# APPLIED GEOMETRIC ALGEBRA

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## 1. Geometric Algebra and Geometric Calculus

****** [blades, k-vectors, reverse, clifs, index notation, einstein convention, metric, gorm, normalization: define "normalize a vector..."]

### 1.1. Geometric Algebra.

Denote by $\mathcal{R}$ the set of real numbers and by $\mathcal{R}^n$ the *n-dimensional real space* defined by $\mathcal{R}^n \doteq \mathcal{R} \otimes \cdots \otimes \mathcal{R}$, where $\otimes$ represents the Cartesian power of $\mathcal{R}$.

Real coordinate space is a vector space, this grants scalar multiplication and vector addition. Geometric product can be used (directly or indirectly, via decomposition into inner and outer products) to calculate lengths, angles, represent rotations, and so on. That is, it transforms real coordinate space into an arena for doing Euclidean geometry. It transforms real spaces into Euclidean spaces.

1.1.1. *Geometric Product.* Let $\mathcal{V}^n$ be an $n$-dimensional linear space and let $a, b, c \in \mathcal{V}^n$. Then, assume a *geometric product of $a$ and $b$*, denoted $ab$, satisfying:

- associativity: $(ab)c = a(bc) = abc\,;$
- left-distributivity: $a(b + c) = ab + ac\,;$
- right-distributivity: $(b + c)a = ba + ca\,;$
- contraction: $aa = a^2 = |a|^2$, where $|a| \in \mathcal{R}^+$ and $|a| = 0$ if and only if $a = 0\,.$

Note that, since $|a| \in \mathcal{R}^+$, thus $a^2 \in \mathcal{R}^+$.

Consider $\mathcal{R}^n$. Then, define the *n-dimensional Euclidean space*, denoted $\mathcal{E}^n$, the set $\mathcal{R}^n$ equipped with a geometric product.

Hereafter, consider the $n$-dimensional linear space $\mathcal{V}^n$ to be a subset of the $n$-dimensional Euclidean space $\mathcal{E}^n$.

1.1.2. *Magnitude of Vectors.* Consider a vector $a \in \mathcal{V}^n$. Then, define the *magnitude of $a$*, denoted $|a|$, by $|a|^2 \doteq aa$. By the contraction axiom, the magnitude of a vector is a scalar.

Consider a vector $\gamma \in \mathcal{V}^n$. Then, call $\gamma$ a *normal vector*, *aka unit vector*, if its magnitude equals unity; *i.e.*, $|\gamma| = 1$.

1.1.3. *Inverse of Vectors.* Consider a non-zero vector $a \in \mathcal{V}^n$. Then, define the *inverse of $a$*, denoted $a^{-1}$, by

$$a^{-1} = \frac{1}{a} \doteq \frac{a}{a^2}\,.$$

Since $a^2$ is a scalar, then $a^{-1}$ maps vectors to vectors.

1.1.4. *Commutator and Anti-commutator Products.* Let $a, b \in \mathcal{V}^n$. Then, define the *anti-commutator product of $a$ and $b$* by $[a, b]_+ \doteq ab + ba$. Similarly, define the *commutator product of $a$ and $b$* by $[a, b]_- \doteq ab - ba$.

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*Key words and phrases.* math notation.

1.1.5. *Symmetric and Anti-symmetric Operators.* Consider a set $\mathcal{S}$. For $a, b \in \mathcal{S}$, let $*$ be a binary operation between $a$ and $b$; *i.e.*, $(a * b) \in \mathcal{S}$. Call the operation *symmetric* if it satisfies $a * b = b * a$. Call the operation *anti-symmetric* if it satisfies $a * b = -b * a$.

1.1.6. *Inner Product of Vectors.* Let $a, b \in \mathcal{V}^n$. Then, define the *inner product of $a$ and $b$* by

$$a \cdot b \doteq [a, b]_+ = ab + ba\,.$$

For *vectors*, the inner product is symmetric. Let $a, b \in \mathcal{V}^n$, then $a \cdot b = b \cdot a$.

The inner product of two vectors results in a scalar. Let $a, b \in \mathcal{V}^n$, then $(a \cdot b) \in \mathcal{R}$.

The inner product of a vector by itself equals its squared. Let $a \in \mathcal{V}^n$, then $a \cdot a = a^2$.

1.1.7. *Outer Product of Vectors.* Let $a, b \in \mathcal{V}^n$. Then, define the *inner product of $a$ and $b$* by

$$a \wedge b \doteq [a, b]_- = ab - ba\,.$$

For *vectors*, the outer product is anti-symmetric. Let $a, b \in \mathcal{V}^n$, then $a \cdot b = -b \wedge a$.

The outer product of a vector by itself equals zero. Let $a \in \mathcal{V}^n$, then $a \wedge a = 0$.

1.1.8. *Canonical Decomposition of the Geometric Product of Vectors.* Let $a, b \in \mathcal{V}^n$. Then, write the geometric product of $a$ and $b$ as the sum of a symmetric and anti-symmetric parts:

$$ab = \frac{1}{2}\,[a, b]_+ + \frac{1}{2}\,[a, b]_- = a \cdot b + a \wedge b\,.$$

Call the last equation the *canonical decompostion of the geometric product of vectors.*

1.1.9. *Parallel, Colinear, Perpendicular and Orthogonal Vectors.* Consider two vectors $a, b \in \mathcal{V}^n$. Then, if $a$ and $b$ are *parallel* or *colinear*, denoted $a \parallel b$, then $a \wedge b = 0$ and thus their product commutes; *i.e.*, $ab = a \cdot b = ba$. On the other hand, if $a$ and $b$ are *perpendicular*, *aka orthogonal*, denoted $a \perp b$, then $a \cdot b = 0$ and thus their product anti-commutes; *i.e.*, $ab = a \wedge b = -ba$.

1.1.10. *Frames.* The vectors of a set $a_1, a_2, \ldots, a_r$ are *linearly independent* if and only if the $r$-blade

$$A_{\bar{r}} = a_1 \wedge a_2 \wedge \cdots \wedge a_r$$

is not zero.

Call an ordered set of vectors $\{a_k : k = 1, 2, \ldots, n\}$ in an $n$-dimensional linear space $\mathcal{V}^n$ a *frame for $\mathcal{V}^n$*, *aka* a *basis for $\mathcal{V}^n$*, if and only if the vectors are linearly independent.

Consider an $n$-dimensional linear space $\mathcal{V}^n$. Then, call an ordered set of vectors $\{\gamma_k; 1 \ldots n\}$ in $\mathcal{V}^n$ a *standard frame for $\mathcal{V}^n$* if the vectors are linearly independent, mutually orthogonal and normal. Mathematically, standard frame elements $\{\gamma_k\}$ satisfy

$$g_{kl} = \gamma_k \cdot \gamma_l = \frac{1}{2}(\gamma_k\gamma_l + \gamma_l\gamma_k)\,,$$

where $g_{kl}$ represent the *metric coefficients.*

1.1.11. *Reciprocal Frames.* Given a standard frame in $\mathcal{V}^n$, then define a *reciprocal frame* $\{\gamma^k = {\gamma_k}^{-1}\}$ by

$$\gamma_k = g_{kl}\gamma^l \qquad \text{or} \qquad \gamma_k \cdot \gamma^l = \delta_k^l \,.$$

To find the reciprocal frame element $\gamma^k$, apply the expression

$$\gamma^k = (-1)^{k-1}\gamma_1 \wedge \cdots \wedge \check{\gamma_k} \wedge \ldots \gamma_n \, i^{-1} \,,$$

where $\check{\gamma_k}$ means that $\gamma^k$ must be omitted from the product.

*Example.* Consider a standard frame $\{\gamma_k; 1 \ldots 3\}$. Then, construct the reciprocal frame $\{\gamma^k\}$.

*Solution.* In this case, the standard frame elements are $\{\gamma_1, \gamma_2, \gamma_3\}$. Thus, the unit pseudoscalar is $i = \gamma_{123}$ and its inverse $i^{-1} = -i$.

To find the reciprocal frame elements, apply the last equation for each $\gamma^k$; *i.e.*,

$$\begin{aligned}
\gamma^1 &= (-1)^{1-1}\gamma_2 \wedge \gamma_3 \,(-\gamma_{123}) = \gamma_1 \,, \\
\gamma^2 &= (-1)^{2-1}\gamma_1 \wedge \gamma_3 \,(-\gamma_{123}) = \gamma_2 \,, \\
\gamma^3 &= (-1)^{3-1}\gamma_1 \wedge \gamma_2 \,(-\gamma_{123}) = \gamma_1 \,.
\end{aligned}$$

In other words, the standard frame elements equal its reciprocal frame elements.

1.1.12. *Coordinates.* Consider a standard frame $\{\gamma_k\}$ for $\mathcal{V}^n$ and its reciprocal frame $\{\gamma^l\}$ such that $g_k^l = \gamma_k \cdot \gamma^l$. Then, for each vector $a \in \mathcal{V}^n$, define a set of *rectangular coordinates* $\{a^k\}$ given by

$$a^k \doteq \gamma^k \cdot a \qquad \text{and} \qquad a = a^k \gamma_k \,.$$

Also, call the coordinates $\{a^k\}$ the *components of $a$ onto the frame* $\{\gamma_k\}$.

In a similar way, define a set of rectangular coordinates $\{a_k\}$ given by

$$a_k \doteq \gamma_k \cdot a \qquad \text{and} \qquad a = a_k \gamma^k \,.$$

Also, call the coordinates $\{a_k\}$ the *components of $a$ onto the frame* $\{\gamma^k\}$.

Note that, since the vector $a$ is a geometric object – independent on any frame, then both representations of $a$ must be equal:

$$a = \gamma_k a^k = \gamma^k a_k \,.$$

1.1.13. *Metric.* Call $\mathcal{E}^n$ the *$n$-dimensional flat space*. The *flat space metric $g$* equals the *Kronecker delta $\delta$*; that is, $g_{kl} = \delta_{kl}$.

In $\mathcal{V}^n$, the *unit pseudoscalar $i$* is given by $i = \gamma_1\gamma_2 \cdots \gamma_n$.

*** convention $\gamma_1\gamma_2 = \gamma_{12}$. Some examples of manipulation of the notation and the anti-symmetric property of outer product. *** find i for 3d and inv i for 3d.

1.1.14. *Clifs.* Let $C \in \mathcal{G}^n$. Then, define the *gorm of $C$* by

$$\operatorname{gorm} C \doteq \langle C^\dagger C \rangle_0 \,.$$

1.2. **Geometric Calculus.** Let $x \in \mathcal{V}^n$ be the position vector, then call a *scalar field $\phi$* a function $\phi : x \mapsto a$, where $a \in \mathcal{R}$; *i.e.*, a scalar field *maps* the position vector to a scalar. Analogously, call a *vector field $\Phi$* a function $\Phi : x \mapsto v$, where $v \in \mathcal{V}^n$; *i.e.*, a vector field *maps* the position vector to a vector. Alternatively, a scalar field can be seen as a function that *assigns* a scalar to every point in $\mathcal{V}^n$, while a vector field can be seen as a function that *assigns* a vector to every point in $\mathcal{V}^n$.

Let $x \in \mathcal{V}^n$ be the position vector and $\{x^k\}$ be the components of $x$ onto a standard frame. Then, agree on the *delta derivative*, $\partial$, notation for partial derivatives:

$$\partial_k = \partial_{x^k} \doteq \frac{\partial}{\partial x^k} \,.$$

Sometimes, the *comma derivative* notation is used instead of the delta derivative. The comma derivative consists on appending to a function a subscript containing a comma and the variable with respect to which the partial derivative is to be taken.

*Example.* Consider a vector $x \in \mathcal{V}^n$ whose components on a given frame are $\left[x^1, x^2, \ldots, x^n\right]$ and let $f$ be a function of $x$; *i.e.*, $f[x] = f[[x_1, x_2, \ldots, x_n]]$. Then, represent the partial derivative of $f$ with respect to the $k$-component of $x$ ($k$th variable) by

$$f_{,k} = f_{,x^k} \doteq \frac{\partial f}{\partial x^k}\,. \qquad \square$$

Consider a reciprocal frame $\left\{\gamma^k\right\}$, then define the geometric derivative $\bigtriangledown$ by

$$\bigtriangledown \doteq \gamma^k \partial_k\,.$$

Treat $\bigtriangledown$ as a vector.

Consider $\phi$ to be a scalar field. Then, define the *gradient of* $\phi$, denoted $\operatorname{grad}\phi$, by

$$\operatorname{grad}\phi \doteq \bigtriangledown\phi = \gamma^k \partial_k \phi = \gamma^k \phi_{,k}\,.$$

*Example.* Consider $\mathcal{E}^3$ and consider a scalar field $\phi : x \mapsto x^2 + y^2 + z^2$. Then, find $\operatorname{grad}\phi$.
*Solution.* The position vector $x$ is implicitly given by the coordinates $[x, y, z]$, thus a standard frame could be defined by $\{\gamma_x, \gamma_y, \gamma_z\}$. However, instead of expanding the geometric derivative onto any frame and then applying it to $\phi$, the solution to the problem is simplified by working directly with geometric algebra. This is done by noticing that $\phi = xx = x^2$. Thus, $\operatorname{grad}\phi = \bigtriangledown x^2 = 2x$. Now, expand $x$ onto the frame to find

$$\operatorname{grad}\phi = 2(x\gamma^x + y\gamma^y + z\gamma^z)\,. \qquad \square$$

*Verification.* The geometric derivative onto the frame $\{\gamma_x, \gamma_y, \gamma_z\}$ takes the form

$$\bigtriangledown = \gamma^k \partial_k = \gamma^x \partial_x + \gamma^y \partial_y + \gamma^z \partial_z\,.$$

Find the gradient of the field by applying geometric derivative to it:

$$\operatorname{grad}\phi = \bigtriangledown\phi = \gamma^x \frac{\partial\phi}{\partial x} + \gamma^y \frac{\partial\phi}{\partial y} + \gamma^z \frac{\partial\phi}{\partial z}\,,$$

that is,

$$\operatorname{grad}\phi = 2(x\gamma^x + y\gamma^y + z\gamma^z)\,,$$

which agrees with the previous result.

## 2. Vector Analysis Summary

2.1. **Basic Concepts.** Call a *vector quantity* $q$ a quantity that has a *magnitude* and a *direction* associated with it.

Here, magnitude means a positive real number and direction is specified relative to some underlying reference frame *To be defined below.* that we regard as fixed.

Call a *vector* an abstract quantity characterized by the two properties magnitude and direction. Thus, two vectors are equal if they have the same magnitude and the same direction.

2.2. **Vector Space.** A *vector space over a field* $\mathcal{F}$ is a set $\mathcal{V}$ together with two binary operations that satisfy the eight axioms listed below. Elements of $\mathcal{V}$ are called *vectors*. Elements of $\mathcal{F}$ are called *scalars*. The first operation, *vector addition*, takes any two vectors $v$ and $w$ and assigns to them a third vector commonly written as $v + w$; call this operation the *sum of* $v$ *and* $w$. The second operation takes any scalar $a$ and any vector $v$ and gives another vector $av$; call this operation the *scalar multiplication of* $v$ *by* $a$.

2.3. **Inner Product Algebra.** Consider three vectors $a, b, c \in \mathcal{E}^n$ and a scalar $\lambda \in \mathcal{R}$. Then, the inner product between $a$ and $b$ satisfies:

- commutativity: $a \cdot b = b \cdot a\,;$
- distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c\,.$
- associativity with scalar multiplication: $(\lambda a) \cdot b = \lambda(a \cdot b)\,.$

## 3. Newtonian Physics

**3.1. A Bit of History.** Newton placed his theory based on universal space and universal time; *i.e.*, space and time are independent on any external influences and on each other; In Newton's theory the position of a particle is represented by one vector $x$ with three components each of which depends on time; *i.e.*, $x[t]$.

Einstein, on the other hand, placed time and space on equal footing with his Theory of special relativity by noting that $ct$, where $c$ is the speed of light, has the dimensions of length. Minkowski, finally, united space and time in one single entity, spacetime, both conceptually and mathematically. In spacetime, the position of a particle is represented by one vector with four components; *i.e.*, the partner of $x$, $y$ and $z$ is not $t$, but rather $ct$.

**3.2. The Geometric Principle.** Physics and geometry are deeply related. So physical objects and physical processes can be modeled using geometric objects and geometric transformations. On the other hand, geometric algebra provides an efficient language to deal with geometry via algebra.

Geometric Principle:

> The laws of physics must all be expressible as geometric (coordinate-independent and reference-frame-independent) relationships – geometric transformations – between geometric objects, which represent physical entities.

There are three different conceptual frameworks for the classical laws of physics and, thus, three different geometric arenas for the laws:

(1) *General Relativity* formulates the laws as geometric relationships between geometric objects in he arena of *curved 4-dimensional spacetime*.

(2) *Special Relativity* is the limit of general relativity in the complete absence of gravity; its arena is *flat, 4-dimensional spacetime*, *aka* Minkowski spacetime.

(3) *Newtonian Physics* is the limit of general relativity when (i) gravity is weak but not necessarily absent, (ii) relative speeds of particles and materials are small compared to the speed of light $c$ and (iii) all stresses (pressures) are small compared to the total density of mass-energy; its arena is *3-dimensional Euclidean space* with time separated off and made universal – by contrast with relativity's reference-frame-dependent time.

The aim is thus to express all physical quantities and laws in a *geometric form*: a form that is independent of any coordinate system or basis vectors.

We shall insist that the Newtonian laws of physics all obey a *Geometric Principle*: they are all geometric relationships between geometric objects, expressible without the aid of any coordinates or bases. An example is the Lorentz force law: $m\mathrm{d}d/\mathrm{d}t = q(E + v \times B)$ – a (coordinate-free) relationship between the geometric (coordinate-independent) vectors $v$, $E$ and $B$ and the scalars (the particle's) mass $m$ and charge $q$; no coordinates are needed for this law of physics, nor is any description of the geometric objects as matrix-like entities. Components are secondary; they only exist after one has chosen a set of basis vectors. Components are an impediment to a clear and deep understanding of the laws of physics. The coordinate-free, component-free description is deeper and – once one becomes accustomed to it – much more clear and understandable.

Besides, coordinate independence and basis independence strongly constrain the laws of physics. This suggests that

> Nature's physical laws *are* geometric and have nothing whatsoever to do with coordinates or components or vector bases.

3.3. **Foundational Concepts.** To lay the geometric foundations for the Newtonian laws of physics an flat, Euclidean space, consider some foundational geometric objects: points, scalars, vectors, (geometric product of vectors), inner product of vectors, distance between points.

The arena for the Newtonian laws is a spacetime composed of the 3-dimensional Euclidean space $\mathcal{E}^3$, called *3-space*, and a *universal time* $t$. Denote *points* (locations) in 3-space by capital script letters, such as $\mathcal{P}$ and $\mathcal{Q}$. These points and the 3-space where they live require no coordinates for their definition.

A *scalar* is a single number associated with a point, say $\mathcal{P}$, in 3-space. We are interested in scalars that represent physical quantities; *e.g.*, temperature $T$. When a scalar is a function of location $\mathcal{P}$ in space – *e.g.*, $T[\mathcal{P}]$, call it a *scalar field*.

A *vector* in 3-space can be thought of as a straight arrow reaching from one point, $\mathcal{P}$, to another, $\mathcal{Q}$; *i.e.*, $\Delta\, x$. Equivalently, $\Delta\, x$ can be thought of as a direction at $\mathcal{P}$ and a number, the *vector's length*. Sometimes, one point $\mathcal{O}$ is selected in 3-space as an "origin" and other points, $\mathcal{P}$ and $\mathcal{Q}$, are identified by their vectorial separations $x_{\mathcal{P}}$ and $x_{\mathcal{Q}}$ from that origin.

The *Euclidean distance* $\Delta\,\xi$ between two points $\mathcal{P}$ and $\mathcal{Q}$ is a scalar that requires no coordinate system for its definition. This distance $\Delta\,\xi$ is also the magnitude (length) $|\Delta\, x|$ of the vector $\Delta\, x$ that reaches from $\mathcal{P}$ to $\mathcal{Q}$ and the square of that length is denoted by

$$|\Delta\, x|^2 = \Delta\, x\, \Delta\, x = \Delta\, x^2 \doteq \Delta\,\xi^2 \,.$$

Of particular importance is the case when $\mathcal{P}$ and $\mathcal{Q}$ are neighboring points and $\Delta\, x$ is a *differential quantity* $\mathrm{d}x$. By traveling along a sequence of such $\{\mathrm{d}x\}$, laying them down tail-at-tip, one after another, we can map out a *curve* to which these $\{\mathrm{d}x\}$ are tangent. The curve is $\mathcal{P}[\lambda]$, with $\lambda$ a *parameter along the curve*, and the vectors that map it out are

$$\mathrm{d}x = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\lambda}\mathrm{d}\lambda\,.$$

The product of a scalar with a vector is still a vector; so if we take the change of location $\mathrm{d}x$ of a particular element of a fluid during a (universal) *time interval* $\mathrm{d}t$ and multiply it by $1/\mathrm{d}t$, then we obtain a new vector, the fluid element's *velocity* $v = \mathrm{d}x/\mathrm{d}t$, at the fluid element's location $\mathcal{P}$. Perfirming this operation at every point $\mathcal{P}$ in the fluid defines the *velocity field* $v[\mathcal{P}]$. Similarly, the sum (or difference) of two vectors is also a vector and so taking the difference of two velocity measurements at times separated by $\mathrm{d}t$ and multiplying by $1/\mathrm{d}t$ generates the *acceleration* $a = \mathrm{d}v/\mathrm{d}t$. Multiplying by the fluid element's (scalar) *mass* $m$ gives the force $f = ma$ that produced the acceleration; dividing by an electrically produced force by the fluid element's charge $q$ gives another vector, the electric field $E = f/q$ and so on. We can define *inner product of pairs of vectors* (*e.g.*, force and displacement) to obtain a new scalar(*e.g.*, work) and *cross products of vectors* to obtain a new vector (*e.g.*, torque). By examining how a differentiable scalar field changes from point to point, we can define its *gradient*. Thus, we can construct all of the standard scalars and vectors of Newtonian physics. What is important is that

> these physical quantities require *no* coordinate system for their definition.

They are geometric (coordinate-independent) objects residing in Euclidean 3-space at a particular time.

We can summarize this by stating that

> the Newtonian physical laws are *all* expressible as geometric relationships between these types of geometric objects and these relationships do *not* depend upon any coordinate system or orientation of axes, nor on any reference frame (on any purported velocity of the Euclidean space in which the measurements are made).

This principle is called the *Geometric Principle* for the laws of physics. It is the Newtonian analog of Einstein's Principle of Relativity.

## 4. Tensors

4.1. **Tensor Algebra without a Coordinate System.** We introduce, in a coordinate-free way, some fundamental concepts of differential geometry: tensors, the inner product, the metric tensor, the tensor product and contraction of tensors.

Because our space is flat, there is a unique way to transport one vector from one location to another while *keeping its length and direction unchanged*. Thus, vectors are unaffected by such a transport. Therefore, a vector is completely determined by its length and direction.

A *rank-$n$ tensor* $T$ is, by definition, a real-valued, linear function of $n$ vectors. We can pictorially see a tensor as a machine that has $n$ slots on top, into which $n$ vectors are inserted, and one slot in its end, out of which a scalar is returned: the value that the tensor $T$ has when evaluated as a function of the $n$ inserted vectors. Notationally, denote tensors by $T$

$$T[-,-,-,-] \qquad [n \text{ slots in which to put the vectors}]\,.$$

If $T$ is a rank-3 tensor (has 3 slots), this its value on the vectors $a$, $b$, $c$ will be denoted by $T[a,b,c]$. Linearity of this function can be expressed as

$$T[ea+fb,c,d] = eT[a,c,d] + fT[b,c,d]\ ,$$

where $e$ and $f$ are real numbers.

Given two vectors $a$ and $b$, the *inner product of $a$ and $B$*, denoted $a \cdot b$, is defined in terms of the squared magnitude by

$$a \cdot b \doteq \frac{1}{4}[(a+b)^2 - (a-b)^2]\,.$$

In Euclidean space, this is the standard inner product.

Because the inner product is a linear function of each of its vectors, regard it as a tensor of rank 2. Then, the inner product is denoted $g-,-$ and is called the *metric tensor*. In other words, the metric tensor $g$ is that linear function of two vectors whose value is given by

$$g[a,b] \doteq a \cdot b\,.$$

Because the inner product is symmetric, then the metric tensor is *symmetric* in its two slots [1]; *i.e.*,

$$g[a,b] = g[b,a]\ .$$

With the aid of the inner product, regard any vector $a$ as a tensor of rank one: the real number that is produced when an arbitrary vector $c$ is inserted into $a$'s slot is

$$a[c] = a \cdot c\,.$$

Second-rank tensors appear frequently in the laws of physics – often in roles where one sticks a single vector into the second slot and leaves the first slot empty, thereby producing a single-slotted entity – a *vector*. An example is a rigid body's (Newtonian) moment-of-inertia tensor $I[-,-]$. Insert the body's angular velocity vector $\omega$ into the second slot and you get the body's angular momentum vector $J[-] = I[-,\omega]$. (In slot-naming notation, every slot could be regarded as an index, so the angular momentum vector equation can be written as $J^j = I_{jk}\omega^k$.)

From three (or any number of) vectors $a, b, c$ we can construct a tensor, their *tensor product* defined by

$$a \otimes b \otimes c[e,f,g] \doteq a[e]\,b[f]\,c[g] = (a \cdot e)(b \cdot f)(c \cdot g)\,.$$

[1] In slot-naming notation, to be developed afterwards, the components of $g$ (in a given frame $\{\gamma_k\}$) are $g_{kl}$; this means that $g$ is "waiting" to accept two vectors – this explains the notation $g[-,-]$. When a vector, say $a$, is plugged in, then $g[-,a]$ and thus $g_{kl}a^l = a_k$; *i.e.*, the result is the reciprocal component of $a$. When two vectors, say $a$ and $b$, are plugged in, then $g[a,b] = a^k g_{kl} b^l = a^k b_k$. In other words, the $g$ "disapears", leaving behind the inner product of $a$ and $b$.

Here, the first expression is the notation for the value of the new tensor, $a \otimes b \otimes c$ *evaluated* on the three vectors $e, f, g$; the middle expression is the ordinary product of three real numbers; the value of $a$ on $e$, the value of $b$ on $f$ and the value of $c$ on $g$ and the third expression is that same product with the htree numbers rewritten as scalar products. Similar definitions can be given for the tensor product of any two or more tensors of any rank; *e.g.*, if $T$ has rank 2 (2 slots) and $S$ has rank 3 (3 slots), then

$$T \otimes S[e, f, g, h, j] \doteq T[e, f]\, S[g, h, j]\ .$$

One last geometric (frame-independent) concept we need is *contraction* introduced by example. From two vectors $a$ and $b$ we can construct the tensor product $a \otimes b$ (a second rank tensor) and we can construct the scalar product $a \cdot b$ (a real number, *aka* a scalar, *aka* a rank-0 tensor). The process of *contraction is the construction of $a \cdot b$ from $a \otimes b$*:

$$\mathrm{cont}(a \otimes b) \doteq a \cdot b\,.$$

It can be shown that any second-rank tensor $T$ can be expressed as a sum of tensor products of vectors; *viz.*,

$$T = a \otimes b + c \otimes d + \cdots\ ;$$

correspondingly, define the contraction of $T$ to be

$$\mathrm{cont}\, T = a \cdot b + c \cdot d + \cdots\ .$$

Note that this contraction lowers the rank of the tensor by two, from 2 to 0. Similarly, for a tenor of rank $n$, one can construct a tensor of rank $n - 2$ by contraction, but in this case, one must specify whch slots are to be contracted. For instance, if $T$ is a third rank tensor, expressible as $T = a \otimes b \otimes c + e \otimes f \otimes g + \cdots$, then the contraction of $T$ on its first and third slots is the rank-1 tensor (vector)

$$\mathrm{cont}_{1,3}(a \otimes b \otimes c + e \otimes f \otimes g + \cdots) = (a \cdot c)b + (e \cdot g)f + \cdots\ .$$

Note that the inner product and tensor contraction lower the rank of tensors, while the tensor product increases it.

All the concepts developed in this section (vectors, tensors, metric tensor, inner product, tensor product and contraction of a tensor) can be carried over, with no change whatsoever, into *any* vector space over the real numbers that is endowed with a concept of squared length, even if the squared length is *not* Euclidean; *e.g.*, the four-dimensional spacetime of special relativity.

4.2. **Component Representation of Tensor Algebra.** In space $\mathcal{E}^3$ of Newtonian physics, there is a unique *orthonormal frame* whose vectors $\{\gamma_x, \gamma_y, \gamma_z\} = \{\gamma_1, \gamma_2, \gamma_3\} = \{\gamma_k\}$ associated with any *Cartesian coordinate system* $\{x, y, z\} = \left\{x^1, x^2, x^3\right\} = \{x_1, x_2, x_3\}$ (In Cartesian coordinates in Euclidean space [2], place indices up or down. It doesn't matter. By definition, in Cartesian coordinates a quantity is the same whether its index is down or up). The frame element $\gamma_k$ has unit length and points along the $x^k$ coordinate direction, which is orthogonal to all the other coordinate directions, so this could be summarized by

$$\gamma_k \cdot \gamma_l = \delta_{kl}\ .$$

Any vector $a \in \mathcal{E}^3$ can be expanded in terms of these frames:

$$a = a^k \gamma_k \qquad \text{and} \qquad a = a_k \gamma^k\ .$$

Adopt Einstein summation convention: repeated indices (in this case $k$) are to be summed (in this $\mathcal{E}^3$ case over $j = 1, 2, 3$). By virtue of the orthonormality of the frame, the components $a_k$ or $a^k$ of $a$ can be computed as the scalar products:

$$a_k = a \cdot \gamma_k \qquad a^k = a \cdot \gamma^k\ .$$

[2] Cartesian coordinates means rectangular coordinates; *i.e.*, the frame elements all measure lengths and are orthonormal to each other. Euclidean space means that the metric is the Euclidean metric; *i.e.*, $g = \delta$.

(The proof of this is straightforward: $a \cdot \gamma_j = (a^k \gamma_k) \cdot \gamma_j = a^k (\gamma_k \cdot \gamma_j) = a^k \delta_{kj} = a_j$. Similarly, for $a^k$.)

Any tensor, say the third-rank tensor $T[-,-,-]$, can be expanded in terms of tensor products of the frame elements:

$$T = T^{ijk} \gamma_i \otimes \gamma_j \otimes \gamma_k \, .$$

The components $T_{ijk}$ of $T$ can be computed from $T$ and the frame elements by the generalization of $a_k = a \cdot \gamma_k$:

$$T_{ijk} = T[\gamma_i, \gamma_j, \gamma_k] \ .$$

*Proof.* Generalize $a_k = a \cdot \gamma_k$:

$$T_{ijk} = T[\gamma_i, \gamma_j, \gamma_k] = T(\gamma_i) T(\gamma_j) T(\gamma_k) = (T \cdot \gamma_i)(T \cdot \gamma_j)(T \cdot \gamma_k) \, . \qquad \square$$

As an important example, the components of the metric are $g_{jk} = g[\gamma_j, \gamma_k] = \gamma_j \cdot \gamma_k = \delta_{jk}$:

$$g_{jk} = \delta_{jk} \qquad \text{in any orthonormal frame in space } \mathcal{E}^3.$$

The components of a tensor product, *e.g.*, $T[-,-,-] \otimes S[-,-]$, are deduced by inserting the frame elements into the slots; they are $T[\gamma_i, \gamma_j, \gamma_k] \otimes S[\gamma_l, \gamma_m] = T_{ijk} S_{lm}$. In words,

> the components of a tensor product are equal to the ordinary arithmetic product of the components of the individual tensors.

In component notation, the inner product of two vectors and the value of a tensor when vectors are inserted into its slots are given by

$$a \cdot b = a^k b_k \qquad \text{and} \qquad T[a, b, c] = T_{ijk} a^i b^j c^k \, ,$$

as one can show using the previous equations. Finally, the contraction of a tensor, say the fourth rank tensor $R[-,-,-,-]$, on two of its slots, say the first and third, has components that are computed from the tensor's own components:

$$\mathrm{comp}(\mathrm{cont}_{1,3} R) = R_{ijik} \, .$$

Note that $R_{ijik}$ is summed on the index $i$, so ti has only two free indices $j$ and $k$ and, thus, is the component of a second rank tensor, as it must be if it is to represent the contraction of a fourth-rank tensor.

4.3. **Slot-Naming Notation.** Consider the rank-2 tensor $F[-,-]$. Define a new tensor $G[-,-]$ to be the same as $F$, but with the slots interchanged; *i.e.*, for two vectors $a$ and $b$ it is true that $G[a, b] = F[b, a]$. To simply indicate that $F$ and $G$ are equal but with the slots interchanged is by the notation $G[-_a, -_b] = F[-_b, -_a]$ or, in index notation, by $G_{ab} = F_{ba}$. This relationship is valid in a particular frame if and only if $G = F$ with slot interchanged is true. Then, look at any "index equation", such as $G_{ab} = F_{ba}$, momentarily as a relationship between components of tensors in a specific frame and then do a quick mind-flip and regard it quite differently: as *a relationship between geometric, frame-independent tensors with the indices playing the roles of names of slots.*

As an example of the power of this *slot-naming notation*, consider the contraction of the first and third slots of a third-rank tensor $T$. In any frame, the components of $\mathrm{cont}_{1,3} T$ are $T_{aba}$. Correspondingly, in slot-naming index notation, denote $\mathrm{cont}_{1,3} T = T_{aba}$. Say that the first and the third slots are "strangling each other", leaving free only the second slot (named $b$) and therefore producing a rank-1 tensor (a vector).

4.4. **Linear Maps.** A *linear map*, *aka linear transformation*, is a function between two *vector spaces* that preserves the operations of vector addition and scalar multiplication. Thus, it *always* maps linear subspaces to linear subspaces – like straight lines to straight lines or to a single point. When a linear map maps from a vector space to itself, it is called *endomorphism* or "linear operator".

*Definition.* Consider two vector spaces $\mathcal{V}$ and $\mathcal{W}$ over the reals $\mathcal{R}$, consider two vectors $x, y \in \mathcal{V}$ and a scalar $\alpha \in \mathcal{R}$. Then, define a *linear map* to be a function $f : \mathcal{V} \mapsto \mathcal{W}$ if it satisfies:

$$\begin{aligned} f[x+y] &= f[x] + f[y] \ , && \text{[additivity]} \\ f[\alpha x] &= \alpha f[x] \ . && \text{[homogeneity of degree 1]} \end{aligned}$$

Equivalently, $f$ is a linear map if it satisfies additivity and homogeneity for any *linear combination of vectors*; *i.e.*, consider a set of vectors $\{\gamma_1, \ldots, \gamma_m\}$ in $\mathcal{V}$ and scalars $\left\{a^1, \ldots, a^m\right\}$ in $\mathcal{R}$. Then, if $f$ is a linear map, it satisfies

$$f\left[a^k \gamma_k\right] = \sum_{k=1}^{m} a^k f[\gamma_k] \ , \tag{4.1}$$

where Einstein summation convention and sigma notation were used [3].

4.5. **Linear Functionals.** A *linear functional*, *aka* linear form, one-form or covector, is a linear map from a vector space to its field of scalars. In $\mathcal{R}^n$, if *vectors* are represented as *column vectors*, then *linear functionals* are represented as *row vectors* and their action on vectors is given by the *dot product*, or the matrix product with the row vector on the left and the column vector on the right. In general, if $\mathcal{V}$ is a vector space over a field $\mathcal{K}$, then a linear functional $f$ is a function from $\mathcal{V}$ to $\mathcal{K}$, which is linear.

More formally, we have the definition:

*Definition.* Define a *linear functional* a function $f : \mathcal{V} \mapsto \mathcal{R}$ if it is a linear map; *i.e.*, if it satisfies eq. (4.1).

[3] In traditional notation: a linear map $f$ satisfies $f\left[a^1\gamma_1 + \cdots + a^m\gamma_m\right] = a^1 f[\gamma_1] + \cdots + a^m f[\gamma_m]$ . for the set $\{\gamma_k\}$ of vectors and the set $\left\{a^k\right\}$ of scalars.

# 5. Examples

5.1. **Geometric Algebra.** Some examples on the usage of geometric algebra applied to physics.

*Example.* Write Lorentz force law using the geometric algebra formalism.

Two changes are needed for Lorentz force law to agree with geometric algebra: change in notation and the replacement of the cross product with the outer product. We solve the replacement of the cross product in two ways.

*Solution.* Notation change: using IUPAC recommendations, the Lorentz force can be written as

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}),$$

where the force $\vec{F}$, the electric field $\vec{E}$, the particle velocity $\vec{v}$ and the magnetic induction $\vec{B}$, *aka* magnetic field, are modeled by vectors in $\mathcal{R}^3$ and the electric charge $Q$ by a scalar in $\mathcal{R}$.

Then, since geometric algebra puts all of its members on equal footing, it is common to denote scalars and vectors by undecorated lower case variables, so $\vec{F} \to f$, $\vec{E} \to e$, $\vec{B} \to b$ and $Q \to q$. With these associations, Lorentz force law, therefore, becomes

$$f = q(e + v \times b). \tag{5.1}$$

Not only does this rewritten equation look more elegant – at least to the writer's eyes, it is actually easier to work with it.

*Solution.* Cross product replacement: Consider eq. (5.1). Using the geometric algebra of space $\mathcal{G}^3$, replace the cross product by its definition in terms of the outer product:

$$v \times b = -i(v \wedge b) = i(b \wedge v) = (b \wedge v)i, \tag{5.2}$$

where $i$ is the unit pseudoscalar in $\mathcal{G}^3$. See that the anti-commutative property of the outer product and the commutativity property [4] of $i$ were used.

Then, plug in the last equalities of eq. (5.2) in Lorentz force law to find,

$$f = q(e + i(b \wedge v)) = q(e + (b \wedge v)i). \qquad \square$$

*Note.* The cross product $v \times b$ was replaced with the outer product $(b \wedge v)i$, via $i$. Algebraically, in $\mathcal{G}^3$, $b \wedge v$ is a bivector and $i$ a trivector, thus their product $(b \wedge v)i$ yields a vector, which agrees with the result of $v \times b$. Geometrically, on the other hand, $v \times b$ *yields* a vector perpendicular to the plane formed by $v$ and $b$, whereas $b \wedge v$ *represents* the plane formed by $b$ and $v$ and thus $(b \wedge v)i$ yields the *dual* of $b \wedge v$; that is, a vector perpendicular to the $b \wedge v$ plane. Therefore, the result, $(b \wedge v)i$, is algebraically and geometrically equivalent to $v \times b$; *i.e.*, no information was lost during the conversion.

The reason for the replacement is that the outer product is more fundamental than the cross product. The cross product exists only in 3-space $\mathcal{V}^3$, while the outer product can be defined in $n$-dimensions. Additionally, the cross product yields a vector perpendicular to its operands, whereas the outer product represents the plane itself formed by its operands; in other words, while the cross product uses local geometrical information to *yield* non-local geometrical information – non-local geometry, the outer product *uses* local geometrical information to *represent* local geometrical information – local geometry.

The next solution refines the replacement of the cross product by using the duality property between vectors and bivectors in $\mathcal{G}^3$.

*Solution.* Considering the identity $v \times b = -(v \wedge b)i$, use $i$ to interchange the outer product with the inner product, via the identity $(x \wedge y)i = x \cdot (yi)$, for vectors $x, y \in \mathcal{G}^3$:

$$v \times b = -v \cdot (bi).$$

Next, since $b$ is a vector in $\mathcal{G}^3$, then $bi$ is its dual – a bivector. Thus, the product $v \cdot (bi)$ anti-commutes:

$$v \times b = (bi) \cdot v.$$

[4] In $\mathcal{G}^3$, $i$ commutes with every other member of the algebra.

Next, replace the last result in Lorentz force law:

$$f = q(e + (bi) \cdot v)\,. \qquad \square$$

5.2. **Problem Solving: Guessing, Dimensional Analysis, Approximate Solutions, Formal Analytic Solutions.** To illustrate various problem solving techniques, we will analyze the motion of a charged particle using Newtonian Physics. We will do so by showing various math and physics methods in different levels of sophistication: guessing, dimensional analysis, approximations and analytic techniques. Finally, we present a final wrapped-up solution.

*Example.* Consider a particle of constant electric charge $q$ and constant mass $m$ moving with velocity $v$ due to an interaction with a constant electromagnetic field. Assuming Newtonian physics, find the rate at which the particle's kinetic energy $k$ changes in time $t$.

*Guess.* As a first approximation to the solution, instead of working with the general case, we go to an specific example by considering the moving particle to be an electron and the electric field to be originated by a proton. The dynamics is described by Lorentz force law.

Let's first analyze the electron-electric field interaction. The proton creates an electric field due to its charge $q_{\mathrm{p}}$. Lorentz force states that the proton's field strength $|e_{\mathrm{p}}|$ is given by $|e_{\mathrm{p}}| \propto |q_{\mathrm{p}}|/r^2$, where $r$ is the distance from the proton's center. Geometrically, this means that $|e_{\mathrm{p}}|$ creates concentric surfaces of equal electric potential in $\mathcal{E}^3$, called *isoelectric surfaces*, just like a static "heat" source forms concentric isothermal surfaces around its center. When something moves towards the proton, it will "pierce" such surfaces. Note that the field strength scales *inversely* with the *squared* distance: for instance, if the distance is *halved*, the field strengthens by a factor of *four*. In other words, the closer to the proton's center, the stronger the interaction with its field becomes. On the other hand, when an electron, with charge $|q_{\mathrm{e}}| < 0$, enters the field, it is "attracted" to the proton's center as the force between them, $|f_{\text{p-e}}| \propto -1/r^2$, increases with decreasing distance. In turn, the electron's velocity $v_{\mathrm{e}}$ increases and so does its kinetic energy $k_{\mathrm{e}} \propto v_{\mathrm{e}}^2$. Therefore, we expect $\dot{k}_{\mathrm{e}} \sim -q_{\mathrm{e}} e_{\mathrm{p}} v_{\mathrm{e}}$. (Notice the negative sign in the expression. It says that the electron looses energy as it falls into the proton! Also, see that $\dot{k}_{\mathrm{e}}$ does not depend on the electron's mass.)

Now, let's analyze the electron-magnetic field interaction. An electron moving in a $b$-field experiences a *sideways* force $f_{\mathrm{m}}$ proportional to (i) the strength of the magnetic "field" $|b|$, (ii) the component of the velocity perpendicular to such field $v_{\mathrm{e}}$ and (iii) the charge of the electron $q_{\mathrm{e}}$; *i.e.*, the second term of the Lorentz force: $f_{\mathrm{m}} = q_{\mathrm{e}} v_{\mathrm{e}} \times b$. Note that $f_{\mathrm{m}}$ is always *perpendicular* to both the $v_{\mathrm{e}}$ and the $b$ that created it, expressed by the (cross) product $(v_{\mathrm{e}} \times b)$. Then, when the electron moves in the field, it traces an helical path in which the helix axis is parallel to the field and in which $v_{\mathrm{e}}$ remains constant. Because the magnetic force is always perpendicular to the motion, the $b$ can do *no* work. It can only do work *indirectly*, via the electric field generated by a changing $b$. This means that, if no work is directly created by the magnetic field, then the change rate of the electron's kinetic energy should not depend directly on it, but rather indirectly, via the electron's velocity: $k_{\mathrm{e}} \propto v_{\mathrm{e}}^2 \implies \dot{k}_{\mathrm{e}} \sim v_{\mathrm{e}}^2$, which has the same dependence as the equation obtained in the electron-electric field analysis.

Finally, because an electron moving towards a proton is an example of a more general case, expect the *form* of the electron-proton case to work for *any* moving charged particle under a constant electromagnetic field. This means that, physically, the change of the particle's kinetic energy $\dot{k}$ should directly depend only on the electric field (and not on the magnetic induction), the particle's charge and its velocity: $\dot{k} \sim qev$. Mathematically, see that, since $e$ and $v$ are both vectors, the product $ev$ must be a product between vectors. The only suitable product is the inner product, *aka* scalar product, because it is the only one to return a scalar; this would agree with the scalar nature of $\dot{k}$. This means, therefore,

$$\dot{k} \sim qe \cdot v\,.$$

We expect this guessed equation to be obtained by formal methods.

For the next solution, we will use dimensional analysis to determine the *functional* form of the model to the phenomenon.

*Dim. Analysis.* To find the *functional form of the physical model* by means of dimensional analysis follow the steps:

(1) Instead of using the SI fundamental dimensions, use the set $\{\mathsf{F}, \mathsf{L}, \mathsf{T}, \mathsf{Q}\}$ of *four* dimensionally independent quantities, where $\mathsf{F}$ represents the dimension of force, $\mathsf{L}$ length, $\mathsf{T}$ time and $\mathsf{Q}$ electric charge.

(2) In the chosen set, the dimensions of the *six* physical quantities that model the phenomenon are $\dim k = \mathsf{FL}$, $\dim t = \mathsf{T}$, $\dim q = \mathsf{Q}$, $\dim e = \mathsf{FQ}^{-1}$, $\dim v = \mathsf{LT}^{-1}$ and $\dim b = \mathsf{FTQ}^{-1}\mathsf{L}^{-1}$.

(3) According to the Buckinham's theorem, *aka*, $\Pi$ theorem, there are $6-4=2$ dimensionless quantities $\Pi$. The first one is $\Pi_1 = k/(tevq)$ and the second one $\Pi_2 = bv/e$.

(4) Finally, the model should have the form:

$$g[\Pi_1, \Pi_2] = g\left[\frac{k}{tevq}, \frac{bv}{e}\right] = 0 \implies \frac{k}{t} = qev\, h\left[\frac{bv}{e}\right] ,$$

where $h$ is a function of $(bv/e)$.

In the last equation, the precise form of the function $h$ must be determined by experimentation or by analytic means. However, dimensional analysis confirms our suspicion: $\dot{k} \sim qev$; *i.e.*, the product $qev$ "lives upstairs" in the equation. The second term, the function $h$, should be equal to a dimensionless parameter $\Pi$ if our guess is to be correct. We will keep $h$, nevertheless, for it may be that our guess is not correct.

For a second approximation, we will use actual equations and will apply to them approximate methods to find the *form* of the physical model. This helps to better understand the physics behind the process by avoiding the distractions of unnecessary constants, numeric factors and complicated notation. Additionally, it helps, as a sketch, to develop and to present the analytic solution.

*Approx. Solution.* First, write the complete set of equations modeling the phenomenon:

$$\begin{aligned} k &= \frac{1}{2}mv^2\,, && \text{[kinetic energy]}\\ f &= q(e + v\times b)\,, && \text{[Lorentz force law]}\\ f &= ma = m\dot{v}\,, && \text{[Newton's second law of motion]} \end{aligned}$$

where the variables were already defined during guessing and dimensional analysis.

Then, drop unnecessary constants and numeric factors, use the secant method to approximate derivatives [5] and treat vectors as scalars [6] to find

$$\begin{aligned} k &\sim mv^2\,, && \text{[approx. kinetic energy]}\\ \dot{k} &\sim k/t \sim mv^2/t \sim (mv/t)v\,, && \text{[approx. kinetic energy time rate change]}\\ f &\sim q(e + vb)\,, && \text{[approx. Lorentz force law]}\\ f &\sim mv/t\,. && \text{[approx. Newton's second law of motion]} \end{aligned}$$

Find the equation of motion by equating Newton's law to Lorentz law: $(mv/t) \sim q(e + vb)$. Plug this equation into the one for $k/t$, via the factor $(mv/t)$:

$$k/t \sim q(e + vb)v \sim qev + qvbv\,.$$

In the last equation, the term $(qvbv)$ is likely to vanish, because $v$ is to enter $(v\times b)$ as $(v\times b)\cdot v$, for $v$ comes from $k \sim mv^2 \sim mv\cdot v$ and thus $(qv\times b)\cdot v \sim (qvbv) = 0$, since $v\times b$ is orthogonal to $v$. Then, the expression would be $k/t \sim qev$ with some product of vectors between $e$ and $v$ – the scalar product. The model could thus be written as $k/t \sim qe\cdot v$. Finally, remembering that $k/t \sim \dot{k}$, then

$$\dot{k} \sim qe\cdot v\,. \qquad \square$$

The equation found by approximate means agrees with our guess and, partially, with dimensional analysis. This increases our confidence in understanding the phenomenon! Besides, all the previous methods have cleared the derivation plan: find $\dot{k}$. Next, find the equation of motion by using the definition of linear momentum, then equating Newton's

[5] In the *secant method*, tangents (derivatives) are replaced by secants; *i.e.*, if $f = f[x]$, then $\mathrm{d}f/\mathrm{d}x \sim f/x$.

[6] This means to replace products between vectors by multiplication between scalars.

law to Lorentz law and leaving $mv$ on one side. Finally, plug in the equation of motion onto $\dot{k}$ and play with products between vectors to arrive to the final solution.

We solve the problem now by presenting a "wordy-version" of the analytic solution: we describe the math derivation in detail.

*Solution.* The particle kinetic energy is $2k = mv^2$. This could be rewritten as

$$2k = mv \cdot v \,,$$

since $v$ is colinear to itself; *i.e.*, its outer product is zero; *viz.*, $v^2 = vv = v \cdot v + v \wedge v = v \cdot v$.

Then, calculate the kinetic energy change rate with time by

$$2k = mv \cdot v \implies 2\dot{k} = m(\dot{v} \cdot v + v \cdot \dot{v}) = m(\dot{v} \cdot v + \dot{v} \cdot v) = 2m\dot{v} \cdot v \,,$$

where the product rule for the differentiation of the inner product $[(f \cdot g)' = f' \cdot g + f \cdot g'$, for vector-valued functions $f$ and $g]$ and the commutativity property of the inner product [for vectors $a$ and $b$, $a \cdot b = b \cdot a$] were used.

Next, one cancels out the numerical factor 2 in both sides of the equality to find that

$$\dot{k} = m\dot{v} \cdot v \,.$$

On the other hand, the particle's motion can be modeled by equating Newton's second law of motion with Lorentz force, since the particle interacts with an electromagnetic field. Thus, we find that

$$\dot{p} = q(e + v \times b) \,,$$

where $p$ is the particle's linear momentum. By definition, $p = mv$, so $\dot{p} = \dot{m}v + m\dot{v} = m\dot{v}$, because mass is constant, $\dot{m} = 0$, then we have that

$$m\dot{v} = q(e + v \times b) \,.$$

Plug in the last equation (equation of motion) into the $\dot{k}$ expression:

$$\dot{k} = qe \cdot v + q(v \times b) \cdot v \,.$$

For vectors $x, y, z$, the product $(x \times y) \cdot z$ is called the *scalar triple product*. This product equals zero whenever $x = z$. In our case, we have that $x = z = v$, or, more precisely, $(v \times b) \cdot v = 0$. Therefore, one finally finds

$$\dot{k} = qe \cdot v \,,$$

the rate at which the particle's kinetic energy changes with respect to time.

This (analytic) solution confirms our guessed model and the approximate solutions. Then, it creates confidence, not only on our intuition, but also on the efficacy of approximate methods.

Finally, we present a more formal solution, suitable for publishing.

*Solution.* Agree on the given hypotheses and on the notation previously established.

To begin, model the motion of the charged particle by equating Newton's second law of motion with Lorentz force law to find the particle equation of motion – the mathematical model for the physical phenomenon:

$$m\dot{v} = q(e + v \times b) \,. \tag{5.3}$$

On the other hand, write the particle's kinetic energy as $2k = mv^2 = mv \cdot v$. Then, calculate the change rate of kinetic energy with respect to time $\dot{k}$ using the properties of the inner product:

$$\dot{k} = m\dot{v} \cdot v \,. \tag{5.4}$$

Plug eq. (5.3) into eq. (5.4) to find that $\dot{k} = qe \cdot v + q(v \times b) \cdot v$. Here, the scalar triple product vanishes, for the vector $(v \times b)$ is orthogonal to the $(v \wedge b)$ plane. This gives, finally,

$$\dot{k} = qe \cdot v \,. \qquad \square$$

The formal solution follows the derivation of the "wordy" solution. They only differ in presentation. In the formal solution,

- the presentation is brief, concise, straight to the point, but not incomplete. It only leaves "obvious details" to be filled in – for instance, nowhere it is written that $\dot{p} = \dot{m}v + m\dot{v} = m\dot{v}$, because under hypotheses, $m$ is constant, so it is "well-known" that $f = ma$, in such a case;
- equations are referred to by proper, technical names (Newton's second law of motion, scalar triple product and so on);
- only "important" equations, derivations and results are displayed, whereas small equations, non-trivial, but small, derivations and partial results are presented in-line – with the running text;
- verbs changed to the English imperative.

## 5.3. **Tensors.**

5.3.1. *Particle Kinetics and Lorentz Force in Geometric Language.* In Newtonian physics, a classical particle of mass $m$ moves through space $\mathcal{E}^3$ as universal time $t$ passes. At time $t$, it is located at some point $x[t]$ (its *position*). The function $x[t]$ represents a curve in space, the particle's *trajectory*. The particle's *velocity* $v[t]$ is the derivative of its position, its *momentum* $p[t]$ is the product of its mass and velocity, its *acceleration* $a[t]$ is the time derivative of its velocity and its *energy* is half its mass times velocity squared:

$$x[t]\ ,\quad v[t] = \dot{x}[t]\ ,\quad p[t] = mv[t]\ ,\quad a[t] = \dot{v}[t] = \ddot{x}[t]\quad \text{and}\quad E[t] = \tfrac{1}{2}mv^2[t]\ .$$

Since points in space $\mathcal{E}^3$ are geometric objects (defined independently of any coordinate system), so are the trajectory, velocity, momentum, acceleration and energy. (Physically, velocity has an ambiguity: it depends on one's standard of rest.)

Newton's second law of motion states that the particle's momentum can change only if a force $f$ acts on the particle and such a change is given by

$$\dot{p} = ma = f\ .$$

If the force is produced by an electric field $E$ and magnetic field $B$, then this law of motion takes the familiar Lorentz-force form:

$$\dot{p} = q(E + v \times B)\ .$$

Note that these laws of motion are geometric relationships between geometric objects, thus independent of any coordinate system.

5.3.2. *Particle Kinetics in Index Notation.* As an example of slot-naming index notation, we can rewrite the equations of particle kinetics as follows:

$$v^i = \dot{x}^i\ ,\quad \dot{p}^i = mv^i\ ,\quad a^i = \dot{v}^i = \ddot{x}^i\ ,\quad E = \frac{1}{2}mv^j v_j\ ,\quad \dot{p}^i = q(E^i + \epsilon^i_{jk}v^j B^k)\ .$$

In the last equation, $\epsilon^i_{jk}$ is the so-called *Levi-Civita tensor*, which is used to produce the cross product.

These equations could be viewed in either of two ways: (i) as the frame-independent geometric laws $v = \dot{x}$, $p = mv$, $a = \dot{v} = \ddot{x}$, $2E = mv^2$ and $\dot{p} = q(E + v \times B)$ written in slot-naming notation or ii) as equations for the components of $\{v, p, a, E, B\}$ in some particular Cartesian coordinate system.

5.3.3. *Meaning of Slot-Naming Index Notation. Example.* Find the components of the tensor $S[a, b, -]$, where $a$ and $b$ are vectors.

*Solution.* In slot-naming notation, $S$ is rank-3, so it has 3 slots, or indices, say $i$, $j$ and $k$. Then, $S^{ijk}$. The vectors $a$ and $b$ occupy the first and the second slots of $S$, respectively; so $a \to a^i$ and $b \to b^j$.

Next, since there is only one empty slot in $S$, then its result, say $c$, must be a rank-1 tensor – a vector, with the index equal to the remaining slot of $S$; *viz.*, $k$. Finally, write the product in slot-index notation as

$$c^k = S^{ijk} a^i b^j \ .$$

In the last expression, since the indices are written using Latin letters, then they all run from 1 to 3. The index $k$ is a free index, whereas $i$ and $j$ are repeated, *aka* dummy, indices; *i.e.*, the (one) tensor equation represents three equations with nine terms.

*Example.* Find the components of the tensor $d = S[a, -, b]$.

*Solution.* Assign the indices $\{i, j, k\}$ to $S$. Then, $S^{ijk}$. The vectors $a$ and $b$ will take the first and third slots of $S$, respectively, while leaving the second slot to $e$:

$$S^{ijk} a^i b^k = d^j \ .$$

Again, the indices are Latin letters, so they all run from 1 to 3. The index $j$ is a free index and $i$ and $k$ are repeated indices.

*Example.* Find the components of the tensor $e = T[-, -, a]$.

*Solution.* Assign the indices $\{i, j, k\}$ to $T$. Then, $T^{ijk}$. The vector $a$ will take the last slot of $T$, while leaving the two first slots to the second-rank tensor $e$:

$$T^{ijk} a^k = e^{ij} \ .$$

The indices are Latin letters, so they all run from 1 to 3. The index $k$ is a repeated index and $i$ and $j$ are both free indices.

*Example.* Convert the expression $a^i b^{ij}$ to geometric, index-free notation.

*Solution.* The product $a^i b^{ij}$ has two variables and three indices, so it comes from the product of a rank-1 tensor, a vector, $a$, and a rank-2 tensor, $b$; *i.e.*, $a \otimes b[-, -]$. But, by definition $a \otimes b[-, -] = b[a, -]$, which gives the result.

*Example.* Convert the expression $a^i b^{ij}$ to geometric, index-free notation.

*Solution.* The term $b^{ij}$ is a rank-2 tensor, so $b[-, -]$ or, in slot naming notation, $b[-_i, -_j]$. Then, $a^i b^{ij}$ represents $a$ taking the first slot, $i$, of $b$; *i.e.*, $a^i b^{ij} = b^{ij} a^i = b[a, -]$.

*Example.* Convert the equation $s^{ijk} = s^{kji}$ to geometric, index-free notation.

*Solution.* This equation represents the same tensor, but with slots interchanged. Then,

$$s^{ijk} = s^{kji} \implies s[-_i, -_j, -_k] = s[-_k, -_j, -_i] \ . \qquad \square$$

*Example.* Convert the equation $a^i b_i = a^i b^j g_{ij}$ to geometric, index-free notation.

*Solution.* This equation is the definition of the inner product for two vectors, $a^i$ and $b^j$, via the metric tensor $g[-, -]$; *i.e.*, $a^i b^j g_{ij} = g[a, b]$.

5.3.4. *Numerics of Component Manipulations. Example.* The third rank tensor $S[-, -, -]$ and vectors $a$ and $b$ have as their only nonzero components $S^{123} = S^{231} = S^{312} = +1$, $a^1 = 3$, $b^1 = 4$, $b^2 = 5$. What are the components of the vector $c = S[a, b, -]$, the vector $d = S[a, -, b]$ and the tensor $W = a \otimes b$?

*Solution.* In component notation, $c^k = T^{ijk} a^i b^j$

5.3.5. *Product of Scalars and Tensors. Example.* Consider $\mathcal{E}^3$ and consider a scalar $s \in \mathcal{R}$. Then, compute the product $sg$ in slot-naming notation.

*Solution.* Denote by $g$ the metric of $\mathcal{E}^3$ and consider a standard frame $\{\gamma_k\}$ in $\mathcal{E}^3$. Then, calculate the components of the metric as $g_{ij} = \delta_{ij}$. Therefore, the product $sg = s\delta_{ij}$, in index notation.

Write the product $s\delta_{ij}$ as $s_{ij}$, since the diagonal components of $\delta_{ij}$ all equal $s$. See this in matrix notation

$$s_{ij} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} . \qquad \square$$

5.4. **Tensor Product – Again!** Consider two vectors $a, b$ in $\mathcal{E}^n$. Consider $a$ to be a rank-one tensor; *i.e.*, a tensor with one slot: $a[-]$. Then, the value of the tensor $a$ when $b$ is inserted in its slot is defined by

$$a[b] \doteq a \cdot b \,,$$

where the result is a real number – by the definition of tensor.

Consider the vectors $a, b, c, e, f, g \in \mathcal{E}^n$. Then, construct a tensor by using the *tensor product* of the vectors, defined by

$$a \otimes b \otimes c[e, f, g] \doteq a[e]\, b[f]\, c[g] \doteq (a \cdot e)(b \cdot f)(c \cdot g) \,.$$

*Remark.* Similar definitions can be given for the tensor product of any two or more tensors of any rank; for instance, if $t$ has rank 2 and $s$ rank 3, then

$$t \otimes s[e, f, g, h, j] \doteq t[e, f]\, s[g, h, j] \ ,$$

where the elements of the set $\{e, f, g, h, j\}$ are vectors in $\mathcal{E}^n$.

5.5. **Component Representation of Tensor Algebra.** Consider the geometric arena for Newtonian physics to be the 3-dimensional (Euclidean) space, $\mathcal{E}^3$, and universal time $t \in \mathcal{R}$. In this space, there is a unique *standard frame*, *aka* ordered set of *orthonormal basis vectors*, $\{\gamma_k; 1 \ldots 3\} \doteq \{\gamma_x, \gamma_y, \gamma_z\}$ associated with any *Cartesian coordinate system* $\{x^k\} \doteq \{x, y, z\}$. The frame element $\gamma_k$ has unit *length* and points along the $x^k$ *coordinate direction*, which is *orthogonal* to all the other coordinate directions. Summarize these frame elements properties by

$$\tfrac{1}{2}(\gamma_k \gamma_l + \gamma_l \gamma_k) = \gamma_k \cdot \gamma_l \doteq \delta_{kl} \,.$$

Expand any vector $u \in \mathcal{E}^3$ as a *linear combination* of the frame elements:

$$u = u^1\gamma_1 + u^2\gamma_2 + u^3\gamma_3 = \sum_{k=1}^{3} u^k \gamma_k \,. \tag{5.5}$$

Agree with *Einstein summation convention* to shorten eq. (5.5) as

$$u \doteq u^k \gamma_k \,.$$

By virtue of the orthonormality of the frame, compute the components $u^k$ of $u$ as the *inner product*

$$u^k = u \cdot \gamma_k \,. \tag{5.6}$$

*Note.* Check this last definition: it is the problematic one in index manipulation! So everything fits, I think $a^k \doteq a \cdot \gamma^k$, so "up" agrees with "up". The problem: it needs the introduction of reciprocal frames and the definition of the mixed metric $\delta_{jk}$! However, with this, we would have:

$$a^j = (a^k \gamma_k) \cdot \gamma_j = a^k (\gamma_k \cdot \gamma_j) = a^k \delta_k^j = a^j \,. \qquad \square$$

*Note.* I think the correct definition would be $u_k = u \cdot \gamma_k$. In this case, analogous to the up-up case, it would be the down agrees with down. The problem: it needs the introduction of reciprocal frames and the decomposition of $u$ onto such a frame. However, with this, we would have:

$$u_k = u^j \gamma_j \cdot \gamma_k = u^j \delta_{jk} = u_k \, .$$

One advantage over the last note is that no mixed metric is needed. Only the one already defined! Besides, another advantage is the generalization of this definition to tensors (see below). Using $u_k = u \cdot \gamma_k$ fits everything!

Expand any tensor, say the rank-3 tensor $t[-,-,-]$ as the *tensor product* of frame elements (analogous to eq. (5.5)):

$$t = t[\gamma_i, \gamma_j, \gamma_k] = t^{ijk} \, \gamma_i \otimes \gamma_j \otimes \gamma_k \, .$$

Compute the *components* $t_{ijk}$ of $t$ on the frame by generalizing of eq. (5.6):

$$t_{ijk} = t[\gamma_i, \gamma_j, \gamma_k] \, . \tag{5.7}$$

*Proof.*

*Remark.* Careful with index position: when the tensor $t$ multiplies the tensor product of the frame elements, then its components go "up" $t^{ijk}$. However, when the tensor components have to be calculated or used, then its components go "down" $t_{ijk}$.

Calculate the *components of the metric* [7] tensor $g$ as

$$g_{jk} = g[\gamma_j, \gamma_k] = \gamma_j \cdot \gamma_k = \delta_{jk} \, .$$

The last equality only holds for any standard frame, *aka* orthonormal basis, in $\mathcal{E}^3$.

Using eq. (5.7), calculate the *components of a tensor product*, *e.g.*. $t[-,-,-] \otimes s[-,-]$ by inserting the frame elements into the slots; *i.e.*,

$$t[\gamma_i, \gamma_j, \gamma_k] \otimes s[\gamma_l, \gamma_m] = t_{ijk} s_{lm} \, .$$

In words,

> the components of a tensor product equal the ordinary arithmetic product of the components of the individual tensors. The position of the indices follows that of the frame elements.

In component notation, compute the *inner product* of two vectors, say $a, b \in \mathcal{E}^3$, and the *value of a tensor* $t$ when vectors, say three vectors $d, e, f \in \mathcal{E}^3$, are inserted into its slots by

$$a \cdot b = g_{ij} a^i b^j \qquad t[d, e, f] = t_{ijk} d^i e^j f^k \, .$$

Calculate the *components of the contraction of a tensor*, say $r[-,-,-,-]$, on two of its slots, say the first and the third, by

$$\operatorname{comp} \operatorname{cont} 1, 3r = r_{ijik} \, .$$

Note that $r_{ijik}$ is summed on the $i$ index, so it has only two free indices: $j$ and $k$. Thus, $r_{ijik}$ are the components of a second rank tensor, as it must be if it is to represent the contraction of a fourth-rank tensor.

### 5.6. **Properties of the Inner Product.**

[7] This is the paramount example of index manipulations with tensors, because it *is* a tensor.

# 6. Notation

## 6.1. General Commands.

- to be defined by: a defby b: $a \doteq b$.
- difference operator: diff a: $\Delta\, a$.
- text in equations: eqtxt.

## 6.2. Sets.

- set: set A: $\mathcal{A}$.
- elements of a set: elset(a,b,c): $\{a, b, c\}$.
- set with a property: setprop(x)(x>0): $\{x : x > 0\}$.
- Cartesian (set) product: set A setprod set B: $\mathcal{A} \otimes \mathcal{B}$.
- Cartesian power: nset An: $\mathcal{A}^n$.
- Dim-grade space (2 is the dimension and 3 is the grade): dgspace V23: $V_3^2$.
- $n$-dim Euclidean space: espace n: $\mathcal{E}^n$.
- $n$-dim Minkowski space: mkspace n: $\mathcal{M}^n$.
- geometric algebra: ga: $\mathcal{G}$.
- geometric algebra on a $n$-dim. linear space $\mathcal{V}^n$: nga n: $\mathcal{G}^n$.
- dimension grade geometric algebra (2 is the dimension and 3 is the grade): dgga 23: $\mathcal{G}_3^2$.
- tuple: tuple(1,2,3): $[1, 2, 3]$.

## 6.3. Functions.

- function definition: fdef(f)(set A cartprod set B)(set R): $f : \mathcal{A} \otimes \mathcal{B} \to \mathcal{R}$.
- function mapping: fmap(f)(x)(x**2): $f : x \mapsto x^2$.
- maps to: x mapsto x**2: $x \mapsto x^2$.
- value at: f vat(x): $f[x]$.
- a binary operation: a bprod b: $a * b$.

## 6.4. Geometric Algebra.

- point: point P: $\mathcal{P}$.
- vector: vec a: $\vec{a}$.
- unit vector: uvec(a): $\hat{a}$.
- omitted vector from a product: ovec(a): $\check{a}$.
- clifs or multivectors: (use capitals) A: $A$.
- pseudoscalar: pscl: $i$.
- better, less typing, use lower-case for vectors $a$ and upper-case for other objects (bivectors, trivectors,...): $A$.
- use i for the pscl: $i$.
- orthogonal: ortho: $a \perp b$.
- parallel: parallel: $a \parallel b$.
- to be perpendicular to: perto(a)(b): $a_{\perp b}$.
- to be parallel to: parto(a)(b): $a_{\parallel b}$.
- to be orthogonal to: ortto(a)(b): $a_{\perp b}$.
- to be colinear to: colto(a)(b): $a_{\parallel b}$.
- projection of $p$ onto $q$: projon pq: $p_{\parallel q}$.
- rejection of $p$ onto $q$: rejon pq: $p_{\perp q}$.
- magnitude: magn(a): $|a|$.
- inverse: inv(a): $a^{-1}$.
- reverse: rev(a): $a^{\dagger}$.
- hodge dual: hdual(a): ${}^*a$.
- anticommutator: acom(a)(b): $[a, b]_+$.
- commutator: com(a)(b): $[a, b]_-$.
- expanded anticommutator: xacom(a)(b): $ab + ba$.
- expanded commutator: xcom(a)(b): $ab - ba$.
- step: step(A1): $\langle A \rangle_1$.
- scalar step: sstep(A): $\langle A \rangle_0$.
- grade operator: Grade A: grade $A$.
- grade: grade A2: $\langle A \rangle_2$.

- scalar grade: sgrade A: $\langle A\rangle_0$.
- clif with step: sclif Ak: $A_{\bar{k}}$.
- even part: even(A): $A_+$.
- odd part: odd(A): $A_-$.
- gorm (geometric norm?): gorm A: gorm $A$.
- expanded gorm: xgorm A: $\langle A^\dagger A\rangle_0$.
- metric: metric: $g$.
- Kronecker delta: kron: $\delta$.
- signature: diag a: diag $a$.
- signature: sign a: sig $a$.
- inner product: iprod: $a \cdot b$.
- outer product: oprod: $a \wedge b$.
- cross product: cprod: $a \times b$.
- cannonical decomposition of the geometric product: cgprod ab: $a \cdot b + a \wedge b$.

6.5. **Geometric Calculus.**

- ordinary one-dim. derivative: dx x: $\mathrm{d}x$.
- ordinary time derivative (dot derivative): dt x: $\dot{x}$.
- ordinary second time derivative (dot-dot derivative): ddt x: $\ddot{x}$.
- expanded partial derivative: xpd Hq: $\dfrac{\partial H}{\partial q}$.
- comma derivative: cder phi k: $\phi_{,k}$.
- geometric derivative: gder(a): $\bigtriangledown a$.
- directional derivative: dder(F)(a): $\bigtriangledown_a F$.
- Laplace operator: lder(a): $\triangle a$.
- D'Alembert operator: dalder(phi): $\Box \phi$.
- gradient: grad(phi): grad $\phi$.
- divergence: div(phi): div $\phi$.
- curl: curl(phi): curl $\phi$.

6.6. **Tensors.**

- tensor: tens T: $\mathsf{T}$.
- (empty) slot: tuple(slot, a, slot): $[-, a, -]$.
- tensor product: a tprod b: $a \otimes b$.
- tensor contraction: tcont(a tprod b): $\mathrm{cont}(a \otimes b)$.
- indexed tensor contraction: itcont(1,2)(a tprod b tprod c): $\mathrm{cont}_{1,2}(a \otimes b \otimes c)$.
- tensor components: tcomp T: comp $T$.
- tensor components in frame: itcomp T(ij): $T_{ij}$.
- tensor components in reciprocal frame: rtcomp T(ij): $T^{ij}$.
- Levi-Civita tensor: lct: $\epsilon$.

6.7. **Index Notation.**

- frame element, vector: fvec: $\gamma$.
- frame: frm(k): $\{\gamma_k\}$.
- indexed frame: ifrm(k)(0)(n): $\{\gamma_k; 0 \ldots n\}$.
- reciprocal frame: rfrm k: $\left\{\gamma^k\right\}$.
- indexed frame vector: ifvec k: $\gamma_k$.
- indexed reciprocal frame vector: rfvec k: $\gamma^k$.
- components of vector in frame: comp vk: $v^k$.
- components of vector in reciprocal frame: rcomp vk: $v_k$.
- metric coefficients in frame: imet kl: $g_{kl}$.
- metric coefficients in reciprocal frame: rmet kl: $g^{kl}$.
- mixed metric coefficients: mmet kl: $g^k_l$.
- kronecker delta coefficients in frame: ikron kl: $\delta_{kl}$.
- kronecker delta coefficients in reciprocal frame: rkron kl: $\delta^{kl}$.
- mixed kronecker coefficients: mkron kl: $\delta^k_l$.
- indexed geometric derivative (in reciprocal frame): igder k: $\partial_k$.

- indexed geometric derivative (in frame): rgder k: $\partial^k$.

6.8. **Various.**

- Iverson brackets: iverson(k=l): $[k = l]_{\mathrm{Iv}}$.
- Poisson brackets: poisson(f,g): $[f, g]_{\mathrm{Ps}}$.

6.9. **Dimensional Analysis.**

- dimension: dim k: $\dim k$.
- unit: unit k: $\operatorname{unit} k$.
- physical dimension: phdim k: $\mathsf{k}$.
- dimensionless quantity: kdim: $\Pi$.
-

6.10. **Mechanics.**

- position vector: pvec vat t: $x[t]$.
- differential position vector: dpvec vat point P $\mathrm{d}x[\mathcal{P}]$.
- Euclidean distance between points: edis: $\xi$.
- kinetic energy: ken vat t: $K[t]$.
- potential energy: pen vat t: $T[t]$.
- Lagrange function: lag: $L$.
- Hamilton function: ham: $H$.