DIMENSIONAL ANALYSIS

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Contents

1. Introduction and Definitions

1.1. **Introduction.** Intro: basic assumption: physical phenomena and physical objects can be modeled by geometric relations (geometric transformations) among physical quantities, where physical quantities are geometric objects that describe the characteristics of physical phenomena and physical objects. principle of dimensional homogeneity for physical laws. pi-theorem. sonin methodology. independence of the laws of physics on the system of units.

1.2. **Definitions.** According to its dimension(s), any physical quantity can be classified into a base, a derived or dimensionless quantity.

- ⋄ A *base quantity* is a physical quantity having its own dimension. Based upon physical grounds, [**?**], convention establishes the working set of base quantities, [**?** , p. 105].
- ⋄ A *derived quantity* is a quantity that may be written in terms of the base quantities as products of powers of the dimensions of the base quantities. That is, the dimension of any quantity Q, $\dim Q$, may be, in general, written in the form a *dimensional product* of the base quantities:
$$\dim Q = \mathsf{L}^{\alpha}\,\mathsf{M}^{\beta}\,\mathsf{T}^{\nu}\,\mathsf{I}^{\delta}\,\Theta^{\epsilon}\,\mathsf{N}^{\zeta}\,\mathsf{J}^{\eta}\,,$$
where the exponents $\alpha, \beta, \ldots, \eta$ are called the *dimensional exponents*. The numerical value of the dimensional exponents can be positive, negative or zero.
- ⋄ A *dimensionless quantity* is a derived quantity whose dimensional exponents are all zero. Thus, if Π represents a dimensionless quantity, then its dimension equals unity; *i.e.*, $\dim \Pi = 1$.

Consider now a physical phenomenon whose model is based upon a set of n physical quantities $Q_0, Q_1, \ldots, Q_n$. Suppose now that interest is placed in a specific quantity, say Q_0. Q_0 is called a *dependent quantity* if its numerical value is uniquely determined when the numerical values of the other quantities on the set are specified; *i.e.*,

$$Q_0 = f\,(Q_1, \ldots, Q_n)\;, \tag{1.1}$$

for some function f. An *independent quantity* is a non-dependent quantity; *i.e.*, $Q_1, \ldots, Q_n$. The set formed by the independent quantities is said to be *complete* if, once their numerical values are specified, no other quantity can affect the value of Q_0 and is said to be *independent* if the numeric value of each element can be adjusted arbitrarily without affecting the value of any other element. **??** is said to be *dimensionally homogeneous* if the dimensions of f equals the dimensions of Q_0.

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Base quantity	Symbol for the dimension
length	L
mass	M
time	T
electric current	I
thermodynamic temperature	Θ
amount of substance	N
luminous intensity	J

TABLE 1. Base quantities and dimensions used in the International System of Quantities, ISQ, [**?** , p. 105]

Consider a set $\mathcal{A}$ consisting of n physical quantities $Q_1, \ldots, Q_n$. Choose k elements of $\mathcal{A}$ to form a new set $\mathcal{B} = \{Q_1, \ldots, Q_k\}$. The set $\mathcal{B}$ is said to be *dimensionally independent* if none of its members has a dimension that can be expressed in terms of the dimensions of the remaining members. Now pick an element of $\mathcal{B}$, say Q_1, then the set $\mathcal{B}$ is said to be *dimensionally complete* if the dimensions of all the remaining quantities $Q_{k+1}, \ldots, Q_n$ *can* be expressed in terms of the dimensions of the subset $\{Q_1, \ldots, Q_k\}$.

In **??**, the function f, relating dependent and independent quantities, is a product of the physical laws governing the physical phenomenon of interest. Dimensional analysis may assist to determine the *form* of f or even a formula for f.

Example 1.1. *Model a classical particle interacting with a field.*

Solution. Consider Newtonian physics; *i.e.*, (flat) three-dimensional Euclidean space $\mathcal{E}^3$ and universal time $t \in \mathbb{R}^+$.

Consider next a particle of mass $m \in \mathbb{R}^+$ interacting with a field and, due to the interaction, assume the particle's position vector $\mathbf{r}$ traces a curve (trajectory) according to $\mathbf{r} : \mathbb{R}^+ \to \mathcal{E}^3$ with map $t \mapsto \mathbf{r}(t)$.

Then, the particle velocity vector is $\mathbf{v}(t) \doteq d\mathbf{r}(t)\,/dt$ and its kinetic energy $2K(t) \doteq m\,(\mathbf{v}(t))^2$. □

Discussion. In the last example, space (physical object) has been modeled by the three dimensional Euclidean space (math object) while time (physical object) by a real number (math object).

The particle mass (physical property of a physical object) has been modeled by a positive real number while particle motion (physical property of a physical object during a physical phenomenon) by a vector function.

Finally, the particle velocity (physical property of a physical object)has been modeled by a vector function (math object) and the particle kinetic energy (physical property of a physical object during a physical phenomenon) by a real number (math object).

2. METHODOLOGY

The first step in performing a dimensional analysis of a physical phenomenon *must* be the specification of the set of base quantities to be used. Thus, herein, the chosen set is the set provided by the International System of Quantities [1], see **??**.

The Buckingham Pi theorem was taken from [**?** , p. 42].

[1] The International System of Quantities, ISQ, forms the base for the International System of Units, *aka* SI.

Theorem 2.1 (Buckingham Pi theorem)**.** *A system described by* n *variables, built from* r *independent dimensions, is also described by* $n - r$ *independent dimensionless groups.*

3. Examples

Remark 3.1 (Nomenclature and Dimensions for the Electromagnetic Field)**.** *According to [**?** , p. 17], the electric field strength* $\mathbf{E}$ *is defined by* $\mathbf{E} = \mathbf{F}/Q$ *and the magnetic flux density* $\mathbf{B}$*, aka magnetic induction, implicitly by* $\mathbf{F} = Q\mathbf{E}$*. Lorenz force is therefore given by* $\mathbf{F} = Q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$*.*

The dimensions of these physical quantities are $\dim \mathbf{E} = \mathsf{MLI}^{-1}\mathsf{T}^{-3}$*,* $\dim \mathbf{B} = \mathsf{MI}^{-1}\mathsf{T}^{-2}$ *and* $\dim Q = \mathsf{IT}$*.*

Example 3.1 (Energy change for charged particle)**.** *Show that, when a charged particle interacts with electric and magnetic fields, its energy changes at a rate* $dK/dt = \mathbf{v} \cdot \mathbf{E}$*. [*sic*]*

Solution. [Dimensional Analysis Solution] Consider a particle with constant electric charge Q and with constant mass m moving through a constant electromagnetic field with electric field strength $\mathbf{E}$ and with magnetic flux density $\mathbf{B}$. Assume Newtonian physics [2], but dismiss gravitation [3].

We seek the form of the change rate of the particle's kinetic energy dK/dt. So, first, write down the definition of kinetic energy:

$$K = 1/2\, m\mathbf{v}^2\,.$$

On the other hand, motion, and therefore kinetic energy, arises due to the sole interaction of the (charged) particle and a (constant) electromagnetic field. So, the equation of motion for this case reads $d\mathbf{p}/dt = Q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Because velocity appears in the kinetic energy equation, rewrite then the equation of motion in velocity terms using the definition of momentum:

$$m\, d\mathbf{v}/dt = Q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})\,.$$

With the equation for kinetic energy and the equation of motion, perform a dimensional analysis of the phenomenon, following Sonin's plan, [**?**]:

(1) Determine the *complete set of physical quantities* [4] governing the physical process. From the equation of the kinetic energy and the equation of motion, find the set of quantities describing the phenomenon $\{K, m, \mathbf{v}, t, Q, \mathbf{E}, \mathbf{B}\}$. Consider hereafter only the magnitudes of vectorial quantities and drop the decoration for such values; *e.g.*, for velocity $\mathbf{v}$, consider v instead of $|\mathbf{v}|$.

(2) Determine the physical dimensions of the quantities in the aforefound set. List the quantities with their respective dimensions, [**?**] and [**?** , 105]: $\dim K = \mathsf{MLT}^{-1}$; $\dim m = \mathsf{M}$; $\dim v = \mathsf{LT}^{-1}$; $\dim t = \mathsf{T}$; $\dim Q = \mathsf{IT}$; $\dim E = \mathsf{MLI}^{-1}\mathsf{T}^{-3}$; and $\dim B = \mathsf{MI}^{-1}\mathsf{T}^{-2}$.

(3) From the complete set of quantities, find a subset of *dimensionally independent* [5] quantities using dimensional considerations. Consider the kinetic energy equation and the equation of

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[2] That is, neglect relativistic effects by assuming that, at any time t, the particle moves at a speed $|\mathbf{v}(t)|$ much smaller than the speed of light c; *i.e.*, $|\mathbf{v}(t)| \ll c$ or, equivalently, $|\mathbf{v}(t)|/c \ll 1$.

[3] Electromagnetic interaction is much stronger than gravitation: if gravitation is taken as 1, then electromagnetic interaction is as 10^{36}.

[4] Consider a nonempty set of physical quantities assumed to describe a physical phenomenon. The set is said to be *complete* if the quantities in the set, and no other quantities, are used to model the phenomenon.

[5] Consider a complete set of at least two physical quantities. *Dimensional independence* means that the physical dimensions of any of the quantities alone cannot be expressed as a function of the dimensions of the others. Call such a quantity *dimensional independent* and call the set of dimensional independent quantities a *dimensional independent set.* For instance, the physical quantities mass, time and length form a dimensional independent set. Consider, for example, time. The dimension of time alone cannot be expressed as a function of mass and length. The same is true for the dimension of mass and of length. Thus, mass, time and length are all dimensional independent quantities. On the other hand, a *dimensionally dependent quantity* is a quantity that is not dimensionally independent. Call a set of dimensionally dependent quantities *dimensionally dependent set.* For instance, consider the set $\{K, m, v\}$. Mass, m, can be expressed as a function of v and K. Additionally, consider the set $\{\mathbf{E}, v, B,\}$. Magnetic flux density, $\mathbf{B}$, can be expressed as a function of E and v. Thus m and B are dimensionally dependent quantities in their respective sets.

motion and the dimensions of the quantities therein appearing. Notice that K, m and v form a dimensionally dependent set. Discard m. Additionally, E, v and B also form a dimensionally dependent set. Discard B. The complete set of dimensionally independent quantities describing the motion of the particle interacting with the electromagnetic field is finally $\{K, v, t, Q, E\}$.

(4) From the complete set of independent quantities, determine the dimensionless similarity parameters [6]. From the set $\{K, v, t, Q, E\}$, the dependent process quantity is K and the independent process quantities are then the elements of $\{v, t, Q, E\}$. According to Vaschy-Buckingham pi theorem (*aka* Π-theorem), [**?**], the number of dimensionless similarity parameters is given by

$$r = m - n = 5 - 4\,,$$

where r represents the number of dimensionless similarity parameters, m the number of physical quantities describing the process and n the number of independent quantities.

Then, find the dimensionless parameter: $\Pi_1 = \Pi = K/tvQE$.

(5) Finally, using the principle of dimensional homogeneity [7] find the form of the model describing the physical process. For the present example, write $f(\Pi) = f(K/tvQE) = 0$. Factor out K/t from the last equation and rearrange the position of the quantities to find

$$K/t = g(QvE)\ .$$

where the formula for the function g cannot be determined any further by dimensional analysis. However, an equation for g can be determined by experimentation or theoretical considerations.

The last equation provides the *form* of the model of the motion of the particle interacting with the electromagnetic field. □

Remark 3.2. *A function for dK/dt was to be found, but instead a function for K/t was given. The issue is that dimensional analysis cannot differentiate dK/dt from K/t, for their dimensions are the same;* viz., $\dim K/t = \dim dK/dt = \mathsf{ML^2T^{-3}}$. *However, the function g can be thought of as the integral of K/t. That is, we can write*

$$dK/dt = h(QvE)\ ,$$

which yields the desired result.

Remark 3.3. *Dimensional analysis reduced the number of quantities to be considered for the model from seven to one;* i.e., *from $\{K, m, v, t, Q, E, B\}$ to $f(K/tvQE) = 0$.*

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[6] Consider a set of n physical variables a_i not all of them zero. A *dimensionless similarity parameter*, *aka* *dimensionless parameter*, Π is a product of functions of the a_is such that the dimension of Π is unity; *i.e.*, $\dim \Pi = 1$.

[7] Consider a set of n dimensionless similarity parameters Π_i, with i running from 1 to n. The *principle of dimensional homogeneity* states that there exists a function f relating the parameters that can be expressed as an equation of the form $f(\Pi_1, \ldots, \Pi_n) = 0$.

4. Non-dimensionalization

Basic tenet: write physical laws and equations in a "universe-friendly" manner; *i.e.*, dimensionless form. (The universe could not care less on the human-based system of units. Moreover, even for us, system of units are non-unique, merely conventional!)

Example 4.1. *A projectile of mass m moves at any time t through air describing a trajectory $x(t)$. Find the equation of motion if it is known that the projectile was launched at the point $x(0) = 0$ with an initial velocity, $v(t) = \dot{x}(t)$, $v(0) = v_0$. Discard drag.*

Solution. The equation of motion is $m\ddot{x}(t) = -mg$ with the boundary conditions $x(0) = 0$ and $\dot{x}(0) = v_0$.

(1) Rewrite the equation of motion: $dx/dt = -g$ subject to $x = 0$ and $dx/dt = v_0$. Note that motion that does depend on mass.

(2) List the dimensions of the physical quantities: $\dim x = L$, $\dim t = T$, $\dim g = M/T^2$ and $\dim v_0 = M/T$.

(3) Note that the dependent physical quantity is x, the independent one t and the (physical) parameters g and v_0.

(4) Non-dimentionalize the dependent and independent quantities as functions of the parameters. Call the quantities found thereby the *characteristic quantities of the physical phenomenon*, in this case, the char. quant. of projectile motion.

   (a) characteristic length: $x_\star$. Note that $\dim v_0^2/g = L$. So define $x/x_\star \doteq v_0^2/g$ or $x_\star = xg/v_0^2$ and $x = x_\star v_0^2/g$.

   (b) characteristic time: $t_\star$. Note that $\dim v_0/g = T$. So define $t/t_\star \doteq v_0/g$ or $t_\star = tg/v_0$.

(5) Find the characteristic derivative operators using the chain rule:

$$\frac{d}{dt} = \left(\frac{dt_\star}{dt}\right)\frac{d}{dt_\star} = \frac{d}{dt}\frac{tg}{v_0}\frac{d}{dt_\star} = \frac{g}{v_0}\frac{d}{dt_\star}\,,$$

and

$$\frac{d^2}{dt^2} = \left(\frac{d}{dt}\right)^2 = \frac{g^2}{v_0^2}\frac{d^2}{dt_\star^2}\,.$$

(6) Replace the characteristic quantities in the equation of motion:

$$\frac{d^2x}{dt^2} = -g \implies \frac{g^2}{v_0^2}\frac{d^2}{dt_\star^2}\left(\frac{x_\star v_0^2}{g}\right) = -g\,,$$
$$\implies \frac{d^2x_\star}{dt_\star^2} = -1\,.$$

Note that the last equation is dimensionless and parameter-free.

(7) Replace the characteristic quantities in boundary conditions of the equation of motion:

$$\frac{d}{dt}x = v_0 \implies \frac{g}{v_0}\frac{d}{dt_\star}\left(\frac{x_\star v_0^2}{g}\right) = v_0\,,$$
$$\implies \frac{dx_\star}{dt_\star} = 1.$$

(8) Rewrite the equation of motion and its boundary conditions:

$$\ddot{x}_\star(t_\star) = -1 \qquad \text{subject to} \qquad \begin{cases} x_\star(0) = 0 \\ \dot{x}_\star(t_\star) = 1\,, \end{cases}$$

where $x_\star = xg/v_0^2$ and $t_\star = tg/v_0$.

The last set of equations yields the required result. □

Example 4.2. *Non-dimensionalise the Navier-Stokes equation:*

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \eta \nabla^2 \mathbf{u}\,,$$

where the physical quantities and their dimensions are mass density $\dim \rho = ML^{-3}$*, partial derivative with respect to time* $\dim \partial_t = T^{-1}$*, velocity* $\dim \mathbf{u} = LT^{-1}$*, geometric derivative (aka "del" or "nabla" operator)* $\dim \nabla = L^{-1}$*, viscosity* $\dim \eta = ML^{-1}T^{-1}$ *and pressure* $\dim p = ML^{-1}T^{-2}$*.*

Solution. For every dimension, choose a representative value. In fluid mechanics, the set is regularly a *lengthscale* $\dim L = L$, *velocity* $\dim U = LT^{-1}$ and *viscosity* $\dim \eta = ML^{-1}T^{-1}$.

Introduce *new dimensionless quantities* (*characteristic physical quantities*) based on the representative values

$$\mathbf{u} = U\mathbf{u}_\star \qquad p = U\eta p_\star / L\,.$$

Scale lengths and times

$$\partial_{t_\star} = L\partial_t / U \qquad \nabla_\star = L\nabla\,,$$

which result in the new equation (multiplying by L^2U^{-1}/η):

$$\rho UL/\eta\,(\partial_{t_\star}\mathbf{u} + \mathbf{u} \cdot \nabla_\star p_\star + \nabla^2{}_\star \mathbf{u})\,.$$

Introduce the Reynolds number $Re = \rho UL/\eta$ to have

$$Re\,(\partial_{t_\star}\mathbf{u} + \mathbf{u} \cdot \nabla_\star p_\star + \nabla^2{}_\star \mathbf{u})\,,$$

which yields the desired result. □

5. Geometric Algebra

5.1. **Test.** Vector: $\mathbf{u} = \mathsf{e}_k u^k$, mapping: $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, set: $\mathcal{A}$, dim.less quantity: Π.

The metric tensor components are given by

$$\mathsf{g} = g_{kl} \doteq \mathsf{e}_k \cdot \mathsf{e}_l \,. \tag{5.1}$$

Consider a n-dimensional Euclidean space, $\mathcal{E}^n$, with a frame $\mathcal{F} = \{\mathsf{e}_k\}$ and consider two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. Then their geometric product $\mathbf{uv}$ is

$$\mathbf{uv} = \mathsf{e}_k u^k \mathsf{e}_l v^l = u^k \mathsf{e}_k \mathsf{e}_l v^l = u^k \delta_{kl} v^l = u^k v_k = u_l v^l \,. \tag{5.2}$$

symmetrical inner product: $\mathbf{u} \cdot \mathbf{v} \doteq \frac{1}{2}(\mathbf{uv} + \mathbf{vu})$.

anti-symmetrical outer product: $\mathbf{u} \wedge \mathbf{v} \doteq \frac{1}{2}(\mathbf{uv} - \mathbf{vu})$.

then, geometric product decomposition:

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \wedge \mathbf{u} \,. \tag{5.3}$$

3+1D treatment: Define

$$\Box \doteq \partial_t + \nabla \tag{5.4}$$

and define

$$R \doteq t + \mathbf{r} \,, \tag{5.5}$$

then

$$V = \partial_t R = 1 + \mathbf{v} \,. \tag{5.6}$$

Useful relationship in 3D: given $\mathbf{u}, \mathbf{v} \in \mathcal{E}^3$, then

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + i \mathbf{u} \times \mathbf{v} \,, \tag{5.7}$$

where the identity $\mathbf{u} \wedge \mathbf{v} = i \mathbf{u} \times \mathbf{v}$ was used. This is due to the equivalence between bivectors and pseudovectors in $\mathcal{E}^3$.

Underline vector: $\underline{\mathbf{v}}$.

Space vector (wavy-underlined vector): $\utilde{\mathbf{v}}$, $\utilde{\mathbf{v}}/|v|$.

Perpendicular and parallel vectors: $\mathbf{b} = \mathbf{b}^{\perp} + \mathbf{b}^{\parallel}$.

Norm of a vector: $|\mathbf{v}|$.

Iverson brackets: $\mathsf{e}_\mu \mathsf{e}_\nu + \mathsf{e}_\nu \mathsf{e}_\mu = 2\,[\mu = \nu]$.

5.2. **Multivector multiplication in 3D.** Multivector multiplication in 3D (works only in 3D): the trick is to express bivectors by pseudovectors: in 3D we need to use only vector multiplication. It is only necessary to *write all bivectors as pseudovectors and treat i as though it were a scalar, employing the rule $i^2 = -1$ whenever necessary.*

Example 5.1. *Consider [8] $A, B \in \mathcal{E}^3$ defined by $A = a\mathsf{e}_1 + b\mathsf{e}_1\mathsf{e}_2$ and $B = c + \mathsf{e}_2\mathsf{e}_3$. Find the product AB.*

[8] Remember that $\mathcal{E}^3$ means the linear space $\mathbb{R}^3$ endowed with the geometric product.

Solution. For $\mathcal{E}^3$, find the unit pseudoscalar: $i = \mathsf{e}_1\mathsf{e}_2\mathsf{e}_3$.

For A and B, replace the basis bivectors by their correspondent pseudovectors: $\mathsf{e}_1\mathsf{e}_2 \to i\mathsf{e}_3$ and $\mathsf{e}_2\mathsf{e}_3 \to i\mathsf{e}_1$; *i.e.*,

$$\begin{aligned} AB &= (a\mathsf{e}_1 + b\mathsf{e}_1\mathsf{e}_2)(c + \mathsf{e}_2\mathsf{e}_3) \\ &= (a\mathsf{e}_1 + bi\mathsf{e}_3)(c + i\mathsf{e}_1)\,. \end{aligned}$$

Multiply the components of A and B term wise:

$$\begin{aligned} AB &= (a\mathsf{e}_1 + bi\mathsf{e}_3)(c + i\mathsf{e}_1) \\ &= ac\mathsf{e}_1 + bci\mathsf{e}_3 + bi^2\mathsf{e}_3\mathsf{e}_1 + ai\mathsf{e}_1^2\,. \end{aligned}$$

Use the identities: $i^2 = -1$ and $\mathsf{e}_1^2 = 1$ to find

$$AB = ac\mathsf{e}_1 + bci\mathsf{e}_3 - b\mathsf{e}_3\mathsf{e}_1 + ai\,.$$

Replace $\mathsf{e}_3\mathsf{e}_1$ by the pseudovector $i\mathsf{e}_2$ and regroup the terms to find

$$AB = ac\mathsf{e}_1 + i(bc\mathsf{e}_3 - b\mathsf{e}_2) + ia\,,$$

which yields the desired result. □

Note 5.1. *The equation $AB = ac\mathsf{e}_1 + i(bc\mathsf{e}_3 - b\mathsf{e}_2) + ia$ represents a multivector with three parts: a vector $ac\mathsf{e}_1$, a bivector $i(bc\mathsf{e}_3 - b\mathsf{e}_2)$ expressed as its dual and the trivector ia.*

5.3. **Dimensions and Units.** Dimensions: $\dim F = \mathsf{LMT}^{-2}$.

Units: unit $F = \mathrm{kg\,m\,s^{-2}}$.

5.4. **Electromagnetism.** Electro magnetic field:

$$F \doteq \mathbf{E} + ic\mathbf{B}\,,$$

where F is the electromagnetic field, $\mathbf{E}$ the electric field, c the speed of light in vacuum and $\mathbf{B}$ the magnetic flux density. Note that F is a multivector, $\mathbf{E}$ a vector and i the unit pseudoscalar of 3D. However, $ic\mathbf{B}$ is a bivector, or rather a paravector (or pseudovector), the *dual* of a bivector.

Electromagnetic source density:

$$J \doteq \frac{1}{\epsilon_0}\rho - Z_0\mathbf{J}\,,$$

where $Z_0 \doteq (\mu_0/\epsilon_0)^{\frac{1}{2}}$ is the characteristic impedance of free space, ρ the charge density and $\mathbf{J}$ the current density.

The link between current density $\mathbf{J}$ and the motion of the charge density: $\mathbf{J} = \rho\mathbf{v}$. In the case of only a single type of charge: $\mathbf{J} = \epsilon_0^{-1}\rho(1 - \mathbf{v}/c)$. $\mathbf{J} = \frac{\rho}{\epsilon_0}(1 - \mathbf{v}/c)$.

5.5. **Geometric Calculus.** Consider $\mathcal{E}^3$, the frame $\{\mathsf{e}_k;\, k : 1 \ldots 3\}$, a scalar function f and a vector function $\mathbf{f}$. Then,

- ◇ the partial derivative is noted as $\partial_k \doteq \partial/\partial k$;
- ◇ the geometric derivative is defined as $\nabla \doteq \mathsf{e}_k\partial_k$;
- ◇ the geometric derivative of a scalar function agrees with the conventional gradient: $\nabla f = \mathsf{e}_k\partial_k f$;
- ◇ the geometric derivative of a vector function results in a scalar and a bivector: $\nabla\mathbf{f} = \nabla \cdot \mathbf{f} + \mathbf{f} \wedge \nabla$;

- $\diamond$ in particular the standard divergence and curl are given respectively by $\nabla \cdot \mathbf{f}$ and $-i\nabla \wedge \mathbf{f}$; and
- $\diamond$ in general, application of a vector operator to a multivector function follows the same rules as for left multiplication (premultiplication) by an ordinary vector.

By the simple process of direct multiplication, ∇^2 is readily shown to be equivalent to the scalar operator $\partial_x^2 + \partial_y^2 + \partial_z^2$ or, in index notation, $\partial_1^2 + \partial_2^2 + \partial_3^2$ or, using Einstein convention, $\partial_k \partial_k$.

As an example, in electromagnetic theory, let us evaluate $\nabla\mathbf{E}$ in free space. This resolves into its scalar and bivector parts as follows

$$\begin{aligned}
\nabla\mathbf{E} &= \nabla \cdot \mathbf{E} + i\nabla \times \mathbf{E}\,, \\
&= \rho\epsilon_0 - i\partial_t \mathbf{B}\,, \\
&= \rho\epsilon_0 - \partial_t B\,.
\end{aligned}$$

where B is a bivector and $i\mathbf{B}$ its dual (para-) vector.

5.6. **Reversion.** Reversion is the expression of the geometric product in reverse order. Let $U = \mathbf{abc}$, then its reversion is $U^\dagger = \mathbf{cba}$.

Another example: let $U = a + \mathbf{b} + \mathbf{cd} + \mathbf{ef} + \mathbf{ghk}$, then

$$\begin{aligned}
U^\dagger &= (a + \mathbf{b} + \mathbf{cd} + \mathbf{ef} + \mathbf{ghk})^\dagger\,, \\
&= a^\dagger + \mathbf{b}^\dagger + (\mathbf{cd})^\dagger + (\mathbf{ef})^\dagger + (\mathbf{ghk})^\dagger\,, \\
&= a + \mathbf{b} + \mathbf{dc} + \mathbf{fe} + \mathbf{khg}\,.
\end{aligned}$$

Reversion properties:

- $\diamond$ let U and V be two multivectors. Then $(U+V)^\dagger = U^\dagger + V^\dagger$ and $(UV)^\dagger = V^\dagger U^\dagger$.
- $\diamond$ scalar and vectors are unaffected by reversal: $(a + \mathbf{u})^\dagger = a + \mathbf{u}$.
- $\diamond$ for any product of vectors $U = \mathbf{u}_1\mathbf{u}_2 \cdots \mathbf{u}_n$, then we have $UU^\dagger = \mathbf{u}_1^2\mathbf{u}_2^2 \cdots \mathbf{u}_n^2$; *i.e.*, a scalar.

5.7. **Theorems.**

Theorem 5.1. *Let* $\mathbf{u}, \mathbf{v} \in \mathcal{E}^n$. *Then* $\mathbf{u} \cdot (i\mathbf{v}) = i(\mathbf{u} \wedge \mathbf{v})$.

Theorem 5.2. *Let* $\mathbf{u} \in \mathcal{E}^n$. *Then*

$$\nabla \cdot (i\mathbf{u}) = i(\nabla \wedge \mathbf{u})\,. \tag{5.8}$$

Proof. In GA, ∇ is treated as a vector. So, working in the LHS of the equality,

$$\begin{aligned}
\nabla \cdot (i\mathbf{u}) &= \mathsf{e}_j \partial_j \cdot (i\mathsf{e}_k u^k)\,, && \text{[expanding terms]} \\
&= \partial_j \mathsf{e}_j \cdot (i\mathsf{e}_k) u^k\,, && \text{[rearranging scalar terms]} \\
&= \partial_j i(\mathsf{e}_j \wedge \mathsf{e}_k) u^k\,, && \text{[by last theorem]} \\
&= i(\mathsf{e}_j \partial_j \wedge \mathsf{e}_k u^k)\,, && \text{[rearranging scalar terms]} \\
&= i(\nabla \wedge \mathbf{u})\,. && \text{[by definitions of } \nabla \text{ and } \mathbf{u}]
\end{aligned}$$

The last equation provides the result. $\square$

5.8. **Lorentz Force.** Lorentz force is the force $\mathbf{f}$ acting on a particle of charge q while it moves with velocity $\mathbf{v}$ through an electric field $\mathbf{E}$ and a magnetic field (magnetic flux density) B. Note that q is modeled by a scalar variable whereas $\mathbf{f}$, $\mathbf{v}$ and $\mathbf{E}$ by vector variables and B by an axial vector. (Remember: an axial vector is the result of a cross product.) Conventionally, Lorentz force is written as

$$\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times B)\,.$$

In GA, on the other hand, B is modeled as a bivector that can be related to its dual, in 3D, by $B = i\mathbf{B}$. So, remembering that the cross product is defined $\mathbf{u} \times \mathbf{v} \doteq -i\mathbf{u} \wedge \mathbf{v}$ or, alternatively, by $\mathbf{u} \wedge \mathbf{v} = i\mathbf{u} \times \mathbf{v}$, we have in the RHS of Lorentz force

$$\begin{aligned}\mathbf{v} \times B &= -i\mathbf{v} \wedge B\,,\\ &= -\frac{1}{2}i(\mathbf{v}B - B\mathbf{v})\,,\\ &= -\frac{1}{2}(i\mathbf{v}B - iB\mathbf{v})\,,\\ &= B \cdot \mathbf{v}\,.\end{aligned}$$

Finally, then, we can write Lorentz force using GA terms

$$\mathbf{f} = q(\mathbf{E} + B \cdot \mathbf{v})\,.$$

5.9. **Maxwell's Equations.** Recalling that, in 3D, the vector derivative ∇ acting on a vector such as $\mathbf{E}$ may be written in terms of divergence and curl as $\nabla\mathbf{E} = \nabla \cdot \mathbf{E} + i\nabla \times \mathbf{E}$. In the case of a bivector like B, however, we have $\nabla B = \nabla(i\mathbf{B}) = i\nabla\mathbf{B}$ where $\mathbf{B}$ is the vector dual to B. Here we have treated ∇ just like any other vector and used the fact that i commutes with all 3D vectors. We may now apply this to Maxwell's equations as follows:

$$\begin{aligned}\nabla\mathbf{E} = \nabla \cdot \mathbf{E} + i\nabla \times \mathbf{E} &= \frac{\rho}{\epsilon_0} - i\partial_t\mathbf{B}\,,\\ \nabla(ci\mathbf{B}) = ic\nabla \cdot \mathbf{B} + i^2c\nabla \times \mathbf{B} &= -c\mu\mathbf{J} - \frac{1}{c}\partial_t\mathbf{E}\,,\end{aligned}$$

where $c\mu_0 = Z_0$ is the characteristic impedance of free space.

Taking first the sum and then the difference of these two equations, equating the bivector B to $i\mathbf{B}$, we may write

$$\left(\nabla + \frac{1}{c}\partial_t\right)(\mathbf{E} + cB) = \frac{\rho}{\epsilon_0} - Z_0\mathbf{J}\,,$$

or, equivalently, in terms of the electromagnetic field $F = \mathbf{E} + cB$ and the electromagnetic source density $J = \dfrac{\rho}{\epsilon_0} - Z_0\mathbf{J}$, we have, more succinctly,

$$\left(\nabla + \frac{1}{c}\partial_t\right)F = J\,.$$

Sometimes an operator, $\triangle = \nabla + \dfrac{1}{c}\partial_t$, is used to further simplify the last equation. With such an operator, we have

$$\triangle F = J\,.$$

The last equation can thus be named "Maxwell equation". This contrast the four equations provided by conventional – Gibbs – vector algebra.

5.10. **Spacetime Treatment.** From now on, we use spacetime conventions:

- ◇ time and space are treated on equal footing: vectors; *i.e.*, time and distance are measure with the same meterstick;
- ◇ speed of light is taken dimensionsless equal to unity; *i.e.*, $c = 1$ and $\dim c = 1$; and
- ◇ impedance of free space is taken dimensionsless equal to unity; *i.e.*, $Z_0 = 1$ and $\dim Z_0 = 1$, with this, $\mu_0 = \epsilon_0 = 1$.

With these conventions, Maxwell equation becomes

$$(\nabla + \partial_t)(\mathbf{E} + B) = \rho - \mathbf{J}\,,$$

that is, all the constants have been normalized.

5.11. **Scalar Wave Equation.** If we multiply both sides of Maxwell equation, $(\nabla + \partial_t)F = J$, by $(\nabla - \partial_t)$, we obtain the scalar wave equation in the GA formalism

$$(\nabla^2 - \partial_t^2)F = J\,.$$

Introducing d'Alembert operator $\Box$, *aka* d'Alembertian or the wave operator, defined by

$$\Box \doteq (\nabla + \partial_t)(\nabla - \partial_t) = \nabla^2 - \partial_t^2\,,$$

we have the scalar wave equation written more succinctly as

$$\Box F = J\,.$$

[In SI units, we have $(\nabla^2 - \frac{1}{c^2}\partial_t^2)F = J$. Thus d'Alembert operator is defined by $\Box \doteq \nabla^2 - \frac{1}{c^2}\partial_t^2$. However, the scalar wave equation remains unchanged, $\Box F = J$.]

5.12. **Multivector Potential.** We are accustomed to two types of electromagnetic potential: one scalar and the other vector. It now seems a certainty that in a geometric algebra over Newtonian space, these may be combined into a single multivector potential A. Recalling that the scalar and vector potentials Φ and $\mathbf{A}$ give rise to scalar wave equations with ρ and $\mathbf{J}$ respectively as sources, we should now find a single wave equation relating A to J. In fact, we should have

$$\Box A = J \implies (\nabla + \partial_t)(\nabla - \partial_t)A = (\nabla + \partial_t)F\,.$$

In asserting this, we are anticipating that there is some suitable linear combination of Φ and $\mathbf{A}$ that will form the requisite multivector potential A and that F must be derivable from it by the process of differentiation. It is clear that $(\nabla + \partial_t)$ factors out of the last equation leaving us with

$$F = (\nabla - \partial_t)A + F'\,,$$

where F' is any solution of the homogeneous (source-free) Maxwell's equation, $(\nabla - \partial_t)F' = 0$. We can therefore view F' as being an externally applied vector + bivector field satisfying some given boundary condition. Taking initially the simple condition $F' = 0$ and recalling that $\Box = \nabla^2 - \partial_t^2$ is a scalar operator, A must have the the same form as J, that is to say, a scalar plus a vector, a particular form of multivector that is known as a *paravector*. By writing A in the form $\Phi + c\mathbf{A}$ we therefore find

$$F = (\nabla - \partial_t)A = (\nabla - \partial_t)(-\Phi + c\mathbf{A})$$

which implies

$$\mathbf{E} + cB = \partial_t\Phi + c\nabla \cdot \mathbf{A} + (-\nabla\Phi - c\partial_t\mathbf{A}) + c\nabla \wedge \mathbf{A}\,.$$

Note that on the RHS of the last equation we have a vector plus a bivector, while on the LHS we have an addition of a scalar, a vector and a bivector.

5.13. **Multivectors.** A multivector, $M \in \mathcal{E}^3$, is a math object of the form

$$M \doteq s + \mathbf{v} + B + T\,,$$

where s is a scalar, $\mathbf{v}$ a vector, B a bivector and T a trivector.

Alternatively, a multivector can be written as

$$M = s + \mathbf{v} + i\mathbf{b} + it = s + \mathbf{v} + i(t + \mathbf{b})\,,$$

where s, t are scalars, $\mathbf{v}, \mathbf{b}$ vectors and i the unit pseudoscalar. The pseudovector $i\mathbf{b}$ is the dual of the bivector B and the pseudoscalar it the dual of the trivector T.

5.13.1. *Multivector Addition.* Let $M_1, M_2 \in \mathcal{E}^3$ be two multivectors defined as $M_1 = s_1 + \mathbf{v}_1 + i\mathbf{b}_1 + it_1$ and $M_2 = s_2 + \mathbf{v}_2 + i\mathbf{b}_2 + it_2$. Then, the addition of $M_1 + M_2$ is defined as

$$\begin{aligned} M_1 + M_2 &\doteq (s_1 + s_2) + (\mathbf{v}_1 + \mathbf{v}_2) + i(\mathbf{b}_1 + \mathbf{b}_2) + i(t_1 + t_2) \\ &= (s_1 + s_2) + (\mathbf{v}_1 + \mathbf{v}_2) + i(t_1 + t_2 + \mathbf{b}_1 + \mathbf{b}_2)\,. \end{aligned}$$

5.13.2. *Multivector Multiplication.* Let $M_1, M_2 \in \mathcal{E}^3$ be two multivectors defined as $M_1 = s_1 + \mathbf{v}_1 + i\mathbf{b}_1 + it_1$ and $M_2 = s_2 + \mathbf{v}_2 + i\mathbf{b}_2 + it_2$. Then, the multiplication of M_1M_2 is given by

$$M_1M_2 = (s_1 + \mathbf{v}_1 + i\mathbf{b}_1 + it_1)(s_2 + \mathbf{v}_2 + i\mathbf{b}_2 + it_2)\,.$$

Term-wise multiply the two multivectors

$$\begin{aligned} M_1M_2 = {} & s_1s_2 + s_1\mathbf{v}_2 + s_1i\mathbf{b}_2 + s_1it_2 + \mathbf{v}_1s_2 + \mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_1i\mathbf{b}_2 + \mathbf{v}_1it_2 \\ & + i\mathbf{b}_1s_2 + i\mathbf{b}_1\mathbf{v}_2 + i\mathbf{b}_1i\mathbf{b}_2 + i\mathbf{b}_1it_2 + it_1s_2 + it_1\mathbf{v}_2 + it_1i\mathbf{b}_2 + it_1it_2\,. \end{aligned}$$

Use the fact that i commutes with every object in $\mathcal{E}^3$ and $i^2 = -1$ to find $\mathbf{v}_1i\mathbf{b}_2 = i\mathbf{v}_1\mathbf{b}_2$, ..., $i\mathbf{b}_1i\mathbf{b}_2 = i^2\mathbf{b}_1\mathbf{b}_2 = -\mathbf{b}_1\mathbf{b}_2$, and so on. With these replacements, we have

$$\begin{aligned} M_1M_2 = {} & s_1s_2 + s_1\mathbf{v}_2 + s_1i\mathbf{b}_2 + s_1t_2i + s_2\mathbf{v}_1 + \mathbf{v}_1\mathbf{v}_2 + i\mathbf{v}_1\mathbf{b}_2 + t_2i\mathbf{v}_1 \\ & + s_2i\mathbf{b}_1 + i\mathbf{b}_1\mathbf{v}_2 - \mathbf{b}_1\mathbf{b}_2 - t_2\mathbf{b}_1 + s_2t_1i + t_1i\mathbf{v}_2 - t_1\mathbf{b}_2 - t_1t_2\,. \end{aligned}$$

Finally, geometric products of vectors can be split into inner and outer products; *e.g.*, $\mathbf{v}_1\mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \wedge \mathbf{v}_2$.

5.13.3. *Multivector Inverse.* Let $M \in \mathcal{E}^3$ be a multivector given by $M = s + \mathbf{v} + i\mathbf{b} + it$. Then, the inverse of M, M^{-1}, is found by

$$M^{-1} = (s - \mathbf{v} - i\mathbf{b} + it)(f + ig)/(f^2 + g^2)\,,$$

where we have the scalars $f = s^2 - v^2 + b^2 - t^2$ and $g = 2(\mathbf{v} \cdot \mathbf{b} - st)$.

The special case of the inverse of a vector is given by

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{v^2}\,,$$

which can be checked through $\mathbf{v}\mathbf{v}^{-1} = \mathbf{v}\mathbf{v}/v^2 = 1$ as required. Hence, we can see that the inverse of any vector is a vector in the same direction with the reciprocal of the length of the original vector.

5.14. **Linear Spaces.** Notation: $\bigwedge^2 \mathcal{E}^3$: linear space of 3-D bivectors.

Notation: $\bigwedge^3 \mathcal{E}^3$: linear space of 3-D trivectors.

Notation: $\mathcal{G}_3 = \mathcal{E}^1 \oplus \mathcal{E}^3 \oplus \bigwedge^2 \mathcal{E}^3 \oplus \bigwedge^3 \mathcal{E}^3$: 3-D geometric algebra: multivectors are linear combinations of scalars $\in \mathcal{E}^1$, vectors $\in \mathcal{E}^3$, bivectors $\in \bigwedge^2 \mathcal{E}^3$ and trivectors $\in \bigwedge^3 \mathcal{E}^3$.

This decomposition introduces a *multivector structure* into Clifford algebra $\mathcal{G}_3$. The multivector structure is unique; *i.e.*, an arbitrary element $u \in \mathcal{G}_3$ can be uniquely decomposed into a sum of k-vectors, the k-vector parts $\langle u \rangle_k$ of u,

$$u = \langle u \rangle_0 + \langle u \rangle_1 + \langle u \rangle_2 + \langle u \rangle_3 \qquad \text{where} \qquad \langle u \rangle_k \in \bigwedge^k \mathcal{E}^3 .$$

5.15. **Inner and Outer Products.** Different identities: let $\mathbf{a}, \mathbf{b} \in \mathcal{E}^3$ (two vectors), then

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \tfrac{1}{2}(\mathbf{ab} + \mathbf{ba}) \\ \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \phi \\ \mathbf{a} \wedge \mathbf{b} &= \tfrac{1}{2}(\mathbf{ab} - \mathbf{ba}) \\ |\mathbf{a} \wedge \mathbf{b}| &= |\mathbf{a}||\mathbf{b}| \sin \phi , \end{aligned}$$

where $0 \leq \phi \leq \tau$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ ($\tau \doteq 2\pi = 360°$).

Combine the identities to have

$$\mathbf{a}^2\mathbf{b}^2 = (\mathbf{a} \cdot \mathbf{b})^2 + |\mathbf{a} \wedge \mathbf{b}|^2 .$$

Left contraction: $\lrcorner$.

Geometric algebra: $\mathcal{G}$.

n-dim Geometric algebra: $\mathcal{G}_n$. n-dim geometric algebra over the field of $\mathbb{R}^n$ $\mathcal{G}_n \doteq \mathcal{G}(\mathcal{E}^n)$.

5.16. **Rotations.**

Theorem 5.3. *Consider a vector* $\mathbf{r} \in \mathcal{E}^2$ *and* i *to be the unit pseudoscalar in* $\mathcal{E}^2$*. Then, left multiplication of* $\mathbf{r}$ *by* i *produces a left turn (rotates* $\mathbf{r}$ *by* $\tau/4$*) and right multiplication of* $\mathbf{r}$ *by* i *produces a right turn (rotates* $\mathbf{r}$ *by* $-\tau/4$*).*

Proof. In $\mathcal{E}^2$, define the orthonormal frame $\{\mathsf{e}_{\mathsf{e}_1,\mathsf{e}_2}\}$. Then, the unit pseudoscalar is $i = \mathsf{e}_1\mathsf{e}_2 = \mathsf{e}_1 \wedge \mathsf{e}_2 = \mathsf{e}_{12}$.

Now, without loss of generality, define $\mathbf{r} = \mathsf{e}_1 + \mathsf{e}_2$. Then, left multiply $\mathbf{r}$ by i

$$\begin{aligned} i\mathbf{r} &= \mathsf{e}_{12}(\mathsf{e}_1 + \mathsf{e}_2) \\ &= \mathsf{e}_{12}\mathsf{e}_1 + \mathsf{e}_{12}\mathsf{e}_2 \\ &= -\mathsf{e}_2 + \mathsf{e}_1 ; \end{aligned}$$

that is, left multiplication produces a left (clockwise) turn.

On the other hand, left multiply $\mathbf{r}$ by i

$$\begin{aligned} \mathbf{r}i &= (\mathsf{e}_1 + \mathsf{e}_2)\mathsf{e}_{12} \\ &= \mathsf{e}_1\mathsf{e}_{12} + \mathsf{e}_2\mathsf{e}_{12} \\ &= \mathsf{e}_2 - \mathsf{e}_1 ; \end{aligned}$$

that is, right multiplication produces a right (counterclockwise) turn.

This ends the proof. □

5.17. **Things.** Consider $\mathcal{E}^n$ and a orthonormal frame $\{\mathsf{e}_k;\, k : 1 \ldots n\}$. Explore $\mathsf{e}_k\mathsf{e}_l + \mathsf{e}_l\mathsf{e}_k = 2\,[k = l]$.

When $k = l$, then

$$\begin{aligned} \mathsf{e}_k\mathsf{e}_k + \mathsf{e}_k\mathsf{e}_k &= 2\,[k = k] \\ 2\mathsf{e}_k\mathsf{e}_k &= 2 \\ \mathsf{e}_k\mathsf{e}_k &= 1 \\ \implies \mathsf{e}_k^2 &= 1\,, \end{aligned}$$

where $\mathsf{e}_k^2 \doteq \mathsf{e}_k\mathsf{e}_k$.

When $k \neq l$, then

$$\begin{aligned} \mathsf{e}_k\mathsf{e}_l + \mathsf{e}_l\mathsf{e}_k &= 0 \\ \mathsf{e}_k\mathsf{e}_l &= -\mathsf{e}_l\mathsf{e}_k\,. \end{aligned}$$