PROPAGATION OF ERRORS IN CALCULATIONS

[Adapted from [1, p. 47]]

DESCRIPTION

Consider the quantities a_ks, with measured values $\hat{a}_k$s and measured uncertainty Δa_ks, and the quantities b_ls, with sample mean values $\bar{b}_l$s and sample standard deviations σ_{b_l}s; *i.e.*, the $\hat{a}_k$s arise from single measurements and the $\bar{b}_l$s from multiple measurements. Consider now a function f that depends on the a_ks and on the a_ls. Then, the most likely value of f is given by

$$\hat{y} = f\left[\hat{a}_k, \bar{b}_l\right] ,$$

whereas its maximum uncertainty by

$$\Delta y = \sum_k \text{abs}\, \partial_{a_k} y \Delta a_k + \sum_l \text{abs}\, \partial_{b_l} y \sigma_{b_l} ,$$

where abs x represents the absolute value of x.

EXAMPLE

Determine the local free fall acceleration by the period of a mathematical pendulum of length $l/\text{m} = 1.00 \pm 0.01$ and period $t/\text{s} = 2.0062 \pm 0.0057$. The length was measured once and the period thousand times (Monte Carlo simulation).

The period of a math pendulum of length l is given by

$$t = 2\pi\sqrt{\frac{l}{g}} ,$$

where g represents the local free fall acceleration.

From the pendulum equation, isolate g to have

$$g = 4\pi^2 \frac{l}{t} .$$

Then, the max uncertainty of g is

$$\Delta g = \text{abs}\, \partial_l g \Delta l + \text{abs}\, \partial_t g \sigma_t = 4\pi^2 \frac{1}{t^2} \Delta l + 8\pi^2 \frac{l}{t^3} \sigma_t .$$

Divide the last equation by the equation for g to have the relative uncertainty (fractional change) of g

$$\frac{\Delta g}{g} = \frac{\Delta l}{l} + 2\frac{\sigma_t}{t} .$$

Replace the given values in the g equation and in the equation for its relative uncertainty to find

$$\begin{aligned} g &= 4\pi^2 \frac{l}{t} = 4\pi^2 \frac{1.00}{2.0062^2} = 9.808696222936447\,, \\ \Delta g &= g\left(\frac{\Delta l}{l} + 2\frac{\sigma_t}{t}\right) \\ &= 9.808696222936447\left(\frac{0.01}{1.00} + 2\frac{0.0057}{2.0062}\right) \\ &= 0.153823746668\,. \end{aligned}$$

Finally, the value for the free fall acceleration is $g/\mathrm{m\,s^{-2}} = 9.809 \pm 0.154$.

[1] Carl von Ossietzky Universität Oldenburg. *Error theory and regression analysis*. Institute of Physics, 2014.