel v$	$uv = +vu = u \cdot v$
$u \perp v$	$uv = -vu = u \wedge v$

TABLE 2. Special cases of geometric product of two vectors

and then add and subtract to solve for

$$\langle uv \rangle_0 = u \cdot v = \frac{uv + vu}{2} = +v \cdot u,$$
$$\langle uv \rangle_2 = u \wedge v = \frac{uv - vu}{2} = -v \wedge u.$$

Beware that the algebraic signs in these equations can alternate as the order of the cliff – multivector – increases.

1.8. **Parallel and Perpendicular.** If vectors u and v are *parallel*, then $u \wedge v = 0$ and $uv = u \cdot v$. If they are *perpendicular*, then $u \cdot v = 0$ and $uv = u \wedge v$. These important special cases are summarized in Tab. 2. where

$$v_{\parallel} = n(n \cdot v)$$

is the *projection* and

$$v_{\perp} = n(n \wedge v)$$

is the *rejection* of v, parallel and perpendicular to n.

NOTE 1. *In Tab. 2, note that when $u \perp v$, then $u \wedge v = uv$. For this reason, when having an* orthogonal *basis, say $\{\sigma_i\}$, one can write $\sigma_i \sigma_j$ instead of $\sigma_i \wedge \sigma_j$ as a shorthand.*

1.9. **Magnitudes.** If v is a vector, then

$$v^2 = vv = v \cdot v + v \wedge v = v \cdot v = |v|^2$$

where the magnitude of v, $|v|$, is its scalar length.

Suppose that the angle between the vectors u and v is θ. Since the inner product is the familiar dot or scalar product, its magnitude

$$|u \cdot v| = |u||v||\cos\theta|$$

is the length of the projection of u on v.

Since the outer product is the dual of the familiar cross or vector product, expect its magnitude

$$|u \wedge v| = |u \times v| = |u||v||\sin\theta|$$

to be the area of the parallelogram framed by u and v.

Consider the trivector

$$T = u \wedge (v \wedge w) = (u \wedge v) \wedge w = u \wedge v \wedge w$$

is the volume of the parallelepiped framed by u on v on w.

1.10. **Geometric Interpretations.** Visualize a scalar s as a point, a vector u as a directed line or arrow, a bivector $B = u \wedge v$ as a directed plane, and a trivector $T = u \wedge v \wedge w$ as a directed solid. Use the outer product to create cliffs – multivectors – of any order.

Think of the outer product $B = u \wedge v$ as the parallelogram formed by sweeping v along u. Think of the outer product $-B = v \wedge u$ as the parallelogram formed by sweeping u along v. Interchanging the vectors reverses the orientation of the area, due to the antisymmetry of the wedge product. Similarly, think of the wedge product $T = u \wedge v \wedge w$ as the parallelepiped formed by sweeping $u \wedge v$ along w. Again, permuting the vectors cycles the orientation.

1.11. **Metric.** Let $g_{kl} = \delta_{kl}$ denote the flat space metric. Given the frame $\{\sigma_k\}$, consider a second *reciprocal frame* $\left\{\sigma^k\right\}$, where

$$\sigma_k = g_{kl}\sigma^l = g_{km}\sigma^m$$

and

$$\sigma^k = g^{kl}\sigma_l = g^{km}\sigma_m.$$

Consistency demands

$$g_{kl}g^{lm} = \delta^k{}_m,$$

so that the metric g_{kl} is its own inverse g^{kl}. Therefore,

$$\sigma^k \cdot \sigma_l = g^{km}\sigma_m \cdot \sigma_l = g^{km}g_{ml} = \delta^k{}_l.$$

and the reciprocal frames are mutually orthonormal.

Expand a vector in both frames,

$$v = v^k\sigma_k = v_l\sigma^l = v_l g^{lk}\sigma_k,$$

where

$$v_k = g^{kl}v_l$$

and similarly

$$v^k = g_{kl}v_l.$$

The basis vectors σ_k and σ^k are the same in flat space. However, they differ in curved spaces, where these raising and lowering relations are more than just index gymnastics.

1.12. **Rotations.** To rotate a vector $v = \sigma_1$ through an angle θ in the plane $B = \sigma_1 \wedge \sigma_2 = \sigma_1\sigma_2$, multiply by the *rotor*

$$R = e^{-\sigma_1\sigma_2\theta/2} = e^{-B\theta/2}$$

on the right and by its *reversion*

$$R^\dagger = e^{-\sigma_2\sigma_1\theta/2} = e^{+B\theta/2}$$

on the left, with the exponentials defined by their infinite series expansions like

$$R^\dagger = e^{+B\theta/2} = \cos\frac{\theta}{2} + \sigma_1\sigma_2 \sin\frac{\theta}{2},$$

as $i^2 = (\sigma_1\sigma_2)^2 = -1$. Hence, the rotated vector is

$$\begin{aligned} v' = RvR^\dagger &= e^{-B\theta/2}ve^{+B\theta/2} \\ &= \left(\cos\frac{\theta}{2} - \sigma_1\sigma_2\sin\frac{\theta}{2}\right)\sigma_1\left(\cos\frac{\theta}{2} + \sigma_1\sigma_2\sin\frac{\theta}{2}\right) \\ &= \sigma_1\cos\theta + \sigma_2\sin\theta. \end{aligned}$$

Such rotation in a plane generalizes to any dimension, whereas the traditional notion of rotation about a line is confined to three dimensions. Rotate any cliff in the same way.

1.13. **Geometric Calculus of Space.**

1.13.1. *Geometric Derivative.* Use the flat space metric $g_{kl} = \delta_{kl}$ to define equivalent basis vectors $\sigma_l = g_{lk}\sigma^k$. With orthonormal frame $\{\sigma^k\}$ and position $x = \sigma_k x^k$, expand the *geometric derivative*, *aka vector derivative*, ∇ in components $\nabla_k = \partial_k$ to get

$$\nabla = \sum_{i=1}^{3} \sigma^k \frac{\partial}{\partial x^k} = \sum_i \sigma^k \frac{\partial}{\partial x^k} \; [i \leq 3] = \sigma^k \partial_k.$$

(Although ∇ is sometimes referred to as the "nabla" operator, pronounce the pseudo-letters ∇ ∇ and ∂ "del" in analogy with pronouncing the Greek letters Δ and δ "delta".) Expand ∇ in alternate notations as

$$\begin{aligned}\nabla &= \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3, \\ &= \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z}.\end{aligned}$$

Apply ∇ to a generic vector field $v(x)$ to get

$$\nabla v = (\sigma^k\partial_k)(\sigma_l v^l),$$

and use $\sigma^k = \sigma_k$, and so on, to expand this to

$$\begin{aligned}\nabla v = &+ \sigma_1\sigma_1\partial_1 v^1 + \sigma_1\sigma_2\partial_1 v^2 + \sigma_1\sigma_3\partial_1 v^3 \\ &+ \sigma_2\sigma_1\partial_2 v^1 + \sigma_2\sigma_2\partial_2 v^2 + \sigma_2\sigma_3\partial_2 v^3 \\ &+ \sigma_3\sigma_1\partial_3 v^1 + \sigma_3\sigma_2\partial_3 v^2 + \sigma_3\sigma_3\partial_3 v^3.\end{aligned}$$

By invoking the anti-commutation and normalization of the basis vectors, segregate the symmetric and antisymmetric parts, to find

$$\begin{aligned}\nabla v = &+ \partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3 \\ &+ \sigma_2\sigma_3(\partial_2 v^3 - \partial_3 v^2) + \sigma_3\sigma_1(\partial_3 v^1 - \partial_1 v^3) + \sigma_1\sigma_2(\partial_1 v^2 - \partial_2 v^1)\end{aligned}$$

or

$$\nabla v = \nabla \cdot v + \nabla \wedge v$$

where the symmetric, scalar part is the *interior derivative* or *divergence*

$$\nabla \cdot v = \partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3,$$

and the antisymmetric, bivector part is the *exterior derivative*

$$\begin{aligned}\nabla \wedge v &= \sigma_2\sigma_3(\partial_2 v^3 - \partial_3 v^2) + \sigma_3\sigma_1(\partial_3 v^1 - \partial_1 v^3) + \sigma_1\sigma_2(\partial_1 v^2 - \partial_2 v^1), \\ &= \mathcal{I}\sigma_1(\partial_2 v^3 - \partial_3 v^2) + \sigma_2(\partial_3 v^1 - \partial_1 v^3) + \sigma_3(\partial_1 v^2 - \partial_2 v^1), \\ &= \mathcal{I}\nabla \times v,\end{aligned}$$

which is the dual of the traditional *curl*

$$\nabla \times v = \sigma_1(\partial_2 v^3 - \partial_3 v^2) + \sigma_2(\partial_3 v^1 - \partial_1 v^3) + \sigma_3(\partial_1 v^2 - \partial_2 v^1)$$

2. More Sample Text

2.1. Kronecker delta. Kronecker delta is defined as

$$\delta_{kl} \equiv [k = l]\,,$$

using *Iverson bracket*.

Example 2.1. Express the Kronecker delta in 2-D and 3-D.

Solution. In 2-D,

$$\delta = [\delta] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \operatorname{diag}(1,1).$$

In 3-D,

$$\delta = [\delta] = \operatorname{diag}(1,1,1).$$

□

2.2. Inner Product in Euclidean Space. Let $u, v \in \mathfrak{E}^3$. Dot product is defined as

$$u \cdot v = u^k v^k.$$

The dot product thus defined hides physics and doesn't work well with Einstein summation convention.

To redefine the dot product, project u and v in the frame $\{\sigma_k\}$, then calculate the dot product, but as a bilinear operation

$$u \cdot v \equiv \sigma_k u^k \cdot \sigma_l v^l = u^k (\sigma_k \cdot \sigma_l) v^l,$$

now, remembering that $\sigma_k \cdot \sigma_l = g_{kl}$ and that [9] $u^k g_{kl} = v_l$,

$$u \cdot v = u^k g_{kl} v^l = u^k v_k = u_l v^l.$$

Finally, for Euclidean flat space, if $u, v \in \mathfrak{E}^n$, then $g_{kl} = \delta_{kl}$, so $u^k \cdot v^l$ could be written as

$$u \cdot v = u^k \delta_{kl} v^l = u^k v_k = u_l v^l.$$

2.3. Metric, Matrices, Transformations. This section was taken from [2, p. 9].

The Lagrangian for the problem is

$$\mathcal{L} = T - U = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - U(r)\,. \tag{6}$$

We can move to spherical coordinates using the definition of $\{x, y, z\}$ in terms of $\{r, \theta, \phi\}$:

$$x = r \sin\theta \cos\phi, \tag{7}$$

$$y = r \sin\theta \sin\phi, \tag{8}$$

$$z = r \cos\theta. \tag{9}$$

Finding the velocities, $\dot{x}$, $\dot{y}$ and $\dot{z}$ and replacing these values in Eq. 6, then the Lagrangian would be

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2 \right) - U(r)\,.$$

And so, innocuously, begins our journey. Let's rewrite Eq. 6 in matrix-vector notation (we'll take potentials that are arbitrary functions of all three coordinates), the kinetic term is the beneficiary here

$$\mathcal{L} = \frac{1}{2} m \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} - U(x, y, z)\,. \tag{10}$$

---

[9] That is, g_{kl} lowers and exchanges indices.

We can make the move to spherical coordinates just by changing our curve coordinates from $\{x, y, z\}$ to $\{r, \theta, \phi\}$ and modifying the matrix:

$$\mathcal{L} = \frac{1}{2}m \begin{bmatrix}\dot{r} & \dot{\theta} & \dot{\phi}\end{bmatrix} \begin{bmatrix}1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta\end{bmatrix} \begin{bmatrix}\dot{r} \\ \dot{\theta} \\ \dot{\phi}\end{bmatrix} - U(r, \theta, \phi)\,. \tag{11}$$

But while it takes up a lot more space on the page, this is not so trivial a statement. Think of it this way – consider two points infinitesimally close together in the two coordinate systems, the infinitesimal distance (squared) between the two points can be written in Cartesian or spherical coordinates:

$$ds^2 = dx^2 + dy^2 + dz^2,$$
$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

These are just statements of the Pythagorean theorem in two different coordinate systems. The *distance* between the two points is the same in both, that can't change, but the representation is different [10].

These distances can also be expressed in matrix-vector form [11]

$$ds^2 = \begin{bmatrix}dx & dy & dz\end{bmatrix} \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}dx \\ dy \\ dz\end{bmatrix} \tag{12}$$

and [12]

$$ds^2 = \begin{bmatrix}dr & d\theta & d\phi\end{bmatrix} \begin{bmatrix}1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta\end{bmatrix} \begin{bmatrix}dr & d\theta & d\phi\end{bmatrix}. \tag{13}$$

The matrix form for infinitesimal length should look familiar (compare with Eq. 10 and Eq. 11), and this isn't so surprising – velocity is intimately related to infinitesimal displacements [13]. If we ask for the distance traveled along a curve parametrized by t in an infinitesimal interval dt, the answer is provided by

$$ds^2 = \left(\frac{dx}{dt}dt\right)^2 + \left(\frac{dy}{dt}dt\right)^2 + \left(\frac{dz}{dt}dt\right)^2 = (dt)^2\,(v \cdot v).$$

We call the matrix in Eq. 12 and Eq. 13 the *metric*. It is often represented not as a matrix but as a *second-rank tensor* and denoted [14] $g_{\mu\nu}$. It tells us, given a coordinate system, how to measure distances [15]. In classical mechanics, we usually go the other way around, as we have done here – we figure out how to measure distances in the new coordinates and use that to find $g_{\mu\nu}$ (actually, we rarely bother with the formal name or matrix, just transform kinetic energies and evaluate the equations of motion).

---

[10] That is, *distance* is an *invariant*, while the values of the components of the vector in various coordinate systems are *not* invariant.

[11] In Cartesian coordinates, x, y and z have dimension of length, [L], so they all measure *lengths*.

[12] In spherical coordinates, only r measures length, whereas θ and ϕ measure angles!

[13] Displacement and velocity share the same metric; *i.e.*, when space coordinates change from one system to another, the components of displacement change and so do the components of velocity, in the exact same way! It's not surprising because of their definitions: if a displacement in space is x, its velocity is $v = dx/dt$.

[14] Greek indices, like μ and ν, run from 1 to 3, in the reference text. In other text books, they run from 0 to 3.

[15] Knowing how to measure distances is very useful in physics. With them, we can calculate displacements, velocities, accelerations, forces, and so on.

Label the vectors appearing in Eq. 12 and Eq. 13 dx^μ, so that the three components associated with $\mu = 1, 2, 3$ correspond to the three components of the vector

$$dx^\mu = \begin{bmatrix} dx & dy & dz \end{bmatrix},$$

for example. Then we can define the *Einstein summation notation* to express lengths. Referring to Eq. 12 for the actual matrix-vector form, we can write [16]

$$ds^2 = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} dx^\mu g_{\mu\nu}\, dx^\nu.$$

The idea behind the notation is that when you have an index appearing twice, as in the top line, the explicit $\sum$ is redundant. The prescription is

> take each repeated index and sum it over the dimension of the space. Rename $x = x^1$, $y = x^2$, $z = x^3$, then
>
> $$dx^\mu\, g_{\mu\nu}\, dx^\nu = dx^1\, g_{11}\, dx^1 + dx^2\, g_{22}\, dx^2 + dx^3\, g_{33}\, dx^3 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

In general, there would be more terms in the sum, the diagonal form of $g_{\mu\nu}$ simplified life *[sic]* (here). The same equation, $ds^2 = dx^\mu\, g_{\mu\nu}\, dx^\nu$, holds if we take $g_{\mu\nu}$ to be the matrix defined in Eq. 13 and $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

In Einstein summation notation, we sum over repeated indices where one is up, one is down (objects like $g_{\mu\nu}\, dx_\mu$ are nonsense and will never appear). The repeated index, because it takes on all values $1 \to D$ (in this case, $D = 3$ dimensions) has no role in labeling a component, and so can be renamed as we wish, leading to statements like [17] (we reintroduce the summation symbols to make the point clear):

$$ds^2 = dx^\mu\, g_{\mu\nu}\, dx^\nu = ds^2 = dx^\alpha\, g_{\alpha\beta}\, dx^\beta.$$

Finally, the explicit form of the metric can be recovered from the *squared line element* (just ds^2 written out). If we are given the line element

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

then, we know that the metric, in matrix form, is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

with coordinate differential

$$dx^\alpha = \begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix}.$$

If, instead, we are given

$$ds^2 = d\rho^2 + \rho^2\, d\phi^2 + dz^2,$$

then we read off the metric

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

---

[16] In summation convention, a repeated index can only appear *twice*, that's why we use μ for the first vector and ν for the second one; *i.e.*, we write $dx^\mu g_{\mu\nu}\, dx^\nu$, instead of $dx^\mu g_{\mu\mu}\, dx^\mu$, even if the vectors at the two sides of the metric are the same vector! – see Summation Convention Rule No. 3 in [4, p. 3]: "It is illegal to use the same dummy index more than twice in a term unless its meaning is made explicit".

[17] There are a few general properties of the metric that we can assume for all metrics considered here: 1) The metric is symmetric, this is a convenient notational device, we have no reason to expect $dxdy \neq dydx$ in a line element. 2) It does not have to be diagonal. 3) It can depend (as with the spherical metric) on position.

and coordinate differential

$$dx^\alpha = \begin{bmatrix} d\rho \\ d\phi \\ dz \end{bmatrix}.$$

2.4. **Metric, Space Metric and Spacetime Metric.**

Definition 2.1 (Metric): *A metric on a set $\mathfrak{X}$ is a function (called the* distance function *or simply* distance*) $d\colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{R}$. For all $x, y, z \in \mathfrak{X}$, this function is required to satisfy the following axioms:*

(1) *non-negativity, or separation: $d(x, y) \geq 0$;*
(2) *identity of indiscernibles, or coincidence: $d(x, y) = 0$, if and only if $x = y$;*
(3) *symmetry: $d(x, y) = d(y, x)$;*
(4) *subadditivity / triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.*

Metric tensor components: given two vectors u and v in $\mathfrak{V}$, the metric tensor, denoted as g, is defined as

$$g(u, v) \equiv u \cdot v.$$

Thus, the metric tensor is a function that takes two vectors are returns their inner product, a scalar; *i.e.*, $g\colon \mathfrak{V} \to \mathfrak{R}$.

Space (Euclidean) metric: to find the components of the metric in Euclidean space, $\mathfrak{E}^n$, calculate the value of the metric with the basis elements as arguments; *i.e.*,

$$g(u, v) = \sigma_i \cdot \sigma_j = \delta_{ij} = \mathrm{diag}\left(\overbrace{1, \ldots, 1}^{n \text{ times}}\right).$$

Then, the components of the metric in $\mathfrak{E}^n$ are equal to the Kronecker delta.

Spacetime (Minkowskean) metric: in spacetime, Minkowski space, $\mathfrak{S}^4$, the basis elements are

$$\sigma_0 = -1; \qquad \sigma_1 = \sigma_2 = \sigma_3 = 1.$$

The components of the metric can therefore be calculated applying its definition to the basis elements:

$$g(\sigma_\mu, \sigma_\nu) = \sigma_\mu \cdot \sigma_\nu = \eta_{\mu\nu},$$

where $\eta_{\mu\nu}$ is thus defined as

$$\eta \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{diag}\,(-1, 1, 1, 1).$$

The diag (η) can be summarized by its signature: $\mathrm{sign}\,\eta = (- + + +)$.

Notice the signature of $\mathfrak{E}^4$ and $\mathfrak{S}^4$. For 4-D space, we have

$$\mathrm{sign}\,\mathfrak{E}^4 = (+ + + +)\,,$$

whereas for spacetime,

$$\mathrm{sign}\,\mathfrak{S}^4 = (- + + +)\,.$$

Because of the "-" in σ_0 of $\mathrm{sign}\,\mathfrak{S}^4$, the spacetime metric is called *pseudo-metric*. This is because it violates the second axiom in the definition of metric, Def. 2.1 – identity of indiscernibles; *i.e.*, given two points $\mathfrak{P}$ and $\mathfrak{Q}$: in Euclidean space, if the distance between $\mathfrak{P}$ and $\mathfrak{Q}$ is zero, $\mathfrak{P} = \mathfrak{Q}$ ($\mathfrak{P}$ and $\mathfrak{Q}$ are the same). In spacetime, if the distance between $\mathfrak{P}$ and $\mathfrak{Q}$ is zero, $\mathfrak{P}$ and $\mathfrak{Q}$ are not necessarily the same.

2.5. **Euclidean Space.** Taken from [1].

2.5.1. *Definition.* Euclidean space in linear. Thus it could be defined as a linear combination of orthogonal (mutually perpendicular) basis vectors. In this way, any point, say $\mathfrak{P}$, could be identified by

$$p = p^i \sigma_i,$$

where p is the vector representation of $\mathfrak{P}$, σ_i are scalar multipliers (one for each dimension) and $\{\sigma_i\}$ are the (orthogonal) basis vectors.

We have two types of mathematical objects: scalars and vectors. The properties of such a space can be defined in terms of the axioms of linear space.

2.5.2. *Quadratic Form.* Euclidean space is quadratic, how can space be both linear and quadratic? We have already seen how vectors and scalar multiplication are linear, some aspects of Euclidean space are quadratic.

For instance the way that we measure distance (the metric) of a space. If we have a two dimensional Euclidean space, where a given point is represented by the vector: $v = (x, y)$ then the distance from the origin is given by the square root of $x^2 + y^2$. Other physical quantities such as the inertia tensor are also related to the square of the distance to a given point.

2.5.3. *Properties of Euclidean Space.* Euclidean space has the following properties:

- There is no preferred origin in euclidean space. Any point would be as good as any other as a choice for the origin;
- There is no preferred direction in Euclidean space;
- There is no specific way to define a point at infinitya
- The 'metric' for Euclidean space: there is a function, for a given space, that defines the distance, d, between points. For Euclidean space, if p and q are two vectors representing, respectively, point $\mathfrak{P}$ and point $\mathfrak{Q}$ then $d(\mathfrak{P}, \mathfrak{Q}) = |p - q|^2 = (p - q) \cdot (p - q)$;
- Euclidean space is flat;
- Euclidean space is linear;
- Euclidean space is continuous (differeniatable);
- Euclidean n-space, $\mathfrak{E}^n$, is the most elementary example of an n dimensional manifold.

2.5.4. *Rotations in Euclidean Space.* The Cartesian coordinate system allows us to specify directions, but what about direction of rotation? Which direction of rotation do we consider positive? This is an arbitrary decision in that it does not matter as long as we are consistent so, on this website, I have chosen to use the right hand rule. This is because that is the convention used by the VRML/x3d standards.

A rotation can be specified by a vector.

If the thumb of the right hand is pointed in the direction of vector, the positive direction of rotation is given by the curl of the fingers.

Rotations can be specified in many ways, we could use axis and angle in which case the positive angle direction is as described here. Another alternative is to use Euler Angles where we will use the right hand rule for the positive angle about each base positive coordinate direction.

2.5.5. *Linear Euclidean Space.* Euclidean space is linear. This linearity could be expressed as

- all points on a Cartesian coordinate system;
- linear combination of orthogonal (mutually perpendicular) basis vectors. So any point, say $\mathfrak{P}$, could be identified by

  $$p = \sigma_i p^i,$$

  where p represents $\mathfrak{P}$, σ_i orthogonal basis vectors (one for each direction) and p^i scalar multipliers (one for each basis vector).

3. Quadratic Forms

Definition 3.1 (Symmetric Matrix): *Let A be a $m \times n$ matrix. A is said to be* symmetric *if is a* square matrix, *$m = n$, and if it is equal to its transpose, $A = A^T$.*

Definition 3.2 (Quadratic Forms): *Let A denote a $n \times n$* symmetric *matrix with real entries and let x denote a $n \times 1$ (column) vector. Then a function $Q \colon \mathfrak{R}^n \to \mathfrak{R}$ is said to be a* quadratic form *if it can be written as*

$$Q(x) \equiv x^T A x.$$

Note that

$$\begin{aligned} Q(x) = x^T A x &= \begin{bmatrix} x^1 & \cdots & x^n \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \\ &= x^i a_{ij} x^j. \end{aligned}$$

Example 3.1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

and a vector x. Calculate the quadratic form.

Solution. See that A is square and symmetric. Thus

$$\begin{aligned} Q(x) &= \begin{bmatrix} x^1 & x^2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \\ &= x^1 x^1 + 2x^1 x^2 + 2x^1 x^2 + x^2 x^2, \\ &= (x^1)^2 + 4x^1 x^2 + (x^2)^2. \end{aligned}$$

□

Definition 3.3 (Bilinear Form): *Let $\mathfrak{V}$ be a vector space and let u and v be in $\mathfrak{V}$. A* bilinear form *is a function $B \colon \mathfrak{V} \times \mathfrak{V} \to \mathfrak{R}$ which is linear in each argument separately;* i.e.,

(1) $B(u + u', v) = B(u, v) + B(u', v)$;
(2) $B(u, v + v') = B(u, v) + B(u, v')$;
(3) $B(\alpha u, v) = B(u, \alpha v) = \alpha B(u, v)$.

Any bilinear form on $\mathfrak{R}^n$ with basis $\mathfrak{B}$ can be expressed as

$$B(x, y) = [x]_{\mathfrak{B}}^T A \, [y]_{\mathfrak{B}} = x^i a_{ij} x^j,$$

where A is a $n \times n$ matrix and $[x]_{\mathfrak{B}}$ denotes the matrix representation of x with respect to the basis $\mathfrak{B}$.

3.1. Coordinate representation.

Let $\mathfrak{C} = \{\sigma_1, \ldots, \sigma_n\}$ be a basis for a finite-dimensional space $\mathfrak{V}$. Define the matrix A by $(A_{ij}) = B(\sigma_i, \sigma_j)$. Then if the $n \times 1$ matrix $[v]_{\mathfrak{C}}$ represents a vector v with respect to this basis, and analogously, $[w]_{\mathfrak{C}}$ represents w, then

$$B(v, w) = [v]_{\mathfrak{C}}^T A \, [w]_{\mathfrak{C}}.$$

Example 3.2. Calculate the quadratic form for the metric of a vector v in $\mathfrak{S}^4$.

Solution. Let $\mathfrak{B} = \{\sigma_i\}$ represent the $\mathfrak{S}^4$ basis in Cartesian coordinates. The representation of the spacetime metric on $\mathfrak{B}$ is then

$$[\eta] = \eta_{ij} = \sigma_i \cdot \sigma_j = \operatorname{diag}(-1, 1, 1, 1).$$

Since v belongs in $\mathfrak{S}^4$, its representation in $\mathfrak{B}$ is $[v]_{\mathfrak{B}}$. The quadratic form can be finally computed as

$$\begin{aligned}
[v]_{\mathfrak{B}}^T [\eta]_{\mathfrak{B}} [v]_{\mathfrak{B}} &= v^i \eta_{ij} v^j, \\
&= \eta_{ij} v^i v^j, \\
&= -v^0 v^0 + v^1 v^1 + v^2 v^2 + v^3 v^3.
\end{aligned}$$

□

4. Fields

4.1. **Vector Fields.** A vector field has a vector value for every point in a space: $v\colon \Re^3 \to \Re^3$. For instance a force field like gravity, if we have a very large mass at the origin, then we can plot the gravitational field, as in the diagram above. Its difficult to find a suitable way to show this, especially in 3 dimensions, an alternative might be to plot lines of constant field force; *i.e.*, $F\colon \Re^3 \to \Re^3$, such that $x \mapsto GmMx/x \cdot x$:

$$F(x) = GmM\frac{x}{x \cdot x}.$$

where x is the position vector.

4.2. **Scalar Fields.** A scalar field gives a scalar value for every point in the space: $\phi\colon \Re^3 \to \Re$. We could represent how some scalar quantity, for example temperature, is defined for each point in the space. We could define a function to show this, $T\colon \Re^3 \to \Re$ such that $(x, y, z) \mapsto x + y^2 + z$, for instance.

One way to illustrate this is to join up all the points with the same values such as contour lines, or isothermals, or isobars, or whatever appropriate to the scalar quantity.

4.3. **Relationship between Scalar and Vector Field.** We can relate this to the vector field example above by choosing the scalar to represent potential, in this case energy potential. In this case the scalar and vector fields are related by

$$W = \int F \cdot\, dx.$$

where W represents work, F a force and dx a "infinitesimal" displacement. (The integral is a line integral.)

Or, we could use its inverse:

$$F = \operatorname{grad} W.$$

4.4. **Generalization.** We can generalize this to other cases as follows:

> grad converts a vector field to a scalar field representing the differential "slope" of the vector field.

or its inverse:

> the line integral converts a scalar field to a vector field.

4.5. **Covector Field.** Covectors are dual objects to vectors. In the section about tensors we thought about them being the rows in matrix made up of columns representing the basis vectors.

In terms of fields, if,

- v is a vector field;
- c is the corresponding covector field;
- s is a scalar field;

then

$$v \cdot c = s;$$

that is, the dot product of the vector field and the covector field is a scalar field. 1-form is a covector field.

4.6. **Tangent Field.** The limiting space when small neighbourhoods of the manifold are taken resulting in a flat n-dimensional space.

4.6.1. *Divergence.* We get a scalar field from a vector field by using the divergence.

5. Vectors and Covectors

5.1. **Vectors.** A vector space is a mathematical structure formed by a collection of elements called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars in this context. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field. The operations of vector addition and scalar multiplication must satisfy certain requirements, called axioms. An example of a vector space is that of Euclidean vectors, which may be used to represent physical quantities such as forces: any two forces (of the same type) can be added to yield a third, and the multiplication of a force vector by a real multiplier is another force vector. In the same vein, but in a more geometric sense, vectors representing displacements in the plane or in three-dimensional space also form vector spaces.

Vector spaces are the subject of linear algebra and are well understood from this point of view, since vector spaces are characterized by their dimension, which, roughly speaking, specifies the number of independent directions in the space. A vector space may be endowed with additional structure, such as a norm or inner product. Such spaces arise naturally in mathematical analysis, mainly in the guise of infinite-dimensional function spaces whose vectors are functions.

Definition 5.1 (Vector space): *A vector space over a field $\mathfrak{F}$ is a set $\mathfrak{V}$ together with two binary operations that satisfy the eight axioms listed below. Elements of $\mathfrak{V}$ are called* vectors. *Elements of $\mathfrak{F}$ are called* scalars. *The first operation,* vector addition, *takes any two vectors v and w and assigns to them a third vector which is commonly written as $v + w$, and called the sum of these two vectors. The second operation takes any scalar a and any vector v and gives another vector av. Multiplication is done by rescaling the vector v by a scalar a, the multiplication is called* scalar multiplication of v by a.

To qualify as a vector space, $\mathfrak{V}$ and the operations of addition and multiplication must adhere to a number of requirements called axioms. In the list below, let u, v and w be arbitrary vectors in $\mathfrak{V}$, and a and b scalars in $\mathfrak{F}$.

Axiom 5.1 (Associativity of addition): $u + (v + w) = (u + v) + w$

Axiom 5.2 (Commutativity of addition): $u + v = v + u$

Axiom 5.3 (Identity element of addition): *There exists an element $0 \in \mathfrak{V}$, called* the zero vector, *such that $v + 0 = v$ for all $v \in \mathfrak{V}$.*

Axiom 5.4 (Inverse elements of addition): *For every $v \in \mathfrak{V}$, there exists an element $-v \in \mathfrak{V}$, called the additive inverse of v, such that $v + (-v) = 0$*

Axiom 5.5 (Distributivity of scalar multiplication with respect to vector addition): $a(u + v) = au + av$

Axiom 5.6 (Distributivity of scalar multiplication with respect to field addition): $(a+b)v = av+bv$

Axiom 5.7 (Compatibility of scalar multiplication with field multiplication): $a(bv) = (ab)v$

Axiom 5.8 (Identity element of scalar multiplication): $1v = v$, *where 1 denotes the multiplicative identity in $\mathfrak{F}$.*

When the scalar $\mathfrak{F}$ is the $\mathfrak{R}$, the vector space is called a *real vector space*. When the scalar field is the complex numbers, it is called a complex vector space. The most general definition of a vector space allows scalars to be elements of any fixed field $\mathfrak{F}$. The notion is then known as a *vector space over* $\mathfrak{F}$. A field is, essentially, a set of numbers possessing addition, subtraction, multiplication and division operations. For example, rational numbers also form a field.

In contrast to the intuition stemming from vectors in the plane and higher-dimensional cases, there is, in general vector spaces, no notion of nearness, angles or distances. To deal with such matters, particular types of vector spaces are introduced.

The requirement that vector addition and scalar multiplication be binary operations includes (by definition of binary operations) a property called closure: that $u+v$ and av are in $\mathfrak{V}$ for all a in $\mathfrak{F}$, and u, v in $\mathfrak{V}$. Some older sources mention these properties as separate axioms.

NOTE 2. *Briefly, a linear space is a set of elements of any kind on which certain operations can be performed (called addition and multiplication by scalars). In defining a linear space, we don't specify the nature of the elements nor do we tell how the operations are to be performed. Instead, we require that the operations have certain properties which we take as axioms for a linear space.*

5.2. **Coordinate spaces.** The first example of a vector space over a field $\mathfrak{F}$ is the field itself, equipped with its standard addition and multiplication. This is the case $n = 1$ of a vector space usually denoted $\mathfrak{F}^n$, known as the *coordinate space* whose elements are n-tuples (sequences of length n):

$$\left(a^1, a^2, \ldots, a^n\right),$$

where the a^i are elements of $\mathfrak{F}$.

5.3. **Coordinate Space - Formal Definition.** Let $\mathfrak{F}$ denote an arbitrary field (such as the $\mathfrak{R}$). For any positive integer n, the space of all n-tuples of elements of $\mathfrak{F}$ forms an n-dimensional vector space over $\mathfrak{F}$ called *coordinate space* and denoted [18] $\mathfrak{F}^n$. An element [19] of $\mathfrak{F}^n$ is written

$$x = \left(x^1, x^2, \ldots, x^n\right)$$

where each x^i is an element of $\mathfrak{F}$. The binary operations, required to satisfy the vector space axioms, on $\mathfrak{F}^n$ are defined by

$$x + y = \left(x^1 + y^1, x^2 + y^2, \ldots, x^n + y^n\right)$$

and

$$\alpha x = \left(\alpha x^1, \alpha x^2, \ldots, \alpha x^n\right).$$

The *zero vector* is given by

$$0 = (0, 0, \ldots, 0)$$

and the additive inverse of the vector x is given by

$$-x = \left(-x^1, -x^2, \ldots, -x^n\right).$$

NOTE 3. *With these definitions, coordinate spaces satisfy the axioms of vector spaces. That is, coordinate spaces were designed to be vector spaces!*

The coordinate space $\mathfrak{F}^n$ comes with a standard basis:

$$\begin{aligned} \sigma_1 &= (1, 0, \ldots, 0), \\ \sigma_2 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ \sigma_n &= (0, 0, \ldots, 0, 1), \end{aligned}$$

[18] It is denoted $\mathfrak{F}^n$ instead of $\mathfrak{V}$, because the vector elements are formed from the elements of $\mathfrak{F}$! So coordinate spaces map scalars to vectors, real numbers in 1-D to vectors in n-D.

[19] A coordinate space takes the field elements, scalars, and uses them to form the vector space elements, vectors!

where 1 denotes the multiplicative identity in $\mathfrak{F}$. To see that this is a basis, note that an arbitrary vector in $\mathfrak{F}^n$ can be written uniquely in the form

$$v = x^i \sigma_i$$

where we have used Einstein summation convention.

When the field $\mathfrak{F}$ is the $\mathfrak{R}$, we have that $\mathfrak{R}^n$.

5.4. **Bases and Dimensions.** *Bases* allow the introduction of coordinates on a vector space. A basis is a (finite or infinite) set $\mathfrak{B} = \{\sigma_i\}_{i \in \mathfrak{I}}$ of vectors σ_i, indexed by some index set $\mathfrak{I}$, that *spans* the whole space and is *linearly independent*. "Spanning the whole space" means that any vector v can be expressed as a finite sum (called a linear combination) of the basis elements [20]

$$v = \sigma_k a^k$$

where the a^k are scalars and σ_k $(k = 1, \ldots, n)$ elements of the basis $\mathfrak{B}$.

Linear independence means that the scalars a^k, which define the coordinates of the vector v with respect to the $\mathfrak{B}$, are uniquely determined.

For example, the coordinate vectors $\sigma_1 = (1, 0, \ldots, 0)$, $\sigma_2 = (0, 1, 0, \ldots, 0)$, to $\sigma_n = (0, 0, \ldots, 0, 1)$, form a basis of $\mathfrak{F}^n$, called *the standard basis*, since any vector $\left(x^1, x^2, \ldots, x^n\right)$ can be uniquely expressed as a linear combination of these vectors

$$\left(x^1, x^2, \ldots, x^n\right) = x^1 (1, 0, \ldots, 0) + x^2 (0, 1, 0, \ldots, 0) + \cdots + x^n (0, 0, \ldots, 0, 1) = x^i \sigma_i.$$

The corresponding coordinates $x^1, x^2, \ldots, x^n$ are just the *Cartesian coordinates* of the vector.

Every vector space has a basis.

5.5. **Linear Maps.** The relation of two vector spaces can be expressed by *linear map* or *linear transformation*. They are functions that reflect the vector space structure; *i.e.*, they preserve sums and scalar multiplication:

$$f(x + y) = f(x) + f(y)$$

and

$$f(ax) = af(x)$$

for all x and y in $\mathfrak{V}$, all a in $\mathfrak{F}$.

An *isomorphism* is a linear map $f : \mathfrak{V} \to \mathfrak{W}$, such that there exists an inverse map $g : \mathfrak{W} \to \mathfrak{V}$, which is a map such that the two possible compositions $f \circ g : \mathfrak{W} \to \mathfrak{W}$ and $f \circ g : \mathfrak{V} \to \mathfrak{V}$ are identity maps. Equivalently, f is both one-to-one (injective) and onto (surjective). If there exists an isomorphism between $\mathfrak{V}$ and $\mathfrak{W}$, the two spaces are said to be *isomorphic*; they are then essentially identical as vector spaces, since all identities holding in $\mathfrak{V}$ are, via f, transported to similar ones in $\mathfrak{W}$, and vice versa via g.

5.6. **Covectors.** Vectors in coordinate spaces take elements with a field structure and create an array (tuple) of these field elements which produces a vector space structure. This is described as a vector *over* a field.

A covector is the dual of this (see Tab. 3); *i.e.*,

- vector: field produces a vector space;
- covector: vector space produces a field.

that is: if we think of a vector as a mapping from a field to a vector space then a covector represents a mapping from a vector space to a field. So how can we construct a covector?

[20] Einstein summation convention in force!

	name	operations	operands
combined structure	vector space	add and scalar mult	scalar and vector
element structure	field	add,subtract, mult and divide	ints, reals, *etc.*.

TABLE 3. Relationships between vector spaces and fields

5.6.1. *Linear Functional.* A linear function like

$$f(x, y, z) = 3x + 4y + 2z$$

has similarities to vectors, for instance we can add them; *i.e.*, if

$$f_1(x, y, z) = 3x + 4y + 2z$$

and

$$f_2(x, y, z) = 6x + 5y + 3z$$

then we can add them by adding corresponding terms

$$f_1(x, y, z) + f_2(x, y, z) = 3x + 4y + 2z + 6x + 5y + 3z = 9x + 9y + 5z.$$

We can also apply scalar multipication, for instance,

$$2f_1(x, y, z) = 6x + 8y + 4z.$$

We can also multiply them together, for instance:

$$f_1(x, y, z)\, f_2(x, y, z) = 18x^2 + 39xy + 20y^2 + 21xz + 22yz + 6z^2,$$

but this product is no longer linear but is quadratic [21].

So these functions have the same properties as vectors (they are *isomorphic* to vectors), however we now want to reverse this and make the functions, f_1, f_2 and f_3, the unknowns and the vector, (x, y, z), is the known:

$$\begin{aligned}
(af_1 + bf_2 + cf_3)(x) &= (af_1)(x) + (bf_2)(x) + (cf_3)(x),\\
(af_1 + bf_2 + cf_3)(y) &= (af_1)(y) + (bf_2)(y) + (cf_3)(y),\\
(af_1 + bf_2 + cf_3)(z) &= (af_1)(z) + (bf_2)(z) + (cf_3)(z)
\end{aligned}$$

where we need to find the *function multipliers* a, b and c.

5.6.2. *Duals.* We can generalise this duality between vectors and covectors to tensors one of the aims of this type of approach is to analyze geometry and physics in a way that is independent of the coordinate system.

The duality shows itself in various ways:

- If vectors are related to columns of a matrix then covectors are related to the rows.
- The dot product of a vector and its corresponding covector gives a scalar.
- When the coordinate system in changed then the covectors move in the opposite way to vectors (contravariant and covariant).
- If a vector is made from a linear combination of basis vectors then a covector is made by combining the normals to planes.
- When we take an infinitesimally small part of a manifold the vectors form the tangent space and the covectors form the cotangent space.

[21] Just like the geometric product: geometric multiplying two vectors produces non-vectors; *viz.*, a scalar and a bivector.

- Vector elements are represented by superscripts and covector elements are represented by subscripts.

5.6.3. *Coordinate Independence.* So lets start with a 3-D global orthogonal coordinate system. First we will start with a coordinate system based on a linear combination of orthogonal basis vectors.

Consider two coordinate systems whose basis vectors are $\{\sigma_k\}$ and $\{\sigma_l\}$. A physical vector, say p, can be represented by either

$$p = p^k \sigma_k$$

or

$$p = p^l \sigma_l,$$

where p is the physical vector being represented in tensor terms, p^k components of the tensor in one coordinate system, p^l components of the tensor in the the other coordinate system, and we have used Einstein summation convention.

So we can transform between the two using:

$$v^k = T_{kl} v^l$$

or

$$\sigma_k = T'_{kl} \sigma_l,$$

where T is a matrix tensor which rotates the vector v to vector v' and T' a matrix tensor which rotates the basis σ_k to the basis σ_l.

5.7. **Linear Functional.** In linear algebra, a *linear functional* or *linear form*, *aka*, *one-form* or *covector* is a *linear map* from a vector space to its field of scalars. In $\mathfrak{R}^n$, if vectors are represented as column vectors, then linear functionals are represented as row vectors, and their action on vectors is given by the dot product, or the matrix product with the row vector on the left and the column vector on the right. In general, if $\mathfrak{V}$ is a vector space over a field $\mathfrak{K}$, then a linear functional f is a function from $\mathfrak{V}$ to $\mathfrak{K}$, which is linear

$$\begin{aligned} f(v+w) &= f(v) + f(w) && \forall v, w \in \mathfrak{V}, \\ f(av) &= af(v) && \forall v \in \mathfrak{V}, \forall a \in \mathfrak{K}. \end{aligned}$$

The set of all linear functionals from $\mathfrak{V}$ to $\mathfrak{K}$, $Hom_k(\mathfrak{V}, \mathfrak{K})$, is itself a vector space over $\mathfrak{K}$. This space is called *the dual space of* $\mathfrak{V}$, or sometimes the algebraic dual space, to distinguish it from the continuous dual space. It is often written $\mathfrak{V}*$ or $\mathfrak{V}'$ when the field $\mathfrak{K}$ is understood.

NOTE 4. *Vector spaces take elements from the underneath scalar field and use them to form vector space elements, vectors. A linear functional takes vector elements and use them to form scalar fields! They are dual operations!*

NOTE 5. *Because vector spaces are linear spaces and because vectors and covectors are duals, then covectors form a linear space too.*

EXAMPLE 5.1 (Linear Functionals in $\mathfrak{R}^n$). Suppose that vectors in the real coordinate space $\mathfrak{R}^n$ are represented as column vectors

$$x = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}.$$

Then any linear functional can be written in these coordinates as a sum of the form

$$f(x) = a_1 x^1 + \cdots + a_n x^n = a_k x^k.$$

This is just the matrix product of the row vector $\begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ and the column vector x:

$$f(x) = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}.$$

NOTE 6. *Vector elements are represented by superscripts, x^k (using the basis σ_k), and covector elements are represented by subscripts, a_k (using the basis σ^k).*

5.7.1. *Basis of the dual space in finite dimensions.*

6. Term-wise Multiplication *vs.* Index Notation

6.1. **Geometric Algebra in a Glance.** Based on [3].

6.1.1. *Basic Axioms.* Let u, v and w be members [22] of $\mathfrak{E}^n$. Assume a geometric product, denoted as uv, that satisfies

$$u(vw) = (uv)w = uvw, \qquad \text{[associative]}$$
$$uv \neq vu. \qquad \text{[not generally commutative]}$$

Assume a right-handed frame $\{\sigma_k;\, k : 1 \ldots n\}$ whose members satisfy the abstract algebra

$$\sigma_k\sigma_l + \sigma_l\sigma_k = 2\delta_{kl}, \tag{14}$$

where δ_{kl} is the Kronecker delta; *i.e.*, $\delta_{kl} = [k = l]$ (Iverson bracket). Note that δ_{kl} is also the (flat) space metric.

Under the geometric product, the basis vectors are thus

- anti-symmetric, *e.g.*, $\sigma_1\sigma_2 = -\sigma_2\sigma_1$; and
- normalized, *e.g.*, $\sigma_3\sigma_3 = (\sigma_3)^2 = 1$.

6.1.2. *Inner and Outer Products of Vectors.*

Definition 6.1 (Inner Product of Vectors): *Let u and v be in $\mathfrak{E}^n$. The* inner product *of u and v, denoted as $u \cdot v$, is defined by*

$$u \cdot v \equiv \frac{uv + vu}{2}. \tag{15}$$

Definition 6.2 (Outer Product of Vectors): *Let u and v be in $\mathfrak{E}^n$. The* outer product *of u and v, denoted as $u \wedge v$, is defined by*

$$u \wedge v \equiv \frac{uv - vu}{2}. \tag{16}$$

Using these definitions, the geometric product can be written as

$$uv = u \cdot v + u \wedge v.$$

This form of the geometric product is called the *fundamental decomposition* of the geometric product for *vectors*.

The inner product of basis vectors can be determined using Eq. 14 and Eq. 15:

$$\sigma_k \cdot \sigma_l = \delta_{kl}.$$

Then, for u and v in $\mathfrak{E}^n$, the geometric product can be written in index notation as

$$\begin{aligned} uv &= (\sigma_k u^k)(\sigma_l v^l), \\ &= \sigma_k\sigma_l u^k v^l, && \text{[rearrenging scalars]} \\ &= (\sigma_k \cdot \sigma_l + \sigma_k \wedge \sigma_l)u^k v^l, && \text{[fundamental decomposition]} \\ &= (\delta_{kl} + \sigma_k \wedge \sigma_l)u^k v^l. && \text{[orthogonal basis]} \end{aligned}$$

Finally, an alternative form to express the inner and outer products in index notation is

$$u \cdot v = \delta_{kl} u^k v^l,$$
$$u \wedge v = \sigma_k \wedge \sigma_l u^k v^l.$$

---

[22] $\mathfrak{E}^n$ denotes an n-dimensional Euclidean space.

6.1.3. *Shorthand notation.* Eq. 14 can be rewritten as

$$\delta_{kl} = \frac{\sigma_k\sigma_l + \sigma_l\sigma_k}{2}.$$

On the other hand, the geometric product of two basis vectors is

$$\sigma_k\sigma_l = \sigma_k \cdot \sigma_l + \sigma_k \wedge \sigma_l = \delta_{kl} + \sigma_k \wedge \sigma_l.$$

Joining these equations together, we have a shorthand notation for basis vectors:

$$\sigma_k\sigma_l = \sigma_k \wedge \sigma_l.$$

Finally, with this notation, we have the index form of the geometric product for $u, v \in \mathfrak{E}^n$.

$$uv = \delta_{kl}u^k v^l + \sigma_k\sigma_l u^k v^l.$$

6.1.4. *Examples.*

Example 6.1. Let u and v be in $\mathfrak{R}^2$. Expand the geometric product uv using term-wise multiplication and index notation.

Solution. [Term-wise multiplication] Expand u and v in components:

$$u = \sigma_1 u^1 + \sigma_2 u^2,$$
$$v = \sigma_1 v^1 + \sigma_2 v^2,$$

where $\{\sigma_1, \sigma_2\}$ is a orthonormal basis for $\mathfrak{R}^2$.

Find the geometric product by term-wise multiplication:

$$\begin{aligned}
uv &= (\sigma_1 u^1 + \sigma_2 u^2)(\sigma_1 v^1 + \sigma_2 v^2) \\
&= \sigma_1 u^1 \sigma_1 v^1 + \sigma_1 u^1 \sigma_2 v^2 + \sigma_2 u^2 \sigma_1 v^1 + \sigma_2 u^2 \sigma_2 v^2 && \text{[left multiplication]} \\
&= \sigma_1\sigma_1 u^1 v^1 + \sigma_2\sigma_2 u^2 v^2 + \sigma_1\sigma_2 u^1 v^2 + \sigma_2\sigma_1 u^2 v^1 && \text{[reordering scalars]} \\
&= u^1 v^1 + u^2 v^2 + \sigma_1\sigma_2 u^1 v^2 + \sigma_2\sigma_1 u^2 v^1 && [\sigma_1\sigma_1 = \sigma_2\sigma_2 = 1] \\
&= u^1 v^1 + u^2 v^2 + \sigma_1\sigma_2 u^1 v^2 - \sigma_1\sigma_2 u^2 v^1 && [\sigma_1\sigma_2 = -\sigma_2\sigma_1] \\
&= u^1 v^1 + u^2 v^2 + \sigma_1\sigma_2(u^1 v^2 - u^2 v^1) && [\text{gathering } \sigma_1\sigma_2] \\
&= u^1 v^1 + u^2 v^2 + i(u^1 v^2 - u^2 v^1) && [i = \sigma_1\sigma_2]
\end{aligned}$$

□

Solution. [Index notation] Let $\{\sigma_k\}$ be a orthonormal basis for $\mathfrak{R}^2$. Then, $u = \sigma_k u^k$ and $l = \sigma_l v^l$. The geometric product is

$$\begin{aligned}
uv &= (\sigma_k u^k)(\sigma_l v^l) \\
&= \sigma_k\sigma_l u^k v^l && \text{[reordering scalars]} \\
&= \sigma_1\sigma_l u^1 v^l + \sigma_2\sigma_l u^2 v^l && [k = 1, 2] \\
&= \sigma_1\sigma_1 u^1 v^1 + \sigma_2\sigma_1 u^2 v^1 + \sigma_1\sigma_2 u^1 v^2 + \sigma_2\sigma_2 u^2 v^2 && [l = 1, 2] \\
&= u^1 v^1 + u^2 v^2 + \sigma_2\sigma_1 u^2 v^1 + \sigma_1\sigma_2 u^1 v^2 && [\sigma_m\sigma_m = 1] \\
&= u^1 v^1 + u^2 v^2 - \sigma_1\sigma_2 u^2 v^1 + \sigma_1\sigma_2 u^1 v^2 && [\sigma_m\sigma_n = -\sigma_n\sigma_m] \\
&= u^1 v^1 + u^2 v^2 + \sigma_1\sigma_2(u^1 v^2 - u^2 v^1) && [\text{gathering } \sigma_1\sigma_2] \\
&= u^1 v^1 + u^2 v^2 + i(u^1 v^2 - u^2 v^1) && [i = \sigma_1\sigma_2]
\end{aligned}$$

□

Solution. [Fundamental decomposition of the geometric product in index notation] Let $\{\sigma_k\}$ be a orthonormal basis for $\mathfrak{R}^2$. Then, $u = \sigma_k u^k$ and $v = \sigma_l v^l$. The geometric product is

$$\begin{aligned} uv &= (\sigma_k u^k)(\sigma_l v^l) \\ &= \sigma_k \sigma_l u^k v^l && \text{[reordering scalars]} \\ &= (\delta_{kl} + \sigma_k \sigma_l) u^k v^l && \text{[index gp and shorthand notation]} \end{aligned}$$

If we expand the implied summations, apply the definition of δ, $\delta_{kl} = [k = l]$, and the anti-symmetric property of the geometric product, $\sigma_1\sigma_2 = -\sigma_2\sigma_1$, then we get the final solution

$$uv = u^1v^1 + \sigma_1\sigma_2(u^1v^2 - u^2v^1). \qquad \text{[gathering } \sigma_1\sigma_2\text{]}.$$

$\square$

7. Geometric Algebra - LaTeX API

7.1. Set Theory.

- set: set A: $\mathfrak{A}$.
- elements of a set: elemset(1,2,3): $\{1, 2, 3\}$.
- cartesian product: cartprod [operator]: A cartprod B: $\mathfrak{A} \times \mathfrak{B}$.
- cartesian power: cartpow An: $\mathfrak{A}^n$.
- real set: realset: $\mathfrak{R}$.
- real space: rspace n: $\mathfrak{R}^n$.
- Euclidean space: espace n: $\mathfrak{E}^n$.
- (Minkowski) spacetime space: stspace 4: $\mathfrak{S}^4$.
- signature of spaces: sign(stspace 4) = tuple(-+++), sign $\mathfrak{S}^4 = (- + ++)$.
- linear space: lspace L: $\mathfrak{L}$.
- dual linear space: dlspace L: $\mathfrak{L}^*$.
- vector space: vecspace V: $\mathfrak{V}$.
- dual vector space: dvecspace V: $\mathfrak{V}^*$.
- basis set: bset B: $\mathfrak{B}$.
- (mathematical) field: field F: $\mathfrak{F}$.
- (geometric) algebra: algebra G: $\mathfrak{G}$.
- n-dim algebra: nalgebra G3: $\mathfrak{G}_3$.

7.2. Tuple.

- tuple: tuple(x,y,z): (x, y, z).

7.3. Functions.

- function: f: f.
- function definition: fdef(g)(set V cartprod W)(realset): $g \colon \mathfrak{V} \times \mathfrak{W} \to \mathfrak{R}$.
- function mapping: fmap(x)(x**2): $x \mapsto x^2$.
- function with arguments: fargs(f)(x,y,z): $f(x, y, z)$.
- function composition [operator]: (f)fcomp(g): $f \circ g$.

7.4. Geometric Objects.

- object: object B: $\mathfrak{B}$.
- point: point P: $\mathfrak{P}$.
- curve (line): curve C: $\mathfrak{C}$.
- plane: plane P: $\mathfrak{P}$.
- volume: vol V: $\mathfrak{V}$.
- vector: vec v: v.
- bivector: bivec B: B.
- trivector: trivec T: T.
- cliff: clif C: C.
- rotor: rtr R: R.
- pseudo-scalar 2d: psclii: i.
- pseudo-scalar 3d: pscliii: $\mathcal{I}$.
- angle: angl(theta): θ.

7.5. Operations.

- magnitude: magn C: $|C|$.
- grade: grade C1: $\langle C \rangle_1$.
- scalar grade: sgrade C: $\langle C \rangle_0$.
- inner product: iprod [operator]: $u \cdot v$.

- outer product: oprod [operator]: $u \wedge v$.
- cross product: oprod [operator]: $u \times v$.
- reverse: rev(vec v): $v^{\dagger}$.

7.6. **Index Notation in Frame.**

- basis vector: bvec: σ.
- (reference) frame: frm k: $\{\sigma_k\}$.
- indexed frame: ifrm(k)(1)(n): $\{\sigma_k;\, k : 1 \ldots n\}$.
- (contravariant) components of vectors, bivectors, trivectors on a frame: comp(v)(1): v^1.
- basis vector on frame (indexed basis vector): ibvec k: σ_k.
- basis bivector on frame (indexed basis bivector): ibbivec(1)(2): σ_{12}.
- basis trivector on frame (indexed basis trivector): ibtrivec(1)(2)(3): σ_{123}.
- (general) basis cliff on frame (indexed basis cliff): ibclif(123): σ_{123}.
- the general basis cliff is useful in index manipulations: $\sigma_{123} = -\sigma_{132} = \sigma_{312} = \cdots$.
- Kronecker delta: kron: δ.
- (indexed) Kronecker delta on frame: ikron(i)(j): δ_{ij}.
- (mixed) Kronecker delta: mkron(i)(j): ${\delta^i}_j$.
- matrix representation of an object: mtxrep(ikron kl): $[\delta_{kl}]$.

7.7. **Index Notation in Dual Frame.**

- (reference) dual frame: dfrm k: $\left\{\sigma^k\right\}$.
- basis vector on dual frame (indexed basis vector): dbvec k: σ^k.
- (covariant) components of vectors, bivectors, trivectors on a dual frame: dcomp(v)(1): v_1.

7.8. **Metric.**

- metric: metric: g.
- (indexed) metric in frame: imetric kl: g_{kl}.
- (indexed) metric in dual frame: dmetric kl: g^{kl}.
- spacetime metric: stmetric: η.
- (indexed) spacetime metric: istmetric(mu)(nu): $\eta_{\mu\nu}$.

7.9. **Classical Mechanics.**

- Lagrangian: lagr: $\mathcal{L}$.
- dot derivative: dotd x: $\dot{x}$.

7.10. **Fields.**

- position vector: pvec x: x.
- scalar field: sfield(phi)(pvec x): $\phi(x)$.
- vector field: vfield(vec F)(pvec x): $F(x)$.
- geometric derivative: dvec(pvec x): ∇x.
- (indexed) geometric derivative: idvec xk: ∂_k.
- gradient [operator]: grad W: $\operatorname{grad} W$.

7.11. **Matrices.**

- matrix: mtx(A): A.
- indexed matrix: imtx(A)(ij): A_{ij}.
- elements of a matrix (explicit matrix representation): emtx(1, 2, 3): $\begin{bmatrix}1 & 2 & 3\end{bmatrix}$.
- transpose of a matrix: trmtx(A): A^T.
- diagonal of a matrix: diag(1,2,3): $\operatorname{diag}(1, 2, 3)$.
- matrix representation: mtxrep(vec v): $[v]$.
- matrix representation of an element in a given basis (set): mtxrepb(vec v)(B): $[v]_{\mathfrak{B}}$.

- determinant: dtr(mtx A): $\det A$.
- elements of a determinant (explicit determinant representation): edtr(1, 2, 3): $\begin{vmatrix} 1 & 2 & 3 \end{vmatrix}$.

7.12. **Dimensional Analysis.**

- phisical dimension of a physical quantity: phdim(L): $[\mathsf{L}]$.

7.13. **Conventions.**

- Iverson brackets: iverson(i = j): $[i = j]$.
- Iverson brackets with sum: iversum(i)(comp xi)(i=j): $\sum_i x^i \; [i = j]$.

7.14. **Various.**

- color something in red: inred(1 + 2): $1 + 2$.
- color something in blue: inblue(1 + 2): $1 + 2$.
- color something in green: ingreen(1 + 2): $1 + 2$.
- *e.g.*, *i.e.*, *etc.*, *aka*, *[sic]*, *viz.*.

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