

APPROXIMATIONS

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ABSTRACT. Only wimps specialize on the general case. Real scientists pursue examples. – Beresford Parlett [?].

1. INTRODUCTION

[...] the principle known as Occam's Razor: *essentia non sunt multiplicanda praeter necessitatem* (hypotheses are not to be multiplied without necessity).

— ATTRIBUTED TO WILLIAM OF OCKHAM, QUOTED BY R.V. JONES, [? , p.95].

Only wimps specialize on the general case. Real scientists pursue examples.

— BERESFORD PARLETT, [?].

[The Art of Approximation in Science and Eng. Sanjoy]

An approximate model can be better than an exact model! An approximate answer is all that we can understand because our minds are a small part of the world itself. So when we represent or model the world, we have to throw away aspects of the world in order for our minds to contain the model. Thus, making useful models means discarding less important information so that our minds may grasp the important features that remains.

1.1. **Praveen.** [Analysis of Strategic Knowledge in Back of the Envelope Reasoning. Praveen K. Paritosh and Kenneth D. Forbus]

All estimation problems ask for a quantitative value for some parameter. Our approach breaks down the problem solving into two distinct processes:

- (1) Strategies: Using strategies to transform the current problem into possibly easier problems.
- (2) Estimation: Coming up with a numeric estimate for a parameter. The number could already be known or similar examples might be used to make an estimate.

With strategies represented as above, there are three syntactic possibilities for a strategy based on what aspect of problem it transforms:

- (1) Object-based.
- (2) Quantity-based.

Next we look at each of these in turn.

1.1.1. *Object-based Strategies.*

- (1) Mereology: The mereology strategy transforms an object into other objects that are its parts. For extensive quantities, for example, the weight of a basket of fruits is the sum of weights of all the fruits and the basket. For intensive quantities, on the other hand, for instance, the density of a mixture is the weighted average of the densities of the constituents.
- (2) Similarity: The similarity strategy transforms the object into other object(s) which are similar to it. For example, if asked for the population of Austria, a reasonable guess could be the population of Switzerland, based on the similarity of the two countries. However, if two objects are similar, it doesn't warrant the inference that

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values of all the quantities for two objects are similar. For example, two similar basketball players might have similar height, but not necessarily two professors.

- (3) **Ontology:** The ontology strategy tries to find other objects from the ontology hierarchy which might be used to guess the quantity in question. In the simplest form, if O is an instance of O_1 , then we can use the knowledge about the class to guess the value for the instance. For example, if we know that Jason Kidd is a point guard, then we can use the knowledge that point guards are relatively shorter than other players on the team to guess his height. If we didn't have information about point guards, we could even use the fact that Jason Kidd is a basketball player to guess his height.

1.1.2. *Quantity-based Strategies.* A quantity-based strategy relates a quantity, Q , to a set of quantities, $\{Q_i\}$, such that the values of these quantities (for the object O) can be combined in a known way to derive the original quantity.

- (1) **Density:** This strategy converts a quantity into a *density quantity* and an *extent quantity*. Here, density is used in a general sense to mean average along any dimension: we talk of electric flux density, population density, *etc.* Rates, averages, and even quantities like teachers per student are considered densities. For example, number of K-8 teachers in US can be estimated by multiplying the number of teachers per student by number of students.
- (2) **Domain Laws:** This strategy converts a quantity into other quantities that are related to it via laws of the domain. Domain laws include laws of physics as well as rules of thumb. For example, Newton's second law of motion, $f = ma$, relates the force on an object to its mass and acceleration. The application of domain laws by the problem solver requires formalizing the assumptions and approximations implicit in the laws. Since we are not interested in an exact answer, but an approximate estimate, aggressively applying approximations to simplify the problem solving becomes crucial. Some of the approximations are:
 - **Geometry:** Assume simplest shape, *e.g.*, consider a spherical cow.
 - **Distribution:** Assume either a uniform distribution, or a Dirac-delta (point mass).
 - **Calculus:** Integrals can be simplified by sums or average multiplied by extent, and differentials by differences.
 - **Algebra:** Use simplification heuristics to reduce the number of unknowns.

1.1.3. *System-based Strategies.* System-based strategies transform both the quantity and the object into other quantities and objects. They represent relationships between quantities of a system as a whole. For example, for a system with no external force, the momentum remains conserved. This translates into a conservation equation that relates the masses and velocities of the parts of the system. It would seem that this effect can be obtained by sequentially applying a quantity-based and an object-based strategy (or *vice versa*) since all the above strategies are compositional. There are two reasons for keeping this a separate type of strategy: 1) it represents a reasoning pattern that is different, 2) sometimes it is much more efficient to apply a system-based strategy; *e.g.*, applying conservation of momentum leads to safely ignoring all the internal forces which need to be made explicit if one is not applying a system-based strategy. The two system-based strategies are:

- (1) **System Laws:** This class consists of physical laws that are applicable to a system as a whole. Many physical quantities remain conserved for a system, *e.g.*, energy, mass, momentum, angular momentum, *etc.* As a result of this, often one can write a balance equation that relates the expressions that denote the value of the quantity in two different states of the system. To write a balance equation, appropriate assumptions about the system in consideration have to be made.
- (2) **Scale-up:** This is often an empirical strategy. A smaller model that works under the same physical laws can be used to estimate the quantity values for a full-scale prototype. To ensure that the scale-up is valid, all the dimensionless groups must be kept the same in the model and the prototype. For example, the Reynolds number is a dimensionless group that corresponds to the nature of flow (laminar,

transient or turbulent), and for a flow model to be valid for scaling up, the Reynolds number must be the same in both situations.

1.2. **Quine.** [Addition: Back of the Envelope (BOTE) Calculations. B. M. Quine]

1.2.1. *What and why?*

- (1) Often – during brain storming, discussions, out in the field – engineers need to make rapid estimates to eliminate candidate solutions, to establish feasibility, to sketch out potential paths to a solution.
- (2) Although most engineers remember key numbers related to their field, no-one has every detail at their fingertips.
- (3) Hence we need to estimate not only the values of numbers we need, but which numbers are appropriate and how to perform the calculation:
- (4) the emphasis here is on “order of magnitude” estimates; *i.e.*, to the nearest factor of 10;
- (5) it is also important to remember that these are rough estimates and to place only appropriate reliance on the results.

1.2.2. *General principles.*

- (1) When you use back-of-the-envelope calculations be sure to recall Einstein’s famous advice: “everything should be made as simple as possible, but no simpler”.
- (2) Don’t worry about specific factors: round to the nearest sensible value; corollary: don’t make numbers more precise than is necessary .
- (3) Guess numbers you don’t know but try to make your guesses good ones and within the bounds of common sense. Common sense requires some education – the accuracy of common sense increases with experience.
- (4) Adjust geometry to suit you. Assume a human is spherical if it helps.
- (5) Extrapolate from what you do know; *e.g.*, use ratios, assume unknown value is same as a similar known quantity
- (6) Use the principle of conservation: what goes in must either come out or stay inside (accumulation). Things are not generally destroyed, so work out where they have gone.
- (7) Ensure formulas are dimensionally correct – *this is a very powerful technique; i.e.*, an expression to tell you the length of something must have overall dimensions of meters.
- (8) Apply a *plausibility* filter: if an answer seems unbelievable, it probably is. You can usually set a range of possible or reasonable values for a quantity that will indicate a major mistake. For instance, speed cannot be faster than speed of light!
- (9) Bound or threshold problems to scope design solutions. For instance, Will it fit in? No it is 100 times too large even with optimistic values.
- (10) Consider your results in context with the assumptions you have made.
- (11) Use two different methods to contrast *Rough Order of Magnitude*, ROM, estimates. For instance, estimation and comparison.

1.2.3. *Uncertainty.*

- (1) Once you have a method to solve the problem, you can include best- or worst-case estimates; *e.g.*, how many light bulbs are there in the U.S.? Somewhere between 10^8 and 10^9 people. Not less than 1 light bulb/person. Likely not more than 10^3 light bulb/person. So the range of the answer is from 10^8 to 10^{12} light bulbs.
- (2) The bounding box: between what values are we sure the answer lies. To get the largest possible overestimate, multiply all the largest possible values and divide by all the smallest possible values. And *vice versa* for the lowest possible underestimate.
- (3) The likely box: similar to the bounding box but using the largest and smallest likely values.

1.3. **Wittich.** [Back of the Envelope Physics. Peter Wittich]

1.3.1. *Goal.* Back of the envelope goal: know what to expect: I want to measure a new quantity. Thus, I need to know roughly what to expect.

1.3.2. *Common estimation tricks.*

- (1) Remember, we want to simplify our calculations. Volume: everything is a square or a sphere. $\pi = 3$, $3^2 = 10$, $3\pi = 10$, $e = 3$, $1\text{ y} = \pi \cdot 10^7\text{ s}$.
- (2) Use the right units. For example, for energy: joules for macroscopic scales; eV for nuclear physics scales.
- (3) Round, round, round. If you think your number is only good to an order of magnitude it doesn't make sense to drag along many significant digits.
- (4) Know some appropriate numbers. If calculating g on the moon, compare it to g on earth.

1.3.3. *What's different compared to the piano tuners?*

- (1) You still need to know physics. You can't use *back of the envelope* to forgo knowledge of physics. You need to know what the base equations are.
- (2) This is where practice comes in. Learn what the relevant approximations are; learn what you can ignore and what you cannot.

1.3.4. *Preliminary conclusions.*

- (1) Doing this kind of estimates is a good tool to have in your toolbox.
- (2) It requires a willingness to estimate quantities and a knowledge of enough physics to know what's important.
- (3) It's a skill that can be practiced.

1.4. **Francis.** ["Back-of-the-Envelope Calculations" Or: The Seven Habits of Highly Effective Astronomers. Paul Francis]

One of the most important skills that any professional astronomer needs is the ability to very quickly get a highly approximate answer to a problem, without getting bogged down in details. This has many purposes:

- (1) Most research projects start off with a bright/crazy idea. Most of these bright ideas turn out to be wrong, irrelevant or unimportant. Given that most ideas do not work, it is vital to be able to tell quickly whether it is worth pursuing a particular idea: if you work out every idea in enormous detail, you will never get to the few interesting ones. This is the most common mistake that young researchers make: they get bogged down in some horribly mathematical detailed calculation, full of nasty integrals and messy algebra, and in the end discover that the effect they are computing is 10^{-27} too small to be detected, something they could have worked out on the back of an envelope right at the beginning if they'd tried, thus saving them a month of work. So,
 whenever starting a research project, try and work out the answer very
 very approximately.

If your idea is crazy, you will quickly find out. Only if this initial quick guesstimate comes out with an interesting answer should you bother doing the calculation in detail.

- (2) No really complex calculation or computer simulation ever gives the right answer first time. Worse still, for real research, there is no answer in the back of the textbook and no higher authority to give you a high mark for getting it right. What this means is that

you should never believe the result of any detailed calculation unless it roughly agrees with some simple approximate estimate that you can do on the back of an envelope and understand fully.

This is a second common mistake of young researchers: they get a result, go out and publish it and do vast amounts of work relying on it, while all the time it was clearly wrong. I see the same problem all the time in undergraduate assignments: someone comes up with an answer that is clearly, wildly, hopelessly wrong; *e.g.*, the mass of some star is 0.7 kg, an asteroid is traveling at 10^{10} m/s – faster than light – or the Earth took 10^{11} y to form (much longer than the age of the universe). So

always check your answers for plausibility.

Even if you don't have time to re-do them properly, you will get credit for saying "this is clearly garbage, but I don't have time to fix it".

- (3) In most real situations, many different effects are operating. For example, consider the elementary physics problem of a falling object. You normally calculate how long it takes to hit the ground considering only gravity, height and maybe air resistance. In reality, however, its course will be influenced by radiation pressure, the Earth's magnetic field, the gravitational pull of Jupiter, the curvature of space-time, coriolis force, cosmic ray bombardment and the perfume being worn by the person who dropped it, to name but a few. Most of these effects are tiny: far too small to have a measurable effect. Nonetheless, many people waste vast amounts of time computing, in tedious detail, these tiny and unimportant effects.

A back-of-the-envelope calculation can often tell you, right away, that one of these effects is far too small to be worth calculating properly.

1.4.1. *Hints for doing Back-of-the Envelope estimates.* Doing guesstimate type calculations is actually far harder than doing things properly in enormous detail. You need

- a really strong grasp of physics,
- intuition about which parts of a problem are important and
- imagination to dream up short-cuts.

The only real way to learn is through practice. What I can give you is a few hints, based on the personal experience of many professional astronomers. Everybody has their own style for doing these simple calculations: you should develop your own, but hopefully these hints will give you somewhere to start.

- (1) Don't worry about factors of 2, π , *etc.* The aim of most approximate calculations is to get an answer that is correct to about an order of magnitude (a factor of ~ 10). So don't worry about little things like factors of two, π or $4/7$. Throw away most constants! The area of a circle is r^2 , the gravitational field of the Earth is 10 m/s^2 and so on.
- (2) Guess numbers. Every professional astronomer should have memorized a bunch of basic numbers, like the typical density of rock, mass of a star, radius of a galaxy, and so on. When you find that some new number is needed, you can often guess its value by comparison with numbers you already know.
- (3) Tinker with the geometry. Feel free to be very cavalier with the geometry of the problem you are working on. For example, the Milky Way galaxy has a complex flattened shape, but for many purposes it can be approximated as a point source. Assume that mountains are square blocks, that asteroid are cubes or spheres: whatever makes the calculation easier.

One specific hint: replace smoothly varying functions (which have to be integrated over) with discrete functions. For example, consider the issue of whether a star passing near the solar system will disturb the planets in their orbits. As the star approaches, its gravitational pull slowly increases, constantly changing in direction, making the calculation of the perturbation of a planet's orbit tricky. Instead, why not just assume that the planet appears from nowhere, popping into existence at a distance from the solar system equal to its closest approach in the real situation. Keep it there for a time roughly comparable to the time needed to pass the solar system, and then make it disappear again. This is now an easy problem to solve: the gravitational pull of the star is always from the same direction and always has the same strength, and the answer won't be too far wrong.

- (4) Use Ratios. Ratios are wonderful things: they avoid the need to work out constants and fiddle with many details. For example, the gravity on the surface of a planet of radius r and density ρ is

$$g = 4G\pi r\rho \propto r\rho.$$

Now, we know g on earth (10 m/s^2). What is g on Mars? Well, Mars is rocky, just like the Earth, so its density is going to be about the same. Its radius is $1/3$

that of the Earth, so on Mars, $g \sim 10/3 \sim 3$. We never needed to know G , or the density and radius of either planet.

- (5) Use Conservation Laws. One of the many wonderful things about Physics is all the lovely conservation laws: conservation of energy, mass, momentum, angular momentum, *etc.* By judicious use of these laws, you can get an approximate answer to many problems, while leaving out all the messy details.

For example, as a giant molecular gas cloud shrinks down to form stars and planets, the details of gas flows, turbulence, shocks and accretion are so complex that not even the world's fastest supercomputers come close to simulating it. Nonetheless, somehow all this messy physics must produce a final solar system with the same angular momentum as the cloud had at the beginning!

- (6) The Method of Dimensions (dimensional analysis). The laws of physics are supposedly universal: they work for everyone. This means that every valid equation should work equally well, regardless of the units you use (as long as they are self consistent). Thus

$$f = ma$$

should work regardless of whether you use newtons, kilograms and meters per second or some other units.

This means that the units of both sides of any valid equation must always be the same. You can often use this fact to work out the form of an equation, without any knowledge of the physics. Play around with the various plausible terms in the equation (which you can usually guess) until you come up with something with the same units (dimensions) on both sides of the equals sign. This will hopefully be correct, apart from some dimensionless constant (like 2 or π).

One hint on using this method: if you are doing a calculation that will have a numerical answer, it is sometimes tempting to substitute numbers in place of symbols early in the analysis. This is bad for two reasons: firstly, it makes it impossible to use the method of dimensions to check your results. Secondly, if you find you got a silly answer, it is hard to see where you went wrong, and to recalculate things. It is almost always best to keep things as symbols right through to the end of the algebra, and only then to substitute in numbers.

- (7) Plausibility Checking. There are several ways to check your results without re-doing a whole tedious and complex calculation. The *method of dimensions*, discussed above, is one such method. Another is to check that your solution gives a correct result in some situation where you know the answer. For example, if you derive an equation which tells you the thickness of the atmosphere on a planet of a given mass and composition, use it to calculate the thickness of the atmosphere of the Earth and see if it comes out right.

Another very powerful method is to check that the functional form of the equation is correct. For example, say that you derived an equation relating the mass m of a star to its luminosity L . Imagine that the equation you derived was

$$l \propto 1.05 + m^3 + 4\pi/(3 - m),$$

with m measured in units of solar masses and l in units of solar luminosities.

Do you think this could be correct? If you look at the equation, you can easily see two problems. Firstly, what happens if m is very small? Say a star with a mass of 1 kg? It clearly should not be very bright. But the equation above says that even if $m = 0$, the luminosity is still more than 5 solar luminosities. This clearly makes no sense: if this equation were correct, then even pebbles would outshine the sun. Secondly, what happens if $m = 3$? The last term in the equation goes to infinity. This is saying that stars with three times the mass of the Sun are infinitely luminous. This doesn't seem to make much sense. Worse: it gives the wrong answer when $m = 1$ (the Sun should presumably have a luminosity of one solar luminosity) and for m just larger than three, it gives a negative answer (how can anything have a negative luminosity?).

So: always check the functional form of your answer to make sure that it is behaving in the correct way (another good reason for leaving your algebra in the

form of symbols right until the end). If they don't seem to make sense, go back and think very hard about what you've done. Sometimes the calculation is right and the answer that you thought made no sense is actually telling you something revolutionary about the universe (this is how the Hawking radiation from black holes was discovered), but usually it is telling you that you've stuffed up your calculation somewhere.

1.4.2. *Conclusions.* I hope I've convinced you that the art of doing back-of-the-envelope calculations is a very valuable one. They cannot take the place of proper calculations, of the sort you are doing in your physics and maths assignments, but most professional astronomers spend more of their time on these guesstimates than they do on full-blown calculations.

Doing these approximate calculations is admittedly very hard: you have to really understand the physics and rote learning of techniques and equations won't help you with these.

1.5. **Serway.** [Physics for Scientists & Engineers, 6 Ed., Serway and Jewett]

It is often useful to compute an approximate answer to a given physical problem even when little information is available. This answer can then be used to determine whether or not a more precise calculation is necessary. Such an approximation is usually based on certain assumptions, which must be modified if greater precision is needed. We will sometimes refer to an order of magnitude of a certain quantity as the power of ten of the number that describes that quantity. Usually, when an order-of-magnitude calculation is made, the results are reliable to within about a factor of 10. If a quantity increases in value by three orders of magnitude, this means that its value increases by a factor of about $10^3 = 1000$. We use the symbol \sim for *is on the order of*. Thus,

$$0.0086 \sim 10^{-2}, \quad 0.0021 \sim 10^{-3} \quad \text{and} \quad 720 \sim 10^3.$$

The spirit of order-of-magnitude calculations, sometimes referred to as *guesstimates* or *ball-park figures*, is given in the following quotation:

Make an estimate before every calculation, try a simple physical argument...
before every derivation, guess the answer to every puzzle.

Inaccuracies caused by guessing too low for one number are often canceled out by other guesses that are too high. You will find that with practice your guesstimates become better and better. Estimation problems can be fun to work as you freely drop digits, venture reasonable approximations for unknown numbers, make simplifying assumptions, and turn the question around into something you can answer in your head or with minimal mathematical manipulation on paper. Because of the simplicity of these types of calculations, they can be performed on a small piece of paper, so these estimates are often called *back-of-the-envelope calculations*.

1.6. **Cooper.** [Making Estimates in Research and Elsewhere. Lance Cooper]

1.6.1. *Why make estimates in science?* The ability to estimate – to within an order of magnitude or so – the size or probability of various quantities is useful in science as well as in many other endeavors:

- (1) to provide a rough check of more exact calculations;
- (2) to provide a rough check of research results or hypotheses;
- (3) to obtain estimates of quantities when other resources aren't available;
- (4) to obtain estimates of quantities that are difficult to measure precisely;
- (5) to obtain estimates of quantities for which no firm theoretical prediction exists: particularly important in interdisciplinary sciences, soft matter, astrophysics;
- (6) to provide bounds for possible design alternatives.

1.6.2. *How? Getting started:*

- (1) Don't panic when you see the problem.
- (2) Write down any fact you do know related to the question.
- (3) Outline one or more possible procedures for determining the answer.
- (4) List the things you'll need to know to answer the question.

- (5) Keep track of your assumptions.

1.6.3. *Guidelines.* Other general guidelines for making order-of-magnitude estimates:

- (1) Make everything as simple as possible!
 - Don't worry about specific values: round numbers to "convenient values"; *e.g.*, $\pi \sim 3$, $8.4 \sim 10$, *etc.*
 - Choose convenient geometries when modeling; *e.g.*, a spherical cow, a cubic grain of sand, *etc.*
 - Make "educated" guesses or even upper and lower bounds of quantities you don't know. Try to make good guesses, and keep track of these guesses, as they will set bounds on the fidelity of your estimate.
 - Use ratios when possible – by comparing the value of one quantity (*e.g.*, force, energy, *etc.*) in comparison to a related quantity – in order to eliminate unknown parameters and get a dimensionless parameter.
 - If possible, exploit plausible scaling behavior of some quantity; *i.e.*, estimate an unknown quantity by assuming it scales linearly – from known values – with some parameter.
- (2) Checking your estimates:
 - Make sure that your estimates and calculations are dimensionally correct! *This is a very powerful tool!*
 - Check the plausibility of your estimate, if possible; *e.g.*, if your answer exceeds the speed of light or the size of the universe, you've got a problem!
 - Check the plausibility of your estimate using an alternate calculation method. Do the two methods agree to within an order of magnitude?
 - Perform a "reality check" on your estimate based on the number and size of the approximations you made.
 - More quantitatively – place "bounds" on your estimate:
 - (a) to obtain an "upper bound" – in equations, put largest estimated values of quantities in the numerator and the smallest estimated values in the denominator;
 - (b) to obtain a "lower bound" – in equations, put smallest estimated values of quantities in the numerator and the largest estimated values in the denominator.

1.7. Lee DeVille. [Lee DeVille. Math 558 - Methods of Applied Mathematics.]

1.7.1. *Advises on DA and nondim.* Make dim analysis before starting solving a problem. Use the problem parameters as scaling quantities. By applying DA, we have to insert much less information into the problem to obtain solutions and, sometimes, don't have to know how to solve differential equations.

Theorem. Let us consider a physical quantity q that depends on the n physical quantities $\{p_1, p_2, \dots, p_n\}$. We have the relationship $q = f[p_1, p_2, \dots, p_n]$. Next, assume that $q = f[p_1, \dots, p_n]$ is a dimensionally homogeneous relation and $\{q_1, q_2, \dots, q_n\}$ are *dimensionally complete*. Then, there exists a function f such that $q = q^* f[\Pi_1, \dots, \Pi_k]$, where $\{\Pi_i\}$ are dimensionless products of the $\{p_i\}$, $\dim q^* = \dim q$ and k is the dimension of the kernel of A .

To apply the theorem, two conditions must be satisfied:

- (1) $q = f[p_1, \dots, p_n]$ must be dimensionally homogeneous; *i.e.*, a valid physics law;
- (2) the set $\{q_1, q_2, \dots, q_n\}$ must be dimensionally complete.

Then the last theorem expresses existence, but not uniqueness. It says that the function f exists, but it may not be unique. Additionally, it doesn't say how to find f . Regularly, f has to be found by theoretical work or by experimentation.

Now, say you have a diff eqn with boundary and initial conditions. To scale the diff eqn replace the dependent and independent quantities, say position x and time t by the product of dimensionless quantities and characteristic quantities: $\bar{x} = x/x_c$ and $\bar{t} = t/t_c$, where \bar{x} and \bar{t} are the dimensionless, scaled physical quantities and x_c and t_c the characteristic quantities. At the end of the process, one ends with several dimensionless quantities made

of characteristic quantities and constants. These parameters should be set to unity, zero or left as they are according to the following rules of thumb.

1.8. Rules of thumb on nondim.

- (1) (always) Make as many nondim quantities equal to one as possible;
- (2) (usually) Make the nondim quantities that appear in the initial or boundary conditions equal to one.
- (3) (usually) If there's a nondim quantity that, if we were to set it equal to zero, would simplify the problem significantly, allow it to remain free and then see when we can make it small (scaling analysis).

2. PRESSURE

Consider a piston of volume v and sectional area a holding an amount n of an ideal gas and consider a force f being applied on a that compresses the fluid.

Such a force generates a pressure p on the gas. See now that p can be viewed as energy density instead of force per unit area. With this view, one finds the external energy applied to the system e_{ext} by

$$e_{\text{ext}} \sim pv.$$

This external stimulus makes the gas to perform pv work, the gas internal response e_{int} , given by

$$e_{\text{int}} \sim nk_{\text{gas}}\theta,$$

where k_{gas} represents the gas constant and θ the gas temperature.

Thus, according to the energy conservation principle, the external stimulus must be balanced by the gas internal response:

$$e_{\text{ext}} \sim e_{\text{int}} \sim nk_{\text{gas}}\theta \implies k_{\text{gas}}\theta \sim \frac{e_{\text{ext}}}{n}.$$

That is, $k_{\text{gas}}\theta$ is a measure of the external energy distributed per amount of gas – molar energy.

On the other hand, since by definition an ideal gas does not interact, its total internal energy equals its kinetic energy alone:

$$e_{\text{int}} \sim mu^2,$$

where m represents the gas mass and u the average velocity of the gas particles. Thus, one finds

$$mu^2 \sim nk_{\text{gas}}\theta \implies \theta \sim \frac{mu^2}{nk_{\text{gas}}} \propto u.$$

Hence, temperature can also be viewed as a measure of the mean particle velocity of the gas particles.

3. ENERGY

Consider a large, thin concrete slab of thickness l that is *setting*. Setting is an exothermic process that releases e_{th} , where $\dim e_{\text{th}} = [\text{E}/\text{TL}^3]$ – thermal power density. The outside surfaces are kept at the ambient temperature, so the temperature of the walls, θ_w , equal the ambient temperature: $\theta_w = \theta_\infty$. What is the maximum internal temperature?

Guess. Since the walls are kept at constant temperature, the process is at steady state. However, temperature ranges spatially through the slab thickness. If one measures the spatial variation by x , then the slab temperature satisfies $\theta = \theta[x]$.

By symmetry, the center temperature coincides with θ_{max} at the slab center, $x = l/2$, and decreases smoothly to a minimum at the walls, $x = 0$ and $x = l$. This symmetry gives room to think about an inverted parabolic temperature distribution inside the slab with the parabola vertex at θ_{max} .

Dim. Analysis. Place a Cartesian coordinate axis running from one wall to the other covering the slab thickness. Let x measure position within $0 \leq x \leq l$. Thus, since the process is at steady state, $\theta = \theta[x]$.

Choose the dimensional set to be $\{\text{E}, \text{L}, \text{T}, \Theta\}$. Hypothesize the quantities governing the phenomenon to be those listed in table 1.

Quantity	Symbol	Dimension
Slab temperature	θ	Θ
Slab thermal conduction coefficient	k	Θ
Slab thickness	l	L
Wall temperature	θ_w	Θ
Setting power density	e_{th}	E/TL^3
Position within the slab	x	L

TABLE 1. Quantities and dimensions affecting the thermal conduction of the concrete slab setting.

As seen in table 1, according to the Pi-theorem, $6 - 4 = 2$ dimensionless quantities can be constructed. The first one:

$$\Pi_1 = \frac{k(\theta - \theta_w)/l^2}{e_{th}},$$

which measures the relationship between energy conduction and energy production. The second dimensionless quantity:

$$\Pi_2 = \frac{x}{l},$$

which is a geometric ratio.

With both dimensionless quantities, one can apply the principle of dimensional homogeneity for physical laws to find

$$\Pi_1 = \phi_\pi[\Pi_2] \implies \frac{k(\theta - \theta_w)}{e_{th}l^2} = \phi_\pi\left[\frac{x}{l}\right].$$

Scale temperatures by means of Π_1 and lengths by Π_2 ; *i.e.*,

$$\bar{\theta} = \Pi_1\theta \quad \text{and} \quad \bar{x} = \Pi_2x.$$

Hence, finally, the equation governing the phenomenon can be written as

$$\bar{\theta} = \phi_\pi[\bar{x}]. \quad (1)$$

where the function ϕ_π cannot be further determined by dimensional analysis.

Approx. Solution. Assuming a parabolic distribution of temperatures, ϕ_π in eq. (1) can be hypothesize to satisfy

$$\bar{\theta} = a\bar{x}^2 + b\bar{x} + c,$$

where $\{a, b, c\}$ are dimensionless quantities to be determined.

Now, we can use a theorem in geometry that states that three points uniquely determine a parabola. Two of these points can be found from the problem statement:

$$\begin{cases} \bar{x} = 0, \bar{\theta} = 0 \\ \bar{x} = 1, \bar{\theta} = 0 \end{cases}.$$

Setting $a = -1/2$ (an inverted parabola) and solving the previous systems of equations, one finds that

$$a = -\frac{1}{2}, \quad b = \frac{1}{2} \quad \text{and} \quad c = 0.$$

Replacing these values in the hypothesized ϕ_π , one has

$$2\bar{\theta} = \bar{x}(1 - \bar{x}).$$

Now, using symmetry, when $\bar{x} = 1/2$, then $\bar{\theta} = \bar{\theta}_{\max}$:

$$\bar{\theta}_{\max} = \frac{1}{8}$$

or, returning to the dimensional quantities, θ_{\max} can be found by

$$\frac{k(\theta_{\max} - \theta_w)}{e_{th}l^2} = \frac{1}{8}. \quad \square$$

4. PROJECTILE MOTION ON VARIABLE GRAVITY

[Lee DeVille. Math 558 - Methods of Applied Mathematics.]

4.1. Dim analysis. Consider that we throw a ball directly upwards from the surface of the earth. What is the maximum height?

The phenomenon depends on Newton's second law of motion and Newton's law of universal gravitation. Name x the position of the ball of mass m at any time t . Let's make the Ansatz that the maximal height x_{\max} depends only on free fall acceleration g and the initial velocity v_0 , so that

$$x_{\max} = f[g, m, v_0] .$$

If this is true, then these quantities must have the same units; *i.e.*,

$$\dim x_{\max} = \dim f[g, m, v_0] .$$

Let us now make the further Ansatz that this function f can be written as a monomial:

$$\dim x_{\max} = \dim g^a m^b v_0^c .$$

Solving the system, we find that

$$x_{\max} = \Pi \frac{v_0^2}{g} .$$

4.2. Scaling. For instance, consider a ball thrown into air with initial speed v_0 from the surface of earth with radius r . Denote by x its position at any time t and g the free fall acceleration on the surface of the planet. The governing equation is

$$\ddot{x} = -\frac{gr}{(x+r)^2} .$$

Rescale space and time by

$$x = x_c \bar{x} \quad \text{and} \quad t = t_c \bar{t} .$$

After the replacement, the last equation becomes

$$\frac{x_c}{gt_c^2} \ddot{\bar{x}} = -\frac{1}{(1 + x_c \bar{x}/r)^2} .$$

Moreover, we also have the initial conditions

$$\bar{x}[0] = 0, \quad \dot{\bar{x}}[0] = v_0 t_c / x_c ,$$

where v_0 is the initial velocity of the projectile. Note that three non-dim quantities show up in the problem:

$$\Pi_1 = \frac{x_c}{gt_c^2}, \quad \Pi_2 = \frac{x_c}{r} \quad \text{and} \quad \Pi_3 = \frac{v_0 t_c}{x_c} .$$

The first is the ratio of the characteristic acceleration of the problem with respect to the gravitational acceleration of the earth; the second is the characteristic size of the problem compared to the earth's radius and the third is the characteristic velocity with respect to the initial velocity of the problem.

Using the rules of thumb on nondim, we choose $\Pi_3 = 1$. We can see that making $\Pi_2 = 0$, the problem changes completely, since the second derivative vanishes; thus it should remain free. We then set $\Pi_1 = 1$. This means that we choose the char scales of the problem as

$$t_c = \frac{v_0}{g} \quad \text{and} \quad x_c = \frac{v_0^2}{g} .$$

This gives

$$\Pi_2 = \frac{x_c}{r} = \frac{v_0^2}{gr} .$$

Now, we expect this to be small if the projectile is our throwing a ball. We don't expect the char height of this problem to be significant when compared to r . Similarly, notice that the other expression is the ratio of the kinetic energy of the ball to the potential energy of the ball at time zero. Since, we don't expect to be able to throw a ball into orbit, we expect this ratio to be small as well. Since Π_2 is small, we denote it by ϵ and we obtain the rescaled, nondim ODE

$$\ddot{\bar{x}} = -\frac{1}{(1 + \epsilon \bar{x})^2}, \quad \bar{x}[0] = 0 \quad \text{and} \quad \dot{\bar{x}}[0] = 1 .$$

Now, we have chosen wisely, since we know that if we set $\epsilon = 0$ in this problem, we have the ODE

$$\ddot{\bar{x}} = -1, \quad \bar{x}[0] = 0 \quad \text{and} \quad \dot{\bar{x}}[0] = 1, \quad (2)$$

which we can solve explicitly as $\bar{x} = -\bar{t}^2/2 + \bar{t}$.

So the question one can (and should!) ask at this point is how much the addition of an ϵ in the problem changes things. More specifically, if we consider the problem

$$\ddot{\bar{x}}^\epsilon = -\frac{1}{(1 + \epsilon\bar{x})^2}, \quad \bar{x}^\epsilon[0] = 0 \quad \text{and} \quad \dot{\bar{x}}^\epsilon[0] = 1,$$

then *how close* are \bar{x}^ϵ and \bar{x}^0 ? This is a useful question, since we know the latter exactly. We need to be careful though about what we mean by close here. Now, for instance, imagine that we know how to write

$$\bar{x}^\epsilon[t] = \bar{x}^0[t] + \epsilon\bar{x}_1[t] + \epsilon^2\bar{x}_2[t] + \dots$$

and we can guarantee that $\bar{x}_1[t]$ is bounded over some time interval. Then, we have a good approximation to the solution \bar{x}^ϵ and we can make it better and better as $\epsilon \rightarrow 0$. If we can do this, then we call the perturbation in Equation 2 a *regular perturbation*.

Finally, just to get a handle on numbers here, let us assume that the initial velocity was 25 m/s. We then have

$$t_c = \frac{v_0}{g} = \frac{25 \text{ m/s}}{9.8 \text{ m/s}^2} \sim 2.6 \text{ s} \quad x_c = \frac{v_0^2}{g} = \frac{625 \text{ m}^2/\text{s}^2}{9.8 \text{ m/s}^2} \sim 63.8 \text{ m}.$$

This gives us the typical time and length scales for the problem. Moreover, notice that

$$\epsilon = \frac{x_c}{r} = \frac{62.8 \text{ m}}{9.8 \text{ m/s}^2} \sim 9.85 \cdot 10^{-6}.$$

Since ϵ is so small compared to unity, our approximation probably works quite well. However, solving the approximate problem is easy: the time of maximum height occurs at $\bar{t} = 1$ (or $t = t_c = 2.6 \text{ s}$) and therefore the maximum height is $\bar{x} = 1$ (or $x_{\max} = x_c/2 = 31.9 \text{ m}$). Second scaling. What if, on the other hand, we had chosen $\Pi_2 = \Pi_3 = 1$? Then, we would have had

$$\bar{x} = r, \quad \bar{t} = \frac{r}{v_0} \quad \text{and} \quad \Pi_1 = \frac{v_0^2}{gr} \sim \frac{v_0^2}{6.25 \cdot 10^7 \text{ m}^2/\text{s}^2}.$$

For human velocities, this is clearly quite small. Again, choosing $v_0 = 25 \text{ m/s}$, we have

$$\epsilon = \Pi_1 = 9.99 \cdot 10^{-6}.$$

This is again small, but plugging the nondim quantities into the equation, we obtain

$$\epsilon\ddot{\bar{x}} = -\frac{1}{(1 + \bar{x})^2}, \quad \bar{x}[0] = 0 \quad \text{and} \quad \dot{\bar{x}}[0] = 1.$$

Now, when we take the limit as $\epsilon \rightarrow 0$, we obtain an equation where the derivatives disappear and in fact is not a differential equation at all. This is actually a *singular perturbation* instead of a *regular* one and thus the former needs special treatment.

5. DIFFUSION EQUATION

[Lee DeVille. Math 558 - Methods of Applied Mathematics.]

5.1. Dim analysis. If we consider the density of, for instance, a chemical in solution, where we denote said concentration at x and t by $c[x, t]$, then the diffusion equation is given by

$$\partial_t c = \gamma \partial_{xx} c, \quad (3)$$

where γ is the mass diffusion coefficient.

Let us imagine that we post this problem on the domain $0 < x < \infty$ and $t > 0$. Moreover, we assume that $c[x, 0] = 0$ for all x (zero concentration at time zero) and we inject the chemical at $x = 0$, so that $c[0, t] = c_0$. We also append the boundary condition $c[\infty, t] = 0$.

Now, assuming that the concentration is a function of $\{x, t, \gamma, c_0\}$, we obtain

$$\frac{c}{c_0} = \phi_\pi \left[\frac{x}{\sqrt{\gamma t}} \right].$$

Thus we have the nondim quantity $\Pi = x/\sqrt{\gamma t}$ and we have $c/c_0 = \phi_\pi[\Pi]$.

Plugging this expression, $c/c_0 = \phi_\pi[\Pi]$, into Equation 3 and its boundary and initial conditions, we obtain

$$\phi_\pi[\Pi] = 1 - \frac{1}{\sqrt{\pi}} \int_0^\Pi \exp[-s^2/4] \, ds$$

and therefore we have the general solution

$$c[x, t] = c_0 \left(1 - \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{\gamma t}} \exp[-s^2/4] \, ds \right).$$

5.2. Scaling. We can add a nonlinear term to the diffusion equation as follows:

$$\partial_t c = \gamma \partial_{xx} c + \gamma c^3 \quad \text{and} \quad c[x, 0] = c_0[x].$$

This is an example of a *reaction-diffusion* equation; the polynomial term is a (local) reaction of the substance whose concentration we are tracking. Rescaling with

$$x = x_c \bar{x}, \quad t = t_c \bar{t} \quad \text{and} \quad c = c_c \bar{c}$$

we obtain

$$\partial_{\bar{t}} \bar{c} = \gamma \frac{t_c}{x_c^2} \partial_{\bar{x}\bar{x}} \bar{c} + \gamma t_c c_c^2 \bar{c}.$$

Thus, we have

$$\Pi_1 = \frac{\gamma t_c}{x_c^2} \quad \text{and} \quad \Pi_2 = \gamma t_c c_c^2$$

as our nondim quantities. This lets us know what parameters we would choose to have (relative) small diffusion or (relative) large diffusion. If $\Pi_1 \ll \Pi_2$; *i.e.*, $\gamma c_c / \gamma x_c^2 \ll 1$, then we can write $\Pi_1 = \epsilon$, $\Pi_2 = 1$ and we have the PDE

$$\partial_{\bar{t}} \bar{c} = \epsilon \partial_{\bar{x}\bar{x}} \bar{c} + \bar{c}^3,$$

or the *small diffusion* scaling regime. This lets us know how we would make this small; *e.g.*, say the constants $\{\gamma, \gamma\}$, we could either take c_c small (small concentrations) or x_c (long lengthscales) to get the small diffusion regime.

Similarly, if $\Pi_1 \gg \Pi_2$, or $\gamma c_c / \gamma x_c^2 \gg 1$, then we can write $\Pi_1 = 1$, $\Pi_2 = \epsilon$ and we have the PDE

$$\partial_{\bar{t}} \bar{c} = \partial_{\bar{x}\bar{x}} \bar{c} + \epsilon \bar{c}^3,$$

or the *small reaction* scaling regime. This can be obtained by looking at large concentrations or really small lengthscales.

6. SCALING ANALYSIS – THE ART OF GETTING SOMETHING FOR NOTHING

[P.C. Chau, Scaling analysis – the art of getting something for nothing. UCSD. 1999]

When we nondim a model equation, there is more to just turning the model to be in terms of dimless quantities. By making better judgment, we can reduce the model to retain only the most salient physical features or the *minimum parametric representation*. This is the essence of *scaling*. At a more advanced level and especially in asymptotic analysis, we learn to scale the equations so they have “better properties”.

We make use of scaling in introductory fluid dynamics to analyze a variety of problems such as creeping flow, boundary layer flow, lubrication and turbulence. Needless to say that we can apply scaling to mass transfer in a boundary layer. We’ll get there in several steps.

6.1. A brief fluids review. As a review, let’s take a quick look at the balance equations that essentially constitute the boundary layer model later. The continuity equation in *dimensional* quantities is ¹

$$\partial_x u^* + \partial_y v^* = 0 \tag{4}$$

and the species continuity with constant binary diffusion coefficient, γ_{AB} , is ²

$$u^* \partial_x c_A^* + v^* \partial_y c_A^* = \gamma_{AB} (\partial_x u^* c_A^* + \partial_y v^* c_A^*) . \tag{5}$$

¹ In vector notation, the continuity equation is $\nabla \cdot \mathbf{v} = 0$.

² In vector notation, the species continuity equation is $\mathbf{v} \cdot \text{grad } c_A = \gamma_{AB} \text{lap } c_A$.

We now consider defining *dimensionless quantities* with some *reference quantities*

$$c = \frac{c_A^*}{c_{\text{ref}}^*}, \quad x = \frac{x^*}{x_{\text{ref}}^*}, \quad y = \frac{y^*}{y_{\text{ref}}^*}, \quad u = \frac{u^*}{u_{\text{ref}}^*} \quad \text{and} \quad v = \frac{v^*}{v_{\text{ref}}^*}.$$

It is not uncommon, then, to choose $x_{\text{ref}}^* = y_{\text{ref}}^* = l$ and $u_{\text{ref}}^* = v_{\text{ref}}^* = u_\infty$. In scaling analysis, however, we like to choose the reference quantities such that each dimensionless quantity is *normalized*³. This is why we like to use the term *normalized quantities*. In this problem, we know that the length scales are very different in the x^* - and y^* -directions: $l \gg \delta$. So we want to define

$$c = \frac{c_A^*}{c_\infty}, \quad x = \frac{x^*}{l}, \quad y = \frac{y^*}{\delta} \quad \text{and} \quad u = \frac{u^*}{u_\infty}, \quad (6)$$

where c_∞ and u_∞ are the *free stream*⁴ concentration and velocity, l is the length of the flat plate and δ is the boundary layer thickness. What is not immediately clear is the choice of reference for v^* . There are many “styles” of approach. We make two suggestions here.

- (1) We go ahead and substitute the nondimensional quantities in eq. (4), leading to the partial step

$$\frac{\partial(uu_\infty)}{\partial(xl)} + \frac{\partial(vv_{\text{ref}}^*)}{\partial(y\delta)} = 0 \implies \frac{u_\infty}{l} \frac{\partial u}{\partial x} + \frac{v_{\text{ref}}^*}{\delta} \frac{\partial v}{\partial y} = 0,$$

from which it should be clear that we need to define $v_{\text{ref}}^* = u_\infty \delta / l$ to arrive at the dimensionless equation

$$\partial_x u + \partial_y v = 0, \quad (7)$$

where all quantities are normalized.

- (2) On a piece of scrap paper, we write down below eq. (4) the order of magnitude of every quantity in each term:

$$\frac{u_\infty}{l} \frac{\partial u}{\partial x} + \frac{v_{\text{ref}}^*}{\delta} \frac{\partial v}{\partial y} = 0 \implies \frac{u_\infty}{l} \frac{o[1]}{o[1]} + \frac{o[v^*]}{\delta} \frac{o[1]}{o[1]} \sim 0$$

or simply

$$\frac{u_\infty}{l} + \frac{o[v^*]}{\delta} \sim 0.$$

If the equation is properly scaled, then all the terms must be *balanced* and thus $o[v^*]/\delta$ should be the same order as u_∞/l . Hence, once again, we come to the conclusion of the need to choose $u_{\text{ref}}^* = u_\infty \delta / l$. In other words, we expect the magnitude of the velocity v^* to be small relative to u_∞ when it is scaled by the ratio δ/l . Thus, we choose to define

$$v = \frac{v^*}{v_{\text{ref}}^*} = \frac{l}{\delta} \frac{v^*}{u_\infty} \quad (8)$$

as the normalized velocity.

Now we use the definitions of the normalized quantities in eq. (6) and eq. (8) in the species continuity equation, eq. (5), and we should find

$$\frac{u_\infty}{l} u \frac{\partial c}{\partial x} + \frac{u_\infty \delta}{\delta l} v \frac{\partial c}{\partial y} = \gamma_{AB} \left(\frac{1}{l^2} \frac{\partial^2 c}{\partial x^2} + \frac{1}{\delta^2} \frac{\partial^2 c}{\partial y^2} \right)$$

and on rearrangement

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\gamma_{AB} l}{u_\infty \delta^2} \left(\frac{\delta^2}{l^2} \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right).$$

Since all the quantities are normalized, the diffusion term in the x -direction must be much smaller than that in the y -direction because of its coefficient $(\delta/l)^2 \ll 1$ and thus we can neglect it. So we can finally write in dimensionless form

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{1}{\Pi_{\text{sc}} \Pi_{\text{re}}^*} \frac{\partial^2 c}{\partial y^2}, \quad (9)$$

³ That is, the dimensionless quantity is bounded between zero and unity, or in other words, it is of $o[1]$. More importantly, as an *art* form, there is no rule to say that we must use normalized quantities to do scaling analysis. Indeed, we can use $u_{\text{ref}}^* = v_{\text{ref}}^* = l$ and $u_{\text{ref}}^* = v_{\text{ref}}^* = u_\infty$ and can still arrive at whatever results that follow. We choose to use normalized quantities because this is a better habit.

⁴ Free stream means out of the boundary layer, where flow is turbulent.

where we have used

$$\frac{\gamma_{AB} l}{u_{\infty} \delta^2} = \frac{\gamma_{AB}}{\nu} \frac{\nu}{u_{\infty} \delta} \frac{l}{\delta} = \frac{1}{\Pi_{sc}} \frac{1}{\Pi_{re}} \frac{l}{\delta} = \frac{1}{\Pi_{sc}} \frac{1}{\Pi_{re}^*},$$

in which Π_{sc} is the *Schmidt number*⁵, Π_{re} the *Reynolds number* and $\Pi_{re}^* = \Pi_{re} \delta / l$ the *reduced Reynolds number* as used in the analysis of, for example, lubrication flow problems.

We are not really done, and we'll come back to the species continuity equation later. The key of this simple review is to remind us that the *scales* of different quantities can be very different in different directions. Second, the dimensionless groups of a problem arise naturally from the balance equation when we nondimensionalize it. This second point should be familiar to you. Furthermore, you may also remind yourself that we can figure out dimensionless groups from the use of dimensional analysis (or the so-called *Buckingham Pi theorem*).

6.2. Laminar flow with heterogeneous reaction at the wall. Let's side-track a bit and look at a problem in which the scaling information comes from the boundary condition. This is the problem of having a chemical reaction only at the wall of a tube. An example where this scenario may arise is electrochemical reactors.

We consider here a steady state laminar flow in a circular tube of radius ρ . The tube length is long such that $l \gg \rho$. The fluid contains a solute A that undergoes an irreversible first-order reaction at the wall of the tube. We want to see the approximations that we can make under *extreme cases*.

At steady state⁶, the mass balance of species A in the pipe flow with axial convection and radial diffusion is

$$u_z^* \frac{\partial c_A^*}{\partial z^*} = \gamma_{AB} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} r^* \frac{\partial c_A^*}{\partial r^*} \right), \quad (10)$$

where we know quite well from fluid mechanics that

$$u_z^* = 2\hat{v} \left(1 - \left(\frac{r^*}{\rho} \right)^2 \right)$$

and \hat{v} is the average velocity. The boundary conditions are as follows:

$$\text{At } z^* = 0, \quad c_A^* = c_{A_0}^* \quad \text{for } 0 \leq r^* \leq \rho. \quad (11)$$

For $0 \leq z^* \leq l$, we also have

$$\frac{\partial c_A^*}{\partial r^*} \quad \text{at } r^* = 0$$

and

$$-\gamma_{AB} \frac{\partial c_A^*}{\partial r^*} = \kappa c_A^* \quad \text{at } r^* = \rho.$$

The first boundary condition is the inlet concentration, the second is the symmetry condition at the center of the tube and the third is the condition at the wall, where the species A is consumed by chemical reaction with first-order rate coefficient κ . As eq. (10) is written, we have taken it for granted that we have performed the scaling as illustrated in the previous section and found that diffusion in the axial direction is not important. The fact that we only have one (axial) convective term is dictated by the kinematics, that $v = [u_z^*, 0, 0]$.

We now consider defining dimless quantities. Again, we follow the advice given in the previous section and define

$$c = \frac{c_A^*}{c_{A_0}^*}, \quad r = \frac{r^*}{\rho} \quad \text{and} \quad z = \frac{z^*}{l}$$

as normalized quantities. The dimless form of the model in eq. (10) is

$$(1 - r^2) \frac{\partial c}{\partial z} = \frac{\gamma_{AB} l}{2\hat{v} \rho} + \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial c}{\partial r} \right). \quad (12)$$

⁵ The Schmidt number is the ratio of momentum diffusivity and mass diffusivity.

⁶ At steady state, concentration is independent on time.

The corresponding dimless boundary conditions are

$$\text{at } z = 0, \quad c = 1 \quad \text{for } 0 \leq r \leq 1$$

and for $0 \leq r \leq 1$,

$$\frac{\partial c}{\partial r} = 0 \quad \text{at } r = 0$$

and

$$-\frac{\partial c}{\partial r} = \Pi_{\text{da}} c \quad \text{at } r = 1, \quad (13)$$

where

$$\Pi_{\text{da}} = \frac{\kappa \rho}{\gamma_{\text{AB}}}$$

is the *Damköhler number* – a relative measure of reaction rate to diffusion rate. [You can further rewrite eq. (12) to be in terms of Π_{sc} and Π_{re} as we did in the last section, but we'll leave that as an exercise for you.]

Next, we make use of the boundary condition at the wall ($r = 1$), which is the only location where the chemical reaction may have an effect in this model.

We first consider when the Damköhler number is much greater than unity. The gradient $\partial_r c$ in eq. (13) should be bounded⁷ Thus, when $\Pi_{\text{da}} \gg 1$, we must have $c \gg 1$. Indeed, in problem in which the value of Π_{da} is very large, we can replace the boundary condition eq. (13) by the approximation

$$c = 0 \quad \text{at } r = 1.$$

If this observation is not clear, then try rearranging eq. (13) as

$$\frac{1}{\Pi_{\text{da}}} \frac{\partial c}{\partial r} + c = 0 \quad \text{at } r = 1.$$

Physically, the idea is that if the chemical reaction rate is much faster than the diffusion rate (or the replenishment of the reactant A at the surface), then the chance that we can still find unconsumed A on the wall is minute. This scenario of $\Pi_{\text{da}} \gg 1$ is also called *mass transfer limited*.

At the other extreme, we have $\Pi_{\text{da}} \ll 1$, meaning that chemical reaction at the wall is much slower than diffusion. Here, we can replace the boundary condition eq. (13) by the approximation

$$\frac{\partial c}{\partial r} \sim 0 \quad \text{at } r = 1,$$

since c is of $o[1]$. We can go further than that!

With the given physics of this problem, where the chemical reaction as at the wall, the largest gradient $\partial_r c$ that we expect to have is at $r = 1$ (the wall). If $\partial_r c \ll 1$ at the wall, then this condition must also be true over the entire tube, not to mention that we know $\partial_r c = 0$ by symmetry at the center. Thus we make the approximation that $\partial_r c \sim 0$ or, in other words, the concentration of A is only a function of z ; *i.e.*, $c = c[z]$ or $c_A = c_A[z^*]$ only.

In such a case, we can write down the differential balance of the tube as

$$\pi \rho^2 \hat{v} \frac{dc_A^*}{dz^*} = -2\pi \rho \kappa c_A^*$$

in place of eq. (10). After cleaning up, we have

$$\hat{v} \frac{dc_A^*}{dz^*} = -\frac{2\kappa}{\rho} c_A^*, \quad 0 \leq z^* \leq l, \quad (14)$$

which can be solved easily with the boundary condition eq. (11). The approximation in eq. (14) is the so-called *plug-flow* model.

We have arrived at the results in the two limits of Π_{da} with a rational approach. In many texts, these approximations are stated as *ad hoc* assumptions and it is not clear on what basis they may be valid.

⁷ We could have said that $o[\partial_r c] = 1$, too. In making this statement, we are making the assertion that since c and r are normaziled, $\partial_r c$ should be reasonably well behaved. The actual value of the gradient may be larger than one, but the chance that it “blows up” as Π_{da} approaches infinity is not high.

6.3. Magnitude of thermal and concentration boundary layers. In this section, it is important to keep in mind that whatever we do is order of magnitude analysis based on scales. how we handle quantities is sloppy compared with when we have to set up the formal boundary layer flow problem. We first need to revisit the problem of hydrodynamic boundary layer. We did the scaling of continuity in eq. (7).

We still have to tackle the momentum balance. For flow over a flat plate such that $\partial_x p = 0$, we can pretty much jump ahead and make use of what we have learned from deriving eq. (9). The analogous intermediate step in the scaling of the momentum equation in the x^* -direction is

$$\frac{u_\infty}{l} u \frac{\partial u}{\partial x} + \frac{u_\infty}{\delta} \frac{\delta}{l} \frac{\partial u}{\partial y} = \nu \left(\frac{1}{l^2} \frac{\partial^2 u}{\partial x^2} + \frac{1}{\delta^2} \frac{\partial^2 u}{\partial y^2} \right)$$

or

$$u \frac{\partial u}{\partial x} + \frac{\delta}{l} \frac{\partial u}{\partial y} = \frac{\nu l}{u_\infty \delta^2} \left(\frac{l^2}{\delta^2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Since $\delta \ll l$,

$$u \frac{\partial u}{\partial x} + \frac{\delta}{l} \frac{\partial u}{\partial y} = \frac{\nu l}{u_\infty \delta^2} \frac{\partial^2 u}{\partial y^2}. \quad (15)$$

At this step, we can diverge on different paths. In scaling for the boundary layer problem, we make the argument that within the boundary layer, the order of magnitude of the viscous term must be balanced by the inertia terms. So we write

$$u \frac{\partial u}{\partial x} + \frac{\delta}{l} \frac{\partial u}{\partial y} \sim \frac{\partial^2 u}{\partial y^2} \quad (16)$$

as a way to denote that the orders of magnitude of all three terms in the properly scaled boundary layer equation are the same⁸. To arrive at eq. (16), we must have the scaled (or reduced) Π_{re} of order 1; this is the reciprocal of the coefficient on the RHS in eq. (15):

$$\frac{u_\infty \delta^2}{\nu l} \sim 1.$$

We can rewrite this order of magnitude estimation as

$$\frac{\delta}{l} \sim \frac{1}{\sqrt{\Pi_{re}}}, \quad \text{with} \quad \Pi_{re} = \frac{u_\infty l}{\nu}. \quad (17)$$

This order of magnitude dependence of the hydrodynamic boundary layer thickness on the Reynolds number is consistent with the result from solving the Blasius equation. However, we hardly did any work to get the physical insight suggested by eq. (17).

We can now move quickly with the concentration boundary layer equation. Based on the exercise that we had earlier with the species continuity, we can quickly drop the term for the diffusion in the x -direction and write

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\gamma_{AB} l}{u_\infty \delta_c^2} \frac{\partial^2 c}{\partial y^2}.$$

Note that we have used the notation δ_c to denote the concentration boundary layer thickness⁹. To have balanced diffusion and convective terms of the same order,

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} \sim \frac{\partial^2 c}{\partial y^2}, \quad (18)$$

we need to have

$$\frac{\gamma_{AB} l}{u_\infty \delta_c^2} \sim 1$$

or

$$\left(\frac{\delta_c}{l} \right)^2 \sim \frac{1}{\Pi_{sc} \Pi_{re}}.$$

⁸ Reminder: if we were actually solving the problem, we would use the complete equations such as eq. (15) or earlier eq. (9). We should mention that in other fluid dynamic problems, we may have $l = \delta$ and the leading coefficient on the RHS of eq. (15) would then be $1/\Pi_{re}$, where $\Pi_{re} = u_\infty l/\nu$.

⁹ Hiding behind this equation is the implicit assumption that we have started the scaling of v and y with δ_c . If you are sharp, you will spot that the actual value of the dimless v and y cannot be the same as when we use the hydrodynamic δ . But then, keep in mind that we are really doing order of magnitude analysis and have no intention in solving these equations. If we do, we would use only one choice of δ in all the equations.

Thus the concentration boundary layer is scaled by both the Reynolds and Schmidt numbers

$$\frac{\delta_c}{l} \sim \frac{1}{\sqrt{\Pi_{sc}\Pi_{re}}}.$$

With eq. (17), we expect that the ratio of the concentration to the hydrodynamic boundary layer is a function of the Schmidt number

$$\frac{\delta_c}{\delta} \sim \frac{1}{\sqrt{\Pi_{sc}}}. \quad (19)$$

With our experience, we can move even quicker with the thermal boundary layer. Skipping all details, we can virtually guess that the balanced terms with the same order of magnitude should be

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} \sim \frac{\partial^2 \theta}{\partial y^2},$$

where θ is used to denote normalized temperature and we also need

$$\frac{\alpha_t l}{u_\infty \delta_t^2} \sim 1.$$

The thermal boundary layer δ_t is scaled by both the Reynolds and Prondtl numbers

$$\frac{\delta_t}{l} \sim \frac{1}{\sqrt{\Pi_{re}\Pi_{pr}}}$$

and the scale difference between the thermal and hydrodynamic boundary layer is

$$\frac{\delta_t}{\delta} \sim \frac{1}{\sqrt{\Pi_{pr}}}. \quad (20)$$

How valid are these results? We should recognize immediately that the variations of the concentrations and thermal boundary layers with respect to the hydrodynamic boundary layer in eq. (19) and eq. (20) are wrong in *liquid* phase transport. An actual analysis would show that the dependence on Π_{sc} and Π_{pr} is to the $-1/3$ power, not $-1/2$. We should be happy, however, that we get so close for doing virtually nothing. This is where most textbooks left off.

6.4. What is the significance of these results? The results of the scaling analysis provide useful physical insight that should help us construct boundary layer models. To do that, we also need an intuitive idea what the probable values of Π_{sc} and Π_{pr} may be. This is where we can make use of physical chemistry. We just give a quick and dirty idea here.

From the kinetic theory of gases, we should find that the diffusion coefficient of a “hard sphere” gas is the same as the kinematic viscosity (momentum diffusivity)

$$\alpha_m = \nu = \frac{1}{3} \hat{v} \lambda,$$

where \hat{v} is the mean speed and λ is the mean free path. The point is that

$$\Pi_{sc} = 1 \quad \text{for an ideal gas.}$$

Similarly, we should find that for a monatomic gas, Π_{pr} comes down to being the ratio of constant pressure *vs* constant volume thermal capacities – to be exact: $\Pi_{pr} = 5/3$. So for estimation, we generally take that ¹⁰

$$\Pi_{pr} \sim 1 \quad \text{for gases.}$$

Hence, for gas phase problems, we can gather that in terms of orders of magnitude

$$\delta \sim \delta_t \sim \delta_c.$$

That is, the concentration and thermal boundary layers are just as *thick* as the hydrodynamic boundary layer and there is no cutting corners in solving the thermal and mass transport problems. We need the full hydrodynamic boundary layer solution. On the other hand, the order of magnitude estimation in the previous section is proper.

¹⁰ For air, Π_{pr} is around 0.7.

With liquids, we should find that (at least from tables in texts or handbooks) ¹¹

$$\Pi_{sc} = o[10^3] \quad \text{and} \quad \Pi_{pr} = o[10^2] . \quad (21)$$

Thus for most liquid phase problems,

$$\delta > \delta_t > \delta_c .$$

From eq. (21), δ_t is very roughly one-tenth that of δ and δ_c is even *thinner*. Thus, in solving liquid phase transport problems in the boundary layer, we can use an approximate velocity profile close to the surface. This is particularly true with mass transport.

6.5. Proper scaling of the concentration boundary layer. With hindsight, with is always 10/10, we can refine the *standard* textbook scaling analysis in the liquid phase. Now that we understand $\delta_c \ll \delta$ in liquids, we should focus very close to the surface ($y = 0$) when we do the scaling of the concentration problem. In this region, the velocity is much less than the free stream u_∞ that we have used previously. A more proper choice of the reference velocity is

$$v^* = \frac{\delta_c}{\delta} u_\infty ,$$

where we have assumed a linear dependence near the surface in the region of δ_c . Similarly, we also should use δ_c is the reference length. Hence, the dimless u and v are now defined as

$$u = \frac{u^*}{u_\infty \delta_c / \delta} \quad \text{and} \quad v = \frac{v^*}{\delta_c} .$$

With these new quantities, we repeat the exercise of scaling the continuity equation that has led to eq. (8). Now, we have

$$\frac{\delta_c}{\delta} \frac{u_\infty}{l} \frac{\partial u}{\partial x} + \frac{v_{ref}^*}{\delta_c} \frac{\partial v}{\partial y} = 0 ,$$

which means that we need to choose and define

$$v_{ref}^* = \frac{\delta_c^2}{\delta} \frac{u_\infty}{l} \quad \text{and} \quad v = \frac{v^* \delta l}{u_\infty \delta_c^2} .$$

The next step is to rescale the concentration boundary equation. The step equivalent to just before eq. (18) is now

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{\gamma_{AB} \delta l}{u_\infty \delta_c^3} \frac{\partial^2 c}{\partial y^2} .$$

To recover eq. (18), we require this time

$$\frac{\gamma_{AB} \delta l}{u_\infty \delta_c^3} \sim 1$$

or

$$\left(\frac{\delta_c}{\delta} \right)^3 \sim \frac{\gamma_{AB} l}{u_\infty \delta^2} = \frac{\gamma_{AB}}{u_\infty \delta} \left(\frac{l}{\delta} \right)^2 = \frac{1}{\Pi_{re} \Pi_{sc}} \left(\frac{l}{\delta} \right)^2 .$$

Finally we make use of the hydrodynamic boundary layer result in eq. (17) and arrive at

$$\frac{\delta_c}{\delta} \sim \Pi_{sc}^{-1/3} ,$$

which is the expected result from more rigorous analyses.

¹¹ It is difficult to generalize the magnitude of Π_{pr} . For water, we should use $\Pi_{pr} = o[10]$. For viscous oils, Π_{pr} can be as large as 10000. At the other extreme, the Π_{pr} of molten or liquid metal (mercury) is much less than one.