APPLIED GEOMETRIC ALGEBRA

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Contents

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1. Geometric Algebra

****** [blades, k-vectors, reverse, clifs, index notation, einstein convention, metric, gorm]

Denote by $\mathcal{R}$ the set of real numbers and call the elements of $\mathcal{R}$ *scalars*. Next, denote by $\mathcal{R}^+$ the set of positive real numbers including zero. Finally, denote by $\mathcal{R}^n$ the *n-dimensional real space* defined by $\mathcal{R}^n := \mathcal{R} \otimes \cdots \otimes \mathcal{R}$, where $\mathcal{R} \otimes \mathcal{R}$ represents the Cartesian product of $\mathcal{R}$ with $\mathcal{R}$.

1.1. **Geometric Product.**

1.2. **Algebra.**
Let $\mathcal{L}^n$ be an n-dimensional linear space and let $a, b, c \in \mathcal{L}^n$. Then, assume a *geometric product of a and b*, denoted ab, satisfying:

(1) associativity: $(ab)c = a(bc) = abc\,$;
(2) left-distributivity: $a(b + c) = ab + ac\,$;
(3) right-distributivity: $(b + c)a = ba + ca\,$;
(4) contraction: $aa = a^2 = |a|^2$, where $|a| \in \mathcal{R}^+$ and $|a| = 0$ if and only if $a = 0\,$.

Note that, since $|a| \in \mathcal{R}^+$, thus $a^2 \in \mathcal{R}^+$.

Since $\mathcal{R}^n$ is an n-dimensional linear space, define then the *n-dimensional Euclidean space*, denoted $\mathcal{E}^n$, as the set $\mathcal{R}^n$ equipped with a geometric product and call the elements of $\mathcal{E}^n$ *vectors*.

Finally, consider hereafter the n-dimensional linear space $\mathcal{V}^n$ to be a subset of the n-dimensional Euclidean space $\mathcal{E}^n$ and also call vectors the elements of $\mathcal{V}^n$.

1.3. **Geometry.**
Since n-dimensional real coordinate spaces are vector spaces, then scalar multiplication and vector addition are granted. The geometric product can thus be used (directly or indirectly, *e.g.*, via decomposition into inner and outer products, see below) to calculate lengths and angles, to represent rotations, reflections and so on. In other words, the geometric product transforms real coordinate spaces into an arena for doing *Euclidean geometry* – it transforms real spaces into Euclidean spaces.

1.4. **Distance Between Vectors – Metric.**

1.4.1. *Metric Definition.*
[Axioms of metric: `https://en.wikipedia.org/wiki/Metric_(mathematics)`.]

Definition. Consider a set $\mathcal{X}$ and the elements $a, x, y, z \in \mathcal{X}$. Then, define a metric on the set $\mathcal{X}$ as a function $d : \mathcal{X} \otimes \mathcal{X} \to \mathcal{R}$ satisfying

(1) *non-negativity or separation axiom:* $d[x, y] \geq 0\,$.
(2) *identity of indiscernibles or coincidence axiom:* $d[x, y] = 0$ *if and only if* $x = y\,$.
(3) *symmetry:* $d[x, y] = d[y, x]\,$.
(4) *subadditivity or triangle inequality:* $d[x, z] \leq d[x, y] + d[y, z]\,$.

Refer to a metric d on $\mathcal{X}$ intrinsic if any two points x and y in $\mathcal{X}$ can be joined by a curve with length arbitrarily close to $d[x, y]$.

Finally, for sets on which an addition $+ : \mathcal{X} \otimes \mathcal{X} \to \mathcal{X}$ is defined, refer to d as a translation invariant metric if it satisfies:

$$d[x, y] = d[x + a, y + a]\ ,$$

for all x, y and a in $\mathcal{X}$.

1.4.2. *Algebra.* Consider two vectors $x, y \in \mathcal{E}^n$. Then, define the *distance between x and y*, denoted d, by

$$d[x, y] = (|x - y|)^{\frac{1}{2}} \ .$$

Call the distance herein defined also *Euclidean metric* or *Euclidean distance*, since it satisfies the axioms of a metric and it is defined in $\mathcal{E}^n$.

The Euclidean metric here defined satisfies the axioms of a metric and thus turns $\mathcal{V}^n$ on a metric space. Additionally, note that the Euclidean metric is intrinsic and translation invariant.

Finally, hereafter, agree on referring to the Euclidean metric simply as metric.

1.4.3. *Geometry.*

1.5. **Magnitude of Vectors.**

1.5.1. *Algebra.* Consider a vector $a \in \mathcal{V}^n$. Then, define the *magnitude of* a, denoted $|a|$, by $|a|^2 := aa$.

By the contraction axiom, the magnitude of a vector is a scalar.

Refer to the magnitude of a vector herein defined also as *Euclidean norm* or *Euclidean length*.

1.5.2. *Geometry – Length.* The magnitude of a vector measures its *length*.

1.5.3. *Geometry – Scalar Multiples.* Consider a scalar $\alpha \in \mathcal{R}$ and vector $a \in \mathcal{V}^n$. Then, define the *scalar multiple* αa the vector whose magnitude is $|\alpha||a|$ and whose direction is

(1) the same as a if $\alpha > 0$ (positive);
(2) undefined if $\alpha = 0$;
(3) the same as $-a$ if $\alpha < 0$.

From the last item, it follows that $-(\alpha a) = (-\alpha)a$.

Consider a scalar $\beta \in \mathcal{R}$. The scalar multiple also satisfies

(1) associativity: $\alpha(\beta a) = (\alpha\beta)a$;
(2) distributivity: $\alpha(a + b) = \alpha a + \alpha b$ and $(\alpha + \beta)a = \alpha a + \beta a$.

1.6. **Distance *vs.* Length.** Consider two fixed points in $\mathcal{V}^n$, say $\mathcal{P}$ and $\mathcal{Q}$, and an arbitrary point $\mathcal{O}$ also in $\mathcal{V}^n$. Then, set the position of $\mathcal{P}$ relative to $\mathcal{O}$ with a vector $x_{\mathcal{P}}$ and the position of $\mathcal{Q}$ relative to $\mathcal{O}$ with a vector $x_{\mathcal{Q}}$. Calculate the Euclidean distance between $\mathcal{P}$ and $\mathcal{Q}$ by

$$d[\mathcal{P}, \mathcal{Q}] = d[x_{\mathcal{P}}, x_{\mathcal{Q}}] = (|x_{\mathcal{P}} - x_{\mathcal{Q}}|)^{\frac{1}{2}} \ .$$

On the other hand, set a vector from $\mathcal{P}$ to $\mathcal{Q}$: the vector $x_{\mathcal{P}\mathcal{Q}}$. The length of such a vector also measures the distance between $\mathcal{P}$ and $\mathcal{Q}$ and is given by the vector magnitude

$$d[\mathcal{P}, \mathcal{Q}] = |x_{\mathcal{P}\mathcal{Q}}| \,.$$

Since the distance between both points is the same, then both methods provide the same answer, but the vector magnitude one is efficient. That is, to calculate distances using the Euclidean distance formula three points are needed, while using the vector magnitude requires only two. The advantage of the vector magnitude method reveals itself when dealing with differential distances $\{\mathrm{d}x\}$ between neighboring points.

1.7. **Normal Vectors.** Consider a vector $\gamma \in \mathcal{V}^n$. Then, call γ a *normal vector* if its magnitude equals unity; *i.e.*, $|\gamma| = 1$. Call a *non-normal vector* a vector whose magnitude is not unity.

Normal vectors are also called *unit vectors*.

Consider next a nonzero, non-normal vector $a \in \mathcal{V}^n$. Then, define the *normalization of* a the map $a \mapsto a/|a|$.

In other words, normalize the vector a, denoting the result as $\hat{a}$, by

$$\hat{a} := \frac{a}{|a|} \,.$$

Since $|a|$ is a scalar, then $\hat{a}$ maps vectors to vectors.

By the contraction axiom, note that $a/|a|$ is a normal vector.

1.8. **Inverse of Vectors.** Consider a non-zero vector $a \in \mathcal{V}^n$. Then, define the *inversion of* a the map $a \mapsto a/a^2$.

In other words, invert a, or calculate the inverse of a, denoting the result as a^{-1}, by

$$a^{-1} = \frac{1}{a} := \frac{a}{a^2}\,.$$

Since a^2 is a scalar, then a^{-1} maps vectors to vectors.

Note that, by the contraction axiom, $aa^{-1} = 1$.

1.9. **Commutator and Anti-commutator Products.** Let $a, b \in \mathcal{V}^n$. Then, define the *anti-commutator product of* a *and* b by $[a,b]_+ := ab + ba$. Similarly, define the *commutator product of* a *and* b by $[a,b]_- := ab - ba$.

1.10. **Symmetric and Anti-symmetric Operators.** Consider a set $\mathcal{S}$. For $a, b \in \mathcal{S}$, let $*$ be a binary operation between a and b; *i.e.*, $(a * b) \in \mathcal{S}$. Call the operation *symmetric* if it satisfies $a * b = b * a$. Call the operation *anti-symmetric* if it satisfies $a * b = -b * a$.

1.11. **Inner Product of Vectors.**

1.11.1. *Algebra.* Let $a, b \in \mathcal{V}^n$. Then, define the *inner product of* a *and* b by

$$a \cdot b := [a,b]_+ = ab + ba\,.$$

For *vectors*, the inner product is symmetric: $a \cdot b = b \cdot a$.

The inner product of two vectors results in a scalar: $(a \cdot b) \in \mathcal{R}$.

The inner product of a vector by itself equals its squared: $a \cdot a = a^2$.

Consider two scalars $\alpha, \beta \in \mathcal{R}$ and a vector $c \in \mathcal{V}^n$. Then, the inner product also satisfies

(1) commutativity (or symmetry): $a \cdot b = b \cdot a\,;$
(2) distributivity over vector addition: $a \cdot (b + c) = a \cdot b + a \cdot c\,;$
(3) associativity with scalar multiplication: $(\alpha a) \cdot b = \alpha(a \cdot b)\,.$

1.11.2. *Geometry.* Consider two nonzero vectors $a, b \in \mathcal{V}^n$. Then, define the *angle between* a *and* b, denoted θ, by

$$|a||b| \cos\theta := a \cdot b\,.$$

Call two vectors *orthogonal*, *aka perpendicular*, denoted $a \perp b$, if $a \cdot b = 0$ and thus their product anti-commutes; *i.e.*, $ab = a \wedge b = -ba$.

1.12. **Outer Product of Vectors.**

1.12.1. *Algebra.* Let $a, b \in \mathcal{V}^n$. Then, define the *outer product of* a *and* b by

$$a \wedge b := [a,b]_- = ab - ba\,.$$

For *vectors*, the outer product is anti-symmetric. Let $a, b \in \mathcal{V}^n$, then $a \cdot b = -b \wedge a$.

The outer product of a vector by itself equals zero. Let $a \in \mathcal{V}^n$, then $a \wedge a = 0$.

1.12.2. *Geometry.* Call two vectors $a, b \in \mathcal{V}^n$ *parallel* or *colinear*, denoted $a \parallel b$, if $a \wedge b = 0$ and thus their product commutes; *i.e.*, $ab = a \cdot b = ba$.

1.13. **Canonical Decomposition of the Geometric Product of Vectors.** Let $a, b \in \mathcal{V}^n$. Then, write the geometric product of a and b as the sum of a symmetric and anti-symmetric parts:

$$ab = \frac{1}{2}\,[a,b]_+ + \frac{1}{2}\,[a,b]_- = a \cdot b + a \wedge b\,.$$

Call the last equation the *canonical decompostion of the geometric product of vectors*.

1.14. **Frames.** The vectors of a set $a_1, a_2, \dots, a_r$ are *linearly independent* if and only if the r-blade

$$A_{\bar{r}} = a_1 \wedge a_2 \wedge \cdots \wedge a_r$$

is not zero.

Call an ordered set of vectors $\{a_k : k = 1, 2, \dots, n\}$ in an n-dimensional linear space $\mathcal{V}^n$ a *frame for* $\mathcal{V}^n$, *aka* a *basis for* $\mathcal{V}^n$, if and only if the vectors are linearly independent.

Consider an n-dimensional linear space $\mathcal{V}^n$. Then, call an ordered set of vectors $\{\gamma_k; 1 \dots n\}$ in $\mathcal{V}^n$ a *standard frame for* $\mathcal{V}^n$ if the vectors are linearly independent, mutually orthogonal and normal. Mathematically, standard frame elements $\{\gamma_k\}$ satisfy

$$g_{kl} = \gamma_k \cdot \gamma_l = \frac{1}{2}(\gamma_k \gamma_l + \gamma_l \gamma_k)\,,$$

where g_{kl} represent the *metric coefficients*.

1.15. **Reciprocal Frames.** Given a standard frame in $\mathcal{V}^n$, then define a *reciprocal frame* $\left\{\gamma^k = \gamma_k{}^{-1}\right\}$ by

$$\gamma_k = g_{kl}\gamma^l \qquad \text{or} \qquad \gamma_k \cdot \gamma^l = \delta_k^l\,.$$

To find the reciprocal frame element γ^k, apply the expression

$$\gamma^k = (-1)^{k-1}\gamma_1 \wedge \cdots \wedge \check{\gamma}_k \wedge \dots \gamma_n\, i^{-1}\,,$$

where $\check{\gamma}_k$ means that γ^k must be omitted from the product.

Example. Consider a standard frame $\{\gamma_k; 1 \dots 3\}$. Then, construct the reciprocal frame $\left\{\gamma^k\right\}$.

Solution. In this case, the standard frame elements are $\{\gamma_1, \gamma_2, \gamma_3\}$. Thus, the unit pseudoscalar is $i = \gamma_{123}$ and its inverse $i^{-1} = -i$.

To find the reciprocal frame elements, apply the last equation for each γ^k; *i.e.*,

$$\begin{aligned}
\gamma^1 &= (-1)^{1-1}\gamma_2 \wedge \gamma_3\,(-\gamma_{123}) = \gamma_1\,,\\
\gamma^2 &= (-1)^{2-1}\gamma_1 \wedge \gamma_3\,(-\gamma_{123}) = \gamma_2\,,\\
\gamma^3 &= (-1)^{3-1}\gamma_1 \wedge \gamma_2\,(-\gamma_{123}) = \gamma_1\,.
\end{aligned}$$

In other words, the standard frame elements equal its reciprocal frame elements.

1.16. **Metric.** Call $\mathcal{E}^n$ the *n-dimensional flat space*. The *flat space metric* g equals the *Kronecker delta* δ; that is, $g_{kl} = \delta_{kl}$.

In $\mathcal{V}^n$, the *unit pseudoscalar* i is given by $i = \gamma_1\gamma_2 \cdots \gamma_n$.

*** convention $\gamma_1\gamma_2 = \gamma_{12}$. Some examples of manipulation of the notation and the anti-symmetric property of outer product. *** find i for 3d and inv i for 3d.

1.17. **Coordinates.** Consider a standard frame $\{\gamma_k\}$ for $\mathcal{V}^n$ and its reciprocal frame $\left\{\gamma^l\right\}$ such that $g_k^l = \gamma_k \cdot \gamma^l$. Then, express any vector $a \in \mathcal{V}^n$ as a linear combination of the frame elements by

$$a \coloneqq a^k \gamma_k\,.$$

Refer to the $\left\{a^k\right\}$ as the *coordinates of a in the frame* $\{\gamma_k\}$.

Find the coordinates of a by

$$a^k = a \cdot \gamma_k\,.$$

1.17.1. *Inner Product.* Consider an n-dimensional vector space $\mathcal{V}^n$ and consider a standard frame $\{\gamma_k\}$ for $\mathcal{V}^n$. Then, write the inner product of a and b as

$$a \cdot b = a^i\gamma_i \cdot b^j\gamma_j = a^i\gamma_i \cdot \gamma_j b^j = a^i g_{ij} b^j = a^i b_j = a_i b^j\,.$$

1.17.2. *Outer Product.* Consider 3-dimensional vector space $\mathcal{V}^3$ and consider a standard frame $\{\gamma_k\}$ for $\mathcal{V}^3$. Then, write the outer product of γ_i and γ_j as

$$\gamma_i \wedge \gamma_j = \epsilon_{ijk}\gamma_k \,,$$

where ϵ is Levi-Civita's tensor defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is a cyclic permutation of } 123\,, \\ -1 & \text{if } ijk \text{ is an anti-cyclic permutation of } 123\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Consider the standard frame to be $\{\gamma_i, \gamma_j, \gamma_k\}$ for $\mathcal{V}^3$. Then, compute the outer product of γ_i and γ_j as

$$\gamma_i \wedge \gamma_j = \epsilon_{ijk}\gamma_k \,.$$

Consider now two vectors $a, b \in \mathcal{V}^3$. Then, write the outer product of a and b as

$$a \wedge b = \epsilon_{ijk} a^i b^j \gamma_k \,.$$

Consider finally a third vector $c \in \mathcal{V}^3$. Then, write the outer product of a, b and c as

$$a \wedge b \wedge c = \epsilon_{ijk} a^i b^j c^k \gamma_i \wedge \gamma_j \wedge \gamma_k = \epsilon_{ijk} a^i b^j c^k i \,,$$

where i is the *unit pseudo-scalar for* $\mathcal{V}^3$.

1.18. **Clifs.** Let $C \in \mathcal{G}^n$. Then, define the *gorm of* C by

$$\operatorname{gorm} C \coloneqq \langle C^\dagger C \rangle_0 \,.$$

2. Geometric Calculus

Let $x \in \mathcal{V}^n$ be the position vector, then call a *scalar field* ϕ a function $\phi : x \mapsto a$, where $a \in \mathcal{R}$; *i.e.*, a scalar field *maps* the position vector to a scalar. Analogously, call a *vector field* Φ a function $\Phi : x \mapsto v$, where $v \in \mathcal{V}^n$; *i.e.*, a vector field *maps* the position vector to a vector. Alternatively, a scalar field can be seen as a function that *assigns* a scalar to every point in $\mathcal{V}^n$, while a vector field can be seen as a function that *assigns* a vector to every point in $\mathcal{V}^n$.

Let $x \in \mathcal{V}^n$ be the position vector and $\{x^k\}$ be the components of x onto a standard frame. Then, agree on the *delta derivative*, ∂, notation for partial derivatives:

$$\partial_k = \partial_{x^k} := \frac{\partial}{\partial x^k} .$$

Sometimes, the *comma derivative* notation is used instead of the delta derivative. The comma derivative consists on appending to a function a subscript containing a comma and the variable with respect to which the partial derivative is to be taken.

Example. Consider a vector $x \in \mathcal{V}^n$ whose components on a given frame are $[x^1, x^2, \dots, x^n]$ and let f be a function of x; *i.e.*, $f[x] = f[[x_1, x_2, \dots, x_n]]$. Then, represent the partial derivative of f with respect to the k-component of x (kth variable) by

$$f_{,k} = f_{,x^k} := \frac{\partial f}{\partial x^k} . \qquad \square$$

Consider a reciprocal frame $\{\gamma^k\}$, then define the geometric derivative ∇ by

$$\nabla := \gamma^k \partial_k .$$

Treat ∇ as a vector.

Consider ϕ to be a scalar field. Then, define the *gradient of* ϕ, denoted $\operatorname{grad}\phi$, by

$$\operatorname{grad}\phi := \nabla\phi = \gamma^k \partial_k \phi = \gamma^k \phi_{,k} .$$

Example. Consider $\mathcal{E}^3$ and consider a scalar field $\phi : x \mapsto x^2 + y^2 + z^2$. Then, find $\operatorname{grad}\phi$.

Solution. The position vector x is implicitly given by the coordinates $[x, y, z]$, thus a standard frame could be defined by $\{\gamma_x, \gamma_y, \gamma_z\}$. However, instead of expanding the geometric derivative onto any frame and then applying it to ϕ, the solution to the problem is simplified by working directly with geometric algebra. This is done by noticing that $\phi = xx = x^2$. Thus, $\operatorname{grad}\phi = \nabla x^2 = 2x$. Now, expand x onto the frame to find

$$\operatorname{grad}\phi = 2(x\gamma^x + y\gamma^y + z\gamma^z) . \qquad \square$$

Verification. The geometric derivative onto the frame $\{\gamma_x, \gamma_y, \gamma_z\}$ takes the form

$$\nabla = \gamma^k \partial_k = \gamma^x \partial_x + \gamma^y \partial_y + \gamma^z \partial_z .$$

Find the gradient of the field by applying geometric derivative to it:

$$\operatorname{grad}\phi = \nabla\phi = \gamma^x \frac{\partial\phi}{\partial x} + \gamma^y \frac{\partial\phi}{\partial y} + \gamma^z \frac{\partial\phi}{\partial z} ,$$

that is,

$$\operatorname{grad}\phi = 2(x\gamma^x + y\gamma^y + z\gamma^z) ,$$

which agrees with the previous result.

3. Newtonian Physics

3.1. **A Bit of History.** Newton placed his theory based on universal space and universal time; *i.e.*, space and time are independent on any external influences and on each other; In Newton's theory the position of a particle is represented by one vector x with three components each of which depends on time; *i.e.*, $x[t]$.

Einstein, on the other hand, placed time and space on equal footing with his Theory of special relativity by noting that ct, where c is the speed of light, has the dimensions of length. Minkowski, finally, united space and time in one single entity, spacetime, both conceptually and mathematically. In spacetime, the position of a particle is represented by one vector with four components; *i.e.*, the partner of x, y and z is not t, but rather ct.

3.2. **The Geometric Principle.** Physics and geometry are deeply related. So physical objects and physical processes can be modeled using geometric objects and geometric transformations. On the other hand, geometric algebra provides an efficient language to deal with geometry via algebra.

Geometric Principle:

> The laws of physics must all be expressible as geometric (coordinate-independent and reference-frame-independent) relationships – geometric transformations – between geometric objects, which represent physical entities.

There are three different conceptual frameworks for the classical laws of physics and, thus, three different geometric arenas for the laws:

(1) *General Relativity* formulates the laws as geometric relationships between geometric objects in he arena of *curved 4-dimensional spacetime*.
(2) *Special Relativity* is the limit of general relativity in the complete absence of gravity; its arena is *flat, 4-dimensional spacetime*, *aka* Minkowski spacetime.
(3) *Newtonian Physics* is the limit of general relativity when (i) gravity is weak but not necessarily absent, (ii) relative speeds of particles and materials are small compared to the speed of light c and (iii) all stresses (pressures) are small compared to the total density of mass-energy; its arena is *3-dimensional Euclidean space* with time separated off and made universal – by contrast with relativity's reference-frame-dependent time.

The aim is thus to express all physical quantities and laws in a *geometric form*: a form that is independent of any coordinate system or basis vectors.

We shall insist that the Newtonian laws of physics all obey a *Geometric Principle*: they are all geometric relationships between geometric objects, expressible without the aid of any coordinates or bases. An example is the Lorentz force law: $m\mathrm{d}d/\mathrm{d}t = q(E + v \times B)$ – a (coordinate-free) relationship between the geometric (coordinate-independent) vectors v, E and B and the scalars (the particle's) mass m and charge q; no coordinates are needed for this law of physics, nor is any description of the geometric objects as matrix-like entities. Components are secondary; they only exist after one has chosen a set of basis vectors. Components are an impediment to a clear and deep understanding of the laws of physics. The coordinate-free, component-free description is deeper and – once one becomes accustomed to it – much more clear and understandable.

Besides, coordinate independence and basis independence strongly constrain the laws of physics. This suggests that

> Nature's physical laws *are* geometric and have nothing whatsoever to do with coordinates or components or vector bases.

3.3. **Foundational Concepts.** To lay the geometric foundations for the Newtonian laws of physics an flat, Euclidean space, consider some foundational geometric objects: points, scalars, vectors, (geometric product of vectors), inner product of vectors, distance between points.

The arena for the Newtonian laws is a spacetime composed of the 3-dimensional Euclidean space $\mathcal{E}^3$, called *3-space*, and a *universal time* t. Denote *points* (locations) in 3-space by capital script letters, such as $\mathcal{P}$ and $\mathcal{Q}$. These points and the 3-space where they live require no coordinates for their definition.

A *scalar* is a single number associated with a point, say $\mathcal{P}$, in 3-space. We are interested in scalars that represent physical quantities; *e.g.*, temperature T. When a scalar is a function of location $\mathcal{P}$ in space – *e.g.*, $T[\mathcal{P}]$, call it a *scalar field*.

A *vector* in 3-space can be thought of as a straight arrow reaching from one point, $\mathcal{P}$, to another, $\mathcal{Q}$; *i.e.*, Δx. Equivalently, Δx can be thought of as a direction at $\mathcal{P}$ and a number, the *vector's length*. Sometimes, one point $\mathcal{O}$ is selected in 3-space as an "origin" and other points, $\mathcal{P}$ and $\mathcal{Q}$, are identified by their vectorial separations $x\mathcal{P}$ and $x\mathcal{Q}$ from that origin.

The *Euclidean distance* Δs^2 between two points $\mathcal{P}$ and $\mathcal{Q}$ is a scalar that requires no coordinate system for its definition. This distance Δs^2 is also the magnitude (length) $|\Delta x|$ of the vector Δx that reaches from $\mathcal{P}$ to $\mathcal{Q}$ and the square of that length is denoted by

$$|\Delta x|^2 = \Delta x \Delta x = \Delta x^2 \coloneqq \Delta s^2 .$$

Of particular importance is the case when $\mathcal{P}$ and $\mathcal{Q}$ are neighboring points and Δx is a *differential quantity* $\mathrm{d}s$. By traveling along a sequence of such $\{\mathrm{d}s\}$, laying them down tail-at-tip, one after another, we can map out a *curve* to which these $\{\mathrm{d}s\}$ are tangent. The curve is $\mathcal{P}[\lambda]$, with λ a *parameter along the curve*, and the vectors that map it out are

$$\mathrm{d}s = \frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\lambda}\mathrm{d}\lambda .$$

The product of a scalar with a vector is still a vector; so if we take the change of location $\mathrm{d}s$ of a particular element of a fluid during a (universal) *time interval* $\mathrm{d}t$ and multiply it by $1/\mathrm{d}t$, then we obtain a new vector, the fluid element's *velocity* $v = \dot{s}$, at the fluid element's location $\mathcal{P}$. Performing this operation at every point $\mathcal{P}$ in the fluid defines the *velocity field* $v[\mathcal{P}]$. Similarly, the sum (or difference) of two vectors is also a vector and so taking the difference of two velocity measurements at times separated by $\mathrm{d}t$ and multiplying by $1/\mathrm{d}t$ generates the *acceleration* $a = \dot{v}$. Multiplying by the fluid element's (scalar) *mass* m gives the force $f = ma$ that produced the acceleration; dividing by an electrically produced force by the fluid element's charge q gives another vector, the electric field $E = f/q$ and so on. We can define *inner product of pairs of vectors* (*e.g.*, force and displacement) to obtain a new scalar(*e.g.*, work) and *cross products of vectors* to obtain a new vector (*e.g.*, torque). By examining how a differentiable scalar field changes from point to point, we can define its *gradient*. Thus, we can construct all of the standard scalars and vectors of Newtonian physics. What is important is that

> these physical quantities require *no* coordinate system for their definition.

They are geometric (coordinate-independent) objects residing in Euclidean 3-space at a particular time.

We can summarize this by stating that

> the Newtonian physical laws are *all* expressible as geometric relationships between these types of geometric objects and these relationships do *not* depend upon any coordinate system or orientation of axes, nor on any reference frame (on any purported velocity of the Euclidean space in which the measurements are made).

This principle is called the *Geometric Principle* for the laws of physics. It is the Newtonian analog of Einstein's Principle of Relativity.

4. Examples

4.1. Geometric Algebra. Some examples on the usage of geometric algebra applied to physics.

Example. Write Lorentz force law using the geometric algebra formalism.

Two changes are needed for Lorentz force law to agree with geometric algebra: change in notation and the replacement of the cross product with the outer product. We solve the replacement of the cross product in two ways.

Solution. Notation change: using IUPAC recommendations, the Lorentz force can be written as

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B}),$$

where the force $\vec{F}$, the electric field $\vec{E}$, the particle velocity $\vec{v}$ and the magnetic induction $\vec{B}$, *aka* magnetic field, are modeled by vectors in $\mathcal{R}^3$ and the electric charge Q by a scalar in $\mathcal{R}$.

Then, since geometric algebra puts all of its members on equal footing, it is common to denote scalars and vectors by undecorated lower case variables, so $\vec{F} \to f$, $\vec{E} \to e$, $\vec{B} \to b$ and $Q \to q$. With these associations, Lorentz force law, therefore, becomes

$$f = q(e + v \times b). \tag{4.1}$$

Not only does this rewritten equation look more elegant – at least to the writer's eyes, it is actually easier to work with it.

Solution. Cross product replacement: Consider eq. (4.1). Using the geometric algebra of space $\mathcal{G}^3$, replace the cross product by its definition in terms of the outer product:

$$v \times b = -i(v \wedge b) = i(b \wedge v) = (b \wedge v)i, \tag{4.2}$$

where i is the unit pseudoscalar in $\mathcal{G}^3$. See that the anti-commutative property of the outer product and the commutativity property [1] of i were used.

Then, plug in the last equalities of eq. (4.2) in Lorentz force law to find,

$$f = q(e + i(b \wedge v)) = q(e + (b \wedge v)i). \qquad \square$$

Note. The cross product $v \times b$ was replaced with the outer product $(b \wedge v)i$, via i. Algebraically, in $\mathcal{G}^3$, $b \wedge v$ is a bivector and i a trivector, thus their product $(b \wedge v)i$ yields a vector, which agrees with the result of $v \times b$. Geometrically, on the other hand, $v \times b$ *yields* a vector perpendicular to the plane formed by v and b, whereas $b \wedge v$ *represents* the plane formed by b and v and thus $(b \wedge v)i$ yields the *dual* of $b \wedge v$; that is, a vector perpendicular to the $b \wedge v$ plane. Therefore, the result, $(b \wedge v)i$, is algebraically and geometrically equivalent to $v \times b$; *i.e.*, no information was lost during the conversion.

The reason for the replacement is that the outer product is more fundamental than the cross product. The cross product exists only in 3-space $\mathcal{V}^3$, while the outer product can be defined in n-dimensions. Additionally, the cross product yields a vector perpendicular to its operands, whereas the outer product represents the plane itself formed by its operands; in other words, while the cross product uses local geometrical information to *yield* non-local geometrical information – non-local geometry, the outer product *uses* local geometrical information to *represent* local geometrical information – local geometry.

The next solution refines the replacement of the cross product by using the duality property between vectors and bivectors in $\mathcal{G}^3$.

Solution. Considering the identity $v \times b = -(v \wedge b)i$, use i to interchange the outer product with the inner product, via the identity $(x \wedge y)i = x \cdot (yi)$, for vectors $x, y \in \mathcal{G}^3$:

$$v \times b = -v \cdot (bi).$$

Next, since b is a vector in $\mathcal{G}^3$, then bi is its dual – a bivector. Thus, the product $v \cdot (bi)$ anti-commutes:

$$v \times b = (bi) \cdot v.$$

Next, replace the last result in Lorentz force law:

$$f = q(e + (bi) \cdot v). \qquad \square$$

[1] In $\mathcal{G}^3$, i commutes with every other member of the algebra.

5. Vector Analysis Summary

5.1. **Basic Concepts.** Call a *vector quantity* q a quantity that has a *magnitude* and a *direction* associated with it.

Here, magnitude means a positive real number and direction is specified relative to some underlying reference frame *To be defined below.* that we regard as fixed.

Call a *vector* an abstract quantity characterized by the two properties magnitude and direction. Thus, two vectors are equal if they have the same magnitude and the same direction.

5.2. **Vector Space.** A *vector space over a field* $\mathcal{F}$ is a set $\mathcal{V}$ together with two binary operations that satisfy the eight axioms listed below. Elements of $\mathcal{V}$ are called *vectors*. Elements of $\mathcal{F}$ are called *scalars*. The first operation, *vector addition*, takes any two vectors v and w and assigns to them a third vector commonly written as $v + w$; call this operation the *sum of v and w*. The second operation takes any scalar a and any vector v and gives another vector av; call this operation the *scalar multiplication of v by a*.

5.3. **Inner Product Algebra.** Consider three vectors $a, b, c \in \mathcal{E}^n$ and a scalar $\lambda \in \mathcal{R}$. Then, the inner product between a and b satisfies:

- commutativity: $a \cdot b = b \cdot a$;
- distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$.
- associativity with scalar multiplication: $(\lambda a) \cdot b = \lambda(a \cdot b)$.

5.4. **Cross Product Algebra.** Consider three vectors $a, b, c \in \mathcal{E}^3$ and a scalar $\lambda \in \mathcal{R}$. Then, the cross product between a and b satisfies:

- anti-commutativity: $a \times b = -b \times a$;
- distributivity: $a \times (b + c) = a \times b + a \times c$;
- associativity with scalars: $(\lambda a) \times b = \lambda(a \times b)$;
- $a \times a = 0$;
- the vector $a \times b$ equals zero if and only if a and b are parallel.

6. Notation

6.1. General Commands.

- to be defined by: a defby b: $a \coloneqq b$.
- difference operator: diff a: Δa.
- text in equations: eqtxt.

6.2. Sets.

- set: set A: $\mathcal{A}$.
- elements of a set: elset(a,b,c): $\{a, b, c\}$.
- set with a property: setprop(x)(x>0): $\{x : x > 0\}$.
- Cartesian (set) product: set A sprod set B: $\mathcal{A} \otimes \mathcal{B}$.
- Cartesian power: nset An: $\mathcal{A}^n$.
- union of sets: set A union set B: $\mathcal{A} \cup \mathcal{B}$.
- intersection of sets: set A inter set B: $\mathcal{A} \cap \mathcal{B}$.
- Dim-grade space (2 is the dimension and 3 is the grade): dgspace V23: V^2_3.
- n-dim Euclidean space: espace n: $\mathcal{E}^n$.
- n-dim Minkowski space: mkspace n: $\mathcal{M}^n$.
- geometric algebra: ga: $\mathcal{G}$.
- geometric algebra on a n-dim. linear space $\mathcal{V}^n$: nga n: $\mathcal{G}^n$.
- dimension grade geometric algebra (2 is the dimension and 3 is the grade): dgga 23: $\mathcal{G}^2_3$.
- tuple: tuple(1,2,3): $[1, 2, 3]$.

6.3. Probability.

- event A: A: A.
- not event A: lnot A: $\neg A$.
- probability of event A occurring: p vat A: $p[A]$.
- probability of event A not occurring: p vat(lnot A): $p[\neg A]$.
- A and B: A land B: $A \wedge B$.
- A or B: A lor B: $A \vee B$.
- A given B (provided B): A given B: $A \,|\, B$.

6.4. Functions.

- function definition: fdef(f)(set A cartprod set B)(set R): $f : \mathcal{A} \otimes \mathcal{B} \to \mathcal{R}$.
- function mapping: fmap(f)(x)(x**2): $f : x \mapsto x^2$.
- maps to: x mapsto x**2: $x \mapsto x^2$.
- function class (calculus): class k: C^k.
- value at: f vat(x): $f[x]$.
- function composition: f fcomp g: $f \circ g$.
- a binary operation: a bprod b: $a * b$.
- derivative operator (on functions): fder f: $\mathrm{D}f$.
- partial derivative operator (on functions): derivative with respect to x of f: fpder xf: $\partial_x f$.

6.5. Sequences and Series.

- sequence: seq ak()(): $\{a\}_k$.
- sequence with limits: seq(a)(k)(k=1)(10): $\{a\}_k^{k=1}$ 10.
- series: serie(ak)(k=1)(n): $\sum_{a_k}^{k=1} n$.
- Fibonacci numbers: fib vat 10: fib[1] 0.

6.6. Geometric Algebra.

- point: point P: $\mathcal{P}$.
- curve: curve C: $\mathcal{C}$.
- surface: surf S: $\mathcal{S}$.
- region: region R: $\mathcal{R}$.
- bound: bound region A: $\partial\mathcal{A}$.
- vector: vec a: $\vec{a}$.
- normal (unit) vector: nvec(a): $\hat{a}$.
- omitted vector from a product: ovec(a): $\check{a}$.
- clifs or multivectors: (use capitals) A: A.
- pseudoscalar: pscl: i.
- better, less typing, use lower-case for vectors a and upper-case for other objects (bivectors, trivectors,...): A.
- use i for the pscl: i.
- orthogonal: ortho: $a \perp b$.
- parallel: parallel: $a \parallel b$.
- to be perpendicular to: perto(a)(b): $a_{\perp b}$.
- to be parallel to: parto(a)(b): $a_{\parallel b}$.
- to be orthogonal to: ortto(a)(b): $a_{\perp b}$.
- to be colinear to: colto(a)(b): $a_{\parallel b}$.
- projection of p onto q: projon pq: $p_{\parallel q}$.
- rejection of p onto q: rejon pq: $p_{\perp q}$.
- magnitude: magn(a): $|a|$.
- inverse: inv(a): a^{-1}.
- reverse: rev(a): $a^{\dagger}$.
- hodge dual: hdual(a): ${}^*a$.
- anticommutator: acom(a)(b): $[a, b]_+$.
- commutator: com(a)(b): $[a, b]_-$.
- expanded anticommutator: xacom(a)(b): $ab + ba$.

- expanded commutator: xcom(a)(b): $ab - ba$.
- step: step(A1): $\langle A\rangle_1$.
- scalar step: sstep(A): $\langle A\rangle_0$.
- grade operator: Grade A: grade A.
- grade: grade A2: $\langle A\rangle_2$.
- scalar grade: sgrade A: $\langle A\rangle_0$.
- clif with step: sclif Ak: $A_{\bar{k}}$.
- even part: even(A): A_+.
- odd part: odd(A): A_-.
- gorm (geometric norm?): gorm A: gorm A.
- expanded gorm: xgorm A: $\langle A^\dagger A\rangle_0$.
- metric: metric: g.
- Kronecker delta: kron: δ.
- signature: diag a: diag a.
- signature: sign a: sig a.
- inner product: iprod: $a \cdot b$.
- outer product: oprod: $a \wedge b$.
- cross product: cprod: $a \times b$.
- cannonical decomposition of the geometric product: cgprod ab: $a \cdot b + a \wedge b$.

6.7. Geometric Calculus.

- ordinary one-dim. derivative: dx x: $\mathrm{d}x$.
- ordinary time derivative (dot derivative): dt x: $\dot{x}$.
- ordinary second time derivative (dot-dot derivative): ddt x: $\ddot{x}$.
- expanded ordinary derivative: xod Hq: $\dfrac{\mathrm{d}H}{\mathrm{d}q}$.
- expanded partial derivative: xpd Hq: $\dfrac{\partial H}{\partial q}$.
- expanded material derivative: xmd phi t: $\dfrac{\partial \phi}{\partial t}$.
- expanded n-order ordinary derivative: nxod 3xt: $\dfrac{\mathrm{d}^3 x}{\mathrm{d}t^3}$.
- expanded n-order partial derivative: nxpd 3xt: $\dfrac{\partial^3 x}{\partial t^3}$.
- comma derivative: cder phi k: $\phi_{,k}$.
- semi-colon: covariant derivative: coder(cntens Aa)(k): $A^a{}_{;k}$.
- material derivative: mder phi t: $\mathrm{D}_t\phi$.
- absolute time derivative: abstder a: $\dot{\bar{a}}$.
- Christoffel symbol: chris abc: Γ^a_{bc}.
- geometric derivative: gder(a): ∇a.
- directional derivative: dder(F)(a): $\nabla_a F$.
- Laplace derivative: lder(a): $\nabla^2 a$.
- Laplace operator: lap a: lap a.
- D'Alembert operator: dalder(phi): $\Box\phi$.
- gradient: grad(phi): grad ϕ.
- divergence: div(phi): div ϕ.
- curl: curl(phi): curl ϕ.
- rotational (curl): rot(phi): rot ϕ.

6.8. Tensors.

- spatial coordinates: scord k: x^k.
- spatial coordinates time derivative: dtscoord k: $\dot{x}^k$.
- tensor: tens T: T.
- (empty) slot: tuple(slot, a, slot): $[-, a, -]$.
- tensor product: a tprod b: $a \otimes b$.
- tensor contraction: tcont(a tprod b): $\mathrm{cont}(a \otimes b)$.
- indexed tensor contraction: itcont(1,2)(a tprod b tprod c): $\mathrm{cont}_{1,2}(a \otimes b \otimes c)$.
- tensor components: tcomp T: comp T.
- covariant tensor components: cotens T(ij): T_{ij}.
- contravariant tensor components: cntens T(ij): T^{ij}.
- Levi-Civita tensor: lct: ϵ.
- covariant tensor time derivative: dtcotens ak: $\dot{a}_k$.
- contravariant tensor time derivative: dtcntens ak: $\dot{a}^k$.

6.9. Index Notation.

- frame element, vector: fvec: γ.
- frame: frm(k): $\{\gamma_k\}$.
- indexed frame: ifrm(k)(0)(n): $\{\gamma_k; 0 \dots n\}$.
- reciprocal frame: rfrm k: $\left\{\gamma^k\right\}$.
- indexed frame vector: ifvec k: γ_k.
- indexed reciprocal frame vector: rfvec k: γ^k.
- components of vector in frame: comp vk: v^k.
- components of vector in reciprocal frame: rcomp vk: v_k.
- metric coefficients in frame: imet kl: g_{kl}.
- metric coefficients in reciprocal frame: rmet kl: g^{kl}.
- mixed metric coefficients: mmet kl: g_l^k.
- kronecker delta coefficients in frame: ikron kl: δ_{kl}.
- kronecker delta coefficients in reciprocal frame: rkron kl: δ^{kl}.
- mixed kronecker coefficients: mkron kl: δ_l^k.

- indexed geometric derivative (in reciprocal frame): igder k: ∂_k.
- indexed geometric derivative (in frame): rgder k: ∂^k.

6.10. Dimensional Analysis.

- dimension: dim k: $\dim k$.
- dimension and system of units (use underscore as lim): sdim(FLT)k: $\dim_{FLT} p = \left[F/L^2\right]$.
- unit: unit k: $\operatorname{unit} k$.
- physical dimension: phdim k: $[k]$.
- dimensionless quantity: kdim: Π.
- characteristic physical quantity: chpq a: a_c.
- scaled physical quantity: scpq x: $\bar{x}$.
- reynolds number: rey: Π_{re}.
- biot number: biot: Π_{bi}.

6.11. Mechanics.

- position vector: pvec vat t: $x[t]$.
- value at time and position vector (for functions): vattpvec: $[t, x[t]]$.
- separation vector between points: svec: s.
- linear momentum: lmom: p.
- kinetic energy: ken vat t: $k[t]$.
- potential energy: pen vat t: $v[t]$.
- Action functional: action: A.
- Lagrange function: lag: L.
- Hamilton function: ham: H.
- Hamilton Kinetic energy: hken: H_{kin}.
- Hamilton potential energy: hpen: H_{pot}.
- Euler-Lagrange Equation: eleqn(q)(i): $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{q}^i}\right) - \dfrac{\partial L}{\partial q^i}$.
- contra-variant vector: cnvec pi: p^i.
- (contra-variant) indexed vector: ivec pi: p^i.
- covariant vector (covector): covec pi: p_i.
- basis vector: bvec: γ.
- natural basis vector: nbvec i: γ_i.
- dual basis vector: dbvec i: γ^i.
- generalized position vector: gpvec: q.
- indexed generalized position: gpos i: q^i.
- indexed generalized velocity: gvel i: $\dot{q}^i$.
- indexed generalized momentum: gmom i: p_i.
- indexed generalized force: gfor i: f_i.

6.12. Transport Phenomena.

- thermodynamic temperature: temp: θ.
- substance: subs A: A.
- flux: flux: j.
- mass flux of substance A: mflux A: j_{A}.
- concentration of substance A: conc A: c_{A}.
- bracket concentration of substance A: bconc Aa: $[\mathrm{A}]^a$.
- chemical amount of substance A: amount A: n_{A}.
- reaction rate of substance A: rrate A: r_{A}.
- time derivative of conc.: dtconc A: $\dot{c}_{\mathrm{A}}$.
- time derivative of chem. amount: dtamount A: $\dot{n}_{\mathrm{A}}$.

6.13. Various.

- Iverson brackets: iverson(k=l): $[k = l]_{\mathrm{iv}}$.
- Poisson brackets: poisson(f,g): $[f, g]_{\mathrm{pb}}$.
- matrix representation: mtrx metric: $[g]$.
- Taylor series generated by f at the point a: tseries(f)(x)(a): $T_\infty f[x\,; a]$.
- Taylor polynomial of degree n generated by f at the point a: ntpol(n)(f)(x)(a): $T_n f[x\,; a]$.
- Fourier series generated by f at the point x: fseries fx: $F_\infty f[x]$.
- partial sums of the Fourier series generated by f at the point x: nfsum(f)(n)(x): $F_n f[x]$.
- Legendre transform of a function: ltrans f: $f_\star$.
- conjugate variable (under Legendre transf.): cvar: $x_\star$.
- average quantity: avg a: $\langle a \rangle$.

6.14. Constants.

- Boltzmann constant: boltz: k_{b}.
- speed of ligth in vacuum: ligth: c.
- Avogadro's number: avog: n_{a}.

6.15. Alphabet.

- Latin minuscules:

$$abcdefghijklmnopqrstuvwxyz\,.$$

- Latin majuscules:

$$ABCDEFGHIJKLMNOPQRSTUVWXYZ\,.$$

- Greek:

$$\alpha\beta\gamma\delta\epsilon\varepsilon\zeta\eta\theta\vartheta\iota\kappa\lambda\mu\nu\xi\pi\varpi$$

- Greek:

$$\rho\varrho\sigma\varsigma\tau\upsilon\phi\varphi\chi\psi\omega\Gamma\Delta\Theta\Lambda\Xi\Pi\Sigma\Upsilon\Phi\Psi\Omega$$