1 AN ALTERNATIVE FORMULATION OF MECHANICS: LAGRANGIAN MECHANICS

Nature is thrifty in all its actions.

— MAUPERTUIS, [1]

Since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.

— EULER, [2]

The laws of movement and of rest deduced from this principle being precisely the same as those observed in nature, we can admire the application of it to all phenomena. The movement of animals, the vegetative growth of plants ... are only its consequences; and the spectacle of the universe becomes so much the grander, so much more beautiful, the worthier of its Author, wen one knows that a small number of laws, most wisely established, suffice for all movements.

— MAUPERTUIS, [1]

1.1 ACTION

[Section taken from [3]]

Action, $\aleph$, is an attribute of a dynamical physical system. It is represented by a mathematical functional that takes the *system trajectory* as its argument and results in a real number. Generally, the action takes different values for different paths. Action has the dimensions of ET.

system trajectory, *aka* system path or system history

1.1.1 *Notation*

The symbol to represent physical action was chosen after a Borge's story: The Aleph [4]. In Borges' story, the Aleph is a point in space that contains all other points. Anyone who gazes into it can see everything in the universe from every angle simultaneously, without distortion, overlapping or confusion.

Aleph or Alef, ℵ, is the first letter of the Hebrew alphabet and the number 1 in Hebrew. Its esoteric meaning in Judaic Kabbalah, as denoted in the theological treatise *Sefer-ha-Bahir*, relates to the origin of the universe, the « primordial one that contains all numbers ». The aleph is also the first letter of the Arabic alphabet, as well as the Phoenician, Aramaic and Syriac alphabets. Aleph is also the first letter of the Persian alphabet. [...] In mathematics, aleph numbers denote the cardinality (or size) of infinite sets. This relates to the theme of infinity present in Borges's story. [...] The aleph also recalls the *monad* as conceptualized by Gottfried Wilhelm Leibniz, the 17th-century philosopher and mathematician. Just as Borges's aleph registers the traces of everything else in the universe, so Leibniz's monad is a mirror onto every other object of the world [4].

1.2 INTRODUCTION

Classical mechanics postulates that *the actual path followed by a physical system is that for which the action is minimized or, more generally, is* stationary; *i.e.*, the action satisfies a variational principle: the *principle of stationary action*. The action is defined by an integral; the classical equations of motion of a system can be *derived* by minimizing the value of that integral.

This simple principle provides deep insights into physics and is an important concept in modern theoretical physics. The equivalence of these two approaches is contained in *Hamilton's principle*: *the differential equations of motion for any physical system can be re-formulated as an equivalent integral equation*. It applies not only to the classical mechanics of a single particle, but also to classical fields such as the electromagnetic and gravitational fields. Hamilton's principle has also been extended to quantum mechanics and quantum field theory.

1.2.1 *Mathematical definition*

Expressed in mathematical language, using the calculus of variations, the *evolution of a physical system* corresponds to a *stationary point* – usually a minimum – of the action. Several different definitions of « the action » are in common use in physics. The action is usually an integral over time. But for action pertaining to fields, it may be integrated over spatial variables as well. In some cases, the action is integrated along the path followed by the physical system.

system evolution: how the system progresses from one state to another.

The action is typically represented as an integral over time, taken along the path of the system between the initial time t_1 and the final time t_2 of the development of the system,

abbreviated notation for integrals in force [5]

$$\aleph = \int_i e_{\mathrm{lag}} = \int_{t_1,t_2} e_{\mathrm{lag}}\,\mathrm{d}t\,,$$

where the integrand e_{lag} is called the *Lagrangian*. For the action integral to be well-defined the trajectory has to be bounded in time and space.

1.2.2 *Action in classical physics*

In classical physics, the term « action » has a number of meanings.

Action

Most commonly, the term is used for a functional $\aleph$ which takes a function of time and (for fields) space as input and returns a scalar. In classical mechanics, the input function is the evolution $q[t]$ of the system between two times t_1 and t_2, where q represent the generalized position. The action $\aleph[q[t]]$ is defined as the integral of the Lagrangian for an input evolution between the two times

$$\aleph[q[t]] = \int_i e_{\text{lag}}[q[t]\,, \dot{q}[t]\,, t] = \int_{t_1,t_2} e_{\text{lag}}[q[t]\,, \dot{q}[t]\,, t]\, \mathrm{d}t\,.$$

where the endpoints of the evolution are fixed and defined as $q_1 = q[t_1]$ and $q_2 = q[t_2]$. According to Hamilton's principle, the true evolution $q_{\text{true}}[t]$ is an evolution for which the action is stationary (a minimum, maximum, or a saddle point). This principle results in the equations of motion in Lagrangian mechanics.

Abbreviated action

Usually denoted as $\aleph_0$, this is also a functional. Here the input function is the *path* followed by the physical system without regard to its parameterization by time. For example, the path of a planetary orbit is an ellipse, and the path of a particle in a uniform gravitational field is a parabola; in both cases, the path does not depend on how fast the particle traverses the path. The abbreviated action is defined as the integral of the generalized momenta along a path in the generalized position

$$\aleph_0 = \int p \cdot q = \int p_k \mathrm{d}q^k\,.$$

According to Maupertuis' principle, the true path is a path for which the abbreviated action is stationary.

1.2.3 *Euler-Lagrange equations for the action integral*

As noted above, the requirement that the action integral be stationary under small perturbations of the evolution is equivalent to a set of differential equations (called *Euler-Lagrange equations*) that may be determined using the calculus of variations. We illustrate this derivation here using only one coordinate, x; the extension to multiple coordinates is straightforward.

Adopting Hamilton's principle, we assume that the Lagrangian e_{lag} (the integrand of the action integral) depends only on the coordinate $x[t]$ and its time derivative $\dot{x}[t]$, and may also depend explicitly on time. In that case, the action integral can be written

$$\aleph = \int_{t_1,t_2} e_{\text{lag}}[x, \dot{x}, t]\, \mathrm{d}t\,,$$

where the initial and final times (t_1 and t_2) and the final and initial positions are specified in advance as $x^1 = x[t_1]$ and $x^2 = x[t_2]$. Let $x_{\text{true}}[t]$ represent the true evolution that we seek, and let x_{per} be a slightly perturbed version of it, albeit with the same endpoints, $x_{\text{per}}[t_1] = x^1$ and

$x_{\text{per}}[t_2] = x^2$. The difference between these two evolutions, which we will call $\epsilon = x_{\text{per}} - x_{\text{true}}$, is small at all times

$$\epsilon[t] = x_{\text{per}}[t] - x_{\text{true}}[t] \ .$$

At the end points, it vanishes; *viz.* $\epsilon[t_1] = \epsilon[t_2]$.

...

The requirement that $\aleph$ be stationary implies that the first-order change must be zero for *any* possible perturbation $\epsilon[t]$ about the true evolution. This can be true only if

$$\mathrm{d}_t \left(\partial_{\dot{x}} e_{\text{lag}}\right) - \partial_x e_{\text{lag}} = 0 \, .$$

The quantity $\partial_{\dot{x}} e_{\text{lag}}$ is called *conjugate momentum for the coordinate x*.

An important consequence of Euler-Lagrange's equations is that if e_{lag} does not explicitly contain coordinate x; *i.e.*, if $\partial_x e_{\text{lag}} = 0$, then the conjugate momentum is constant in time: $\partial_{\dot{x}} e_{\text{lag}}$. In such cases, the coordinate x is called a *cyclic coordinate*, and its conjugate momentum is conserved.

1.2.4 *Classical fields*

The action principle can be extended to obtain the equations of motion for fields, such as the electromagnetic field or gravitational field. The Einstein equation utilizes the Einstein-Hilbert action as constrained by a variational principle. The trajectory (path in spacetime) of a body in a gravitational field can be found using the action principle. For a free falling body, this trajectory is a geodesic.

1.2.5 *Conservation laws*

Implications of symmetries in a physical situation can be found with the action principle, together with Euler-Lagrange's equations, which are derived from the action principle. An example is Noether's theorem, which states that to every continuous symmetry in a physical situation there corresponds a conservation law (and conversely). This deep connection requires that the action principle be assumed.

2 NEWTONIAN MECHANICS

[Based upon [6]]

2.1 Newton's laws of motion

2.1.1 *Introduction*

Newton's *Principia* sums up the fundamental principles of classical mechanics based on previous knowledge and his ideas. A particle's motion, for instance, can be found applying Newton's *three laws of motion*. These laws and the motto *focus on forces* started a whole program of research for future scientists. Newton's method for finding particles' motion is:

Newton's second law of motion: give me a particle, tell me the forces applied on it and I'll tell you how it moves.

- given a collection of particles, acted upon by a collection of forces, draw a nice diagram, with the particles as points and the forces as arrows;
- added up the forces and
- apply Newton's famous $f = ma$ to figure out where the particle's velocities are heading next.

Post Newtonian researchers found out Newton's method unsatisfactory, since

- it's messy and inelegant;
- it's hard to model extended objects, rather than point particles;
- it obscures certain features of dynamics – chaos theory took over 200 years to discover – and
- it's unclear the relationship between Newton's classical laws and quantum physics.

To resolve these issues, modelers established a new formalism, bringing new perspectives on Newton's ideas by reformulating them using more powerful techniques. Simultaneously, such a formalism provides an elegant viewpoint that reveals the basic principles underling Newton's familiar laws of motion: it pries open $f = ma$ to reveal the structures and symmetries that lie beneath.

Better techniques result into an immediate practical advantage to quantify certain complicated phenomena with relative ease.

Moreover, the formalism has become the basis for *all* of fundamental modern physics. Every theory of Nature is best described in the newly developed language. Finally, it also provides the bridge between the classical and the quantum world.

However, there are phenomena in Nature for which these formalism is not particularly useful: *dissipative systems*, for example, are not so well suited to these new techniques.

2.1.2 *Newtonian mechanics: single particle*

Particle: an object of insignificant size; *e.g.*, an electron, a tennis ball or a planet. The validity of this statement depends on the context: to a first approximation, the earth can be treated as a particle when computing its orbit around the sun. But if you want to understand its spin, it must be treated as an extended object.

The motion of a particle of *mass* m at the *position* q is modeled by *Newton's Second Law*:

$$f = \dot{p}\,, \tag{2.1}$$

wherein $f = f[q, \dot{q}]$ represents the *force* that, in general, can depend on both the position q and the velocity $\dot{q}$ and $p = m\dot{q}$ represents the *momentum*. Equation (2.1) reduces to $f = ma$ when mass is constant, $dm = 0$.

friction forces depend on velocity.

The goal of classical dynamics: given positions and velocities at an initial time $t = t_0$, integrate eq. (2.1) to determine $q[t]$ for all t, as long as f remains finite.

Equation (2.1) holds only in an *inertial frame*: a frame where a *free particle* with constant mass travels in a straight line:

free particle: particle subject to no force

$$q = q_0 + vt\,. \tag{2.2}$$

Newton's first law states that such frames exist.

An inertial frame is *not* unique: there are an infinite number of inertial frames. Let $\mathcal{F}$ be an inertial frame. Then, there are ten *linearly independent transformations* $\mathcal{F} \to \mathcal{F}'$ such that $\mathcal{F}'$ is also an inertial frame; *i.e.*, if eq. (2.2) holds in $\mathcal{F}$, then it also holds in $\mathcal{F}'$. These are

- three rotations: $q' = Oq$, where O is a 3×3 orthogonal matrix;
- three spatial translations: $q' = q + q_k$, for a constant position q_k;
- three boosts: $q' = q + v_k t$, for a constant velocity v_k;
- one time translation: $t' = t + t_k$, for a constant time t_k.

They will be important later, where we will see that these symmetries of space and time are the underlying reason for conservation laws.

Invariant: if you have a particle moving in a straight line and apply any or all of the transformations to eq. (2.2), then you end up with a particle also moving in a straight line.

As a parenthetical remark, recall from special relativity that *Einstein's laws of motion* are invariant under *Lorentz transformations*, which, together with translations, make up the *Poincaré group*. We can recover the Galilean group from the Poincaré group by taking the speed of light to infinity.

Angular momentum

We define the angular momentum ϕ of a particle and the torque τ acting upon it as

$$\phi = q \times p \qquad \text{and} \qquad \tau = q \times f \tag{2.3}$$

Note that, unlike linear momentum, both ϕ and τ depend on where we take the origin: *we measure angular momentum with respect to a particular point*. Let us cross both sides of eq. (2.1) with q. Using the fact that $\dot{q}$ is parallel to p, we can write $\mathrm{d}_t\,(q \times p) = q \times \dot{p}$. Then we get a version of Newton's second law that holds for angular momentum:

$$\tau = \dot{\phi}\,. \tag{2.4}$$

2.1.3 *Conservation laws*

From eq. (2.1) and eq. (2.4), two important conservation laws follow immediately:

- If $f = 0$, then p is constant throughout the motion and
- if $\tau = 0$, then ϕ is constant throughout the motion.

Notice that $\tau = 0$ does not require $f = 0$, but only $q \times f = 0$. This means that f must be parallel to q. This is the definition of a *central force*. As written above in terms of forces and torques, these conservation laws appear trivial. Afterwards, we'll see how they arise as a property of the symmetry of space as encoded in the Galilean group.

2.1.4 *Energy*

Recall the definitions of energy. Firstly define the *kinetic energy* e_{kin} as

$$2e_{\mathrm{kin}} = m\dot{q} \cdot \dot{q}\,. \tag{2.5}$$

Suppose hereafter that the mass is constant. Compute the change of kinetic energy with time: $\mathrm{d}_t e_{\mathrm{kin}} = \dot{p} \cdot \dot{q} = f \cdot \dot{q}$. If the particle travels from position q_1 at time t_1 to position q_2 at time t_2, then this change in kinetic energy is given by

$$e_{\mathrm{kin}}[t_2] - e_{\mathrm{kin}}[t_1] = \int_{t_1,t_2} \frac{\mathrm{d}e_{\mathrm{kin}}}{\mathrm{d}t}\mathrm{d}t = \int_{t_1,t_2} f \cdot \dot{q}\,\mathrm{d}t = \int_{q_1,q_2} f \cdot \mathrm{d}q\,, \tag{2.6}$$

where the final expression involving the integral of the force over the path is called the *work done by the force*. So see that the *work done is equal to the change in kinetic energy*. From now on we will mostly focus on a very special type of force known as a *conservative force*: a force that depends only on position q rather than velocity $\dot{q}$ and is such that the work done is independent of the path taken. In particular, for a closed path, the work done vanishes.

$$\oint f \cdot \dot{q} = 0 \iff \nabla \times f = 0\,. \tag{2.7}$$

It is a deep property of Euclidean (flat) space $\mathcal{E}^3$ that this property implies we may write the force as

$$f = -\nabla e_{\mathrm{pot}} \tag{2.8}$$

for some potential $e_{\mathrm{pot}}[q]$. Systems which admit a potential of this form include gravitational, electrostatic and interatomic forces. *When we have* *a conservative force, we necessarily have a conservation law for energy*. To see this, return to eq. (2.6), which now reads

$$e_{\text{kin}}[t_2] - e_{\text{kin}}[t_1] = -\int_{q_1,q_2} \nabla e_{\text{pot}} \cdot \mathrm{d}q = -e_{\text{pot}}[t_2] + e_{\text{pot}}[t_1] \;, \quad (2.9)$$

or, rearranging things,

$$e_{\text{kin}}[t_1] + e_{\text{pot}}[t_1] = e_{\text{kin}}[t_2] + e_{\text{pot}}[t_2] \equiv e \qquad (2.10)$$

So $e = e_{\text{kin}} + e_{\text{pot}}$ is also a *constant of motion*. It is the energy. When the energy is considered to be a function of position q and momentum p it is referred to as the *Hamiltonian* e_{ham}. Afterwards, we will be seeing much more of the Hamiltonian.

2.1.5 *Examples*

THE SIMPLE HARMONIC OSCILLATOR This is a one-dimensional system with a force proportional to the distance x to the origin: $f[x] = -kx$. This force arises from a potential $2e_{\text{pot}} = kx^2$. Since $f \neq 0$, momentum is *not* conserved (the object oscillates backwards and forwards) and, since the system lives in only one dimension, angular momentum is not defined. But energy $2e = m\dot{x}^2 + kx^2$ is conserved.

THE DAMPED SIMPLE HARMONIC OSCILLATOR We now include a friction term so that $f[x, \dot{x}] = -kx - \gamma\dot{x}$. Since f is not conservative, energy is not conserved. This system loses energy until it comes to rest.

a conservative force depends only on position, rather than on position and velocity.

PARTICLE MOVING UNDER GRAVITY Consider a particle of mass m moving in three dimensions under the gravitational pull of a much larger particle of mass m'. The force is $fq^2 = -k_{\text{ne}}mm'\hat{q}$, which arises from the potential $e_{\text{pot}}q = -k_{\text{ne}}mm'$. Again, the linear momentum p of the smaller particle is not conserved, but the force is both central and conservative, ensuring the particle's total energy e and the angular momentum ϕ are conserved.

3 LAGRANGIAN MECHANICS

[Based upon [6]]

3.1 The principle of least action

Part of the power of the Lagrangian formulation over the Newtonian approach is that it does away with vectors in favor of more general coordinates. Let's write the positions of n particles with coordinates q^i, where $i = 1, \ldots, 3n$. Then Newton's equations read

$$\dot{p}_i = -\partial_{q^i} e_{\text{pot}} \,, \tag{3.1}$$

where $p_i = m_i \dot{q}^i$. The *number of degrees of freedom* of the system is said to be $3n$. These parameterize a $3n$-dimensional space known as the *configuration space* $\mathcal{C}$. Each point in $\mathcal{C}$ specifies a configuration of the system; *i.e.*, the positions of *all* n particles. *Time evolution* gives rise to a curve in $\mathcal{C}$, *vide* fig. 1.

3.1.1 *Lagrangian*

Define the Lagrangian to be a function of the positions q^i and the velocities $\dot{q}^i$ of all the particles, given by

$$e_{\text{lag}} = e_{\text{kin}} - e_{\text{pot}} \implies e_{\text{lag}}\left[q^i, \dot{q}^i\right] = e_{\text{kin}}[\dot{q}] - e_{\text{pot}}[q] \,, \tag{3.2}$$

where e_{kin} is the kinetic energy and e_{pot} the potential energy. Note the minus sign between energies! To describe the principle of least action, we consider all smooth paths $q^i[t]$ in $\mathcal{C}$ with fixed end points so that

$$q^i[t_{\text{i}}] = q^i_{\text{initial}} \qquad \text{and} \qquad q^i[t_{\text{f}}] = q^i_{\text{final}} \,. \tag{3.3}$$

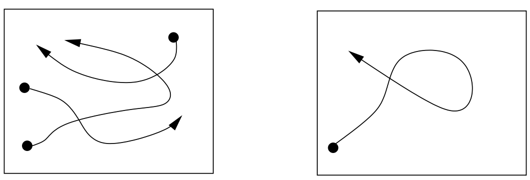

Figure 1 The path of particles in real space (on the left) and in configuration space (on the right).

Of all these possible paths, only one is the true path taken by the system. Which one? To each path, let us assign a number called the action $\aleph$ defined as

$$\aleph = \int_{t_\mathrm{i},t_\mathrm{f}} e_\mathrm{lag} \implies \aleph\big[q^i[t]\big] = \int_{t_\mathrm{i},t_\mathrm{f}} e_\mathrm{lag}\big[q^i, \dot{q}^i\big]\, \mathrm{d}t\,. \tag{3.4}$$

The action is a functional; *i.e.*, a function of the path, which is itself a function. The *principle of least action* is the following result:

the actual path taken by the system is an extremum of $\aleph$.

The requirement that $\aleph$ be extremum implies that the first-order change must be zero for any possible perturbation about the true evolution. This can be true only if

$$\mathrm{d}_t\left(\partial_{\dot{q}^i} e_\mathrm{lag}\right) - \partial_{q^i} e_\mathrm{lag} = 0 \tag{3.5}$$

for each $i = 1, \ldots, 3n$. These equations are known as *Lagrange's equations* (or *Euler-Lagrange's equations*). It can be proved that Lagrange's equations are equivalent to Newton's.

Let's now define an operator, *Euler-Lagrange's operator* $\natural$ to keep notation tidy:

$$\natural_{q\dot{q}} \doteq \mathrm{d}_t\left(\partial_{\dot{q}}\right) - \partial_q\,. \tag{3.6}$$

The musical symbol $\natural$, called natural, represents the unaltered pitch of a note.

Using this operator, Lagrange's equations can be rewritten as

$$\natural_{q^i\dot{q}^i} e_\mathrm{lag} = 0\,. \tag{3.7}$$

3.1.2 *Remarks on the principle of the least action*

- This is an example of a variational principle.
- The principle of *least* action is a slight misnomer. The proof only requires that $\partial\aleph = 0$, and does not specify whether it is a maxima or minima of $\aleph$. So *Principle of stationary action* would be a more accurate, but less catchy, name. It is sometimes called « Hamilton's principle ».
- *All* the fundamental laws of physics can be written in terms of an action principle. This includes electromagnetism, general relativity, the standard model of particle physics and attempts to go beyond the known laws of physics such as string theory.
- There is a beautiful generalization of the action principle to quantum mechanics.
- Back to classical mechanics, there are two very important reasons for working with Lagrange's equations rather than Newton's. The first is that Lagrange's equations hold in any coordinate system, while Newton's are restricted to an inertial frame. The second is the ease with which we can deal with constraints in the Lagrangian system. We'll look at these two aspects in the next two subsections.

3.1.3 *Changing coordinate systems*

Lagrange's equations hold in any coordinate system. This follows immediately from the action principle, which is a statement about paths and *not* about coordinates.

Paths are *coordinate-independent* geometric objects.

REFERENCES

Standing on the shoulder of giants

— NEWTON, NEWTON:WIKIQUOTE

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A QUOTES

The laws of physics must all be expressible as geometric (coordinate-independent and reference-frame-independent) relationships between geometric objects, which represent physical entities.

— BLANDFORD AND THORNE, [7, PART I, P. III]

The idea of dimensional analysis is that units [...] are artificial. The universe cares not for our choice of units. Valid physical laws must have the same form in any system of units. Only dimensionless quantities – pure numbers – are the same in every unit system, so we write equations in a universe-friendly, dimensionless form. Often, there is only one such form. Then, without doing any work, we have solved the problem.

— MAHAJAN, [8, P. 27]

We believe that technology is at its very best and it's more empowering when it simply disappears.

— IVES, [WE BELIEVE THE SAME ABOUT MATHEMATICS.]

In almost every quantitative problem, the analysis simplifies when you follow the proverbial advice of doing first things first. First approximate and understand the most important effect – the big part – then refine your analysis and understanding. This procedure of successive approximation or « taking out the big part » generates meaningful, memorable, and usable expressions.

— MAHAJAN, [9, P. 77]

DOCUMENT REVISION HISTORY

The following table describes the changes to « An alternative formulation of mechanics ».

VERSION	DATE	NOTES
0.0.1	27/12/2013	First release
0.1.0	05/08/2014	Changes in libraries and file organization
0.1.0	19/08/2014	Current release