MATHEMATICAL MODELING

DIEGO HERRERA

1. General Geometric Algebra

Taken from [1].

Linear space: set of elements which is closed under addition and scalar multiplication.

Vector space: linear space plus a defined geometric product.

1.1. **Geometric Product.**

Let $\mathfrak{V}^n$ be a n-dim vector space over the real numbers. The geometric product of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathfrak{V}^n$ is defined by four basis axioms:

(1) Associative rule: $\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c}$,
(2) Left distributive rule: $\mathbf{a}(\mathbf{b}+\mathbf{c}) = \mathbf{ab}+\mathbf{ac}$,
(3) Right distributive rule: $(\mathbf{b}+\mathbf{c})\mathbf{a} = \mathbf{ba}+\mathbf{ca}$,
(4) Contraction rule: $\mathbf{aa} = (\mathbf{a})^2 = |\mathbf{a}|^2$,

where $|\mathbf{a}|$ is a real number called the *magnitude* or *length* of $\mathbf{a}$ and $|\mathbf{a}| = 0$ iff $\mathbf{a} = \mathbf{0}$. Both distributive rules are needed, since multiplication is *not* commutative.

$\mathfrak{V}^n$ is closed under addition but not under multiplication (contraction rule: one begins with a vector and ends up with a real number!) By multiplication and addition, vectors in $\mathfrak{V}^n$ generate a larger linear space $\mathfrak{G}_n = (\mathfrak{V}^n)$, the geometric algebra of $\mathfrak{V}^n$ which is closed under addition and multiplication.

The contraction rule determines a measure of distance between vectors in $\mathfrak{V}^n$, called *Euclidean geometric algebra*. Thus, $\mathfrak{V}^n$ can be regarded as a n-dim *Euclidean space*, $\mathfrak{E}^n$.

Additionally, the contraction rule determines the *inverse* of vectors

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{aa}} = \frac{\mathbf{a}}{|\mathbf{a}|^2},$$

for $\mathbf{a} \neq \mathbf{0}$.

1.2. **Inner and Outer Products.**

The geometric product $\mathbf{ab}$ can be decomposed into a symmetric and an antisymmetric part defined by

$$\mathbf{a}\cdot\mathbf{b} := \frac{\mathbf{ab}+\mathbf{ba}}{2} \qquad \text{and}$$

$$\mathbf{a}\wedge\mathbf{b} := \frac{\mathbf{ab}-\mathbf{ba}}{2}.$$

thus

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}.$$

The last equation is called the *fundamental decomposition of the geometric product for vectors*.

The product $\mathbf{a}\cdot\mathbf{b}$ is the conventional Euclidean inner product, whereas the product $\mathbf{a}\wedge\mathbf{b}$ is the Grassmann product. The outer product is the antisymmetric part that generates *bivectors*. Geometrically, $\mathbf{a}\wedge\mathbf{b}$ represents a directed plane segment.

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The geometric interpretation of the geometric product:

- ⋄ $\mathbf{ab} = -\mathbf{ba}$ iff $\mathbf{a} \cdot \mathbf{b} = 0$ or $\mathbf{ab}$ anticommutes iff $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal*,
- ⋄ $\mathbf{ab} = \mathbf{ba}$ iff $\mathbf{a} \wedge \mathbf{b} = 0$ or $\mathbf{ab}$ commutes iff $\mathbf{a}$ and $\mathbf{b}$ are *colinear*.
- ⋄ $\mathbf{a}^{-1}\mathbf{b} =$ measures the relative direction and magnitude of $\mathbf{a}$ and $\mathbf{b}$.
- ⋄ transformation of $\mathbf{a}$ into $\mathbf{b}$: $\mathbf{a}(\mathbf{a}^{-1}\mathbf{b}) = \mathbf{b}$.

Outer product of k-vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ generate a new entity $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n$ called a k-blade. The integer k is called the *step of the blade* or the *grade of the blade*.

A linear combination of k-blades with the *same* step is called a k-vector. Therefore, every k-blade is a k-vector. But, that every k-vector is a k-blade only holds for $\mathfrak{G}_n$ with $n \leq 3$.

From a vector $\mathbf{a}$ and a k-vector A_k, the outer product gives a $(k+1)$-vector $\mathbf{a} \wedge A_k$:

$$\mathbf{a} \wedge A_k = \frac{\mathbf{a}A_k + (-1)^k A_k \mathbf{a}}{2} = (-1)^k A_k \wedge \mathbf{a}.$$

The inner product, on the other hand, gives

$$\mathbf{a} \cdot A_k = \frac{\mathbf{a}A_k - (-1)^k A_k \mathbf{a}}{2} = (-1)^{k+1} A_k \cdot \mathbf{a}.$$

Thus, the outer product is a “step-up” operation, while the inner product, a “step-down” operation.

The decomposition of the geometric product into step-down and step-up parts:

$$\mathbf{a}A_k = \mathbf{a} \cdot A_k + \mathbf{a} \wedge A_k,$$
$$A_k\mathbf{a} = A_k \cdot \mathbf{a} + A_k \wedge \mathbf{a}.$$

1.3. **Identities.** Let $\langle A \rangle_k$ be a k-vector A that has a step k, let $\mathbf{a}$, $\mathbf{b}$, ... be vectors and A, B, ... be quantities of any step.

The outer product is associative and antisymmetric. Associative:

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C = A \wedge B \wedge C,$$

Antisymmetric: it reverses the sign under interchange of any two vectors in a multiple outer product:

$$A \wedge \mathbf{a} \wedge B \wedge \mathbf{b} \wedge C = -A \wedge \mathbf{b} \wedge B \wedge \mathbf{a} \wedge C.$$

The inner product is *not* strictly associative:

$$\langle A \rangle_r \cdot (\langle B \rangle_s \cdot \langle C \rangle_t) = (\langle A \rangle_r \cdot \langle B \rangle_s) \cdot \langle C \rangle_t,$$

for $r + t \leq s$.

The inner product relates to the outer product by

$$\langle A \rangle_r \cdot (\langle B \rangle_s \cdot \langle C \rangle_t) = (\langle A \rangle_r \wedge \langle B \rangle_s) \cdot \langle C \rangle_t,$$

for $r + s \leq t$.

Another identity:

$$\mathbf{a} \cdot (\mathbf{b} \wedge C) = (\mathbf{a} \cdot \mathbf{b})C + \mathbf{b} \wedge (\mathbf{a} \cdot C).$$

A reduction formula:

$$\mathbf{a} \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \ldots \wedge \mathbf{b}_r) = \sum_{k=1}^{r} (-1)^{r+1} (\mathbf{a} \cdot \mathbf{b}_k) \mathbf{b}_1 \wedge \ldots \wedge \check{\mathbf{b}}_k \wedge \ldots \wedge \mathbf{b}_r,$$

where $\check{\mathbf{b}}_k$ is the vector omitted from the product.

Other useful identities:

$$\begin{aligned}(\mathbf{a}_r \wedge \ldots \wedge \mathbf{a}_1) \cdot (\mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_r) &= (\mathbf{a}_r \wedge \ldots \wedge \mathbf{a}_2) \cdot [\mathbf{a}_1 \cdot (\mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_r)] \\ &= \sum_{k=1}^{r} (-1)^{k+1} (\mathbf{a}_1 \cdot \mathbf{b}_k)(\mathbf{a}_r \wedge \ldots \wedge \mathbf{a}_2) \cdot (\mathbf{b}_1 \wedge \ldots \wedge \check{\mathbf{b}}_k \wedge \ldots \wedge \mathbf{b}_r).\end{aligned}$$

Any matrix elements can be represented as inner products between two sets of vectors:

$$a_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j.$$

The determinant of a matrix can then be defined by

$$\det [a_i j] = (\mathbf{a}_r \wedge \ldots \wedge \mathbf{a}_1) \cdot (\mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_r).$$

1.4. **Parenthesis Reduction Convention.** To reduce parentheses there's a operator precedence convention

(1) outer products first,
(2) inner products then,
(3) geometric product finally.

Example:

$$\begin{gathered}(A \wedge B)C = A \wedge BC \neq A \wedge (BC),\\ (A \cdot B)C = A \cdot BC \neq A \cdot (BC),\\ A \cdot (B \wedge C) = A \cdot B \wedge C \neq (A \cdot B) \wedge C.\end{gathered}$$

With such a convention, the most useful special case of the reduction formula becomes

$$\begin{gathered}\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b},\\ \mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = \mathbf{a} \cdot \mathbf{b}\mathbf{c} - \mathbf{a} \cdot \mathbf{c}\mathbf{b}.\end{gathered}$$

1.5. **Additive Structure of $\mathfrak{G}_n$.** The vectors of a set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ are *linearly independent* iff the r-blade

$$\langle A \rangle_k = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_r$$

is not zero. Every such a r-blade determines a unique r-dim subspace $\mathfrak{V}^r$ of the vector space $\mathfrak{V}^n$ consisting of all vectors $\mathbf{a}$ in $\mathfrak{V}^n$ which satisfy

$$\mathbf{a} \wedge \langle A \rangle_k = 0.$$

The blade $\langle A \rangle_k$ is a *directed volume* for $\mathfrak{V}^r$ with magnitude (scalar volume) $|\langle A \rangle_k|$.

An ordered set of vectors $\{\mathbf{a}_k : k = 1, 2, \ldots, n\}$ in $\mathfrak{V}^n$ is a *basis* for $\mathfrak{V}^n$ iff it generates a non-zero n-blade $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n$. The n-blades are called *pseudoscalars of* $\mathfrak{V}^n$ *or* $\mathfrak{G}_n$. They make up a 1-dimensional *linear* space, so one may choose a unit pseudoscalar $\mathfrak{I}$ and write

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n = \alpha \mathfrak{I},$$

where α is a scalar.

Solve for the scalar volume $|\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n|$ and express it as a determinant

$$\alpha = (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n)\mathfrak{I}^{-1} = (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n) \cdot \mathfrak{I}^{-1}.$$

The pseudoscalars $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_n$ and $\mathfrak{I}$ are said to be the same orientation iff α is positive. Thus, the *choice of* $\mathfrak{I}$ *assigns an orientation to* $\mathfrak{V}^n$ and $\mathfrak{V}^n$ *is said to be oriented*. The opposite orientation is obtained by selecting $-\mathfrak{I}$ as unit pseudoscalar.

From any basis $\{\mathbf{a}_r\}$ for $\mathfrak{V}^n$, one can generate a basis for the linear space $\mathfrak{G}_n^r$ of r-vectors by forming all independent outer products of r-vectors from the set $\{\mathbf{a}_r\}$. The number of ways it can be done is given by the binomial coefficient $\binom{n}{r}$. Therefore, $\mathfrak{G}_n^r$ is a linear space of dimension $\binom{n}{r}$. The entire geometric algebra $\mathfrak{G}_n = \mathfrak{G}(\mathfrak{V}^n)$ can be described as a sum of subspaces with different grade; *i.e.*,

$$\mathfrak{G}_n = \mathfrak{G}_n^0 + \mathfrak{G}_n^1 + \cdots + \mathfrak{G}_n^r + \cdots + \mathfrak{G}_n^n = \sum_{r=0}^{n} \mathfrak{G}_n^r.$$

Thus $\mathfrak{G}_n$ is a linear space of dimension

$$\dim \mathfrak{G}_n = \sum_{r=0}^{n} \dim \mathfrak{G}_n^r = \sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

The elements of $\mathfrak{G}_n$ are called *multivectors* or quantities or *cliffs*. Any multivector A can be expressed *uniquely* as a linear combination of its r-vector parts $\langle A\rangle_k$; *i.e.*,

$$A = \langle A\rangle_0 + \langle A\rangle_1 + \cdots + \langle A\rangle_n = \sum_{r=0}^{n} \langle A\rangle_r.$$

Thus, it can be said that

- ◇ the outer product moves a k-vector up one step, and
- ◇ the inner product moves a k-vector down one step.

1.6. **Symmetric Under Duality Transformation.** The *dual* $A\mathfrak{I}$ *of a multivector* A is obtained by multiplication with the unit pseudoscalar $\mathfrak{I}$.

This operation maps scalars into pseudoscalars, vectors into $(n-1)-vectors$ and viceversa. It transforms r-vector $\langle A\rangle_k$ into a $(n-r)$-vector $\langle A\rangle_r\mathfrak{I}$.

For any arbitrary multivector, the decomposition gives the term-by-term equivalence

$$A\mathfrak{I} = \sum_{r=0}^{n} \langle A\rangle_r \mathfrak{I} = \sum_{r=0}^{n} \langle A\mathfrak{I}\rangle_{n-r}.$$

The duality transformation of $\mathfrak{G}_n$ interchanges up and down; therefore, it must interchange inner and outer products:

$$(\mathbf{a} \cdot \langle A\rangle_r)\mathfrak{I} = \mathbf{a} \wedge (\langle A\rangle_r \mathfrak{I}).$$

This can be solved to empress the inner product in terms of the outer product and two duality transformations:

$$\mathbf{a} \cdot \langle A\rangle_r = [\mathbf{a} \wedge (\langle A\rangle_r \mathfrak{I})]\mathfrak{I}^{-1}.$$

1.7. **Reversion and Scalar Parts.** Reversion facilitates computations. *Reversion* changes the order of vector factors in any multivector:

$$(\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_r)^\dagger = \mathbf{a}_r\mathbf{a}_{r-1} \ldots \mathbf{a}_1.$$

It follows that $\mathbf{a}^\dagger = \mathbf{a}$ and $\alpha^\dagger = \alpha$.

The reverse of the r-vector part of any multivector A is given by

$$\langle A^\dagger\rangle_r = \langle A\rangle_r{}^\dagger = (-1)^{\frac{r(r-1)}{2}} \langle A\rangle_r.$$

Moreover,

$$(AB)^\dagger = B^\dagger A^\dagger,$$
$$(A+B)^\dagger = A^\dagger + B^\dagger.$$

The operation of selecting the scalar part of a multivector is so important that it deserves an own symbol: $\langle A\rangle = \langle A\rangle_0$.

This operation generalizes the selection of the real part of a complex number and corresponds to computing the trace of a matrix in matrix algebra:

$$\langle ABC\rangle = \langle BCA\rangle.$$

1.8. **Scalar Product and Multiplicative Inverse.** *Scalar product* is defined for all multivectors in the 2^n-dim linear space $\mathfrak{G}_n$ by

$$\langle A^\dagger B\rangle = \sum_{r=0}^{n} \langle \langle A\rangle_r{}^\dagger \langle B\rangle_r\rangle = \langle B^\dagger A\rangle.$$

For every multivector A, this determines a scalar *magnitude* or *modulus* $|A|$ given by

$$|A|^2 = \langle A^\dagger A\rangle = \sum_r |\langle A\rangle_r|^2 = \sum_r \langle A\rangle_r{}^\dagger \langle A\rangle_r.$$

The measure of the “size” of a multivector has the Euclidean property

$$|A|^2 \geq 0$$

with $|A| = 0$ iff $A = 0$. Accordingly, the scalar product is said to be *Euclidean* or *positive definite*.

Every r-vector has a multiplication inverse:

$$\langle A\rangle_r{}^{-1} = \frac{\langle A\rangle_r}{|\langle A\rangle_r|^2}$$

1.9. **Involution.** An *automorphism* of an algebra $\mathfrak{G}_n$ is a one-to-one mapping of $\mathfrak{G}_n$ onto $\mathfrak{G}_n$ which preserves the algebraic operation of addition and multiplication. An *involution* $\underline{\alpha}$ is an automorphism which equals the identity mapping when applied twice. Thus, for an element A in $\mathfrak{G}_n$,

$$\underline{\alpha}^2(A) = \underline{\alpha}(\underline{\alpha}(A)) = A.$$

Geometric algebra has two fundamental involutions:

(1) reversion: $A^\dagger = \underline{\alpha}(A)$,
(2) main involution: $A^* = \underline{\alpha}(A)$.

Main involution, denoted $A^* = \underline{\alpha}(A)$, can be defined by the following procedure:

Any multivector A can be decomposed into a part

$$A_+ = \langle A\rangle + \langle A\rangle_2 + \cdots$$

called the *even step of* A and into a part

$$A_- = \langle A\rangle_1 + \langle A\rangle_3 + \cdots$$

called the *odd step of* A, thus

$$A = A_+ + A_-.$$

Now, the *involute of* A can be defined by

$$A^* = A_+ - A_-.$$

It follows that, for geometric product of any multivectors, one has

$$(AB)^* = A^* B^*.$$

References

[1] David Hestenes. New foundations for mathematical physics. on-line, http://geocalc.clas.asu.edu/html/NFMP.html, 1998.