AXIOMATIC GEOMETRIC ALGEBRA

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1. Geometric Algebra

1.1. **Motivation to Define the Geometric Product.** Consider two vectors a and b in 2D and consider an orthonormal frame $\{\gamma_k\}$. Then, find the components of the vectors on the frame: $a = \gamma_k a^k$ and $b = \gamma_l b^l$, where the indices k and l run from 1 to 2 (Einstein summation convention in force!).

Expand a onto the frame, expand the k index and multiply every term by b:

$$ab = (\gamma_k a^k)b = \gamma_1 a^1 b + \gamma_2 a^2 b\,.$$

Expand b onto the frame, expand the l index and multiply every term:

$$ab = \gamma_1 a^1(\gamma_l b^l) + \gamma_2 a^2(\gamma_l b^l) = \gamma_1 a^1 \gamma_1 b^1 + \gamma_1 a^1 \gamma_2 b^2 + \gamma_2 a^2 \gamma_1 b^1 + \gamma_2 a^2 \gamma_2 b^2\,.$$

Rearrange the vector components (scalars) and the terms:

$$ab = a^1 b^1 \gamma_1\gamma_1 + a^2 b^2 \gamma_2\gamma_2 + a^1 b^2 \gamma_1\gamma_2 + a^2 b^1 \gamma_2\gamma_1\,.$$

Agree that $\gamma_1\gamma_1 = \gamma_2\gamma_2 = 1$ and that $\gamma_1\gamma_2 = -\gamma_2\gamma_1$ to have

$$ab = (a^1 b^1 + a^2 b^2) + (a^1 b^2 - a^2 b^1)\gamma_1\gamma_2\,.$$

The first parenthesis holds the result of the traditional inner product between a and b; the second parenthesis, on the other hand, holds the result of the not-so-traditionally-known outer product between a and b.

Denote the inner product of a and b by $a \cdot b$ and the outer product of a and b by $a \wedge b$.

Note:

(1) From the geometric product, the inner and the outer product follow.

(2) The inner product is symmetric ($a \cdot b = b \cdot a$), while the outer product is anti-symmetric ($a \wedge b = -b \wedge a$).

(3) The inner product of two vectors yields a scalar, whereas the outer product a bivector.

(4) Therefore, write the geometric product between a and b, denoted ab, as the sum of a scalar, symmetric part (the inner product) and a bivector, anti-symmetric part (the outer product); *i.e.*,

$$ab = a \cdot b + a \wedge b\,.$$

(5) The geometric product is neither symmetric nor anti-symmetric, but is not commutative:

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b\,.$$

That is, in general, $ab \neq ba$.

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1.2. **Basic Axioms.** *Axiom.* Let $\mathcal{V}^n$ be an n-dimensional linear space and let $a, b, c \in \mathcal{V}^n$. Then, assume a *geometric product of* a *and* b, denoted ab, satisfying:

- ◇ associativity: $(ab)c = a(bc) = abc\,;$
- ◇ left-distributivity: $a(b + c) = ab + ac\,;$
- ◇ right-distributivity: $(b + c)a = ba + ca\,;$
- ◇ contraction: $aa = a^2 = |a|^2$, where $|a| \in \mathcal{R}^+$ and $|a| = 0$ if and only if $a = 0\,.$

Note. Since $|a| \in \mathcal{R}^+$, thus $a^2 \in \mathcal{R}^+$.

Notation. Denote by $\mathcal{R}$ the *set of real numbers* and by $\mathcal{R}^n$ the *n-dimensional real space* defined by $\mathcal{R}^n \equiv \mathcal{R} \otimes \cdots \otimes \mathcal{R}$, where $\otimes$ represents the Cartesian power of $\mathcal{R}$.

Definition. Consider $\mathcal{R}^n$. Then, define the *n-dimensional Euclidean space*, denoted $\mathcal{E}^n$, the set $\mathcal{R}^n$ equipped with a geometric product.

1.3. **Magnitude of Vectors.** *Definition.* Consider a vector $a \in \mathcal{E}^n$. Then, define the *magnitude of* a, denoted $|a|$, by

$$|a|^2 \equiv aa\,.$$

By the contraction axiom, the magnitude of a is a scalar.

1.4. **Inverse of Vectors.** *Definition.* Consider a non-zero vector $a \in \mathcal{E}^n$. Then, define the *inverse of* a, denoted a^{-1}, by

$$a^{-1} \equiv \frac{a}{a^2}\,.$$

Notation. The inverse of a vector a may be also noted by $1/a$; *i.e.*, $1/a \equiv a^{-1}$.

Motivation. Begin with the trivial identity $a = a$. Multiply both sides by a and use the contraction axiom: $aa = aa = a^2$. Since $a^2 \in \mathcal{R}$ and $a \neq 0$, then divide both sides by a^2 to have $aa/a^2 = 1$. Call $a/a^2 = a^{-1}$ the inverse of a. The element a^{-1} is called the inverse because it behaves like the multiplicative inverse of real numbers; *viz.*, if $r \in \mathcal{R}$ and if $r \neq 0$, then $rr^{-1} = 1$.

1.5. **Commutator and Anticommutator Products.** *Definition.* Let $a, b \in \mathcal{E}^n$. Then, define the *anti-commutator product of* a *and* b by

$$[a, b]_+ \equiv ab + ba\,.$$

Similarly, define the *commutator product of* a *and* b by

$$[a, b]_- \equiv ab - ba\,.$$

1.6. **Inner Product of Vectors.** *Definition.* Let $a, b \in \mathcal{V}^n$. Then, define the *inner product of* a *and* b by

$$a \cdot b \equiv \frac{1}{2}\,[a, b]_+ = \frac{1}{2}(ab + ba)\,.$$

Definition. Consider a set $\mathcal{S}$. For $a, b \in \mathcal{S}$, let $\diamond$ be a binary operation between a and b; *i.e.*, $(a \diamond b) \in \mathcal{S}$. Call the operation *symmetric* if it satisfies $a \diamond b = b \diamond a$. Call the operation *anti-symmetric* if it satisfies $a \diamond b = -b \diamond a$.

Theorem. For *vectors*, the inner product is *symmetric*.

Proof. Let $a, b \in \mathcal{E}^n$. We need to prove that $a \cdot b = b \cdot a$. By definition $2(a \cdot b) = ab + ba$, but $ab + ba = ba + ab = 2(b \cdot a)$. Therefore, $a \cdot b = b \cdot a$. □

Theorem. The inner product of two vectors results in a scalar.
Proof. Consider two vectors $a, b \in \mathcal{E}^n$. We need to prove $(a \cdot b) \in \mathcal{R}$. Then, add the vectors to form the vector $a + b$. Square the vector sum to find $(a+b)^2 = (a+b)(a+b)$. Multiply term-wise the sum to have $(a+b)^2 = a^2 + ab + ba + b^2$. Since $ab + ba = 2a \cdot b$, then $(a+b)^2 = a^2 + 2(a \cdot b) + b^2$. By the contraction axiom, all the squared terms are manifestly scalars, therefore, the inner product $a \cdot b$ must also be a scalar. □

Theorem. The inner product of a vector by itself equals its squared.
Proof. Let $a \in \mathcal{E}^n$. We need to prove $a \cdot a = a^2$. Then, find the inner product of a with a by $a \cdot a$. By the definition of the inner product and by the contraction axiom, $2(a \cdot a) = aa + aa = 2aa$. Therefore, $a \cdot a = aa = a^2$. □

1.7. **Outer Product of Vectors.** *Definition.* Consider two vectors $a, b \in \mathcal{E}^n$. Then, define the *outer product of a and b* by

$$a \wedge b \equiv \frac{1}{2}\,[a, b]_- = \frac{1}{2}(ab - ba)\,.$$

Theorem. The outer product of two vectors is anti-symmetric.
Proof. Consider two vectors $a, b \in \mathcal{E}^n$. We need to prove that $a \wedge b = -b \wedge a$. By definition, write the outer product of a and b: $2(a \wedge b) = ab - ba$. Next, write the outer product of b and a: $2(b \wedge a) = ba - ab$. Add the last equations to find $a \wedge b + b \wedge a = 0$, which implies $a \wedge b = -b \wedge a$. □

Theorem. The outer product of a vector by itself equals zero.
Proof. Let $a \in \mathcal{E}^n$. We need to prove $a \wedge a = 0$. By definition of outer product, $2(a \wedge a) = aa - aa$. By the contraction axiom, $aa = a^2 \in \mathcal{R}$. Thus, $a \wedge a = a^2 - a^2$. Therefore, $a \wedge a = 0$. □

1.8. **Canonical Decomposition of the Geometric Product.** Consider two vectors $a, b \in \mathcal{E}^n$. Then, write the geometric product of a and b as the sum of a symmetric and anti-symmetric parts:

$$ab = \frac{1}{2}\,[a, b]_+ + \frac{1}{2}\,[a, b]_+ = a \cdot b + a \wedge b\,. \tag{1.1}$$

Call the last equation the *fundamental decompostion of the geometric product of vectors*.

1.9. **Blades and k-Vectors.** *Definition.* The anti-symmetric product of k vectors $a_1, a_2, \ldots, a_k$ generates an entity $a_1 \wedge a_2 \wedge \cdots \wedge a_k$ called a *k-blade*. Call the integer k the *step of the blade* or the *grade of the blade*.

In other words, a k-blade is the outer product of k vectors.

Definition. A linear combination of blades with the same step is called a k-vector.

Note. By definition, every k-blade is a k-vector. Every k-vector is a k-blade holds ony in geometric algebras $\mathcal{G}^n$ with $n \leq 3$.

A clif (multivector) in $\mathcal{V}^n \subset \mathcal{E}^3$: $C = s + v + B + T$, where s is a scalar, v a vector, B a bivector and T a trivector.

Grade: let $\mathcal{V}^n$ be an n-dimensional linear space, $\mathcal{G}^n = \mathcal{G}[\mathcal{V}^n]$ and $C \in \mathcal{G}^n$. Then, C can be decomposed as

$$C = \langle C \rangle_0 + \langle C \rangle_1 + \cdots + \langle C \rangle_n\,.$$

Projection of vector onto vector: let $p, q \in \mathcal{V}^n$. Define the projection of p onto q, denoted $p_{\parallel q}$, by

$$p_{\parallel q} \equiv q \frac{p \cdot q}{q \cdot q}\,.$$

Two vectors p and q are parallel if and only if their product commutes; *i.e.*, $p \parallel q \iff pq = p \cdot q$.

Two vectors p and q are perpendicular if and only if their product anti-commutes; *i.e.*, $p \perp q \iff pq = p \wedge q$.

Two vectors p and q are orthogonal if and only if their product anti-commutes; *i.e.*, $p \perp q \iff pq = p \wedge q$.

Let $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\} = \{\gamma_k; 1 \ldots 4\}$ be a set of orthonormal spacelike vectors.

Denote the reverse of a clif C by $C^\dagger$.

Let $C \in \mathcal{G}^n$. Define the gorm of C, denoted $\operatorname{gorm} C$, by

$$\operatorname{gorm} C \equiv \langle C^\dagger C \rangle_0 \,.$$

Consider fourth-dimensional Minkowski spacetime, $\mathcal{M}^4$. In $\mathcal{M}^4$, consider a frame $\{\gamma_\mu; 0 \ldots 4\}$ whose elements satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \,,$$

where the elements of g have the signature $\operatorname{sig} g_{\mu\nu} = [- + + +]$.

Given the frame $\{\gamma_\mu\}$, expand any vector $u \in \mathcal{M}^4$ as a linear combination of the frame vectors by

$$u = u^0 \gamma_0 + u^1 \gamma_1 + u^2 \gamma_2 + u^3 \gamma_3 = u^\mu \gamma_\mu \,,$$

for suitable scalars u^μ. Call the scalars u^μ the components of u on $\{\gamma_\mu\}$.

Let $u, v \in \mathcal{M}^4$, then write the inner product of u and v by

$$u \cdot v = u^\mu \gamma_\mu \cdot v^\nu \gamma_\nu = u^\mu (\gamma_\mu \cdot \gamma_\nu) v^\nu = u^\mu g_{\mu\nu} v^\nu \,.$$

Expand the indices μ and ν to have

$$u \cdot v = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3 \,.$$

Given the frame $\left\{\gamma^k = 1/\gamma_k\right\}$, expand any vector $u \in \mathcal{M}^4$ as a linear combination of the frame vectors by

$$u = u_0 \gamma^0 + u_1 \gamma^1 + u_2 \gamma^2 + u_3 \gamma^3 = u_\mu \gamma^\mu \,,$$

for suitable scalars u_μ. Call the scalars u_μ the components of u on $\{\gamma^\mu\}$.

Let $u, v \in \mathcal{M}^4$, then write the inner product of u and v by

$$u \cdot v = u^\mu g_{\mu\nu} v^\nu = u^\mu v_\mu \,.$$

Expand the index μ to have

$$u \cdot v = -u^0 v_0 + u^1 v_1 + u^2 v_2 + u^3 v_3 \,.$$

Let $u, v \in \mathcal{M}^4$, then write the inner product of u and v by

$$u \cdot v = u^\mu g_{\mu\nu} v^\nu = u_\nu v^\nu \,.$$

Expand the index ν to have

$$u \cdot v = -u_0 v^0 + u_1 v^1 + u_2 v^2 + u_3 v^3 \,.$$

Consider a to be in an n-dimensional linear space $\mathcal{V}^n$. Then, the reverse of a is $a^\dagger$. Therefore, $\operatorname{gorm} a = a^\dagger a = aa$. Expand this to find $aa = a \cdot a + a \wedge a$, but $a \wedge a = 0$. Therefore, the gorm of a vector a equals the inner product $a \cdot a$.

Consider $x \in \mathcal{M}^4$ where $x = ct\gamma_0 + x^k \gamma_k$. Then, the gorm of x is given by

$$\operatorname{gorm} x = x^\dagger x = xx = x \cdot x = x^\mu \gamma_\mu \cdot x^\nu \gamma_\nu = x^\mu x^\nu \eta_{\mu\nu} = -c^2 t^2 + x^1 x^1 + x^2 x^2 + x^3 x^3 \,.$$

Let A be a clif of grade r and B a clif of grade s, then the geometric product is liable to contain terms of all grades from $|r - s|$ to $|r + s|$ counting by twos.

Example. Let $\{\gamma_k; 1 \ldots 4\}$ be an orthonormal frame of spacelike vectors and let $A \equiv \gamma_1 \wedge \gamma_2$ and $B \equiv (\gamma_2 + \gamma_3) \wedge (\gamma_4 + \gamma_1)$. Calculate AB, $A \cdot B$ and $A \wedge B$.

Solution. Both A and B are homogeneous of grade 2, then expect terms of grade 0, 2 and 4. Next, calculate the product using the properties of the frame vectors and the shorthand notation $\gamma_{kl} = \gamma_k \wedge \gamma_l$:

$$AB = (\gamma_1 \wedge \gamma_2)((\gamma_2 + \gamma_3) \wedge (\gamma_4 + \gamma_1)) = (\gamma_{12})(\gamma_{24} + \gamma_{21} + \gamma_{34} + \gamma_{31}) = 1 + \gamma_{14} + \gamma_{23} + i \,,$$

where $i = \gamma_{1234}$.

Finally, take the corresponding grades of AB to find the inner and outer products

$$\begin{aligned} A \cdot B &= \langle AB \rangle_0 = 1\,, \\ &= \langle AB \rangle_2 = \gamma_{14} + \gamma_{23}\,, \\ A \wedge B &= \langle AB \rangle_4 = i\,, \end{aligned}$$

which yields the desired results. □

The inner product is not strictly associative. It obeys the associative relation:

$$A_{\bar{r}} \cdot (B_{\bar{s}} \cdot C_{\bar{t}}) = (A_{\bar{r}} \cdot B_{\bar{s}}) \cdot C_{\bar{t}} \qquad \text{for } r + t \leq s\,.$$

However, it relates to the outer product by

$$A_{\bar{r}} \cdot (B_{\bar{s}} \cdot C_{\bar{t}}) = (A_{\bar{r}} \wedge B_{\bar{s}}) \cdot C_{\bar{t}} \qquad \text{for } r + s \leq t\,.$$

Other identity relating inner and outer products is needed to carry out step reduction explicitly:

$$a \cdot (b \wedge C) = (a \cdot b) C + b \wedge (a \cdot C)\,.$$

In particular, by iterating the last equation, one can derive the reduction formula

$$a \cdot (b_1 \wedge b_2 \wedge \cdots b_r) = \sum_{k=1}^{r} (-1)^{r+1} (a \cdot b_k) b_1 \wedge \cdots \wedge \check{b}_k \wedge \cdots \wedge b_r\,,$$

where $\check{b}_k$ means that the kth vector b_k is omitted from the product.

Precedence convention: outer products take precedentec over (are performed before) inner products, and both inner and outer products take precedence over the geometric products. For example, the simplest (and consequently the most common and useful special case of the reduction formula is the identity

$$a \cdot (b \wedge c) = (a \cdot b) c - (a \cdot c) b\,.$$

The precedence convention allow us to write this as

$$a \cdot b \wedge c = a \cdot bc - a \cdot cb\,.$$

The vectors of a set $\{a_1, a_2, \ldots, a_r\}$ are *linearly independent* if and only if the r-blade

$$A_{\bar{r}} = a_1 \wedge a_2 \wedge \cdots \wedge a_r$$

is not zero.

The blade $A_{\bar{r}}$ is a *directed volume* for $\mathcal{V}^r$ with magnitude (scalar volume) $|A_{\bar{r}}|$.

An ordered set of vectors $\{a_k : k = 1, 2, \ldots, n\}$ in $\mathcal{V}^n$ is a *basis for* $\mathcal{V}^n$ if and only if it generates a nonzero n-blade $a_1 \wedge a_2 \wedge \cdots \wedge a_n$. The n-blades are called *pseudoscalars* or $\mathcal{V}^n$ or $\mathcal{G}^n$. They make up a 1-dimensional linear space, so we may choose a unit pseudoscalar i and write

$$a_1 \wedge a_2 \wedge \cdots \wedge a_n = \alpha i\,,$$

where α is a scalar.

From any basis $\{a_r\}$ for $\mathcal{V}^n$ we can generate a basis for the linear space $\mathcal{G}^n_r$ of r-vectors by forming all independent outer products of r vectors from the set $\{a_k\}$. The number of ways this can be done is given by the binomial coefficient $\binom{n}{r}$. Therefore, $\mathcal{G}^n_r$ is a linear space of dimension $\binom{n}{r}$. The entire geometric algebra $\mathcal{G}^n = \mathcal{G}[\mathcal{V}^n]$ can be described as a sum of subspaces with different grade; *i.e.*,

$$\mathcal{G}^n = \mathcal{G}^n_0 + \mathcal{G}^n_1 + \cdots + \mathcal{G}^n_r + \cdots + \mathcal{G}^n_n = \sum_{r=0}^{n} \mathcal{G}^n_r\,.$$

Thus, $\mathcal{G}^n$ is a linear space of dimension 2^n.

The elements of $\mathcal{G}^n$ are called *clifs* or *multivectors*. Any multivector A can be expressed uniquely as a sum of its r-vector parts $A_{\bar{r}}$; *i.e.*,

$$A = A_{\bar{0}} + A_{\bar{1}} + \cdots + A_{\bar{r}} + \cdots + A_{\bar{n}} = \sum_{r=0}^{n} A_{\bar{r}} \, .$$

The outer product with a vector "moves" a k-vector up, while the inner product moves it down. $\mathcal{G}^n$ is symmetric under a *duality transformation*. The *dual* iA of a multivector A is obtained by multiplication with the unit pseudoscalar i. This operation transforms on r-vector $A_{\bar{r}}$ into an $(n-r)$-vector $A_{\bar{r}}i$. The duality transformation interchanges clifs, so it must interchange inner and outer products. This is expressed algebraically by the identity

$$(a \cdot A_{\bar{r}})i = a \wedge (A_{\bar{r}}i) \, .$$

Example. As a special case, show that the unit pseudoscalar interchanges inner and outer products in $\mathcal{G}^3$.

Solution. Consider a vector space $\mathcal{V}^3$ and the geometric algebra $\mathcal{G}^3 = \mathcal{G}^3\left[\mathcal{V}^3\right]$ on $\mathcal{V}^3$. Consider a frame $\{\gamma_k; 1 \ldots 3\}$ in $\mathcal{G}^3_1$ and let $u \in \mathcal{G}^3_1$ be a vector and $B \in \mathcal{G}^3_2$ a bivector.

Next, calculate the unit pseudoscalar: $i = \gamma_1\gamma_2\gamma_3$.

Before calculating the product uB, remember that if a clif A of grade $r = \operatorname{grade} A$ is multiplied by a clif B of grade $s = \operatorname{grade} B$, then the product AB is liable to contain terms of grades from $|r-s|$ to $|r+s|$ counting by twos. In particular, since u is a vector, then $r = \operatorname{grade} u = 1$ and, since B is a bivector, then $s = \operatorname{grade} B = 2$. Thus, the product uB will contain terms of grades from $|r-s| = |1-2| = 1$ to $|r+s| = |1+2| = 3$. The inner product is always the lowest part of uB; *i.e.*, $u \cdot B = \langle uB \rangle_1$ and the outer product is always the highest part of uB; *i.e.*, $u \wedge B = \langle uB \rangle_3$.

Then calculate the inner and outer products of u and B by writing their definitions

$$\langle uB \rangle_1 = u \cdot B = \frac{1}{2}(uB - Bu) = -B \cdot u \, , \text{and}$$
$$\langle uB \rangle_3 = u \wedge B = \frac{1}{2}(uB + Bu) = +B \wedge u \, .$$

In $\mathcal{G}^3$, bivectors are duals of bivectors and i commutes with every member of $\mathcal{G}^3$; then, because of the duality transformation, set, without lost of generality, $B = vi$ to find

$$u \cdot (vi) = \frac{1}{2}(uvi - viu) = \frac{1}{2}(uvi - vui) = \frac{1}{2}(uv - vu)i = u \wedge vi \, ,$$
$$u \wedge (vi) = \frac{1}{2}(uvi + viu) = \frac{1}{2}(uvi + vui) = \frac{1}{2}(uv + vu)i = u \cdot vi \, ,$$

where in the last two equations, the definition of inner and outer products for *vectors* were used. This yields the desired result. □

The last two formulas can be generalized to any clif M:

$$u \cdot (Mi) = u \wedge Mi$$
$$u \wedge (Mi) = u \cdot Mi \, .$$

The operation of *reversion* reverses the order of *vector* factors in any clif. Reversion can thus be defined by

$$(a_1 a_2 \cdots a_r)^\dagger = a_r a_{r-1} \cdots a_1 \, .$$

It follows that $a^\dagger = a$ and $\alpha^\dagger = \alpha$ for vector a and scalar α.

It can be proved that

$$(AB)^\dagger = B^\dagger A^\dagger \, ,$$
$$(A+B)^\dagger = A^\dagger + B^\dagger \, .$$

The operation of selecting the scalar part of a clif is noted by

$$\langle A \rangle_0 \equiv A_{\bar{0}} \, .$$

It has the permutation property

$$\langle ABC\rangle_0 = \langle BCA\rangle_0 \, .$$

A natural *scalar product* is defined for all clifs in the 2^n-dimensional linear space $\mathcal{G}^n$ by

$$\langle A^\dagger B\rangle_0 \equiv \langle B^\dagger A\rangle_0 \, .$$

For every clif A this determines a scalar *magnitude* $|A|$ given by

$$|A|^2 = \langle A^\dagger A\rangle_0 = \sum_r |A_{\bar r}|^2 = \sum_r {A_{\bar r}}^\dagger A_{\bar r} \, .$$

Note that the magnitude of A has the Euclidean property

$$|A|^2 \geq 0 \, ,$$

with $|A| = 0$ if and only if $A = 0$. Thus, the scalar product is said to be *Euclidean*. The multivector is said to be *unimodular* if $|A| = 1$.

An important consequence is that every r-vector has a multiplicative inverse given by

$${A_{\bar r}}^{-1} = \frac{{A_{\bar r}}^\dagger}{|A_{\bar r}|^2} \, .$$

However, some clifs of mixed step have no inverses.

If vectors u and v are *parallel* or *colinear*, then $u \wedge v = 0$ and therefore $uv = u \cdot v$. If they are *perpendicular* or *orthogonal*, then $u \cdot v = 0$ and thus $uv = u \wedge v$.

If n is a unit vector and v is a general vector, then

$$v = n^2 v = nnv = n(nv) = n(n \cdot v + n \wedge v) = v_{\parallel n} + v_{\perp n} \, ,$$

where

$$v_{\parallel n} \equiv n(n \cdot v)$$

is the *projection of v on n* and

$$v_{\perp n} \equiv n(n \wedge v)$$

is the *rejection of v from n.*

Note that the projection is parallel to n, whereas the rejection is perpendicular to n.

If v is a vector, then

$$v^2 = vv = v \cdot v + v \wedge v = v \cdot v = |v|^2 \, ,$$

where the *magnitude of v*, denoted by $|v|$, is v scalar length.

To rotate a vector $v = \gamma_1$ through an angle θ in the plane $B = \gamma_1 \wedge \gamma_2 = \gamma_1\gamma_2 = \gamma_{12}$, then multiply by the *rotor*

$$R = \exp -\gamma_1\gamma_2\theta/2 = e^{-B\theta/2}$$

on the right and by the *rotor reversion*

$$R^\dagger = \exp -\gamma_2\gamma_1\theta/2 = e^{+B\theta/2}$$

on the left, with the exponentials defined by their infinite series expansions

$$R^\dagger = e^{+B\theta/2} \equiv \cos\left[\frac{\theta}{2}\right] + \gamma_1\gamma_2 \sin\left[\frac{\theta}{2}\right]$$

as $i^2 = -1$. Hence, the rotated vector is

$$v' = RvR^\dagger = e^{-B\theta/2} v e^{+B\theta/2} = e^{-\gamma_{12}\theta/2} v e^{+\gamma_{12}\theta/2} = \gamma_1 \cos\theta + \gamma_2 \sin\theta \, .$$

Rotate any multivector in the same way.

Note. In other formalisms, rotation is done via a rotation axis – an axis perpendicular to the plane where the vector lies, whereas in geometric algebra the vector v lies on the plane B, yet it is rotated by the rotor R which is a function of the plane itself; *i.e.*, the concept of rotation a vector around a rotation axis has been replaced by the concept of rotating a vector around the plane in which it lies. In other words, *external* geometry has been replaced by *local* geometry and rotation *axes* have been replaced by rotation *planes*. There are two advantages to the rotation planes approach: local geometry and generalization to n-dimensions.

Exercise. Consider $\mathcal{E}^n$ and an orthonormal frame $\{\gamma_k; 1 \dots n\}$ in $\mathcal{E}^n$. Consider a reciprocal frame $\{\gamma^k\}$ defined by $\gamma^k \cdot \gamma_l = g_{kl}$. Then, given a vector $v \in \mathcal{E}^n$, how does it transform when we change to the reciprocal frame?

Solution. The frame is orthonormal, so, by definition, its elements satisfy $\gamma_k \cdot \gamma_l = g_{kl}$.

On the other hand, the vector v is a geometric object. Thus, it is independent of the coordinate system used to represent it; *i.e.*, it must be the same in both frames:

$$x_k \gamma^k = x^k \gamma_k \,.$$

Inner multiply both sides with γ_l to find

$$x_k \gamma^k \cdot \gamma_l = x^k \gamma_k \cdot \gamma_l$$

or, equivalently,

$$x_l = x^k g_{kl} \,,$$

where the identity $x_k \gamma^k \cdot \gamma_l = x_k g_{kl} = x^l$ was used. This yields the desired result. □

Consider $\mathcal{E}^n$ and a frame $\{\gamma_k; 1 \dots n\}$, the elements of the reciprocal frame $\{\gamma^k\}$ are constructed by applying

$$\gamma^k = (-1)^{(k-1)} \gamma_1 \wedge \cdots \wedge \check{\gamma}_k \wedge \cdots \wedge \gamma_n i^{-1} \,,$$

where $\check{\gamma}_k$ means that the vector γ_k must be omitted from the product and i represents the unit pseudoscalar.

Example. Construct the reciprocal frame elements for the three-dimensional Euclidean space, $\mathcal{E}^3$.

Solution. Consider in $\mathcal{E}^3$ an orthonormal frame $\{\gamma_k\}$. Then, construct the *reciprocal frame elements* $\{\gamma^k\}$ applying the following procedure:

(1) Since the frame elements are orthonormal, then $\gamma_k \gamma_l + \gamma_l \gamma_k = 2 g_{kl}$ where $g_{kl} = \delta_{kl}$; thus, $\gamma_k \gamma_l = \gamma_k \wedge \gamma_l$.

(2) Define the shorthand notation: $\gamma_{kl} \equiv \gamma_k \gamma_l$.

(3) Find $i = \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_1 \gamma_2 \gamma_3 = \gamma_{123}$ and thus $i^2 = ii = \gamma_{123123} = -1$.

(4) Find $i^{-1} = 1/i = i/ii = -i$.

(5) Apply, for each k, the equation

$$\gamma^k = (-1)^{(k-1)} \gamma_1 \wedge \cdots \wedge \check{\gamma}_k \wedge \cdots \wedge \gamma_n i^{-1} \,.$$

For $k = 1$,

$$\gamma^1 = (-1)^{(1-1)} \gamma_2 \wedge \gamma_3 (-i) = (1) \gamma_2 \wedge \gamma_3 (-\gamma_{123}) = \gamma_1 \,.$$

Similarly, for $k = 2$, then $\gamma^2 = \gamma_2$ and, for $k = 3$, then $\gamma^3 = \gamma_3$.

This yields the desired result. □

Note. As a conclusion of the previous construction: for $\mathcal{E}^3$, the reciprocal frame elements equal the frame elements.

In geometric algebra, a blade is a generalization of the concept of scalars and vectors to include simple bivectors, trivectors, *etc*. Specifically, a k-blade is any object that can be expressed as the outer product of k vectors, and is of grade k.

In detail:

- ◇ A 0-blade is a scalar. The inner product of two vectors a and b is a 0-blade, denoted by $a \cdot b$.
- ◇ A 1-blade is a simple vector.
- ◇ A 2-blade is a simple bivector. Linear combinations of 2-blades also are bivectors, but need not be simple, and are hence not necessarily 2-blades. A 2-blade may be expressed as the outer product of two vectors a and b: $a \wedge b$.

- ◇ A 3-blade is a simple trivector, that is, it may expressed as the outer product of three vectors a, b and c: $a \wedge b \wedge c$.
- ◇ In a space of dimension n, a blade of grade $n-1$ is called a *pseudovector*.
- ◇ The highest grade element in a space is called a *pseudoscalar*, and in a space of dimension n is an n-blade.
- ◇ In a space of dimension n, there are $k(n-k)+1$ dimensions of freedom in choosing a k-blade, of which one dimension is an overall scaling multiplier.
- ◇ In a n-dimensional spaces, there are blades of grade 0 through n. A vector subspace of finite dimension k may be represented by the k-blade formed as an outer product of all the elements of a basis for that subspace.

2. Calculus

Consider $\mathcal{E}^n$ with an orthonormal frame $\{\gamma_k; 1 \dots n\}$ of definite positive signature. Then, define a *reciprocal frame* by $\left\{\gamma^k = 1/\gamma_k\right\}$; *i.e.*,

$$\gamma^k = \frac{1}{\gamma_k} = \frac{\gamma_k}{\gamma_k \gamma_k} = \gamma_k \,,$$

where $\gamma_k \gamma_k = 1$ because the frame elements are normal. That is to say, the frame elements equal the reciprocal frame elements if the frame is orthonormal with definite positive signature.

For each spacelike point $x \in \mathcal{E}^n$, an orthonormal frame $\{\gamma_k\}$ and its reciprocal frame $\left\{\gamma^k = \gamma_k\right\}$ determine a set of *rectangular coordinates* $\left\{x^k\right\}$ given by

$$x^k \equiv \gamma^k \cdot x \qquad \text{and} \qquad x = x^k \gamma_k \,.$$

Analogously, the *reciprocal rectangular coordinates* $\{x_k\}$ are given by

$$x_k \equiv \gamma_k \cdot x \qquad \text{and} \qquad x = x_k \gamma^k \,.$$

Define the *geometric derivative*, denoted $\bigtriangledown$, by its components in the following way:

$$\bigtriangledown \equiv \sum_{k=1}^{n} \gamma^k \frac{\partial}{\partial x^k} = \gamma^k \partial_k \,.$$

Note the position of the indices $\partial_k = \partial / \partial x^k$. Treat $\bigtriangledown$ as a vector.

Apply $\bigtriangledown$ to a generic vector field (*i.e.*, to a generic vector-valued function of the position vector x) $v = v[x]$ to get

$$\begin{aligned} \bigtriangledown v &= (\gamma^k \partial_k)(\gamma_l v^l) = (\gamma^k \gamma_l)(\partial_k v^l) = (\gamma^k \cdot \gamma_l + \gamma^k \wedge \gamma_l)\partial_k v^l \,, \\ &= \gamma^k \cdot \gamma_l \partial_k v^l + \gamma^k \wedge \gamma_l \partial_k v^l = \gamma^k \partial_k \cdot \gamma_l v^l + \gamma^k \partial_k \wedge \gamma_l v^l \,, \end{aligned}$$

or

$$\bigtriangledown v = \bigtriangledown \cdot v + \bigtriangledown \wedge v \,.$$

Call the symmetric, scalar part the *interior derivative* or *divergence* and call the anti-symmetric bivector part the *exterior derivative*.

Note that the exterior derivative is the dual of the traditional *curl*:

$$\bigtriangledown \wedge v = i \bigtriangledown \times v \,,$$

where i is the unit pseudoscalar. However, unlike the curl, which exists in only $\mathcal{E}^3$, the exterior derivative generalizes to any number of dimensions.

Example. Consider $\mathcal{E}^3$. Then, calculate $\partial_l x^k$.

Solution. Since both indices are free indices, expand each one as

$$\begin{aligned}
\partial_1 x^1 &= \frac{\partial x}{\partial x} = 1\,, \quad \partial_1 x^2 = \frac{\partial y}{\partial x} = 0\,, \quad \partial_1 x^3 = \frac{\partial z}{\partial x} = 0\,,\\
\partial_2 x^1 &= \frac{\partial x}{\partial y} = 0\,, \quad \partial_2 x^2 = \frac{\partial y}{\partial y} = 1\,, \quad \partial_2 x^3 = \frac{\partial z}{\partial y} = 0\,,\\
\partial_3 x^1 &= \frac{\partial x}{\partial z} = 0\,, \quad \partial_3 x^2 = \frac{\partial y}{\partial z} = 0\,, \quad \partial_3 x^3 = \frac{\partial z}{\partial z} = 1\,.
\end{aligned}$$

This result can be generalized to any number of dimensions (this will be proved in the next section). □

2.1. **Differentiating Scalar Fields.** A scalar field is a scalar-valued function of the position; *i.e.*, a function that takes position as argument and returns a scalar.

Theorem. Show that $\partial_l x^k = \delta^k_l$.
Solution. There are two cases: $k = l$ and $k \neq l$.

For $k = l$, then

$$\partial_l x^k = \partial_l x^l = \frac{\partial x^l}{\partial x^l} = 1\,.$$

For $k \neq l$, then

$$\partial_l x^k = \frac{\partial x^k}{\partial x^l} = 0\,.$$

Therefore, $\partial_l x^k = \delta^k_l$, which yields the desired result. □

Given a *constant* vector $a = \gamma^k a_k$, consider a *scalar* field $\phi = \phi[x]$

$$\phi = a \cdot k = \gamma^k a_k \cdot \gamma^l x_l = a_k \gamma^k \cdot \gamma^l x_l = a_k g^{kl} x_l = a^k x_k\,.$$

whose gradient is

$$\operatorname{grad}(a \cdot x) = \nabla(a \cdot x) = \gamma^l \partial_l (a^k x_k) = \gamma^l a^k \partial_l x_k = \gamma^l a^k \delta^k_l = \gamma^l a_l = a\,.$$

This is analogous to $\mathrm{d}(3x)/\mathrm{d}x = 3$ in ordinary one-dim. calculus. Set $a = \gamma^k$ to find that the gradient of the position vector is the corresponding frame element; *i.e.*,

$$\operatorname{grad}(a \cdot x) = \operatorname{grad}(\gamma^k \cdot x) = \nabla(\gamma^k \cdot x) = \nabla(x \cdot \gamma^k) = \nabla\, x^k = \gamma^k\,,$$

where the definition $x^k \equiv x \cdot \gamma^k$ and the symmetry of the inner product were used. This is analogous to $\mathrm{d}x/\mathrm{d}x = 1$.

Next consider the scalar field

$$\theta = \theta[x] = x^2 = xx = x \cdot x = \gamma_k x^k \cdot \gamma_l x^l = x^k \gamma_k \cdot \gamma_l x^l = x^k g_{kl} x^l = x_l x^l = x^l x_l\,,$$

whose gradient is

$$\begin{aligned}
\operatorname{grad}\theta = \nabla\, x^2 = \nabla\, xx = \gamma^k \partial_k (x^l x_l) &= \gamma^k (\partial_k x^l) x_l + \gamma^k x^l (\partial_k x_l)\,,\\
&= \gamma^k \delta^l_k x_l + \gamma^k x^l \delta_{kl} = \gamma^k x_k + \gamma^k x_k = 2\gamma^k x_k = 2x\,,
\end{aligned}$$

where the product rule was used. This is analogous to $\mathrm{d}x^2/\mathrm{d}x = 2x$.

2.2. **Differentiating Vector Fields.** A vector field is a vector-valued function of the position; *i.e.*, a function that takes position as argument and returns a vector.

Consider a vector field $v = v[x]$. The *interior derivative* or *divergence* of the vector field v is the symmetric

$$\nabla \cdot v = \gamma^k \partial_k \cdot \gamma_l v^l = \partial_k \gamma^k \cdot \gamma_l v^l = \partial_k \delta^k_l v^l = \partial_k v^k\,.$$

The *exterior derivative* of the vector field v is the anti-symmetric

$$\nabla \wedge v = \gamma^k \partial_k \wedge \gamma_l v^l = \gamma^k \wedge \gamma_l \partial_k v^l = \gamma^k \wedge \gamma^l \partial_k v_l\,.$$

Combine both, the interior and exterior, derivatives to have

$$\nabla\, v = \nabla \cdot v + \nabla \wedge v = \partial_k v^k + \gamma^k \wedge \gamma^l \partial_k v_l\,.$$

Example. Calculate the geometric derivative of the position vector x.

Solution. Apply the geometric derivative directly to the position vector x:

$$\bigtriangledown v = \bigtriangledown \cdot v + \bigtriangledown \wedge v = 3 + 0 = 3\,,$$

which is the dimension of the space. This is analogous to $\mathrm{d}x/\mathrm{d}x = 1$ in ordinary one-dimension calculus. □

2.3. **Div, Grad, Curl and All That.**

2.3.1. *Abstract Definitions.* In $\mathcal{E}^n$ consider a frame $\{\gamma_k\}$ whose elements satisfy $[\gamma_k, \gamma_l]_+ = 2g_{kl}$; *i.e.*, the frame is orthonormal. Then, define a reciprocal frame $\{\gamma^k\}$ by $\{\gamma^k = 1/\gamma_k\}$ whose elements satisfy $\gamma_l = g_{lk}\gamma^k$; *i.e.*, the reciprocal frame is also orthonormal. With the reciprocal frame and position $x = \gamma_k x^k$, expand the *geometric derivative*, denoted $\bigtriangledown$, in components $\bigtriangledown_k \equiv \partial_k$ to get

$$\bigtriangledown \equiv \gamma^k \partial_k\,.$$

Thus, regard $\bigtriangledown$ as a vector.

Apply $\bigtriangledown$ to a generic vector field (a vector-valued function of the position vector) $v = v[x]$ to get

$$\bigtriangledown v = \bigtriangledown \cdot v + \bigtriangledown \wedge v\,.$$

Call the scalar part of the product $\bigtriangledown v$ the *interior derivative of* v or *divergence of* v. Denote it by $\operatorname{div} v$. Call the bivector part of the product $\bigtriangledown v$ the *exterior derivative of* v.

In $\mathcal{E}^3$, use the unit pseudoscalar i to define the *curl of* v, denoted $\bigtriangledown \times v$, by

$$\bigtriangledown \times v = -i(\bigtriangledown \wedge v)\,.$$

That is, the curl of v is defined as the dual of the exterior derivative of v.

Apply $\bigtriangledown$ to a generic scalar field (a scalar-valued function of the position vector) $\phi = \phi[x]$ to get $\bigtriangledown \phi$. Call the vector $\bigtriangledown \phi$ the *gradient of* ϕ. Denote it by $\operatorname{grad} \phi$; *i.e.*, $\operatorname{grad} \phi = \bigtriangledown \phi$.

Define the *Laplacian of* ϕ, denoted $\bigtriangleup \phi$, by

$$\bigtriangleup \phi = \bigtriangledown \cdot \bigtriangledown \phi\,.$$

2.3.2. *Index Notation.* Consider a vector field $v = v[x]$. Then, expand $\operatorname{div} v$ as

$$\operatorname{div} v = \bigtriangledown \cdot v = \gamma^k \partial_k \cdot \gamma_l v^l = \partial_k \gamma^k \cdot \gamma_l v^l\,.$$

Use the identity $\gamma^k \cdot \gamma_l = g^k_l$ to find

$$\operatorname{div} v = \bigtriangledown \cdot v = \partial_k g^k_l v^l = \partial_k v^k\,.$$

Consider a vector field $v = v[x]$. Then, expand $\operatorname{curl} v$ as

$$\operatorname{curl} v = -i(\bigtriangledown \wedge v) = -i(\gamma^k \partial_k \wedge \gamma_l v^l) = -i\partial_k v^l (\gamma^k \wedge \gamma_l)\,.$$

Consider a scalar field $\phi = \phi[x]$. Then, expand $\operatorname{grad} \phi$ as

$$\operatorname{grad} \phi = \bigtriangledown \phi = \gamma^k \partial_k \phi\,.$$

Consider a scalar field $\phi = \phi[x]$. Then, expand $\bigtriangleup \phi$ as

$$\bigtriangleup \phi = \bigtriangledown \cdot \bigtriangledown \phi = \gamma^k \partial_k \cdot \gamma^l \partial_l \phi = \partial_k \gamma^k \cdot \gamma^l \partial_l \phi\,.$$

Note that $\gamma^k \cdot \gamma^l = g^{kl}$; thus, the Laplacian becomes

$$\bigtriangleup \phi = \partial_k g^{kl} \partial_l \phi = \partial^l \partial_l \phi = \partial_k \partial^k \phi\,.$$

Use the comma derivative notation to find

$$\bigtriangleup \phi = \sum_{k=1}^{n} \phi_{,kk}\,.$$

2.3.3. *Examples.* Some examples.

Example. Considering the position vector $x \in \mathcal{E}^n$ and a scalar field $\phi = \phi[x]$, find the Laplacian of ϕ.

Solution. Solve the example using various notational systems: vector, abstract, comma derivative and traditional.

- Vector notation: find the Laplacian of ϕ by applying its definition
$$\triangle\,\phi = \operatorname{div}(\operatorname{grad}\phi) = \triangledown \cdot \triangledown\,\phi\,.$$

- Abstract notation (index notation and Einstein summation convention): consider a frame $\{\gamma_k\}$ whose elements satisfy $\gamma_k \cdot \gamma_l = g_{kl}$; *i.e.*, $\{\gamma_k\}$ is an orthonormal coordinate system. Define then a reciprocal frame via $\left\{\gamma^k = 1/\gamma_k\right\}$. Next use the definition of the geometric derivative to find $\triangledown = \gamma^k \partial_k$. Thus, calculate the Laplacian of ϕ:
$$\triangle\,\phi = \triangledown \cdot \triangledown\,\phi = \gamma^k \partial_k \cdot \gamma^l \partial_l \phi = \partial_k \gamma^k \cdot \gamma^l \partial_l \phi\,.$$
Note that, by definition, the reciprocal frame elements satisfy $\gamma^k \cdot \gamma^l = g^{kl}$; then, the Laplacian becomes
$$\triangle\,\phi = \partial_k g^{kl} \partial_l \phi\,.$$
Use the metric coefficients to raise and lower indices:
$$\triangle\,\phi = \partial_k \partial^k \phi = \partial^l \partial_l \phi\,.$$

- Comma derivative notation: In orthonormal frames, both derivatives are equal; *i.e.*, $\partial^k = \partial_l$. Therefore, write the Laplacian of ϕ as
$$\triangle\,\phi = \sum_{k=1}^{n} \phi_{,kk} = \sum_k \phi_{,kk}\,,$$
where $\phi_{,k} \equiv \partial_k$.

- Traditional notation: Use the comma derivative to expand the Laplacian of ϕ in $\mathcal{E}^3$, as example:
$$\triangle\,\phi = \sum_k \phi_{,kk} = \phi_{,11} + \phi_{,22} + \phi_{,33}\,.$$
Use traditional notation to replace the indices; *i.e.*, $\left[x^1, x^2, x^3\right] \to [x, y, z]$:
$$\triangle\,\phi = \phi_{,11} + \phi_{,22} + \phi_{,33} = \frac{\partial\partial\phi}{\partial x^1 \partial x^1} + \frac{\partial\partial\phi}{\partial x^2 \partial x^2} + \frac{\partial\partial\phi}{\partial x^3 \partial x^3} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\,.$$

Vector notation is to be preferred among the other alternatives: it yields compact – elegant – equations, is at the highest level of abstraction, thus requires the least amount of hypotheses, and is independent of any coordinate system, thus holds in *any* coordinate system. Use vector notation to *present* mathematical statements – axioms, definitions, theorems and so on – and results to exercises.

Abstract notation is useful, on the other hand, in *proving* mathematical statements and in solving *particular* problems. Use abstract notation to work throughout solutions to problems.

Finally, traditional notation maybe used to *cross check* results. □