

PROPAGATION OF ERRORS IN CALCULATIONS

[Adapted from [1, p. 47]]

DESCRIPTION

Consider the quantities a_k s, with measured values \hat{a}_k s and measured uncertainty Δa_k s, and the quantities b_l s, with sample mean values \bar{b}_l s and sample standard deviations σ_{b_l} s; *i.e.*, the \hat{a}_k s arise from single measurements and the \bar{b}_l s from multiple measurements. Consider now a function f that depends on the a_k s and on the b_l s. Then, the most likely value of f is given by

$$\hat{y} = f[\hat{a}_k, \bar{b}_l] ,$$

whereas its maximum uncertainty by

$$\Delta y = \sum_k \text{abs}(\partial_{a_k} y) \Delta a_k + \sum_l \text{abs}(\partial_{b_l} y) \sigma_{b_l} ,$$

where $\text{abs } x$ represents the absolute value of x .

EXAMPLE

Determine the local free fall acceleration by the period of a mathematical pendulum of length $l/\text{m} = 1.00 \pm 0.01$ and period $t/\text{s} = 2.0062 \pm 0.0057$. The length was measured once and the period thousand times (Monte Carlo simulation).

The period of a math pendulum of length l is given by

$$t = 2\pi \sqrt{\frac{l}{g}} ,$$

where g represents the local free fall acceleration.

From the pendulum equation, isolate g to have

$$g = 4\pi^2 \frac{l}{t^2} .$$

Then, the max uncertainty of g is

$$\Delta g = \text{abs}(\partial_l g) \Delta l + \text{abs}(\partial_t g) \sigma_t = 4\pi^2 \frac{1}{t^2} \Delta l + 8\pi^2 \frac{l}{t^3} \sigma_t .$$

Divide the last equation by the equation for g to have the relative uncertainty (fractional change) of g

$$\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2 \frac{\sigma_t}{t} .$$

Replace the given values in the g equation and in the equation for its relative uncertainty to find

$$\begin{aligned}
 g &= 4\pi^2 \frac{l}{t^2} = 4\pi^2 \frac{1.00}{2.0062^2} = 9.808696222936447, \\
 \Delta g &= g \left(\frac{\Delta l}{l} + 2 \frac{\sigma_t}{t} \right) \\
 &= 9.808696222936447 \left(\frac{0.01}{1.00} + 2 \frac{0.0057}{2.0062} \right) \\
 &= 0.153823746668.
 \end{aligned}$$

Finally, the value for the free fall acceleration is $g/\text{m s}^{-2} \sim 9.809(154)$.

REFERENCES

- [1] Carl von Ossietzky Universität Oldenburg. *Error theory and regression analysis*. Institute of Physics, 2014.