GEOMETRIC ALGEBRA FOR ENGINEERS

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Contents

1. Introduction

Based on [6].

In applied physics, we mainly work with two mathematical objects: scalars and vectors. Scalars are used to represent physical quantities with magnitude but no direction, such as mass, speed, temperature, and so on. Vectors, on the other hand, are used to represent quantities with both magnitude and direction.

Working with scalars is like working with real numbers: addition.

Vectors, however, respond to different rules than scalar algebra. First of all, they are a set of .

Although their algebras seem to be very different, they are both related.

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The idea of using scalars and vectors is to represent not only physical quantities, but also physical processes. It turns out that geometry, algebra and physics are deeply related.

2. Vector Algebra at a Glance

We take a brief tour on how to manipulate vectors, vector algebra.

2.1. Basic Concepts and Operations – Vector Spaces. In three-dimensions, 3-D, a vector, say $\vec{v}$, is a mathematical object presented as

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k},$$

where v_x, v_y and v_z are real numbers – *aka*, scalars – called the components of v and the vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit (length) vectors pointing in the direction of each axis of the Cartesian coordinate system. For this reason, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are called the Cartesian basis vectors. Note that the scalars show magnitude, whereas the unit vectors, direction.

Two vectors are said to be equal if their respective components are equal; that is, $\vec{u}$ and $\vec{v}$ are equal if

$$u_x = v_x, \qquad u_y = v_y, \qquad \text{and} \qquad u_z = v_z.$$

Adding two vectors, or the vector addition of, $\vec{u}$ and $\vec{v}$ results in another vector, $\vec{u}+\vec{v}$, given by

$$\vec{u} + \vec{v} = (u_x + v_x)\hat{\imath} + (u_y + v_y)\hat{\jmath} + (u_z + v_z)\hat{k}.$$

There is a special vector, $\vec{0}$, whose all components are 0, such that, for any vector $\vec{v}$, we have that $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$.

Scalars and vectors interact with each other, creating a new vector, $a\vec{v}$, via scalar multiplication:

$$a\vec{v} = av_x\hat{\imath} + av_y\hat{\jmath} + av_z\hat{k}.$$

where a is a scalar and $\vec{v}$ a vector.

As a notational aid, we can use scalar multiplication and vector addition to define vector subtraction: given two vectors $\vec{u}$ and $\vec{v}$, vector subtraction is given by

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}.$$

In modern vector algebra, a set whose elements are vectors and that has defined addition and scalar multiplication is called a vector space. We would say then that 3-D space is a vector space.

2.2. Gibbs Vector Algebra. J.W. Gibbs algebra is the one regularly thought in basic engineering courses. It consists of the 3-D vector space with two additional operations: the dot product and the cross product. The set and the operations form an algebra called Gibbs vector algebra.

The dot product of two vectors, $\vec{u}$ and $\vec{v}$ in 3-D space, denoted as $\vec{u}\cdot\vec{v}$, can be found by applying the formula

$$\vec{u}\cdot\vec{v} = u_xv_x + u_yv_y + u_zv_z.$$

The result of the dot product is always a scalar, for this reason, the dot product is also called scalar product.

The cross product of $\vec{u}$ and $\vec{v}$ results in another vector, denoted as $\vec{u}\times\vec{v}$, defined by

$$\vec{u}\times\vec{v} = (u_yv_z - u_zv_y)\hat{\imath} + (u_zv_x - u_xv_z)\hat{\jmath} + (u_xv_y - u_yv_x)\hat{k}.$$

The result of the cross product is a vector, this is why, it is also called vector product.

2.3. Grassmann Algebra. Hermann Grassmann took the 3-D vector space and defined yet another vector operation: the wedge product. The set and the operations form an algebra called Grassmann algebra.

Given two vectors, $\vec{u}$ and $\vec{v}$, the wedge product, denoted as $\vec{u}\wedge\vec{v}$, is defined by

$$\vec{u}\wedge\vec{v} = (u^xv^y - u^yv^x)(\hat{\imath}\wedge\hat{\jmath}) + (u^zv^x - u^xv^z)(\hat{k}\wedge\hat{\imath}) + (u^yv^z - u^zv^y)(\hat{\jmath}\wedge\hat{k}).$$

The entities $\hat{\imath}\wedge\hat{\jmath}$, $\hat{k}\wedge\hat{\imath}$ and $\hat{\jmath}\wedge\hat{k}$ are called bivectors and form a basis for the space generated by them.

2.4. **Cross Product and Wedge Product.**

3. Notation, Notation, Notation!

Before diving into geometric algebra, we need to define some notational conventions, because...

3.1. **Current Notation.**
The traditional notation used in basic treatments of vectors is often verbose. For instance, we have seen that any vector, $\vec{v}$, in 3-D is noted as

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} \tag{1}$$

and scalar multiplication as

$$a\vec{v} = av_x\hat{\imath} + av_y\hat{\jmath} + av_z\hat{k}. \tag{2}$$

In order to have a more compact, elegant notation, we hereafter adopt three pieces of notation.

3.2. **Index Notation.**
The terms in Eq. 1 can be written using a common index for components and unit vectors if we agree to

- index the directions of space as $x \to x^1$, $y \to x^2$ and $z \to x^3$ (note that the indices are written as *super*-scripts for the components of the *position vector*);
- use the lowercase Greek letter gamma, γ, to denote basis vectors;
- rename the Cartesian basis vectors: $\hat{\imath} \to \gamma_x \to \gamma_1$, $\hat{\jmath} \to \gamma_y \to \gamma_2$ and $\hat{k} \to \gamma_z \to \gamma_3$ (note that the indices are written as *sub*-scripts for *basis vectors*);
- use *super*-scripts for the *components* of any vector; *e.g.*, for $\vec{u}$, $u_x \to u^1$.

With this convention, let us rewrite Eq. 1:

$$\vec{v} = \gamma_1 v^1 + \gamma_2 v^2 + \gamma_1 v^1 = \sum_{k=1}^{3} \gamma_k v^k,$$

where we have used the *sigma notation* to shorten the sum.

We can compact our notation even further by agreeing in the next set of conventions.

3.3. **Einstein Summation Convention.**
Einstein summation convention seeks to economize the usage of sigma notation by agreeing with three rules, [6, 8]:

(1) Repeated, doubled, raised and lowered indices, called *dummy indices*, in quantities multiplied together are implicitly summed – doubled indices are also called *paired indices*.
(2) Indices that are *not* summed over, called *free indices*, are allowed to take all possible values unless otherwise stated.
(3) It is illegal to use the same dummy index more than twice in a term unless its meaning is made explicit.

For example, using Einstein Summation Convention, we could rewrite Eq. 1

$$\vec{v} = \gamma_k v^k,$$

where we have just dropped the $\sum$ sign.

3.4. **Vector Decoration.**
Vectors are traditionally decorated in different ways: $\vec{v}$, $\hat{v}$, $\boldsymbol{v}$, and so on. The idea is to make them distinguishable from scalars.

However, since index notation provides with a way of distinguishing scalars from vector components, we will drop vector decoration all together. For instance, we could rewrite Eq. 2 in index notation

$$av = a\gamma_k v^k,$$

where a is a scalar (no index), v^k are vector components and γ_k are vectors (one index). Moreover, γ_k is a unit basis vector (lowercase Greek letter gamma).

Confusion could arise when writing vectors and scalars together; *e.g.*, the result of scalar multiplication: av. For this reason, we will agree that

- scalars will be usually noted with the first letters of the Roman alphabet (a, b, c);

- indices will be usually noted with the middle letters of the Roman alphabet (k, l, m); and
- vectors will be usually noted with the last letters of the Roman alphabet (u, v, w).

4. Exploratory Math

> Too much mathematical rigor teaches *rigor mortis*: the fear of making an unjustified leap even when it lands on a correct result. Instead of paralysis, have courage – shoot first and ask questions later. Although unwise as public policy, it is a valuable problem-solving philosophy.
>
> — SANJOY MAHAJAN, [7]

The main goal of this document is to briefly describe geometric algebra. The problem for newcommers is that in the standard literature the material is presented in a formal, abstract way, see, for instance, [3, 5]. This section, on the other hand, aims to provide a basic understanding on the origins of the object matter.

4.1. Vector Products.

Given two scalars, a and b, multiplication can be done as ab or ba. Moreover, if we add another scalar, we can multiply them as abc, cba, cab, and so on, without changing the result. Vectors, however, have different multiplication rules. To start, there is not one product, but at least three: the product of a vector and a scalar, the dot product of two vectors and the cross product of two vectors, as reviewed in Sec. 2. Each one comes with a receipt to find out the desired result. Scalar multiplication takes a scalar and distributes it through the components of a vector, resulting in another vector.

The dot product accepts two vectors, joins the components of one vector with the respective component of another vector and returns a scalar. If the vectors are dot multiplied in reverse order, their result is the same: $u \cdot v = u \cdot v$; *i.e.*, dot product is commutative.

Finally, the cross product takes two vectors, joins components in a prescribed way, multiplies this result by a unit vector, adds all the so found vectors and returns a vector. It is worth remembering that the cross product is defined only in 3-D, even if the multiplicands live in 1- or 2-D. If the vectors are cross multiplied in reverse order, their result is *not* the same: $u \times v \neq u \wedge v$; *i.e.*, cross product is *not* commutative – anticommutative.

After reviewing their definitions, one could ask: "what about term-wise multiplication?" That is, if we have two vectors, u and v, in 3-D, what is the result [1] of $u^k\gamma_k v^l\gamma_l$?

Let's find out the result by exploring its math.

4.2. Geometric Product.

Let u and v be two vectors in 3-D. Their term-wise multiplication is given by

$$u \otimes v = u^k\gamma_k v^l\gamma_l, \tag{3}$$

where $\otimes$ is the term-wise product of u and v. Let's agree to call this operation *the geometric product of* u *and* v and to denote it by juxtaposition of the operands; *i.e.*, instead of writing $u \otimes v$, we will write uv.

Let's expand the indices in Eq. 3, remembering that k and l both run from 1 to 3 and noting that we are allowed to reorder only scalars or terms, but not unit vectors – the geometric product,

[1] Note that, even though u and v belong to 3-D – their indices both will run from 1 to 3, we have to use different labels, k and l, for them, to be in accordance with Rule 3 of Einstein summation convention, see Sec. 3.3. Because of this rule, we write $u^k\gamma_k v^l\gamma_l$, instead of $u^k\gamma_k v^k\gamma_k$. Just remember, every time a vector is noted in index notation, use a different index label.

as the cross product, is perhaps anticommutative:

$$
\begin{aligned}
uv &= +\,(\gamma_k u^k)(\gamma_l v^l), \\
&= +\,u^k v^l \gamma_k \gamma_l, && \text{[reordering scalars]} \\
&= +\,u^1 v^l \gamma_1 \gamma_l + u^2 v^l \gamma_2 \gamma_l + u^3 v^l \gamma_3 \gamma_l, && [k\text{: 1 to 3}] \\
&= +\,u^1 v^1 \gamma_1 \gamma_1 + u^1 v^2 \gamma_1 \gamma_2 + u^1 v^3 \gamma_1 \gamma_3 \\
&\quad + u^2 v^1 \gamma_2 \gamma_1 + u^2 v^2 \gamma_2 \gamma_2 + u^2 v^3 \gamma_2 \gamma_3 \\
&\quad + u^3 v^1 \gamma_3 \gamma_1 + u^3 v^2 \gamma_3 \gamma_2 + u^3 v^3 \gamma_3 \gamma_3. && [l\text{: 1 to 3}]
\end{aligned}
$$

Notice the patterns of terms in the last equality:

(1) $u^1 v^1 \gamma_1 \gamma_1 + u^2 v^2 \gamma_2 \gamma_2 + u^3 v^3 \gamma_3 \gamma_3$,
(2) $u^1 v^2 \gamma_1 \gamma_2 + u^2 v^1 \gamma_2 \gamma_1$,
(3) $u^1 v^3 \gamma_1 \gamma_3 + u^3 v^1 \gamma_3 \gamma_1$,
(4) $u^2 v^3 \gamma_2 \gamma_3 + u^3 v^2 \gamma_3 \gamma_2$.

We can retrieve the dot product from the first pattern if we set

$$\gamma_1 \gamma_1 = \gamma_2 \gamma_2 = \gamma_3 \gamma_3 = 1,$$

and we can retrieve the *components* of the cross product if we set

$$
\begin{aligned}
\gamma_1 \gamma_2 &= -\gamma_2 \gamma_1, \\
\gamma_2 \gamma_3 &= -\gamma_3 \gamma_2, \\
\gamma_3 \gamma_1 &= -\gamma_1 \gamma_3.
\end{aligned}
$$

or, in other words, if we use a cyclic order for the indices of the basis vectors: $1 \xrightarrow{+} 2 \xrightarrow{+} 3 \xrightarrow{+} 1$. Equivalently, choosing a cyclic order for the indices means, in 3-D, to use a right-handed orthogonal basis set. We will return to the issue on choosing the index order a bit later.

The choices on the basis elements that we have made to retrieve the dot product and cross product *from* the geometric product can be neatly summarized [6] by

$$\gamma_k \gamma_l + \gamma_l \gamma_k = 2\delta_{kl},$$

where δ_{kl} is the Kronecker delta.

Let's, then, rewrite the geometric product of two vectors in 3-D with our agreed conventions

$$
\begin{aligned}
uv = &+ u^1 v^1 + u^2 v^2 + u^3 v^3 \\
&+ (u^1 v^2 - u^2 v^1)\gamma_1 \gamma_2 \\
&+ (u^2 v^3 - u^3 v^2)\gamma_2 \gamma_3 \\
&+ (u^3 v^1 - u^1 v^3)\gamma_3 \gamma_1.
\end{aligned}
$$

This means we can express the geometric product in terms of the dot and outer products by

$$uv = u \cdot v + u \wedge v.$$

The advantage of the last equation is its generality: no components are involved, only vectors.

4.3. **Fundamental Geometric Product Decomposition.** Now that we know how to calculate the geometric product of u and v, let's calculate vu, and see what happens:

$$\begin{aligned}
vu &= +\,(\gamma_k v^k)(\gamma_l u^l) && \text{[vector expansion]}\\
&= +\,\gamma_k \gamma_l v^k u^l && \text{[scalar rearrangment]}\\
&= +\,\gamma_1\gamma_l v^1 u^l + \gamma_2\gamma_l v^2 u^l + \gamma_3\gamma_l v^3 u^l && [k\text{: 1 to 3}]\\
&= +\,\gamma_1\gamma_1 v^1u^1 + \gamma_1\gamma_2 v^1u^2 + \gamma_1\gamma_3 v^1u^3\\
&\quad + \gamma_2\gamma_1 v^2u^1 + \gamma_2\gamma_2 v^2u^2 + \gamma_2\gamma_3 v^2u^3\\
&\quad + \gamma_3\gamma_1 v^3u^1 + \gamma_3\gamma_2 v^3u^2 + \gamma_3\gamma_3 v^3u^3 && [l\text{: 1 to 3}]\\
&= +\,(v^1u^1 + v^2u^2 + v^3u^3)\\
&\quad + (v^1u^2 - v^2u^1)\gamma_1\gamma_2\\
&\quad + (v^2u^3 - v^3u^2)\gamma_2\gamma_3\\
&\quad + (v^3u^1 - v^1u^3)\gamma_3\gamma_1. && \text{[scalar rearrangment]}
\end{aligned}$$

Comparing the equations for the geometric product of uv and vv, we have

$$vu = v \cdot u + v \wedge u = u \cdot v - u \wedge v.$$

In other words, the dot product is commutative, while the outer product is anticommutative. Additionally, note that the last equation does not involve components, thus it is general.

With these findings, we have two equations

$$uv = u \cdot v + u \wedge v \tag{4}$$
$$vu = u \cdot v - u \wedge v. \tag{5}$$

Adding Eq. 4 and Eq. 5, we have

$$\begin{aligned}
uv + vu &= 2(u \cdot v) + 0\\
u \cdot v &= \frac{uv + vu}{2},
\end{aligned}$$

thus, we get the symmetric part of the geometric product.

On the other hand, subtracting Eq. 5 from Eq. 5, we have

$$\begin{aligned}
uv + vu &= 0 + 2(u \wedge v)\\
u \wedge v &= \frac{uv - vu}{2},
\end{aligned}$$

thus, we get the antisymmetric part of the geometric product.

Under these new facts, we can define the inner product and the outer product *from* the geometric product:

$$\begin{aligned}
u \cdot v &\equiv \frac{uv + vu}{2},\\
u \wedge v &\equiv \frac{uv - vu}{2}.
\end{aligned}$$

Again, notice that the last equation is general, since no components are required.

4.4. **Problems with the Geometric Product.** In the last subsection we have emphasized two terms in the last paragraphs:

- "we can retrieve the *components* of the cross product". We did not say "we can retrieve the cross product". To see what we mean, compare the first component of the cross product of two vectors, u and v in 3-D, as seen in Sec. 2:

$$(u_y v_z - u_z v_y)\hat{\imath} = (u^2 v^3 - u_3 v_2)\gamma_1 \qquad \text{[index notation]}$$

  with the correspondent component calculated with the geometric product:

$$(u^2 v^3 - u^3 v^2)\gamma_2\gamma_3.$$

  What is $\gamma_2\gamma_3$? And, if it is something meaningful, what is the relation between $\gamma 1$ and $\gamma_2\gamma_3$ under the geometric product, then? We will elaborate that a bit later.
- "the choices [...] to retrieve the dot product and cross product *from* the geometric product". Because we can retrieve the two products from one, the geometric product is more fundamental to vector algebra that the other two products.

4.4.1. *Kronecker Delta to Raise and Lower Indices.* Consider a reciprocal frame $\mathfrak{D} = \left\{\gamma^k;\ k : 1 \ldots 3\right\}$ defined by

$$\gamma_k = g_{kl}\gamma^l,$$

where g_{kl} are the components of the metric on the frame $\mathfrak{F}$.

Because we are dealing with flat space, we have that the metric equals the flat metric, $g_{kl} = \delta_{kl}$, and thus the dual frame elements are given by

$$\gamma_k \equiv \delta_{kl}\gamma^l.$$

Notice the effect of δ on γ^l: δ_{kl} changes γ_l into γ^k by exchanging the indices, $k \to l$, and then it "disappears" [metric reduction? or reduction by metric? index gymnastics? or another name!?].

Equivalently, the dual frame elements can be defined, implicitly, by

$$\gamma_k \cdot \gamma^l = \delta^k{}_l.$$

Any vector v can then be projected onto $\mathfrak{D}$ by

$$v_k = \gamma^k \cdot v,$$

where v_k are the components of v onto $\mathfrak{D}$.

The representation of v onto $\mathfrak{D}$ can be written as

$$v = \gamma^k v_k.$$

With the aid of reciprocal frames and the metric, we can ease the computation of the inner product of u and v:

$$\begin{aligned}
u \cdot v &= \gamma_k u^k \cdot \gamma_l v^l && \text{[vector representation on frame]}\\
&= \gamma_k \cdot \gamma_l u^k v^l && \text{[scalar rearrangement]}\\
&= \delta_{kl} u^k v^l, && \text{[metric definition]}\\
&= u^k v_k. && \text{[metric reduction]}
\end{aligned}$$

Finally, the last equality, $u^k v_k$, gives an easier method to expand the inner product of two vectors:

$$u \cdot v = u^k v_k = u^1 v_1 + u^2 v_2 + u^3 v_3. \qquad [k\text{: 1 to 3}]$$

Just remember: .

4.5. **Alternative Calculation for the Geometric Product.** We have seen that's possible to calculate the geometric product of two vectors uv, expanded as $\gamma_k u^k$ and $\gamma_l v^l$, by allowing the indices k and l run from 1 to 3.

An alternative way to calculate the geometric product of u and v is as follows [9]:

$$\begin{aligned}
uv &= +\,(\gamma_k u^k)v, && [u \text{ expansion}] \\
&= +\,(\gamma_1 u^1 + \gamma_2 u^2 + \gamma_3 u^3)v && [k\text{: 1 to 3}] \\
&= +\,\gamma_1 u^1 v + \gamma_2 u^2 v + \gamma_3 u^3 v && [v \text{ from the rigth}] \\
&= +\,\gamma_1 u^1(\gamma_l v^l) + \gamma_2 u^2(\gamma_l v^l) + \gamma_3 u^3(\gamma_l v^l) && [v \text{ expansion}] \\
&= +\,\gamma_1 u^1(\gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3) \\
&\quad + \gamma_2 u^2(\gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3) \\
&\quad + \gamma_3 u^3(\gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3) && [l\text{: 1 to 3}] \\
&= + \cdots . && [\text{continue the calculation}]
\end{aligned}$$

This is, expand the first vector and let it run from 1 to 3. Then expand the second vector and let it run from 1 to 3. Next, do algebraic manipulations, remembering to keep the order of the basis vectors. Finally, present the result.

5. The Algebra Bit in Geometric Algebra

5.1. **Vector Spaces.** Adapted from [1].

Notation 1. *The set of real numbers is denoted by* $\mathfrak{R}$. *The elements of* $\mathfrak{R}$ *are called* real numbers *or* scalars.

Definition 5.1 (Real Vector Space): *Consider a non-empty set* $\mathfrak{V}$ *of objects, called* vectors. *The set* $\mathfrak{V}$ *is called a* vector space over $\mathfrak{R}$, *or* real vector space, *if it satisfies the following ten axioms listed in three groups. (In the following, consider* u, v, w *to be in* $\mathfrak{V}$ *and consider* a, b *to be in* $\mathfrak{R}$*;* i.e., *scalars)*

(1) *Closure axioms*

Axiom 5.1 (Closure under addition): *For every pair* u *and* v *there corresponds a unique vector in* $\mathfrak{V}$ *called the sum of* u *and* v, *denoted by* $u + v$.

Axiom 5.2 (Closure under multiplication by real numbers): *For every vector* u *and for every scalar* a *there corresponds a vector in* $\mathfrak{V}$ *called the product of* a *and* u, *denoted by* au.

(2) *Axioms for addition*

Axiom 5.3 (Commutative law): $u + v = v + u$.

Axiom 5.4 (Associative law): $(u + v) + w = u + (v + w)$.

Axiom 5.5 (Existence of zero element): *There is a vector in* $\mathfrak{V}$, *denoted by* 0, *such that, for every* u, $u + 0 = u$.

Axiom 5.6 (Existence of negatives): *For every* u, *there is a vector* $(-1)u$, *such that* $u + (-1)u = 0$.

(3) *Axioms for multiplication by scalars*

Axiom 5.7 (Associative law): $a(bu) = (ab)u$.

Axiom 5.8 (Distributive law for addition in $\mathfrak{V}$): $a(u + v) = au + av$.

Axiom 5.9 (Distributive law for addition of scalars): $(a + b)u = au + av$.

Axiom 5.10 (Existence of identity): *for every* u, $1u = u$.

Definition 5.2 (Real Coordinate Space):

Let $\mathfrak{R}$ denote the set of real numbers. For any positive integer n, the set of all n-tuples of real numbers forms an n-dimensional vector space over $\mathfrak{R}$, denoted $\mathfrak{R}^n$, called *real coordinate space*.

An element of $\mathfrak{R}^n$ is written

$$v = \left(v^1, v^2, \ldots, v^n\right),$$

where the real numbers x^k are called the *coordinates of* v.

Two vectors, u and v, in $\mathfrak{R}^n$, are said to be *equal* if their correspondent coordinates are equal; *i.e.*, u and v are equal if and only if the following n equations are true

$$u^1 = v^1, \qquad u^2 = v^2, \qquad \ldots, \qquad u^n = v^n.$$

The vector space $\mathfrak{R}^n$ comes with a *standard basis*

$$\begin{aligned} \gamma_1 &= (1, 0, \ldots, 0)\,, \\ \gamma_2 &= (0, 1, \ldots, 0)\,, \\ &\vdots \\ \gamma_n &= (0, 0, \ldots, n)\,. \end{aligned}$$

An arbitrary vector in $\Re^n$ can then be uniquely written in the form

$$v = \gamma_k v^k,$$

where we have used Einstein summation notation.

6. More Formal Geometric Algebra

Chain of ideas for this section:

Physics and geometry are deeply related [2, 3]. So physical objects and physical processes can be modeled using geometric objects and geometric transformations. On the other hand, geometric algebra provides an efficient language to deal with geometry via algebra [3].

Based upon [6, 4], this section explodes the relationship between algebra, geometry and physics via the description of geometric algebra in a more formal way.

Space means 3D space.

Tenet: (physical) space as a 3-d real coordinate space. why Euclidean? why not just 3-d real space? or only 3d real coordinate space?

real numbers, scalars, vectors in math and in physics.

Real coordinate space is a vector space, this grants scalar multiplication and vector addition.

Geometric product can be used (directly or indirectly, via decomposition into inner and outer products) to calculate lengths, angles, represent rotations, and so on. That is, it transforms real coordinate space into an arena for doing Euclidean geometry. It transforms real coordinate spaces into Euclidean spaces.

Explain the fundamental decomposition of the geometric product (decomposition into inner and outer products).

Real coordinate spaces, and therefore Euclidean spaces, come by definition with a standard basis – orthonormal basis.

By a theorem of vector spaces, any vector can be represented as a linear combination of the basis elements.

Explain Iverson brackets.

Explain what is meant by flat space metric.

dig into the abstract algebra thing: $\gamma_k\gamma_k = (\gamma_k)^2 = [k = k] = 1$.

Explain Einstein summation convention.

Show the overall structure and the interaction of its elements: (abstract) linear space, vector space, real coordinate space, euclidean space, frame. Algebra, geometric product, geometric algebra, euclidean structure.

Generation of the euclidean space: real number line generates a (linear) vector space called real coordinate space. Geometric product gives structure (geometric algebra) to the real coordinate space to turn it into euclidean space. Euclidean space is the arena onto which Euclidean geometry is done, and, by our basic tenet, physical object and processes can be modeled.

Final note on the writing style: since the formal mathematical parlance includes standard parts such as axioms, definitions, theorems, and so forth, that may seem odd or heavy to read to an engineer, I adopted a semi-formal style that uses straight imperative voice, instead. This way of speaking, heavily influenced by [6], is hoped to be understood by engineers.

6.1. Geometric Algebra of Space.

Herein, $\mathfrak{R}^3$ represents the three-dimensional real coordinate space, whereas $\mathfrak{E}^3$, the three-dimensional Euclidean space.

6.1.1. *Basic Tenet.*

The three-dimensional Euclidean space, $\mathfrak{E}^3$, provides a model for space.

6.1.2. *Geometric Product Axioms.* For vectors u, v and w in $\mathfrak{R}^3$, consider a *geometric product* uv having the following properties

$$(uv)w = u(vw)\,, \qquad \text{[associative]} \tag{6}$$
$$u(v+w) = uv + uw\,, \qquad \text{[left distributive]} \tag{7}$$
$$(v+w)u = vu + wu\,, \qquad \text{[rigth distributive]} \tag{8}$$
$$vv = v^2 = |v|^2\,, \qquad \text{[contraction]} \tag{9}$$

where $|v|$ is a real scalar called the *magnitude of* v.

(I think it would be better to start off with this phrase as a hypothesis, and show that it is true – thesis) Under the geometric product, $\mathfrak{R}^3$ thus becomes the three-dimensional Euclidean space, $\mathfrak{E}^3$.

6.1.3. *Geometric Product Decomposition.* Define a symmetric *inner product*

$$u \cdot v \equiv \frac{uv + vu}{2} = v \cdot u$$

and an antisymmetric *outer product*

$$u \wedge v \equiv \frac{uv - vu}{2} = v \wedge u\,.$$

Then, call

$$uv = u \cdot v + u \wedge v$$

the *fundamental decomposition of the geometric product for* vectors.

6.1.4. *Space Frame.* Consider the Kronecker delta (Iverson brackets)

$$\delta_{kl} = [k = l]\ .$$

Denote the $\mathfrak{E}^3$ frame by $\mathfrak{F} = \{\gamma_k;\, k : 1 \ldots 3\}$. The frame $\mathfrak{F}$ is also called *standard basis of* $\mathfrak{E}^3$ and its elements, $\{\gamma_k\}$, also called *basis vectors*.

The elements of $\mathfrak{F}$ thus satisfy the *abstract algebra*

$$\gamma_k \gamma_l + \gamma_l \gamma_k = 2\delta_{kl}\ ,$$

Under the geometric product, the frame elements are therefore *antisymmetric* and *normalized*.

6.1.5. *Metric.* Adopt Einstein summation convention.

Find the *flat space metric elements* by

$$g_{kl} \equiv \gamma_k \cdot \gamma_k = \delta_{kl}\ .$$

6.1.6. *Vector Decomposition.* Project any v in $\mathfrak{E}^3$ onto $\mathfrak{F}$ by

$$v^k = \gamma_k \cdot v\ ,$$

where v^k are the *components of* v *onto* $\mathfrak{F}$. The components of v onto $\mathfrak{F}$ are also called the *rectangular coordinates of* v.

Write thus the *representation of* v *onto* $\mathfrak{F}$ by

$$v = \gamma_k v^k\ .$$

6.1.7. *Geometric Product Decomposition.* Since the elements of $\mathfrak{F}$, $\{\gamma_k\}$, are orthogonal, write

$$\gamma_k \wedge \gamma_l = \gamma_k \gamma_l \,.$$

Represent u and v onto $\mathfrak{F}$ to calculate their geometric product:

$$uv = \gamma_k u^k \gamma_l v^l = u \cdot v + u \wedge v \,,$$

where

$$u \cdot v = \delta_{kl} u^k v^l$$

and

$$u \wedge v = \gamma_k \gamma_l u^k v^l \,.$$

6.1.8. *Multivectors.* By successive additions and multiplications, the vectors in $\mathfrak{R}^3$ generate an (geometric) algebra, $\mathfrak{G}_3$, called *space algebra*, whose elements are called *multivectors*.

6.2. **Calculus.** symbols
nabla: ∇.
delta: $\boldsymbol{\Delta}$.
triangle: $\triangle$.
triangledown: $\triangledown$.
square: $\square$.
bigtriangledown: $\bigtriangledown$.
bigtriangleup: $\bigtriangleup$.
dual of a multivector in 3d: dual(clif A pscliii): $A\mathcal{I}$.
dual of a multivector in 2d: dual(clif A psclii): Ai.
vectors in other notation systems:

- arrows: $\vec{x}$,
- bold typeface: $\boldsymbol{x}$,
- bold typeface and roman font: $\mathbf{x}$.

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