MATHEMATICAL MODELING FOR PHYSICAL SCIENCES

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Abstract. Abstract goes here :)

1. Guide

This section provides with guidelines to approach the mathematical modeling of physical phenomena.

1.1. **Problem background.** A mathematical model of a physical phenomenon begins with the *problem background*, a description or introduction to the object. It should contain

- a *description* of the essential features of the physical process and
- an identification of the *objectives*, the key questions requiring answers.

Answering what, who, where, how and why questions guides to write down the description. Additionally, including graphical illustrations aids not only in the description, but in the definition of physical quantities and the establishment of hypotheses, as well.

1.2. **Problem formulation.** The *problem formulation* aims to:

- identify key physical processes;
- interpret these processes mathematically;
- establish a mathematical model – governing equations and suitable initial and boundary conditions;
- state clearly the assumptions.

The formulation must be based on sound physical principles, experimental facts or laws expressed in mathematical terms. As a guide, then, define the physical framework (geometry, kinematics, dynamics, thermal transfer and so on), state a dimensional set and then define the physical quantities, constants, parameters, coefficients and provide their dimensions in the chosen set.

Additionally, it is in this stage where all the quantities involved in the problem are clearly and unambiguously defined. Refer to them and also to the considered physical processes by proper names. Use standard names and symbology to the object domain.

A more formal approach is to begin with educated guessing, followed by dimensional analysis, order of magnitude analysis [1], analysis of extreme cases, simplifications and ends with a restricted model. The end result may be less accurate to fit experimental data, but less complex and thus more understandable. If fitting is not satisfactory, one can relax simplifications, one at a time, until a desired, or required, agreement is found.

1.3. **Analysis.** Once the physical and mathematical models and their assumptions have been proposed, one regularly faces a set of equations, probably differential equations together with initial and boundary conditions. The next step is then to analyze the set by

- non-dimensionalize the equations, included initial and boundary conditions;
- making analogies with other related problems or phenomena, as the case of mass, energy and momentum transport and
- relying on analytic and numeric methods, obtaining solutions (results).

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[1] Order of magnitude analysis is preceded by dimensional analysis, since *only* the comparison of *dimensionless* quantities is meaningful!

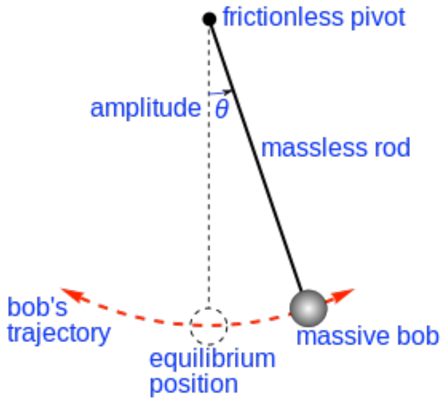

FIGURE 1. Schema of a simple gravitational pendulum

In the case of obtaining analytic solutions to differential equations, it is necessary to verify that they satisfy the differential equations object to the initial and boundary conditions.

As a final step, uncertainty analysis should be performed, to obtained ranges of validity, instead of punctual solutions.

1.4. **Results.** The final step of modeling physical phenomena is to present results, give conclusions and discussion them. Specifically, one should

- interpret results with respect to the original physical process and objectives;
- verify assumptions by confronting the results with reference values or experimental data and
- identify the solution limitations and possible extensions.

2. Background

In this section, we set the description of a physical phenomenon: the motion of a gravitational pendulum. We focus on answer the questions: what is a pendulum? what is a gravitational pendulum? How the pendulum is set into motion? What keeps the pendulum moving? What forces act on the pendulum that may damp its motion? We also provide a bit of some historical information about it.

2.1. **Description.** A *pendulum* is a mechanical system consisting of a bob hanging by a rod attached to a pivot. A *gravitational pendulum* is a pendulum object only to gravitational interactions. Finally, a *simple gravitational pendulum* is a gravitational pendulum consisting of a massive bob hanging by a massless rod attached to a frictionless pivot. Figure 1 depicts a simple gravitational pendulum.

For all the pendulums, at any time t, the angle made by the rod with respect to the vertical, the pendulum equilibrium position, is called the *pendulum displacement*, θ; whereas the maximum displacement is referred to as the *pendulum amplitude*. The *pendulum trajectory*, $\theta[t]$, on the other hand, is found by joining the different θ at their corresponding t. Lastly, the *pendulum angular velocity*, $\dot{\theta}$, is defined as the time derivative of θ.

Returning to the physical description, a gravitational pendulum is set into motion by:

(1) moving the bob from its equilibrium position at rest an amplitude, θ_0, at an initial time;
(2) applying a force that imprints an angular velocity to the bob at an initial time, $\dot{\theta}_0$, or
(3) displacing the bob to θ_0 and then applying $\dot{\theta}_0$ at an initial time.

Once motion starts, the system acquires *kinetic energy*, e_{kin}, then balanced by *gravitational potential energy*, e_{pot}. This restoring energy causes the system to oscillate about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the *pendulum period*, τ. The interplay between both energies continues indefinitely, unless an external force, such as a *damping force*, stops the pendulum from moving. Friction at the pivot or fluid drag, provided a partial or total pendulum submersion in a viscous fluid, are examples of damping forces. Finally, buoyancy is another force that comes into play by effectively reducing the bob weight.

Historically, Galileo Galilei studied pendulums *ca.* 1600. He postulated that they are *isochronos* – period is independent of amplitude. Then, by further studies, Christiaan Huygens proposed that the pendulum period depends on the square of its length, l, and free fall acceleration, g, by founding that

$$\tau = 2\pi\sqrt{\frac{l}{g}}\,.$$

This equation is known as *Huygens' law for the period*. See that Huygens' law is consistent with Galilei's isochronos postulate – nowadays know to be an approximation for small amplitudes.

2.2. **Objective.** The aim herein is to find a closed form of a mathematical function to predict the trajectory of a gravitational pendulum. A closed form may perhaps not be found when modeling a real gravitational pendulum, thus restrictions based on sound physical arguments would need be made. Moreover, the pendulum period is also set as a goal.

3. Physical processes

There are two main classes of gravitational pendulum motion: *undamped motion* – no frictional forces nor drag considered – and *damped motion* – friction and drag acknowledged. In both cases, however, the interplay between the pendulum kinetic energy and gravitational potential need be accounted, since it drives motion.

To begin to find the mathematical model, we estimate the period of a simple gravitational pendulum by using educated guessing. This stage will sketch and, hopefully, backup more formal theoretical and mathematical discoveries.

Next, to uncover the relationships among the physical quantities that may affect the pendulum motion, we firstly propose such quantities, then join them as dimensionless quantities and use finally physical arguments to restrain the physical model. The last step will pave the path to a simple, however accurate, mathematical model.

3.1. **Educated guessing.** Before performing lengthy theoretical calculations, we use simple physical considerations to estimate some pendulum quantities. In concrete, we present an assessment for the pendulum period by approximating its tangencial acceleration and its oscillation distance. We apply Newtonian mechanics arguments to the case of a simple gravitational pendulum.

[Figure 2 source: Sanjoy Mahajan, Order of Magnitude Physics A Textbook with Applications to the Retinal Rod and to the Density of Prime Numbers. PhD Thesis. California Institute of Technology Pasadena, California. 1998]

Consider fig. 2. The pendulum bob is object of a force $f \sim mg\sin[\theta_0]$ that accelerates it at [2] $a \sim g\sin[\theta_0] \sim g\theta_0$. Then, in time τ, the bob moves a distance $a\tau^2 \sim g\theta_0\tau^2$. On the other hand, to complete a cycle, the bob needs to travel a distance $\lambda \sim l\theta_0$, so $g\theta_0\tau^2 \sim l\theta_0$. Hence, the estimation of τ is thus

$$\tau \sim \sqrt{\frac{l}{g}}\,.$$

Additionally, to cross-check, we can estimate a typical bob velocity and with it approximate the period. First, the maximum potential energy is $e_{\mathrm{pot}} \sim mgh$, where [3] $h = l\,(1-\cos[\theta_0]) \sim l\theta_0^2$. On the other hand, the maximum kinetic energy is given

[2] The first term of the Taylor series for $\sin[\theta]$ is θ, with an error of order θ^3.
[3] The first term of the Taylor series for $(1-\cos[\theta])$ is $\theta^2/2$, with an error of order θ^4.

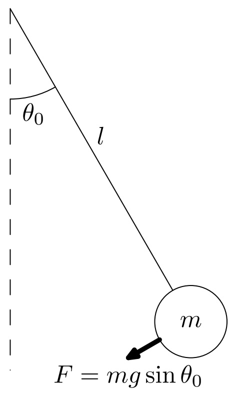

FIGURE 2. A pendulum bob of mass m hangs from a massless rope of length l. The bob is released from rest at an angle θ_0.

Quantity	Symbol	Dimension
Bob displacement	θ	1
Bob amplitude	θ_0	1
Bob mass	m_{bob}	M
Bob density	ρ_{bob}	M/L^3
Bob diameter	d_{bob}	L
Rod length	l_{rod}	L
Rod mass	m_{rod}	M
Torque at pivot	τ	ML^2/T
Pivot friction coefficient	α	1
Fluid density	ρ_{fl}	ML^3
Fluid dynamic viscosity	μ	M/LT
Time	t	T
Free fall acceleration	g	L/T^2

TABLE 1. Physical quantities involved in the motion of a gravitational pendulum

by $e_{\text{kin}} \sim mv^2$. Since a simple pendulum is undamped, the maximum kinetic energy equals the maximum potential energy. Hence, the maximum velocity can be found by $mv^2 \sim mgl\theta_0^2$, which yields $v \sim \theta_0\sqrt{gl}$. Finally, the period is then $\tau \sim \lambda/v$ or

$$\tau \sim \sqrt{\frac{l}{g}}\,,$$

as estimated using force and acceleration.

Note that the estimated period accords with Huygens's law and Galilei's isochronos observation.

3.2. **Dimensional analysis.** In this section, we show how to use dimensional analysis to reduce model complexity. We do this by considering first damped pendulum motion to then going gradually to undamped motion by reasoning physically.

3.2.1. *Dimensional analysis.* Since the problem belongs to dynamics, choose the dimensional set $\{\mathsf{L}, \mathsf{M}, \mathsf{T}\}$, with a cardinality of three. Next, as in table 1, list the possible physical quantities that may influence the pendulum motion along with their symbols and dimensions [4] in the chosen set.

A first approach may be to model the pendulum displacement by a function f of the form

$$\theta = f[\theta_0, t, g, l_{\text{rod}}, m_{\text{rod}}, \tau, \alpha, m_{\text{bob}}, \rho_{\text{bob}}, d_{\text{bob}}, \rho_{\text{fl}}, \mu]\ .$$

[4] The model for the friction at the pivot is $\tau = \alpha mgr$, where α is the friction coefficient, m the mass supported by the pivot and r the radius of the axis or rod supporting the pivot.

This complex relationship can be organized by means of dimensional analysis.

Firstly, since ρ_{bob}, m_{bob} and d_{bob} are related, discard mass; wherewith there are 12 physical quantities and 3 independent dimensions. Thus, according to the Pi-theorem, $12 - 3 = 9$ dimensionless quantities, $\{\Pi_i\}$, can be formed:

$$\Pi_1 = \theta\,,\ \Pi_2 = \theta_0\,,\ \Pi_3 = t\sqrt{\frac{g}{l}}\,,$$
$$\Pi_4 = \alpha\,,\ \Pi_5 = \frac{m_{\text{rod}}}{m_{\text{bob}}}\,,\ \Pi_6 = \frac{(m_{\text{bob}} + m_{\text{rod}})\,g l_{\text{rod}}}{\tau}\,,$$
$$\Pi_7 = \frac{d}{l}\,,\ \Pi_8 = \frac{\rho_{\text{fl}}}{\rho_{\text{bob}}}\,,\ \Pi_9 = \frac{\rho_{\text{fl}} d\sqrt{lg}}{\mu}\,.$$

See that Π_1 contains the quantity being sought, θ, Π_2 the quantity that originates motion, θ_0, and Π_3 the (independent) quantity against which to confront motion, t. Now, again using the Pi-theorem, the desired *dimensionless* function, ϕ_π, has the *form*:

$$\theta = \phi_\pi\left[\theta, \theta_0, t\sqrt{\frac{g}{l}}, \alpha, \frac{m_{\text{rod}}}{m_{\text{bob}}}, \frac{(m_{\text{bob}} + m_{\text{rod}})\,g l_{\text{rod}}}{\tau}, \frac{d}{l}, \frac{\rho_{\text{fl}}}{\rho_{\text{bob}}}, \frac{\rho_{\text{fl}} d\sqrt{lg}}{\mu}\right]\,. \qquad (1)$$

To further reduce the complexity of the mathematical model, restrain the physical model by working on the pendulum and by making assumptions. Under such hypotheses, we will go from a damped motion case to an undamped motion case.

3.2.2. *Assumptions and their mathematical interpretation.* First, consider a *frictionless pivot*. Proper lubrication of the pivot and the rod reduces friction. With this, the frictional torque term disappears, $\Pi_4 = \Pi_6 = 0$.

Next, consider a *massless, inflexible rod*. The rod may be built of a strong material; strong enough to support the bob without elongating. This allows the construction of a very thin rod, wherewith the ratio of masses vanishes, $\Pi_5 = 0$.

Consider a rod length much greater than the bob length. This is possible since we build the pendulum with a very strong rod. Therefore, the ratio d/l can be discarded, $\Pi_7 = 0$.

Consider an non-buoyant fluid by encasing the pendulum and surrounding it by air. The bob density will then be greater than air density. This implies a non-buoyant fluid. Thus, $\Pi_8 = 0$.

Consider finally air as an *inviscid fluid*. An inviscid fluid is a fluid with no viscosity, resulting thus in no drag. Hence, $\Pi_9 = 0$.

Then, after having restrained the physical model, we seek a mathematical function of the form

$$\theta = \phi_\pi\left[\theta_0, t\sqrt{\frac{g}{l}}\right]\,. \qquad (2)$$

Neither dimensional analysis nor order of magnitude analysis can help to find the functional form of ϕ_π. It must be found by a more refined analysis or by experimentation. Nevertheless, based on sensible considerations, we have reduced the complex physical model by passing from 13 dimensional quantities to 3 dimensionless quantities. In the end, however, only confrontation with experimental data will support or disprove the reductions we have done.

Finally, for the sake of mathematical purposes, we combine all the previous assumptions by defining a *simple gravitational pendulum*:

> a simple gravitational pendulum is a pendulum composed of a massive bob hanging by a massless and inflexible rod attached to a frictionless pivot. Under the influence of gravitational interactions, the pendulum swings through an inviscid fluid of negligible density.

3.2.3. *Notes.* Equation (2) forms the minimum combination of dimensionless quantities, since it involves the dependent quantity, θ, the independent quantity, t, the quantity that originates motion, θ_0, and the quantity that keeps the motion, g. As an additional, and welcome, side effect, dimensional analysis tells us that θ does not depend on g alone, but rather on the quotient g/l, which includes the only pendulum property: its length.

On the other hand, considering an undamped system implies that mechanical energy *must* be conserved, for only kinetic energy turns into gravitational potential energy and *vice versa*. Hence, Lagrange's and Hamilton's formulations of mechanics can be used instead of Newton's to analyze the system.

Finally, eq. (2) may seem, at first sight, a very restricted model. It is, nevertheless, a practical one: a longcase clock pendulum. Such a clock consists of case full of air holding inside a heavy bob hanging by a light and inflexible rod attached to a lubricated pivot.

3.3. **Mathematical model.** In this section, we deduce the equation of motion for a simple gravitational pendulum by means of Lagrange's formulation mechanics to a pendulum that is set into motion by displacing the bob an initial angle θ_0 from rest, $\dot{\theta}_0 = 0$.

Consider a simple gravitational pendulum composed of a bob of mass m and a rod of length l. Let θ be the bob displacement for any time t, the amplitude be θ_0, the initial velocity be $\dot{\theta}_0 = 0$ and, finally, g be the free fall acceleration. Then, find the equation of motion for the pendulum.

Using θ as the generalized position and $\dot{\theta}$ as the generalized velocity, write down the Lagrangian, e_{lag}, for the system:

$$e_{\text{lag}} = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\left(1 - \cos[\theta]\right) . \tag{3}$$

Find next the generalized momentum, p_θ, and its temporal change, $\dot{p}_\theta$:

$$p_\theta = \partial_{\dot{\theta}} e_{\text{lag}} = m l^2 \dot{\theta} \implies \dot{p}_\theta = \mathrm{d}_t \partial_{\dot{\theta}} e_{\text{lag}} = m l^2 \ddot{\theta} . \tag{4}$$

Calculate then the generalized force, f_θ:

$$f_\theta = \partial_\theta e_{\text{lag}} = -mgl \sin[\theta] .$$

Replace the generalized force and the temporal change of the generalized momentum in Euler-Lagrange's equation:

$$f_\theta = \dot{p}_\theta \implies m l^2 \ddot{\theta} + mgl \sin[\theta] = 0 .$$

Since $ml > 0$, divide the last equation through ml^2 to have

$$\ddot{\theta} + \frac{g}{l} \sin[\theta] = 0 .$$

Finally, rewrite the last equation by joining to it the initial conditions:

$$\begin{cases} \ddot{\theta}[t] + \dfrac{g}{l} \sin[\theta[t]] = 0 , \\ \theta[0] = \theta_0 , \\ \dot{\theta}[0] = 0 , \end{cases} \tag{5}$$

which yields the equation of motion for a simple gravitational pendulum.

4. Analysis

In this section, we solve eq. (5).

4.1. **Non-dimensionalization.** The independent quantity is t, the dependent quantity θ and the parameters are θ_0, l and g. Since θ is already dimensionless, non-dimensionalize t to the dimensionless time $\bar{t}$ by using Π_3 as scaling factor, a characteristic time:

$$\bar{t} = \Pi_3 = t\sqrt{\frac{g}{l}} \implies t = \bar{t}\sqrt{\frac{l}{g}} .$$

Find next the $\bar{t}$ differentials

$$\mathrm{d}t = \mathrm{d}\bar{t}\sqrt{\frac{l}{g}} \implies \mathrm{d}t^2 = \mathrm{d}\bar{t}^2 \frac{l}{g} .$$

Replacing the last expressions in eq. (5) and dividing the result through $g/l (> 0)$, find the dimensionless and parameter-free differential equation:

$$\begin{cases} \ddot{\theta} + \sin[\theta] = 0 , \\ \theta[0] = \theta_0 , \\ \dot{\theta}[0] = 0 , \end{cases} \tag{6}$$

where the derivatives are to be taken with respect to $\bar{t}$.

4.2. **Analytic solution.** Equation (6) is a non-linear, second-order ordinary differential equation. Linearize it by means of the *small-angle approximation* [5]:

$$\begin{cases} \ddot{\theta} + \theta = 0\,, \\ \theta[0] = \theta_0\,, \\ \dot{\theta}[0] = 0\,. \end{cases}$$

The solution to this equation is

$$\theta\left[\bar{t}\right] = \theta_0 \cos\left[\bar{t}\right]\,, \tag{7}$$

or, returning to the dimensional quantity t,

$$\theta[t] = \theta_0 \cos\left[t\sqrt{\frac{g}{l}}\right]\,, \tag{8}$$

which solves eq. (5) for $\theta \ll 1$.

5. Results

Although physically and mathematically restrained with respect to the original problem, an undamped pendulum, eq. (8) does provide a closed form function to predict the displacement of a simple gravitational pendulum with respect to time.

Hereafter, we discuss this equation under physical grounds and confront its predictions with experimental data.

5.1. **Theoretical discussion.** In this section, we investigate the physical consequences of eq. (8).

[consistency on the description of motion: when particle moves to the right, the force points to the left (grav. potential energy restores kinetic energy), and *vice versa*.

circular motion: amplitude describes a circle, since inflexible rod]

5.1.1. *Analogies with other phenomena.* In classical *simple harmonic motion*, the period of the motion, τ, is the time required for a complete oscillation and defined by

$$\tau = \frac{2\pi}{\omega}\,,$$

where ω is the motion *natural frequency*.

The motion of a simple gravitational pendulum, described by eq. (8), is an instance of simple harmonic motion, where θ_0 is the semi-amplitude of the oscillation and where the natural frequency is

$$\omega = \sqrt{\frac{g}{l}}\,.$$

The period of the pendulum motion, for the outward and return, is thus

$$\tau = 2\pi\sqrt{\frac{l}{g}}\,, \tag{9}$$

which is Huygens's law for the period.

Note that *only* under the small-angle approximation, the period is independent of the amplitude; *i.e.*, *isochronism* – the property Galileo discovered.

5.1.2. *Momentum conservation.* Momentum of a system is conserved if no forces act on the system, thus $\dot{p} = 0$ holds. Since we departed from the hypothesis that gravity drives the simple gravitational pendulum, eq. (8) should *not* preserve momentum [6].

Find the pendulum angular velocity by differentiating eq. (7), the dimensionless form of eq. (8), with respect to $\bar{t}$:

$$\dot{\theta} = \mathrm{d}_{\bar{t}}\theta = -\theta_0 \sin\left[\bar{t}\right]\,. \tag{10}$$

Next, replace the pendulum angular velocity in eq. (4), the pendulum momentum:

$$p_\theta = ml^2\dot{\theta} = -ml^2\theta_0 \sin\left[\bar{t}\right]\,.$$

[5] The *small-angle approxmation* means to take the first term of the Taylor series for $\sin[\theta]$ when $\theta \ll 1$; *i.e.*, $\sin[\theta] \sim \theta$ for $\theta \ll 1$. The incurred error is of order θ^3.

[6] In the grand scheme of things, momentum *is* conserved, but to see this, we would need to add Earth's momentum to the pendulum's.

Finally, calculate the momentum time derivative:

$$\dot{p}_\theta = -ml^2\theta_0 \cos\left[\bar{t}\right] .$$

Since $\dot{p}_\theta$ is *not* overall zero, momentum is *not* conserved during motion. Thus, eq. (8) physical considerations regarding momentum.

5.1.3. *Energy conservation.* In Hamilton's formulation of mechanics, the Hamiltonian of a system equals the system total energy. Thus, if the total energy is conserved, then the Hamiltonian time derivative must be null. Equivalently, it can be proved that [7]

> if the Hamiltonian does not explicitly depend on time, then total energy is conserved.

Particularly, in the case of the simple pendulum, eq. (8) *should* conserve total energy, because it was deduced by hypothesizing an undamped system.

Firstly, write down the Hamiltonian, e_{ham}, of the system:

$$e_{\text{ham}} = e_{\text{kin}} + e_{\text{pot}} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1 - \cos[\theta]\right) .$$

Replace $(1 - \cos[\theta])$ by the first term of its Taylor series:

$$2e_{\text{ham}} = ml^2\dot{\theta}^2 + mgl\theta^2 .$$

Plug eq. (7) and eq. (10) into the last equation to have

$$2e_{\text{ham}} = ml\theta_0^2\left(l\sin^2\left[\bar{t}\right] + g\cos^2\left[\bar{t}\right]\right) .$$

Since the Hamiltonian e_{ham} *does* depend on time, eq. (8) does *not* satisfy the energy conservation principle. The small-angle approximation originates this discrepancy.

5.2. **Experimental data and reference values.** No experimentation was specifically made for the writing of the present document. However, some reference values were found in the internet [source!].

5.2.1. *Experiment.* In [source!], the experimental set-up consisted of a pendulum with a spherical, stainless-steel-made bob of mass $m_{\text{bob}} = 100.0\,\text{m}$ hanged of a stainless steel rod whose length was varied during experimentation; however, it was assumed to be inflexible and massless. The rod was connected to a well lubricated pivot. The pendulum was set into motion by displacing the bob an amplitude $\theta_0 = 10.00°$ from rest. Finally, the pendulum was encased and surrounded by air at room temperature.

5.2.2. *Verification of assumptions.* Before confronting experimental numbers with predictions of eq. (8), it is necessary to verify that all the assumptions made to deduce eq. (8) are satisfied.

First, consider the *frictionless pivot* assumption. The experiment was done by properly lubricating the pivot-rod joint. So the assumption holds. Some useful numbers to back-up this assumption: a dry and clean joint of steel pivot and steel rod has a friction coefficient of 0.80, while when lubricated 0.16.

Consider the *massless, inflexible rod* assumption. [source!] does not report numbers to support this assumption. We take it as satisfied.

Again, the diameter of the bob was not reported. But, we can estimate it by considering the stainless steel density equal to $7750\,\text{kg/m}^3$. Then, considering a spherically shaped bob, the diameter would be

$$d = \sqrt[3]{\frac{6m_{\text{bob}}}{\pi\rho_{\text{bob}}}} = \sqrt[3]{\frac{6 \times 100.00 \times 10^{-3}}{\pi \times 7750}} = 2.2910\,\text{cm} .$$

With this number, we can calculate the d/l ratio for the smaller case of l analyzed: 100.00 cm. Then, $d/l = 0.022\,91$, which can be ignored.

Then, consider the non-buoyant fluid assumption. If the system was at 15 °C, at sea level, then $\rho_{\text{air}} = 1.225\,\text{kg/m}^3$. Hence,

$$\frac{\rho_{\text{air}}}{\rho_{\text{steel}}} = \frac{1.225}{7750} = 0.158 \times 10^{-4} ,$$

[7] This theorem is useful for it saves computing the time derivative of the Hamiltonian.

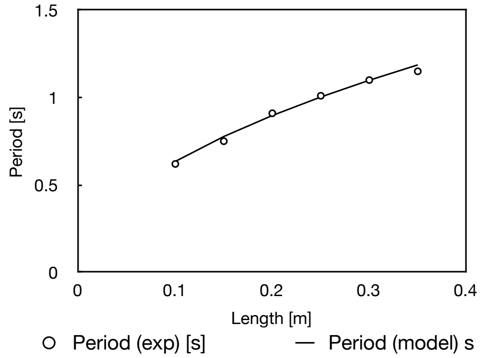

FIGURE 3. Model values for the simple gravitational pendulum and experimental data gathered from a real gravitational pendulum

which can be discarded.

Finally, consider if the hypothesis of taken air as an inviscid fluid was satisfied by plugging some typical values into Π_9:

$$\frac{\rho_{\mathrm{air}} d_{\mathrm{bob}} \sqrt{lg}}{\mu} = \frac{1.225 \times 2.9011 \times 10^{-2} \sqrt{1.00 \times 9.80665}}{1.983 \times 10^{-5}} \sim 5611\,,$$

See that inertial forces, $\rho_{\mathrm{air}} d_{\mathrm{bob}} \sqrt{lg}$, are much larger than viscous forces, μ.

Since the experimental set-up was such that satisfied all the hypotheses leading to eq. (8), we can, therefore, use this equation to model the simple pendulum in this case.

5.2.3. *Analysis of experimental results.* The gathered experimental together with the model predictions of eq. (8) are presented in fig. 3.

Using these data, fig. 3, one finds that the *coefficient of determination*, R^2, between model and experimental figures is 0.9876, while the relative error 2.03 %. Both numbers show agreement between model and experiment, thus eq. (8) is taken to correctly represent real pendulums.

5.2.4. *Conclusions.* A closed mathematical function between the simple pendulum displacement and time was found, eq. (8). This model agrees with the momentum conservation but not with energy conservation due to the small-angle approximation. However, when its predictions are confronted with experimental data, the model figures agree with physical reality if the assumptions leading to it are satisfied by experimental set-ups.

Finally, if more accuracy is required or in cases where less agreement is found when applying eq. (8) to a real pendulum, then the model can be extended by relaxing the assumptions made in section 3.2.2. For instance, if the pendulum swings through a viscous fluid, such as a liquid, then the dimensionless quantities Π_8 and Π_9 should be included. Or, if there is little care in lubricating the pivot, then Π_4, Π_5 and Π_6 should be further studied.

APPENDIX A. SIMPLE HARMONIC MOTION

Only wimps study the general case. Real scientists work through examples.

— BERESFORD PARLETT,
?

In this section, we present a small account of the simple harmonic motion of a spring-mass system – a simple harmonic oscillator.

A.1. **Background.** We give hereafter a brief description of the harmonic motion, emphasizing in simple harmonic motion. Then, we set the main objectives of the current analysis.

Quantity	Symbol	Dimension
Displacement	x	L
Initial displacement	x_0	L
Spring stiffness	k	$\mathsf{F/L}$
System mass	m	$\mathsf{FT^2/L}$
Time	t	T

TABLE 2. Physical quantities involved in the motion of a harmonic oscillator

A.1.1. *Description.* A *harmonic oscillator* is a system that, when displaced from its equilibrium position, experiences a restoring force, f, proportional to a displacement, x:

$$f = -kx\,,$$

where k as a strictly positive constant.

If f is the only force acting on the system, then the system is called a *simple harmonic oscillator* and its motion is said to be a *simple harmonic motion.* Note that the force depends only on the position, thus it can be written as the gradient of a *potential*, e_{pot}; *i.e.*, as $f = -\operatorname{grad} e_{\text{pot}}$. Additionally, since there are no other forces present – such as drag, buoyancy, gravity and so on –, mechanical energy is conserved.

An instance of a harmonic oscillator is a *spring-mass system.* In such a system, f is given by *Hooke's law* and k is a constant factor characteristic of the spring, its *stiffness*. Regularly, the system is set into motion by stretching or contracting the mass together with the spring a distance x_0 from the mass equilibrium position, called the *amplitude*, the maximum displacement, with null initial velocity, $\dot{x} = 0$.

A.1.2. *Objective.* The goal is to obtain a closed form mathematical function to predict the amplitude for a simple harmonic oscillator, as well as a formula to predict its period.

A.2. **Physical processes.** We go now into a more physical and mathematical approach to the analysis of the simple harmonic oscillator.

A.2.1. *Educated guessing.* As a first approach to analyze the spring-mass system, we estimate the oscillator [8] period.

Consider an oscillator composed of a mass m and a spring of stiffness k. After having been set into motion by displacing the oscillator a distance x_0, m experiences a force $f \sim kx_0$ by the spring, which tries to restore the oscillator to its equilibrium position. This force accelerates the oscillator at $\ddot{x} \sim kx_0/m$. During a time τ, the oscillator travels a distance $\ddot{x}\tau^2 \sim kx_0\tau^2/m$. On the other hand, to complete a cycle, the oscillator has to travel a distance $x \sim 2x_0 \sim x_0$. Now, equating both distances, one finds that $kx_0\tau^2/m \sim x_0$, which leads finally to an estimate of the oscillator period

$$\tau \sim \sqrt{\frac{m}{k}}\,.$$

It can be seen that τ dependency on k and m is not linear. Moreover, the last equation implies that simple harmonic motion is *isochronous*; *i.e.*, the period and frequency are independent on the amplitude.

A.2.2. *Dimensional analysis.* We would like to find the form of a dimensionless function of the physical quantities affecting the oscillator (spring-mass) motion.

The problem belongs to mechanics, so we choose the dimensional set $\{\mathsf{F}, \mathsf{L}, \mathsf{T}\}$, with cardinality of three. Using this set, consider table 2 as a list of hypothesized physical quantities affecting oscillator motion.

There are five physical quantities and three base dimensions. Thus, according to the Pi-theorem, $5 - 3 = 2$ dimensionless quantities can be formed:

$$\Pi_1 = \frac{x}{x_0} \qquad \text{and} \qquad \Pi_2 = t\sqrt{\frac{k}{m}}\,. \tag{11}$$

[8] Hereafter, oscillator will refer to a spring-mass system.

Then, again by the Pi-theorem, we seek a dimensionless function, ϕ_π, of the form

$$\Pi_1 = \phi_\pi[\Pi_2] \implies \frac{x}{x_0} = \phi_\pi\left[t\sqrt{\frac{k}{m}}\right].$$

The closed form of ϕ_π must be found by theory.

A.2.3. *Mathematical model.* Since the problem involves forces, we use Newton's formulation of mechanics to find the equation of motion for the simple harmonic oscillator.

Consider a simple harmonic oscillator consisting of a mass m connected to a spring of stiffness k set into motion by initially displacing the mass a distance x_0 from rest. Then, find the equation of motion for the oscillator displacement x for any time t.

Apply Newton's second law of motion to the oscillator to find

$$m\ddot{x} = -kx\,,$$

where $\ddot{x}$ is the oscillator acceleration produced by the restoring force $f = -kx$.

Since $m > 0$, divide the last equation through m to have

$$\ddot{x} + \frac{k}{m}x = 0\,.$$

Lastly, join the initial conditions to the last equation to have the equation of motion for the simple harmonic oscillator:

$$\begin{cases} \ddot{x}[t] + \omega^2 x[t] = 0\,, \\ x[0] = x_0\,, \\ \dot{x}[0] = 0\,, \end{cases} \tag{12}$$

where ω is defined as

$$\omega = \sqrt{\frac{k}{m}}$$

and is called the *oscillator natural frequency*.

A.3. **Analysis.** Now, we solve eq. (12) to find a closed form of $x[t]$.

A.3.1. *Non-dimensionalization.* Consider eq. (12). The independent quantity is t, the dependent one x and the parameters are k and m.

Non-dimensionalize x using Π_1 as a scaling factor or characteristic displacement, found in eq. (11):

$$\overline{x} = \Pi_1 = \frac{x}{x_0} \implies x = x_0\overline{x}\,,$$

with differentials

$$\mathrm{d}x = x_0\mathrm{d}\overline{x} \qquad \text{and} \qquad \mathrm{d}^2x = x_0\mathrm{d}^2\overline{x}\,.$$

Then, non-dimensionalize t using Π_2 as a characteristic time – see eq. (11):

$$\overline{t} = \Pi_2 = t\sqrt{\frac{k}{m}} \implies t = \overline{t}\sqrt{\frac{m}{k}}\,,$$

with differentials

$$\mathrm{d}t = \mathrm{d}\overline{t}\sqrt{\frac{m}{k}} \qquad \text{and} \qquad \mathrm{d}t^2 = \mathrm{d}\overline{t}^2\frac{m}{k}\,.$$

Replacing $\overline{x}$, $\overline{t}$ and their differentials in eq. (12) gives

$$\begin{cases} \ddot{\overline{x}}[\overline{t}] + \overline{x}[\overline{t}] = 0\,, \\ \overline{x}[0] = 1\,, \\ \dot{\overline{x}}[0] = 0\,. \end{cases} \tag{13}$$

where the derivatives are to be taken with respect to $\overline{t}$. Note that the equation of motion is dimensionless and parameter free.

A.3.2. *Analytic solution.* Equation (13) is a second-order, linear ordinary differential equation whose solution is given by

$$\overline{x} = \cos\left[\overline{t}\right] ,$$

or, in dimensional form,

$$x = x_0 \cos\left[t\sqrt{\frac{k}{m}}\right] = x_0 \cos[t\omega] . \tag{14}$$

A.4. **Results.** The oscillator natural frequency is related to the *temporal frequency*, f, by

$$\omega = 2\pi f .$$

In turn, f is related to the oscillator period, τ, by

$$\tau = \frac{1}{f} \implies \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} ,$$

which agrees with what was guessed in appendix A.2.1; *i.e.*, simple harmonic motion is *isochronous*.

On the other hand, since no other force but Hooke's law force acts on the system, energy must be conserved. To verify energy conservation, find first x in dimensional form:

$$\overline{x} = \cos\left[\overline{t}\right] \implies x = x_0 \cos[t\omega] .$$

Then, calculate the oscillator velocity

$$v = \dot{x} = -x_0\omega \sin[t\omega] .$$

With this result, compute the oscillator kinetic energy:

$$2e_{\text{kin}} = mv^2 = mx_0^2\omega^2 \sin^2[t\omega] .$$

But $\omega^2 = k/m$, hence

$$2e_{\text{kin}} = kx_0^2 \sin^2[t\omega] .$$

On the other hand, determine the potential energy by means of Hooke's law force, f. Since f depends only on x, it can be written as the gradient of a potential: $f = -\operatorname{grad} e_{\text{pot}}$. Which, after integration, in one dimension, gives

$$2e_{\text{pot}} = kx^2 .$$

Replacing x and ω in the last equation results in

$$2e_{\text{pot}} = kx_0^2 \cos^2[t\omega] .$$

The total energy of the oscillator, e, is the addition of kinetic and potential energies, thus

$$2e = kx_0^2 \left(\sin^2[t\omega] + \cos^2[t\omega]\right) \implies e = \frac{1}{2}kx_0^2 ,$$

which is a constant. Therefore, energy *is* conserved by eq. (14) during motion of a simple harmonic oscillator.

Appendix B. Dimensional analysis

B.1. **Dorsey.** [Dimensional Analysis, Alan Dorsey]

B.1.1. *Introduction.* The first step in modeling any physical phenomena is the identification of the relevant physical quantities and the relationships among these quantities via known physical laws. For many complex phenomena, where *ab initio* theories are difficult or impossible to construct, modeling methods are indispensable and one of the most powerful modeling methods is *dimensional analysis*. Here we will use dimensional analysis to *solve* problems or at least to infer some information about the solution.

The basic *principle* is that

> physical laws do not depend upon arbitrariness in the choice of the basic units of measurement.

In other words, if a physical law is valid in a chosen system of units, then it's valid in all systems. Consider, for instance, the angular frequency of small oscillations of a mathematical pendulum of length l and mass m:

$$\omega = \sqrt{\frac{g}{l}}\,,$$

where g is the free fall acceleration. The last equation can be derived applying Newton's second law of motion to the pendulum. However, we can deduce it from dimensional considerations alone. What can ω depend upon? It is reasonable to assume that the relevant quantities are m, l and g. Now suppose that we change the system of units so that the unit of mass is changed by a factor of M, the unit of length by a factor of L and the unit of time by a factor of T. With this change of units, the units of frequency will change by a factor of T^{-1}, the units of velocity by a factor of LT^{-1} and the units of acceleration by a factor of LT^{-2}. Therefore, the units of the quantity g/l will change by T^{-2} and those of $(g/l)^{1/2}$. Consequently, the ratio

$$\Pi = \frac{\omega}{\sqrt{g/l}}$$

is invariant under a change of units; Π is called a *dimensionless quantity*. Since it doesn't depend upon the quantities $\{m, g, l\}$, it is in fact a constant. Therefore, from dimensional considerations alone, we find that

$$\omega = \Pi\sqrt{\frac{g}{l}}\,.$$

A few comments are in order:

(1) the frequency is independent of the mass of the pendulum bob;
(2) the constant Π cannot be determined from dimensional analysis alone.

These results are typical to dimensional analysis:

> uncovering often unexpected relations among the quantities, while at the same time falling to pin down numerical constants.

Indeed, to fix the numerical constants we need a real *theory* of the phenomena in question, which goes beyond dimensional analysis.

B.1.2. *Dimensions.* In the previous discussion, note that if the units of length are changed by a factor of L and the units of time by a factor of T, then the units of velocity change by a factor of LT^{-1}. We call LT^{-1} the *dimensions* of the velocity; it tell us the factor by which the numerical value of the velocity changes under a change in the units (within the $\{\mathsf{L}, \mathsf{M}, \mathsf{T}\}$ class). We denote the dimensions of a physical quantity, say, ϕ by $\dim\phi$; thus, $\dim v = [\mathsf{LT}^{-1}]$. A dimensionless quantity would have $\dim\Pi = [1]$; *i.e.*, its numerical value is the same in all systems of units within a given class. What about more complicated quantities such as force? From Newton's second law, $f = m\ddot{\xi}$, so that $\dim f = \dim m \dim\ddot{\xi} = [\mathsf{ML}/\mathsf{T}^2]$. Proceeding in this way, we can easily construct the dimensions of any physical quantity; some of the more commonly encountered quantities are included in Table 1.

We see that all of the dimensions in Table 1 are *power law monomials* of the form (in the $\{\mathsf{L}, \mathsf{M}, \mathsf{T}\}$ class):

$$\dim\phi = \Pi\mathsf{L}^a\mathsf{M}^b\mathsf{T}^c\,,$$

where Π and $\{a, b, c\}$ are constants. In fact, this is a general result that can be proven mathematically, see Barenblatt's book [...]. This property is often called *dimensional homogeneity* and is really the key to dimensional analysis. To see why this is useful, consider again the determination of the period of a point pendulum, in a more abstract form. We have for the dimensions $\dim\omega = [\mathsf{T}^{-1}]$, $\dim g = [\mathsf{L}/\mathsf{T}^2]$, $\dim l = [\mathsf{L}]$ and $\mathsf{m} = [\mathsf{M}]$. If ω is a function of $\{g, l, m\}$, then its dimensions must be a power law monomial of the dimensions of these quantities. We then have

$$\begin{aligned}\dim \omega &= [\mathsf{T}^{-1}]\\ &= \dim g^a \dim l^b \dim m^c\\ &= \left(\mathsf{L}/\mathsf{T}^2\right)^a \mathsf{L}^b\mathsf{M}^c\\ &= \mathsf{L}^{a+b}\mathsf{T}^{-2a}\mathsf{M}^c\,,\end{aligned}$$

with $\{a, b, c\}$ constants determined by comparing the dimensions on both sides of the equation. We see that

$$a + b = 0\,, \qquad -2a = -1 \quad \text{and} \quad c = 0\,.$$

The solution is then $a = 1/2$, $b = -1/2$ and $c = 0$ and we recover that $\omega = \Pi\sqrt{g/l}$.

A set of quantities $\{a_1, a_2, \ldots, a_k\}$ is said to have *independent dimensions* if none of these quantities have dimensions that can be represented as a product of powers of the dimensions of the remaining quantities. For instance, density, $\dim\rho = [\mathsf{M}/\mathsf{L}^3]$, velocity and force have independent dimensions, so that there is no product of powers of these quantities that is dimensionless. On the other hand, density, velocity and pressure, $\dim p = \mathsf{M}/\mathsf{L}\mathsf{T}^2$, are *not* independent for we can write $\dim p = \dim\rho \dim v^2$; *i.e.*, $p/\rho v^2$ is a dimensionless quantity.

Now suppose that we have a relationship between a quantity a, which is being determined in some experiment (which we will refer to as the *governed quantity*), and a set of quantities $\{a_1, \ldots, a_k\}$ that are under experimental control (the *governing quantities*, that is of the form

$$a = f[a_1, \ldots, a_k; a_{k+1}, \ldots, a_n]\,, \tag{15}$$

where $\{a_1, \ldots, a_k\}$ have independent dimensions. For example, this would mean that the dimensions of the governed quantity a is determined by the dimensions of $\{a_1, \ldots, a_k\}$, while all of the a_s's with $s > k$ can be written as products of powers of the dimensions of $\{a_1, \ldots, a_k\}$; *e.g.*, $a_{k+1}/a_1^p \cdots a_k^r$ would be dimensionless, with $\{p, \ldots, r\}$ an appropriately chosen set of constants. With this set of definitions, it is possible to prove that the last equation can be written as

$$a = a_1^p \cdots a_k^r \phi_\pi\left[\frac{a_{k+1}}{a_1^{p_{k+1}} \cdots a_k^{r_{k+1}}}\,, \ldots\,, \frac{a_n}{a_1^{p_n} \cdots a_k^{r_n}}\right]\,, \tag{16}$$

with ϕ_π some function of *dimensionless quantities only*. The great simplification is that while the function f in Equation 15 was a function of n variables, the function ϕ_π in Equation 16 is *only a function of* $n-k$ *variables.* Equation 16 is a mathematical statement of *Buckinham's* Π*-Theorem* – the central result of dimensional analysis. Dimensional analysis cannot supply us with the dimensionless function ϕ_π – we need a real theory for that.

As a simple example of how this works, let's return to the pendulum, but this time we'll assume that the mass can be distributed, so that we relax the condition of the mass being concentrated at a point. The governed quantity is the frequency ω; the governing quantities are g, l, (which can be interpreted as the distance between the pivot point and the center of mass), m and the moment of inertia about the pivot point, i. Since $\dim i = [\mathsf{M}\mathsf{L}^2]$, the set $\{g, m, l, i\}$ is not independent; we can choose as our independent quantities $\{g, m, l\}$ as before, with i/ml^2 a dimensionless quantity. In the nottion developed above, $n = 4$ and $k = 3$. Therefore, dimensional analysis tells us that

$$\omega = \sqrt{\frac{g}{l}}\phi_\pi\left[\frac{i}{ml^2}\right]\,,$$

with ϕ_π some function that cannot be determined from dimensional analysis alone; we need a *theory* in order to determine it.

B.2. **Examples.**

B.2.1. *Oscillations of a star.* A star undergoes some mode of oscillation. How does the frequency ω of oscillation depend upon the properties of the star? The first step is the identification of the physically relevant quantities. Certainly the density ρ and the radius r are important; we'll also need the gravitational constant $g_{\rm new}$, which appears in Newton's law of universal gravitation. We could add the mass m to the list, but if we assume that the density is constant as a first approximation, then $m = \rho 4\pi r^3/3$ and the mass is redundant. Therefore, ω is the governed quantity, with dimensions $\dim \omega = [\mathsf{T}^{-1}]$ and $\{\rho, r, g_{\rm new}\}$ are the governing quantities, with dimensions $\dim \rho = [\mathsf{M}/\mathsf{L}^3]$, $\dim r = [\mathsf{L}]$ and $\dim g_{\rm new} = [\mathsf{L}^3/\mathsf{MT}^2]$. You can easily check that $\{\rho, r, g_{\rm new}\}$ have independent dimensions (in the $\{\mathsf{L}, \mathsf{M}, \mathsf{T}\}$ class, l brings $[\mathsf{L}]$, ρ brings $[\mathsf{M}]$ and $g_{\rm new}$ brings $[\mathsf{T}]$); therefore, $n = 3$ and $k = 3$, so the function ϕ_π is simply a constant in this case. Next, determine the exponents. Then, we have

$$\omega = \Pi \sqrt{g_{\rm new} \rho}\,,$$

with Π a dimless constant. We see that the frequency of oscillation is proportional to the square root of the density and independent of the radius. Once again, the determination of Π requires a real theory of stellar oscillation, but the interesting dependence upon the physical quantities has been obtained from dimensional considerations alone.

B.2.2. *Energy in a nuclear explosion.* We next turn to a famous example worked out by G. I. Taylor. In a nuclear explosion there is an essentially instantaneous release of energy e in a small region of space. This produces a spherical shock wave, with the pressure inside the shock wave thousands of times greater that the initial air pressure, which may be neglected. How does the radius r of this shock wave grow with time t? The relevant governing variables are e, t, and the initial air density ρ_0. This set of quantities has independent dimensions, so $n = 3$ and $k = 3$. We next determine the exponents to find

$$r = \Pi \frac{e^{3/5} t^{2/5}}{\rho_0^{1/5}}\,,$$

with Π an undetermined dimless constant. If we could plot the radius r of the shock as a function of time t on a log-log plot, then the slope of the line should be $2/5$. The intercept of the graph would provide information about the energy e released in the explosion, if the constant Π could be determined. By solving a model shock-wave problem Taylor estimated Π to be about 1; he was able to take declassified movies of nuclear tests and, using his model, infer the yield of the bombs. This data, of course, was strictly classified; it came as a surprise to the American intelligence community that these data were essentially publicly available to those well versed in dim. analysis.

B.2.3. *Solution of the diffusion equation.* Dimensional analysis can also be used to solve certain types of partial differential equations. If this seems too good to be true, it isn't. Here we will concentrate on the solution of the diffusion equation.

We'll start by deriving the one-dimensional diffusion equation. Let $\tau[x, t]$ represent the temperature of a metal bat at a point x at time t. The first step is the derivation of a *continuity equation* for the thermal flow on the bar. Let the bar have a cross sectional area a, so that the infinitesimal volume of the bar between x and $x + \Delta x$ is $a\Delta x$. The amount of thermal energy contained in this volume is $c_p \tau a \Delta x$, with c_p the specific thermal capacity at constant pressure *per unit volume*; it has dimensions $\dim c_p = [\mathsf{M}/\mathsf{LT}^2\Theta]$. In a time interval $\mathrm{d}t$, this energy changes by an amount $c_p \partial_t \tau a \Delta x \mathrm{d}t$ due to the change in temperature. This change in energy must come from somewhere and is the result of a flux of energy $q[x, t]$ through the area a (q is thermal energy flowing through a unit area per unit time). Into the left side of the volume, an amount of energy $qa\mathrm{d}t$ flows an a time $\mathrm{d}t$; on the right hand side of the volume, a quantity $a\,(q + \partial_x q \Delta x)\,\mathrm{d}t$ flows out in a time $\mathrm{d}t$, so that the net accumulation of thermal energy in the volume is $-a\partial_x q \Delta x \mathrm{d}t$. Equating the two expressions for the rate of change of the thermal energy in the volume $a\Delta x$, we find

$$c_p \partial_t \tau = -\partial_x q\,,$$

which is the equation of continuity. It is a mathematical expression of the conservation of energy in the volume $a\Delta x$. We supplement this with a phenomenological law of energy conduction known as *Fourier's law*: the thermal flux is proportional to the negative of the local temperature gradient (thermal energy flows from a hot reservoir to a cold reservoir):

$$q = -\kappa \partial_x \tau ,$$

with κ the *thermal conductivity* of the metal bar. The thermal conductivity is usually measured in units of $\mathrm{W\,cm^{-1}\,K^{-1}}$ and has dimensions $\dim \kappa = [\mathsf{ML}/\mathsf{T}^3\Theta]$. Combining the last equations, we obtain the *diffusion equation*, often called the *heat* equation:

$$\partial_t \tau = d \partial^2_{x^2} \tau ,$$

where $d = \kappa/c_p$ is the *thermal diffusivity* of the metal bar; it has dimensions $\dim d = [\mathsf{L}^2/\mathsf{T}]$. The last equation is the diffusion equation for thermal energy. It usually results from combining a continuity equation with an empirical law that expresses a current of flux in terms of some local gradient.

Suppose that the bar is very long, so that we can consider the idealized case of an infinite bar. At an initial time $t = 0$, we add an amount of thermal energy q_0 (with $\dim q_0 = [\mathsf{ML}^2/\mathsf{T}^2]$) at some point of the bar, which we will arbitrarily call $x = 0$. We could do this, for instance, by briefly holding a match to the bar. The energy is conserved at all times, so that

$$c_p a \int_{\infty}^{\infty} \tau[x,t]\,\mathrm{d}x = q_0 .$$

How does this thermal energy diffuse away from $x = 0$ as a function of time t; *i.e.*, what is $\tau[x,t]$? We first identify the important parameters. The temperature τ certainly depends upon x, t and the diffusivity d; we see from the last equation that it also depends upon the initial conditions through the combination $q = q_0/c_p a$. We have then $n = 4$ dimensions. These dimensions are not independent, for the quantity $x/(dt)^{1/2}$ is dimensionless, so that $k = 3$. We will choose as our governing quantities $\{t, d, q\}$. Now express τ in terms of these quantities using dimensional analysis, so the solution of the diffusion equation is of the form

$$\tau[x,t] = \frac{q}{(dt)^{1/2}} \phi_\pi \left[\frac{x}{(dt)^{1/2}} \right] ,$$

with ϕ_π a function that we still need to determine. The important point is that ϕ_π is only a function of the combination $x/(dt)^{1/2}$ and not x and t separately. To determine ϕ_π, let's introduce the dimensionless variable $z = (dt)^{1/2}$. Now use the chain rule to calculate various derivatives of τ to have:

$$\frac{\mathrm{d}^2\phi_\pi[z]}{\mathrm{d}z^2} + \frac{z}{2}\frac{\mathrm{d}\phi_\pi[z]}{\mathrm{d}z} + \frac{1}{2}\phi_\pi[z] = 0 .$$

Note that

> dimensional analysis has reduced the problem from the solution of a partial differential equation in two quantities to the solution of an ordinary differential equation in one quantity!

The normalization condition, boundary condition, becomes

$$\int_{\infty}^{\infty} \phi_\pi[z]\ \mathrm{d}z = 1 .$$

The last equation is an exact differential

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\phi_\pi}{\mathrm{d}z} + \frac{z}{2}\phi_\pi \right) = 0 ,$$

which we can integrate once to obtain

$$\frac{\mathrm{d}\phi_\pi}{\mathrm{d}z} + \frac{z}{2}\phi_\pi = c ,$$

where c is an integration constant.

However, since any physically reasonable solution would have both $\phi_\pi \to 0$ and $\mathrm{d}_x\phi_\pi \to 0$ as $x \to \infty$, the integration constant must be zero. We now need to solve a first order differential equation. The solution of such an equation is

$$\phi_\pi[z] = h \exp\left[-z^2/4\right] ,$$

with h a constant. To determine h, we use the normalization condition

$$h \int_{\infty}^{\infty} \exp\left[-z^2/4\right] \, \mathrm{d}z = h \, (4\pi)^{1/2} = 1 \, ,$$

where the integral (known as a *Gaussian integral*) can be found in integral tables. Therefore, $h = 1/\left(4\pi\right)^{1/2}$. Finally, returning to our original quantities, we have

$$\tau[x, t] = \frac{q}{(4\pi d t)^{1/2}} \exp\left[-x^2/4dt\right] \; .$$

This is the complete solution for the temperature distribution in a one-dimensional bar due to a point source of thermal energy.