SYNOPSIS OF GEOMETRIC ALGEBRA

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Taken from [?].

Generally useful relations and formulas for the geometric algebra $\mathcal{G}_3$ of Euclidean (3-)space, $\mathcal{E}^3$, are listed here.

For vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ... and scalars α, β, ..., the Euclidean geometric algebra has the following properties [1]

(1) associativity: $\mathbf{a}(\mathbf{bc}) = (\mathbf{abc}) \qquad \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.
(2) commutivity: $\alpha\mathbf{b} = \mathbf{b}\alpha \qquad \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
(3) distributivity: $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac} \qquad (\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca}$.
(4) linearity: $\alpha(\mathbf{b} + \mathbf{c}) = \alpha\mathbf{b} + \alpha\mathbf{c} = (\mathbf{b} + \mathbf{c})\alpha$.
(5) contraction: $\mathbf{a}^2 = \mathbf{aa} = |\mathbf{a}|^2$.

Contraction makes the algebra Euclidean!

The multiplicative inverse of a nonzero vector $\mathbf{a}$ is given by $\mathbf{a}^{-1} = \mathbf{a}/\mathbf{a}^2$.

The inner (symmetric) product of two vectors is defined as $\mathbf{a} \cdot \mathbf{b} \equiv (\mathbf{ab} + \mathbf{ba})/2$.

The outer (antisymmetric) product of two vectors is defined as $\mathbf{a} \wedge \mathbf{b} \equiv (\mathbf{ab} - \mathbf{ba})/2$.

The geometric product $\mathbf{ab}$ is related to the inner product $\mathbf{a} \cdot \mathbf{b}$ and to the outer product $\mathbf{a} \wedge \mathbf{b}$ by

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \wedge \mathbf{a} = 2\mathbf{a} \cdot \mathbf{b} - \mathbf{ba}\,.$$

Also notice $\mathbf{aa} = \mathbf{a} \cdot \mathbf{a}$, $\mathbf{a} \wedge \mathbf{a} = 0$ and $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$.

The geometric interpretation of the geometric product $\mathbf{ab}$ is that

$$\begin{aligned} \mathbf{ab} &= -\mathbf{ba} \quad \text{iff} \quad \mathbf{a} \cdot \mathbf{b} = 0\,, && \text{[orthogonal vectors]} \\ \mathbf{ab} &= +\mathbf{ba} \quad \text{iff} \quad \mathbf{a} \wedge \mathbf{b} = 0\,. && \text{[colinear vectors]} \end{aligned}$$

That is, the geometric product anticommutes for orthogonal (perpendicular) vectors and commutes for colinear vectors.

For any multivectors A, B, C, ..., the scalar part of the geometric product satisfies $\langle AB\rangle_0 = \langle BA\rangle_0$.

Selectors without a grade subscript select for the scalar part, so that $\langle \ldots \rangle = \langle \ldots \rangle_0$.

Reversion satisfies $(AB)^\dagger = B^\dagger A^\dagger$, $\mathbf{a}^\dagger = \mathbf{a}$ and $\langle A^\dagger \rangle = \langle A \rangle^\dagger = \langle A \rangle$.

The unit right-handed pseudoscalar $i$satisfies $i^2 = -1$, $\mathbf{a}i = i\mathbf{a} = \mathbf{a} \cdot i$. Additionally, $i^{-1} = -i$.

The vector cross product $\mathbf{a} \times \mathbf{b}$ is implicitly defined as $\mathbf{a} \wedge \mathbf{b} = i(\mathbf{a} \times \mathbf{b}) = i\mathbf{a} \times \mathbf{b}$ or given explicitly by $\mathbf{a} \times \mathbf{b} = -i(\mathbf{a} \wedge \mathbf{b})$.

Inner and outer products are related by the duality relations: $\mathbf{a} \wedge (i\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})i$ and $\mathbf{a} \cdot (i\mathbf{b}) = (\mathbf{a} \wedge \mathbf{b})i = \mathbf{b} \times \mathbf{a}$.

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[1] Remember: herein the geometric algebra of $\mathcal{E}^n$ is a linear space with a (geometric) product defined. k-vectors, k-blades and multivectors are generated from vectors (elements of $\mathcal{E}^n$), scalar multiplication, vector addition (linear combinations) and vector (geometric) multiplication.

Every multivector A can be uniquely expressed in the expanded form

$$A = \alpha + \mathbf{a} + i\mathbf{b} + i\beta = \sum_{k=0}^{3} \langle A \rangle_k \,,$$

where the k-vector parts are $\langle A \rangle_0 = \alpha$, $\langle A \rangle_1 = \mathbf{a}$, $\langle A \rangle_2 = i\mathbf{b}$ and $\langle A \rangle_3 = i\beta$.

The even part is a quaternion of the form $\langle A \rangle_+ = \alpha + i\mathbf{b}$.

The conjugate $\widetilde{A}$ of A is defined by

$$\widetilde{A} = \langle A^\dagger \rangle_+ - \langle A^\dagger \rangle_- = \alpha - \mathbf{a} - i\mathbf{b} + i\beta.$$

Precedence Convention: to avoid the usage of too many parentheses, adopt the product precedence convention: 1) outer products, 2) inner products and 3) geometric products.

Algebraic identities:

$$\mathbf{a} \cdot$$

Tensor: A

Tensor product: $\mathsf{A} \otimes \mathsf{B}$