APPROXIMATIONS

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ABSTRACT. Only wimps specialize on the general case. Real scientists pursue examples. – Beresford Parlett [1].

1. PRESSURE

Consider a piston of volume v and sectional area a holding an amount n of an ideal gas and consider a force f being applied on a that compresses the fluid.

Such a force generates a pressure p on the gas. See now that p can be viewed as energy density instead of force per unit area. With this view, one finds the external energy applied to the system e_{ext} by

$$e_{\text{ext}} \sim pv\,.$$

This external stimulus makes the gas to perform pv work, the gas internal response e_{int}, given by

$$e_{\text{int}} \sim nk_{\text{gas}}\theta\,,$$

where k_{gas} represents the gas constant and θ the gas temperature.

Thus, according to the energy conservation principle, the external stimulus must be balanced by the gas internal response:

$$e_{\text{ext}} \sim e_{\text{int}} \sim nk_{\text{gas}}\theta \implies k_{\text{gas}}\theta \sim \frac{e_{\text{ext}}}{n}\,.$$

That is, $k_{\text{gas}}\theta$ is a measure of the external energy distributed per amount of gas – molar energy.

On the other hand, since by definition an ideal gas does not interact, its total internal energy equals its kinetic energy alone:

$$e_{\text{int}} \sim mu^2\,,$$

where m represents the gas mass and u the average velocity of the gas particles. Thus, one finds

$$mu^2 \sim nk_{\text{gas}}\theta \implies \theta \sim \frac{mu^2}{nk_{\text{gas}}} \propto u\,.$$

Hence, temperature can also be viewed as a measure of the mean particle velocity of the gas particles.

2. ENERGY

Consider a large, thin concrete slab of thickness l that is *setting*. Setting is an exothermic process that releases e_{th}, where $\dim e_{\text{th}} = [\mathsf{E}/\mathsf{TL}^3]$ – thermal power density. The outside surfaces are kept at the ambient temperature, so the temperature of the walls, θ_{w}, equal the ambient temperature: $\theta_{\text{w}} = \theta\infty$. What is the maximum internal temperature?

Guess. Since the walls are kept at constant temperature, the process is at steady state. However, temperature ranges spatially through the slab thickness. If one measures the spatial variation by x, then the slab temperature satisfies $\theta = \theta[x]$.

By symmetry, the center temperature coincides with θ_{max} at the slab center, $x = 1/2$, and decreases smoothly to a minimum at the walls, $x = 0$ and $x = l$. This symmetry gives room to think about an inverted parabolic temperature distribution inside the slab with the parabola vertex at θ_{max}.

Date: October 28, 2013.
Key words and phrases. mathematical modeling.

Quantity	Symbol	Dimension
Slab temperature	θ	Θ
Slab thermal conduction coefficient	k	Θ
Slab thickness	l	L
Wall temperature	θ_{w}	Θ
Setting power density	e_{th}	$\mathsf{E/TL}^3$
Position within the slab	x	L

TABLE 1. Quantities and dimensions affecting the thermal conduction of the concrete slab setting.

Dim. Analysis. Place a Cartesian coordinate axis running from one wall to the other covering the slab thickness. Let x measure position within $0 \leq xl$. Thus, since the process is at steady state, $\theta = \theta[x]$.

Choose the dimensional set to be $\{\mathsf{E}, \mathsf{L}, \mathsf{T}, \Theta\}$. Hypothesize the quantities governing the phenomenon to be those listed in table 1.

As seen in table 1, according to the Pi-theorem, $6 - 4 = 2$ dimensionless quantities can be constructed. The first one:

$$\Pi_1 = \frac{k\left(\theta - \theta_{\mathrm{w}}\right)/l^2}{e_{\mathrm{th}}}\,,$$

which measures the relationship between energy conduction and energy production. The second dimensionless quantity:

$$\Pi_2 = \frac{x}{l}\,,$$

which is a geometric ratio.

With both dimensionless quantities, one can apply the principle of dimensional homogeneity for physical laws to find

$$\Pi_1 = \phi_\pi[\Pi_2] \implies \frac{k\left(\theta - \theta_{\mathrm{w}}\right)}{e_{\mathrm{th}}l^2} = \phi_\pi\left[\frac{x}{l}\right]\,.$$

Scale temperatures by means of Π_1 and lengths by Π_2; *i.e.*,

$$\overline{\theta} = \Pi_1\theta \qquad \text{and} \qquad \overline{x} = \Pi_2 x\,.$$

Hence, finally, the equation governing the phenomenon can be written as

$$\overline{\theta} = \phi_\pi[\overline{x}]\,. \tag{1}$$

where the function ϕ_π cannot be further determined by dimensional analysis.

Approx. Solution. Assuming a parabolic distribution of temperatures, ϕ_π in eq. (1) can be hypothesize to satisfy

$$\overline{\theta} = a\overline{x}^2 + b\overline{x} + c\,,$$

where $\{a, b, c\}$ are dimensionless quantities to be determined.

Now, we can use a theorem in geometry that states that three points uniquely determine a parabola. Two of these points can be found from the problem statement:

$$\begin{cases} \left[\overline{x} = 0, \overline{\theta} = 0\right] \\ \left[\overline{x} = 1, \overline{\theta} = 0\right]\,. \end{cases}$$

Setting $a = -1/2$ (an inverted parabola) and solving the previous systems of equations, one finds that

$$a = -\frac{1}{2}\,, \qquad b = \frac{1}{2} \qquad \text{and} \qquad c = 0\,.$$

Replacing these values in the hypothesized ϕ_π, one has

$$2\overline{\theta} = \overline{x}\left(1 - \overline{x}\right)\,.$$

Now, using symmetry, when $\overline{x} = 1/2$, then $\overline{\theta} = \overline{\theta_{\max}}$:

$$\overline{\theta_{\max}} = \frac{1}{8}$$

or, returning to the dimensional quantities, $\theta_{\max}$ can be found by

$$\frac{k\left(\theta_{\max} - \theta_{\mathrm{w}}\right)}{e_{\mathrm{th}}l^2} = \frac{1}{8}\,. \qquad \square$$

References

[1] Michael Berry. Two-state quantum asymtotics. *Annals of the New York Academy of Sciences*, 755, April 1995.