TENSOR ALGEBRA AND TENSOR CALCULUS

DIEGO HERRERA

Contents

1. Component-free Tensor Algebra

A rank-n tensor is, by definition, a real-valued, linear function of n vectors. Pictorially, we shall regard a tensor, say T, as a box with n slots in its top, into which are inserted n vectors, and one slot in its end, out of which rolls computer paper with a single real number printed on it: the value that the tensor T has when evaluated as a function of the n inserted vectors. We shall denote the tensor by sans-serif character T. For instance, if T is a 3-rank tensor (3 slots), then its value on the vectors a, b and c will be denoted $\mathsf{T}(a, b, c)$. Linearity of this function can be expressed as

$$\mathsf{T}(\alpha a + \beta b, c, d) = \alpha \mathsf{T}(a, c, d) + \beta \mathsf{T}(b, c, d) \ ,$$

where α and β are real numbers.

The *squared length* $(a)^2 \equiv a^2$ of a vector a is the squared distance between the points at its tail and its tip – calculated by the *Euclidean norm* [1]. The *inner product* $a \cdot b$ of two vectors is defined in terms of the squared length by [2]

$$a \cdot b \equiv \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right] \ .$$

In Euclidean space, this is the standard inner product, familiar from elementary geometry. Because the inner product $a \cdot b$ is a linear function of each of its vectors, we can regard it as a tensor of rank 2. When so regarded, the inner product is denoted $\mathsf{g}(\_, \_)$ and is called the *metric tensor*. In other words, the metric tensor g is that linear function of two vectors whose value is given by

$$\mathsf{g}(a, b) \equiv a \cdot b \, . \tag{1}$$

Notice that, because $a \cdot b = b \cdot a$, the metric tensor is *symmetric* in its two slots; *i.e.*, one gets the same real number independently of the order in which one inserts the two vectors into the slots [3]: $\mathsf{g}(a, b) = \mathsf{g}(b, a)$.

With the aid of the inner product, we can regard any vector a as a tensor of rank one: the real number that is produced when an arbitrary vector c is inserted into a's slot is

$$a(c) \equiv a \cdot c \, .$$

Date: March 10, 2013.

Key words and phrases. tensor algebra tensor calculus index notation.

[1] If $\mathcal{P}$ represents a point in $\mathfrak{E}^3$, then a vector p can be associated to it. This vector can be represented, in its turn, as $p = (p_1, p_2, p_3)$. Then the *Euclidean norm*, $|p|$, (Euclidean length or magnitude) of the point is defined by the formula $|p|^2 = p_1^2 + p_2^2 + p_3^2$.

[2] In components, the inner product of $a, b \in \mathfrak{E}^3$ is given by $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$.

[3] In slot-naming notation, to be developed afterwards, the components of g (in a given Cartesian coordinate system with basis elements e_i) are g_{ij}; *i.e.*, g is "waiting" to accept two vectors – this explains the notation $\mathsf{g}(\_, \_)$. When a vector, say a, is plugged in, $\mathsf{g}(\_, a)$, then $g_{ij} a^j$ or a^i; that is, the result is the dual of a. When two vectors, say a and b, are plugged in, then $\mathsf{g}(a, b) = a^i g_{ij} b^j = a^i b_i$; *i.e.*, g "disapears", leaving behind the inner product of a and b!

Second-rank tensors appear frequently in the laws of physics – often in roles where *one sticks a single vector into the second slot and leaves the first slot empty thereby producing a single-slotted entity, a vector*. A familiar example is a rigid body's (Newtonian) moment-of-inertia tensor $\mathsf{I}(\_,\_)$. Insert the body's angular velocity vector Ω into the second slot, and you get the body's angular momentum vector $J(\_) = \mathsf{I}(\_,\Omega)$. Another example is the stress tensor of a solid, a fluid, a plasma, or a field.

From three (or any number of) vectors a, b, c we can construct a tensor, their *tensor product*, defined as follows:

$$a \otimes b \otimes c(e,f,g) \equiv a(e)\, b(f)\, c(g) = (a \cdot e)(b \cdot f)(c \cdot g)\,. \tag{2}$$

Here the first expression is the notation for the value of the new tensor, $a \otimes b \otimes c$ evaluated on the three vectors e, f, g; the middle expression is the ordinary product of three real numbers, the value of a on e, the value of b on f and the value of c on g; and the third expression is that same product with the three numbers rewritten as scalar products. Similar definitions can be given (and should be obvious) for the tensor product of any two or more tensors of any rank; for example, if T has rank 2 and S has rank 3, then

$$\mathsf{T} \otimes \mathsf{S}(e,f,g,h,i) = \mathsf{T}(e,f)\, \mathsf{S}(g,h,i)\ .$$

One last geometric (*i.e.*, frame-independent) concept we shall need is *contraction*. We shall illustrate this concept first by a simple example, then give the general definition. From two vectors a and b we can construct the tensor product $a \otimes b$ (a second-rank tensor), and we can also construct the scalar product $a \cdot b$ (a real number, *i.e.*, a scalar – a rank-0 tensor). The process of contraction is the construction of $a \cdot b$ from $a \otimes b$

$$\mathrm{cont}(a \otimes b) \equiv a \cdot b\,.$$

One can show fairly easily using component techniques that any second-rank tensor T can be expressed as a sum of tensor products of vectors, $\mathsf{T} = a \otimes b + c \otimes d + \cdots$; and correspondingly, it is natural to define the contraction of T to be $\mathrm{cont}\,\mathsf{T} = a \cdot b + c \cdot d + \cdots$. Note that this contraction process lowers the rank of the tensor by two, from 2 to 0. Similarly, for a tensor of rank n one can construct a tensor of rank $n-2$ by contraction, but in this case one must specify which slots are to be contracted. For example, if T is a third rank tensor, expressible as $\mathsf{T} = a \otimes b \otimes c + e \otimes f \otimes g + \cdots$, then the contraction of T on its first and third slots is the rank-1 tensor (vector)

$$\mathrm{cont}_{1,3}(a \otimes b \otimes c + e \otimes f \otimes g + \cdots) \equiv (a \cdot c)b + (e \cdot g)f + \cdots\,.$$

All the concepts developed in this section (vectors, tensors, metric tensor, inner product, tensor product, and contraction of a tensor) can be carried over, with no change whatsoever, into *any* vector space over the real numbers [4] that is endowed with a concept of squared length – for example, to the four-dimensional *spacetime* of special relativity.

A note on notation: points: $\mathcal{P}$, vectors: a, tensors: T.

2. Component Representation of Tensor Algebra

In the Euclidean 3-space of Newtonian physics, there is a unique set of *orthonormal basis vectors* [5] $\{\mathsf{e}_x, \mathsf{e}_y, \mathsf{e}_z\} \equiv \{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\} = \left\{\mathsf{e}^1, \mathsf{e}^2, \mathsf{e}^3\right\}$ associated with any *Cartesian coordinate system* $\{x, y, z\} \equiv \left\{x^1, x^2, x^3\right\} \equiv \{x_1, x_2, x_3\}$. [In Cartesian coordinates in Euclidean space, by definition, a quantity is the same whether its index is down or up.] The basis vector e_j points along the x_j

[4] If the vector space's scalars are complex numbers, as in quantum mechanics, then slight changes are needed.

[5] Because of their special nature – unit length and orthogonal, we note the basis elements with sans-serif font e, like tensors; although, they are *vectors*.

coordinate direction, which is orthogonal to all the other coordinate directions, and it has unit length, so

$$\mathsf{e}_j \cdot \mathsf{e}_k = \delta_{jk} \,. \tag{3}$$

Any vector a in 3-space can be expanded in terms of this basis,

$$a = a_1\mathsf{e}_1 + a_2\mathsf{e}_2 + a_3\mathsf{e}_3 = \sum_{j=1}^{3} a_j\mathsf{e}_j = a_j\mathsf{e}_j \,.$$

In the last equality, we have adopted Einstein summation convention: repeated indices (in this case j) are to be summed (in this 3-space case over $j = 1, 2, 3$). So $a = \sum_{j=1}^{3} a_j\mathsf{e}_j$ becomes $a = a_j\mathsf{e}_j$.

By virtue of the orthonormality of the basis, the components a_j of a can be computed as the scalar product

$$a_j = a \cdot \mathsf{e}_j \,. \tag{4}$$

(The proof [6] of this is straightforward $a \cdot \mathsf{e}_j = (a_k\mathsf{e}_k) \cdot \mathsf{e}_j = a_k(\mathsf{e}_k \cdot \mathsf{e}_j) = a_k\delta_{jk} = a_j$.)

Any tensor, say the third-rank tensor $\mathsf{T}(\_,\_,\_)$, can be expanded in terms of tensor products of the basis vectors:

$$\mathsf{T} = T_{ijk}\mathsf{e}_i \otimes \mathsf{e}_j \otimes \mathsf{e}_k \,.$$

The components T_{ijk} of T can be computed from T and the basis vectors by the generalization

$$T_{ijk} = \mathsf{T}(\mathsf{e}_i \otimes \mathsf{e}_j \otimes \mathsf{e}_k) \,. \tag{5}$$

(This equation can be derived using the orthonormality of the basis in the same way as **??** was derived.) As an important example, the components of the metric are $g_{jk} = \mathsf{g}(\mathsf{e}_j, \mathsf{e}_k) = \mathsf{e}_j \cdot \mathsf{e}_k = \delta_{jk}$ [where the first equality is the method **??** of computing tensor components, the second is the definition **??** of the metric, and the third is the orthonormality relation **??**]:

$$g_{jk} = \delta_{jk} \qquad \text{in any orthonormal basis in 3-space.}$$

The components of a tensor product, *e.g.*, $\mathsf{T}(\_,\_,\_) \otimes \mathsf{S}(\_,\_)$, are easily deduced by inserting the basis vectors into the slots [**??**]; they are $\mathsf{T}(\mathsf{e}_i, \mathsf{e}_j, \mathsf{e}_k) \otimes \mathsf{S}(\mathsf{e}_l, \mathsf{e}_m) = T_{ijk}S_{lm}$ [*cf.* **??**]. In words, the components of a tensor product are equal to the ordinary arithmetic product of the components of the individual tensors.

In component notation, the inner product of two vectors and the value of a tensor when vectors are inserted into its slots are given by

$$a \cdot b = a_i b_i \,, \qquad \mathsf{T}(a, b, c) = T_{ijk}a_i b_j c_k \,.$$

as one can easily show using previous equations. Finally, the contraction of a tensor [say, the fourth rank tensor $\mathsf{R}(\_,\_,\_,\_)$] on two of its slots [say, the first and third] has components that are easily computed from the tensor's own components:

$$\text{comp}(\text{cont}_{1,3}\,\mathsf{R}) = R_{ijik} \,. \tag{6}$$

Note that R_{ijik} is summed on the i index, so it has only two free indices, j and k, and thus is the component of a second rank tensor, as it must be if it is to represent the contraction of a fourth-rank tensor.

[6] In the sum, $x_i\delta_{ij}$ the effect of the metric is to substitute j for i in x_i; *i.e.*, from x_i to x_j.

3. Slot-Naming Index Notation

We now pause, in our development of the component version of tensor algebra, to introduce a very important new viewpoint: Consider the rank-2 tensor $\mathsf{F}(\_,\_)$. We can define a new tensor $\mathsf{G}(\_,\_)$ to be the same as F, but with the slots interchanged; *i.e.*, for any two vectors a and b it is true that $\mathsf{G}(a,b) = \mathsf{F}(b,a)$. We need a simple, compact way to indicate that F and G are equal except for an interchange of slots. The best way is to give the slots names, say α and β – *i.e.*, to rewrite $\mathsf{F}(\_,\_)$ as $\mathsf{F}(\_{\alpha},\_{\beta})$ or more conveniently as $F_{\alpha\beta}$; and then to write the relationship between G and F as $G_{\alpha\beta} = F_{\beta\alpha}$. "NO!" some readers might object. This notation is indistinguishable from our notation for components on a particular basis. "GOOD!" a more astute reader will exclaim. The relation $G_{\alpha\beta} = F_{\beta\alpha}$ in a particular basis is a true statement if and only if $\mathsf{G} = \mathsf{F}$ *with slots interchanged* is true, so why not use the same notation to symbolize both? This, in fact, we shall do. We shall ask our readers to look at any *index equation* such as $G_{\alpha\beta} = F_{\beta\alpha}$ like they would look at an Escher drawing: momentarily think of it as a relationship between components of tensors in a specific basis; then do a quick mind-flip and regard it quite differently, as a relationship between geometric, basis-independent tensors with the indices playing the roles of names of slots. This mind-flip approach to tensor algebra will pay substantial dividends.

As an example of the power of this *slot-naming index notation*, consider the contraction of the first and third slots of a third-rank tensor T. In any basis the components of $\mathrm{cont}_{1,3}\,\mathsf{T}$ are T_{aba}; *cf.*, **??**. Correspondingly, in slot-naming index notation we denote $\mathrm{cont}_{1,3}\,\mathsf{T}$ by the simple expression T_{aba}. We say that the first and third slots are "strangling each other" by the contraction, leaving free only the second slot (named b) and therefore producing a rank-1 tensor – a vector.

4. Examples

Example 4.1 (Particle Kinetics and Lorentz Force in Geometric Language). *In this section we shall illustrate our geometric viewpoint by formulating Newton's laws of motion for particles.*

Solution. In Newtonian physics, a classical particle [7] moves through Euclidean 3-space as universal time t passes. At time t the particle is located at some point $\mathcal{X}(t)$ (its *position*). The function $x(t)$ represents a curve in 3-space, the particle's *trajectory*. The particle's *velocity* $v(t)$ is the time derivative of its position, its *momentum* $p(t)$ is the product of its *mass* m and velocity, its *acceleration* $a(t)$ is the time derivative of its velocity, and its *energy* K is half its mass times velocity squared:

$$v(t) = \frac{dx(t)}{dt}\,, \qquad p(t) = mv(t)\ , \qquad a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}\,, \qquad K(t) = \frac{1}{2}m(v(t))^2\,. \tag{7}$$

Since points in 3-space are geometric objects (defined independently of any coordinate system), so also are the trajectory $x(t)$, the velocity, the momentum, the acceleration and the energy. (Physically, of course, the velocity has an ambiguity; it depends on one's standard of rest.)

Newton's second law of motion states that the particle's momentum can change only if a force F acts on it, and that change is given by $dp(t)\,/dt = ma(t) = F$.

If the force is produced by an electric field E and magnetic field B, then this law of motion takes the familiar Lorentz-force form

$$dp(t)\,/dt = q\,(E + v \times B)\ . \tag{8}$$

(Here we have used the vector cross product, which will not be introduced formally until later). Obviously, these laws of motion are geometric relationships between geometric objects. □

Example 4.2 (Particle Kinetics in Index Notation). *Rewrite the equations of particle kinetics in slot-naming index notation.*

Solution.

$$\begin{aligned} v_i(t) &= \frac{dx_i(t)}{dt}\,, \quad p_i(t) = mv_i(t)\ , \quad a_i(t) = \frac{dv_i(t)}{dt} = \frac{d^2v_i(t)}{dt^2}\,, \\ K(t) &= \frac{1}{2}mv_iv_i\,, \quad dp_i(t)\,/dt = q\,(E_i + \epsilon_{ijk}v_jB_k)\ . \end{aligned} \tag{9}$$

(In the last equation ϵ_{ijk} is Levi-Civita tensor, which is used to produce the cross product; we shall learn about it below.)

?? can be viewed in either of two ways:

(1) as the basis-independent geometric laws (**??** and **??**) written in slot-naming index notation; or
(2) as equations for the components of v, p, a, E and B in some particular Cartesian coordinate system.

□

Example 4.3 (Numerics of Component Manipulations). *The third rank tensor* $\mathsf{S}(\_,\_,\_)$ *and vectors* a *and* b *have as their only nonzero components* $S_{123} = S_{231} = S_{312} = 1$, $a_1 = 3$, $b_1 = 4$ *and* $b_2 = 5$. *What are the components of the vector* $c = \mathsf{S}(a, b, \_)$, *the vector* $d = \mathsf{S}(a, \_, b)$ *and the tensor* $\mathsf{W} = a \otimes b$?

[7] A classical particle is an object whose mass does not interact with any gravitational field.

Solution. [Partial solution: In component notation, $c_k = S_{ijk}a_ib_j$, where (of course) we sum over the repeated indices i and j. This tells us that $c_1 = S_{231}a_2b_3$ because S_{231} is the only component of S whose last index is a 1; and this in turn implies that $c_1 = 0$ since $a_2 = 0$. Similarly, $c_2 = S_{312}a_3b_1 = 0$ (because $a_3 = 0$). Finally, $c_3 = S_{123}a_1b_2 = 1 \cdot 3 \cdot 5 = 15$. Also, in component notation $W_{ij} = a_ib_j$, so $W_{11} = a_1b_1 = 3 \cdot 4 = 12$ and $W_{12} = a_1b_2 = 3 \cdot 5 = 15$.] □

Example 4.4. *Assuming Newtonian physics, consider a particle of constant mass m interacting with some field. Suppose that this interaction causes the particle to change its position r while time t passes. Finally, consider that the particle's position can be described as a function of the form $r(t)$. Calculate then the rate at which the particle's kinetic energy changes.*

Solution. While tracing the curve $r(t)$, the particle changes its position with a velocity $v(t) = dr(t)\,/dt$. The particle's kinetic energy K is thus $K(t) = 1/2\,m(v(t))^2$.

To calculate the kinetic energy change rate, rewrite the kinetic energy equation by decomposing $(v(t))^2$: $K(t) = 1/2\,m\,(v(t) \cdot v(t))$. Then differentiate both sides:

$$\frac{d}{dt}K(t) = \frac{1}{2}m\frac{d}{dt}v(t) \cdot v(t)\ .$$

Apply Leibniz rule:

$$dK(t)\,/dt = 1/2\,m\,(dv(t)\,/dt \cdot v(t) + v(t) \cdot dv(t)\,/dt)\,.$$

Use the commutative property of the inner product [8] on the right-hand side of the last equation:

$$dv(t)\,/dt \cdot v(t) + v(t) \cdot dv(t)\,/dt = v(t) \cdot dv(t)\,/dt + v(t) \cdot dv(t)\,/dt = 2v(t) \cdot dv(t)\,/dt\,.$$

Finally, rewrite the rate at which kinetic energy changes as

$$dK(t)\,/dt = m\,v(t) \cdot dv(t)\,/dt\,,$$

which gives the desired result. □

Solution. Alternatively, consider the particle's kinetic energy K equation $2K(t) = m(v(t))^2$. Differentiate both sides using directly the properties of the inner product:

$$2\frac{d}{dt}K(t) = 2m\,v(t) \cdot \frac{d}{dt}v(t)\ .$$

The rate at which kinetic energy changes is therefore

$$dK(t)\,/dt = m\,v(t) \cdot dv(t)\,/dt\,,$$

yielding finally the desired result. □

Example 4.5. *Assuming Newtonian physics, consider a particle of constant mass m interacting with some field. Suppose that this interaction causes the particle to change its position r while time t passes. Finally, consider that the particle's position can be described as a function of the form $r(t)$. Calculate then the rate at which the particle's momentum changes.*

Solution. While tracing the curve $r(t)$, the particle changes its position with a velocity $v(t) = dr(t)\,/dt$. The particle's momentum p is thus $p(t) = mv(t)$.

To calculate the rate at with momentum changes, differentiate the last equation:

$$dp(t)\,/dt = d/dt(mv(t)) = m\,dv(t)\,/dt + v(t)\ dm/dt\,.$$

Since the particle's mass is constant, then $dm/dt = 0$ and therefore the momentum change rate simplifies to

$$dp(t)\,/dt = m\,dv(t)\,/dt\,,$$

[8] Consider two vectors a and b and their inner product $a \cdot b$. Saying that the inner product is commutative means $a \cdot b = b \cdot a$.

which gives the desired result. $\square$

Note 4.1 (Nomenclature and Dimensions for the Electromagnetic Interaction)**.** *According to [**?** , p. 16, 17], electric charge, aka quantity of electricity, is denoted by Q. The electric field strength is denoted by E and defined by $E = F/Q$ while the magnetic flux density, aka magnetic induction, is denoted by B and defined by $F = QE$. Using these symbols, Lorenz force therefore reads $F = Q\,(E + v \times B)$.*

The dimensions of these physical quantities are $\dim Q = \mathsf{IT}$, $\dim E = \mathsf{MLI}^{-1}\mathsf{T}^{-3}$ *and* $\dim B = \mathsf{MI}^{-1}\mathsf{T}^{-2}$.

Example 4.6 (Energy change for charged particle)**.** *Without introducing any coordinates or basis vectors, show that, when a charged particle interacts with electric and magnetic fields, its energy changes at a rate $dK/dt = v \cdot E$. [*sic*]*

Solution. Consider a particle with constant electric charge Q and with constant mass m moving through a constant electromagnetic field with electric field strength E and with magnetic flux density B. Assume Newtonian physics [9], but dismiss gravitation [10].

At any time t, the particle changes its position $r(t)$ with a velocity $v(t) = dr(t)\,/dt$ due to the interaction with the field. The particle then gains kinetic energy $K(t)$ at a rate

$$dK(t)\,/dt = mv(t) \cdot dv(t)\,/dt\,.$$

Since movement is caused only by a Lorenz force, the equation of motion therefore reads $m\,dv(t)\,/dt = Q\,(E + v(t) \times B)$. Pre inner multiply this equation of motion by $v(t)$ to have

$$mv(t) \cdot dv(t)\,/dt = Q\,v(t) \cdot E\,,$$

where the vectorial identities $v(t) \cdot v(t) \times B = B \cdot v(t) \times v(t)$ and $v(t) \times v(t) = 0$ have been used.

Equate the equation of motion to the change rate of the particle kinetic energy:

$$dK(t)\,/dt = Q\,v(t) \cdot E\,,$$

which yields the desired result. $\square$

Verification. Use the principle of dimensional homogeneity to verify the equation for the change rate of the particle kinetic energy. On the LHS, $\dim(dK(t)\,/dt) = \mathsf{ML}^2\mathsf{T}^{-3}$. On the RHS, $\dim(Q\,v(t) \cdot E) = \mathsf{ML}^2\mathsf{T}^{-3}$. Thus, the equation for the change rate is dimensionally homogeneous. $\square$

v, $v(t)$, $v'(t)$, $v(t)'$, $v'(t)'$ and $v(t) \equiv \dot{r}(t)$, where $\dot{r}(t) \equiv dr(t)\,/dt$.

[9] That is, neglect relativistic effects by assuming that, at any time t, the particle moves at a speed $|v(t)|$ much smaller than the speed of light c; *i.e.*, $|v(t)| \ll c$ or, equivalently, $|v(t)|/c \ll 1$.

[10] Electromagnetic interaction is much stronger than gravitation: if gravitation is taken as 1, then electromagnetic interaction is as 10^{36}.