MATHEMATICAL MODELING FOR PHYSICAL SCIENCES

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ABSTRACT. Abstract goes here :) In the following, the introduction to each section contains the action plan, while the subsections are the plan applied to an example: the motion of a gravitational pendulum.

CONTENTS

1. PROBLEM BACKGROUND

A mathematical model of a physical phenomenon begins with the *problem background*, a description or introduction to the object. Specifically, the background should contain:

- a *description* of the essential features of the physical process and
- an identification of the *objectives*, the key questions requiring answers.

Answering what, who, where, how and why questions guides to write down the description. Additionally, including graphical illustrations aids not only in the description, but in the definition of physical quantities and the establishment of hypotheses, as well.

1.1. **Description.** Phenomenon: motion of a gravitational pendulum.

- What is a pendulum? A pendulum is a mechanical system consisting on a bob hanging by a rod attached, in turn, to a pivot.
- What is a gravitational pendulum? A gravitational pendulum is a pendulum object to gravitational interactions only.
- How is the pendulum set into motion? There are basically three ways of setting a gravitational pendulum into motion:
  (1) by moving the bob from its equilibrium position to an initial angle, θ_0, at time $t = 0$;
  (2) by applying a force that imprints an angular velocity to the bob, $\dot{\theta}_0$, at time $t = 0$;
  (3) or by both at the same time $t = 0$.
- What keeps the pendulum moving? Once the pendulum is swinging, gravitational action keeps it moving, since an interplay between kinetic energy and gravitational potential energy is established. Kinetic energy is impressed by the initial displacement or initial forces, while the gravitational potential tries to restore the bob to its equilibrium position.
- What forces act on it to damp its motion? Friction on the pivot or drag when the pendulum is partially or totally submerged in a viscous fluid damp the pendulum motion.

1.2. **Objective.** To seek for a mathematical relation, perhaps in a closed form, to predict the pendulum amplitude variation with time; *i.e.*, deduce a function of the form $\theta[t]$.

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2. Problem formulation

The *problem formulation* aims to:

- identify key physical processes;
- interpret these processes mathematically;
- establish a mathematical model – governing equations and suitable initial conditions and boundary conditions;
- state clearly the assumptions.

Additionally, the formulation must be based on sound physical principles, experimental facts or laws expressed in mathematical terms.

As a guide, define the physical framework (geometry, kinematics, dynamics, thermal transfer and so on), state a dimensional set and then define the physical quantities, constants, parameters, coefficients and state their dimensions.

Usually, one begins with educated guessing, followed by dimensional analysis, order of magnitude analysis, analysis of extreme cases and ends with model restrictions. The end result is a less accurate model, but less complex as well. Notice that order of magnitude analysis is preceded by dimensional analysis, since *only* the comparison of *dimensionless* quantities is meaningful!

2.1. Physical processes.

In the case of a swinging gravitational pendulum, there are two main cases to study:

- free pendulum motion – where no frictional forces and no drag are taken into account and
- damp pendulum motion – where frictional forces or drag are considered.

In both cases, however, the interplay between kinetic energy and gravitational potential must be regarded, since it drives motion. It can be seen, finally, that the problem domain is that of dynamics.

2.2. Mathematical interpretation.

The mathematical model wishes to find a function to predict the pendulum amplitude variation with time: $\theta = \theta[t]$.

First, one can hypothesize that the pendulum bob hangs by a massless, frictionless and inflexible rod. This implies that the system center of gravity will coincide with the bob center of gravity (massless rod), that no damping due to friction will happen and that the bob will trace a circular orbit of radius equal to the length of the rod (inflexible rod).

Next, let us list of the possible physical quantities that may influence the pendulum motion together with their symbols and dimensions:

- pendulum: amplitude, $\dim\theta = [1]$, initial amplitude, $\dim\theta_0 = \dim\theta[0] = [1]$;
- rod: length, $\dim l = [\mathsf{L}]$;
- bob: mass, $\dim m_{\mathrm{bob}} = [\mathsf{M}]$, density, $\dim\rho_{\mathrm{bob}} = [\mathsf{M}/\mathsf{L}^3]$, diameter, $\dim d_{\mathrm{bob}} = [\mathsf{L}]$;
- fluid: density, $\dim\rho_{\mathrm{fl}} = [\mathsf{M}/\mathsf{L}^3]$, dynamic viscosity, $\dim\mu = [\mathsf{M}/\mathsf{LT}]$;
- others: time, $\dim t = [\mathsf{T}]$, free fall acceleration, $\dim g = [\mathsf{L}/\mathsf{T}^2]$.

Since the problem belongs to dynamics, the chosen dimensional set was $\{\mathsf{L},\mathsf{M},\mathsf{T}\}$, with cardinality of three.

Because m_{bob}, ρ_{bob} and d_{bob} are related, we can keep two of the three quantities. We keep the density and the diameter. Then, there are 6 physical quantities and 3 independent dimensions. Thus, according to the Pi-theorem, the number of dimensionless quantities that can be formed is $9 - 3 = 6$. These quantities are

$$\Pi_1 = \theta\,,\quad \Pi_2 = \theta_0\,,\quad \Pi_3 = d/l\,,\quad \Pi_4 = \rho_{\mathrm{fl}}/\rho_{\mathrm{bob}}\,,\quad \Pi_5 = t\sqrt{g/l}\,,\quad \Pi_6 = \rho_{\mathrm{fl}} d\sqrt{lg}/\mu\,.$$

Again, according to the Pi-theorem, the mathematical function we seek is of the *form*:

$$\theta = \psi\left[\theta_0, \frac{d}{l}, \frac{\rho_{\mathrm{fl}}}{\rho_{\mathrm{bob}}}, t\sqrt{\frac{g}{l}}, \rho_{\mathrm{fl}} d\frac{\sqrt{lg}}{\mu}\right]\,.$$

Now, we use order of magnitude analysis to restrain the physical model so to simplify the mathematical model. We assume:

- $l \gg d$, then $d/l \to 0$. The validity of this assumption depends on one's tolerance; *e.g.*, if the bob diameter is, say, 5 cm and the rod length 1 m, then the ratio $d/l = 0.05$, which might be discarded for some purposes.

- $\rho_{\text{bob}} \gg \rho_{\text{fl}}$, then $\rho_{\text{fl}}/\rho_{\text{bob}} \to 0$. Say, for instance, that the bob is made of steel and swings through air at 15 °C and at sea level, then

$$\rho_{\text{air}}/\rho_{\text{steel}} = 1.225\,\text{kg/m}^3/7750\,\text{kg/m}^3 \sim 0.158 \times 10^{-4}\,,$$

which can be ignored. However, if the steel bob swings through oil with $\rho_{\text{oil}} = 850\,\text{kg/m}^3$, then

$$\rho_{\text{oil}}/\rho_{\text{steel}} = 850\,\text{kg/m}^3/7750\,\text{kg/m}^3 \sim 0.110\,,$$

which may be important for some applications.
- an inviscid fluid. An inviscid fluid is a fluid with no viscosity, then

$$\rho_{\text{fl}} d\sqrt{lg}/\mu = 0\,.$$

The validity of this assumption will also depend on the circumstances.

Then, after having restrained the physical model, we seek for a mathematical function of the form

$$\theta = \psi\left[\theta_0, t\sqrt{\frac{g}{l}}\right]\,.$$

For mathematical purposes, we can combine all the previous assumptions by defining a *simple gravitational pendulum* as a pendulum composed of a massive bob of mass m hanging by a massless, frictionless and inflexible rod of length l attached to a frictionless pivot. Under the influence of gravitational interactions, the pendulum swings through an inviscid fluid of negligible density.

The last definition leads to a very restricted physical model. However, this will allow us to finally find a closed form for the pendulum amplitude.

2.3. **Mathematical model.** Consider a simple gravitational pendulum composed of a bob of mass m and a rod of length l. Let θ be the amplitude of the pendulum for any time t. Let the initial angle be θ_0 and the initial velocity be $\dot{\theta}_0 = 0$. Finally, let g be the free fall acceleration. Then, find the equation of motion for the pendulum.

Using θ as the generalized position and $\dot{\theta}$ as the generalized velocity, write the Lagrangian for the system:

$$e_{\text{lag}} = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos[\theta]\,.$$

Find next the generalized momentum and its temporal change

$$\partial_{\dot{\theta}} e_{\text{lag}} = ml^2\dot{\theta} \implies \mathrm{d}_t \partial_{\dot{\theta}} e_{\text{lag}} = ml^2\ddot{\theta}\,.$$

Calculate the generalized force:

$$\partial_\theta e_{\text{lag}} = -mgl\sin[\theta]\,.$$

Replace the generalized force and the temporal change of the generalized momentum in the Euler-Lagrange equation to find:

$$\Box_{\theta\dot{\theta}} e_{\text{lag}} = ml^2\ddot{\theta} + mgl\sin[\theta] = 0\,.$$

Since $ml > 0$, divide the last equation through ml^2 to have

$$\ddot{\theta} + \frac{g}{l}\sin[\theta] = 0\,,$$

object to the initial conditions $\theta[0] = \theta_0$ and $\dot{\theta}[0] = 0$.

Note that we deduced the equation by means of Lagrangian mechanics. Writing down the Lagrangian was possible due to the fact that the system was assumed to be conservative; *i.e.*, neither friction nor drag were considered.

2.4. **Assumptions.** The equation of motion for the pendulum was found using the following assumptions:

3. Analysis

- Non-dimensionalize
- Analogies with other related problems
- Use analytic and numeric methods to obtain solutions/results.

In the case of analytic solutions, verify that they satisfy the model equations.
Verify assumptions, uncertainty analysis.

4. Results

Results/Conclusions/Discussion.

- Interpret results with respect to the original physical process and objectives.
- Identify limitations and extensions

Verify assumptions, confront with reference values and experimental data.