Adapted from [1, 2]

AVERAGE VALUE OF A SEQUENCE

Consider a *sequence* of numbers $a = [a_k]_1^n$. The *average value of the sequence*, $\bar{a}$, is the arithmetic average of the sequence series:

$$\bar{a} = \frac{1}{n} \sum_k a_k \,.$$

Apply the same concept to find *function* average values.

AVERAGE VALUE OF A FUNCTION

Consider a function f integrable over the interval $a \leq x \leq b$. Then, estimate the function average value $\bar{f}_{\text{est}}$ by partitioning the interval into subintervals of width $\triangle x = (b - a)/n$, by picking a point x_k in each subinterval, by calculating the function values $\{f[x_k]\}$ at each x_k and by averaging such values:

$$\bar{f}_{\text{est}} = \frac{1}{n} \sum_k f[x_k] \;.$$

Note that, as n increases, the estimate improves – a hint to work with calculus.

To begin with, multiply and divide the last equation by $\triangle x$, then use $n\triangle x = (b - a)$ to have

$$\bar{f}_{\text{est}} = \frac{1}{b - a} \sum_k f[x_k] \triangle x \,.$$

Calculate next the average value of f by taking the limit of the last equation:

$$\bar{f} = \frac{1}{b - a} \lim_{n \to \infty} \sum_{k=1}^{n} f[x_k] \triangle x = \frac{1}{b - a} \int_{[a,b]} f[x] \, \mathrm{d}x \,.$$

Finally, define the *average value* of a function f integrable over the interval $i = [a, b]$ as

N.B. : functional notation for the integral, [3, p. 69]

$$\bar{f} \doteq \frac{1}{b - a} \int_i f \,;$$

that is, the average value of an integrable function over an interval equals the integral of the function divided by the size of the interval.

[v.0.1.0 16:39 05/08/2014]

MONTE-CARLO INTEGRATION

Monte-Carlo integration is a procedure to estimate a value for the integral of a function over an interval not by partitioning the interval and picking values at the subintervals, but by *randomly* picking numbers within the whole interval and with them calculating function values. The process of picking random numbers within the interval is called *random sampling*.

Integration

Suppose we wish to estimate the value of the integral $l = \int_i f$, where f represents an integrable function over the interval $i = [a, b]$.

First, *randomly* choose n points $\{x_k\}$ within $a \leq x \leq b$ and use these to estimate the average value of f

$$\bar{f}_{\text{est}} = \frac{1}{n} \sum_{k=1}^{n} f\left[x_k\right]$$

then, an estimate for the value of the integral becomes

$$l_{\text{est}} = (b - a)\, \bar{f}_{\text{est}}\,.$$

This is Monte-Carlo integration.

Integration uncertainty

The *central limit theorem* of probability theory gives an estimate for the *uncertainty* in Monte-Carlo integration.

Suppose the average value of a function f is estimated by random sampling n numbers, *aka* the *sample size*,

$$\bar{f}_{\text{est}} = \frac{1}{n} \sum_{k=1}^{n} f\left[x_k\right]\ .$$

Then, the *variance* of the estimated average is

$$\text{var}\, \bar{f}_{\text{est}} = \frac{\sigma^2}{n}\,,$$

where σ is the variance of f.

Measure the uncertainty u by the standard deviation:

$$u = \frac{\sigma}{\sqrt{n}}\,.$$

Note that the uncertainty goes to zero like $1/\sqrt{n}$; *i.e.*, for example, to decrease the uncertainty by a factor of 1000, increase the sample size by a factor of 1000000.

Example

Estimate the value of the integral

N.B. : The reference value is $\left(1 - \exp\left[-2\pi\right]\right)/2 \sim 0.4990663$.

$$l = \int_{[0,\tau]} \exp\left[-x\right] \sin\left[x\right]\ \mathrm{d}x\,,$$

where $\tau \doteq 2\pi$.

With the aid of computer generated (pseudo) random numbers (see appendix A for details), it was possible to estimate the value of the integral and its uncertainty as 0.498 19(37) with a sample size of 100 000. The result of the Monte-Carlo integration differs in 0.2% from the reference value of 0.499 066 3.

[1] Q. Fang, *Integral Properties and Average Value* (2014).

[2] Unknown, *Monte-Carlo Integration Simulation*, AMTH142 (2007).

[3] T. M. Apostol, *Calculus, One-Variable Calculus with an Introduction to Linear Algebra* (Xerox, 1967).

A LISTINGS

This section contains the Ruby code used to approximate the integral in the example of Monte-Carlo integration.

The code was written, tested and run using `ruby 2.1.2p95 (2014-05-08 revision 45877) [x86_64-darwin13.0]`.

```
#!/usr/local/bin/ruby
#--
# Have faith in the way things are.
#
# monte-carlo.rb
# date: 2014.08.05
#++

# == Description
# Estimate integral: int{exp(-x)sin(x) dx}{0 to tau} by Monte-Carlo
#
# == Algorithm
# define the sample size = n
# randomly choose n points within [0, tau], use them to calculate f(x)
# and store the f(x)s in an array
# estimate the function average value
# estimate the function integral value
# estimate the uncertainty
#
# == Author
# rimbaud1854
#
# == Copyright
# Copyright (c) 2014 rimbaudcode
# Licensed under GPLv3+. No warranty provided.

include Math
require 'descriptive_statistics'

sample_size = 100_000
tau         = 2.0 * PI

low_bound = 0.0
up_bound  = tau
interval  = (low_bound..up_bound)

function_values = []
sample_size.times do
    random_point = rand interval
    function_values << (exp(-random_point) * sin(random_point))
  end

average_function = function_values.mean
average_integral = (up_bound - low_bound) * average_function
```

```
uncertainty      = function_values.standard_deviation / sqrt(sample_size)
result           = [average_integral, uncertainty]

p result
```