A BRIEF INTRODUCTION TO GEOMETRIC ALGEBRA

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Contents

Date: August 22, 2012.
Key words and phrases. Geometric Algebra, scalars, vectors, bivectors, cliffs.

1. Quick Review

Let's review quickly what we already know about vectors and what we can do with them.

1.1. The Engineering Vector.

In engineering, a vector, say a 3-D vector $\vec{v}$, is thought of as a triplet of mathematical objects joint in an entity that looks like

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{z}, \tag{1}$$

where v_x, v_y and v_z are *the components of* $\vec{v}$ and the vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{z}$ are *the elements of the orthonormal basis of the 3-D space*.

The components of $\vec{v}$ are also called *scalar components*, since they are elements of the set of the real numbers; *i.e.*, v_x, v_y and v_z are real numbers.

Orthonormal basis means that the vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{z}$ form an angle of 90° with one another [1] (*orthogonality*), have unit length (*normalization*), and *every* vector of $\mathbb{R}^3$ can be *uniquely* decomposed as shown in Eq. 1. Such a descomposition is also called *the representation of* $\vec{v}$ *in* $\mathbb{R}^3$. The basis set of $\mathbb{R}^3$, *aka the 3-D basis*, is the set that has $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ as elements.

1.2. Vector Operations.

In basic curses, we learn that, with vectors, we can do addition, subtraction, multiplication, but not division, at least not between vectors. Addition and subtraction are presented *component wise*; *e.g.*, for two vectors $\vec{u}$ and $\vec{v}$, we have

$$\vec{u} + \vec{v} = (u_x + v_x)\hat{\imath} + (u_y + v_y)\hat{\jmath} + (u_z + v_z)\hat{z},$$
$$\vec{u} - \vec{v} = (u_x - v_x)\hat{\imath} + (u_y - v_y)\hat{\jmath} + (u_z - v_z)\hat{z}.$$

Addition and subtraction of vectors with scalars is not defined. So, a thing such as $a + \vec{v}$, where $a \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^3$ is not even mentioned.

So far, so good. What is left is to review the interesting operation of multiplication. Interesting, because it comes in three flavors:

(1) Multiplication with scalars: if $a \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^3$, then $a\vec{v}$ is another vector defined as
$$a(v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}) = av_x\hat{\imath} + av_y\hat{\jmath} + av_z\hat{k}.$$

(2) Dot product: let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$. Then the *dot product of* $\vec{u}$ *and* $\vec{v}$ is defined as
$$\vec{u} \cdot \vec{v} = u_xv_x + u_yv_y + u_zv_z.$$
The dot product is also called *scalar product*, since its result is a scalar.

(3) Cross product: let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$. Then the *cross product of* $\vec{u}$ *and* $\vec{v}$ is given by
$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{\imath} + (u_3v_1 - u_1v_3)\hat{\jmath} + (u_1v_2 - u_2v_1)\hat{k}.$$
The cross product is also called *vector product*, since it results in another vector.

Of the three definitions, that of the cross product is the weirdest, because it seems arbitrary [2]. Although cumbersome, did you notice the symmetry in the scalar components of $\vec{u}\times\vec{v}$, $(u_2v_3-u_3v_2)$, $(u_3v_1 - u_1v_3)$, and $(u_1v_2 - u_2v_1)$?

One last thing about the cross product: it is only defined for vectors in 3D, no more, no less. This seems not to be a problem, since in engineering we do not work in more than 3D; but it severely and unnecessarily limits theory.

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[1] As regularly used, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{z}$ point in every increasing direction of Cartesian axes. $\hat{\imath}$ points in the direction of increasing x's, $\hat{\jmath}$ in the direction of increasing y's, and $\hat{k}$ in the direction of increasing z's and they are placed one after another using the *right-hand rule*.

[2] Under its limits of validity, the cross product is nevertheless well behaved, mathematically consistent, and thus useful. Besides, it has a proper geometrical interpretation that we will see later.

2. Exploratory Maths

3. Mathematical Model

In mathematics, analytic geometry (also called Cartesian geometry) describes every point in three-dimensional space by means of three coordinates. Three coordinate axes are given, usually each perpendicular to the other two at the origin, the point at which they cross. They are usually labeled x, y, and z. Relative to these axes, the position of any point in three-dimensional space is given by an ordered triple of real numbers, each number giving the distance of that point from the origin measured along the given axis, which is equal to the distance of that point from the plane determined by the other two axes.

Other popular methods of describing the location of a point in three-dimensional space include cylindrical coordinates and spherical coordinates, though there is an infinite number of possible methods. See Euclidean space.

`http://en.wikipedia.org/wiki/Three-dimensional_space`.

`http://en.wikipedia.org/wiki/Euclidean_space`.

4. The Algebra Bit in Geometric Algebra

The previous sections were all about reviewing past knowledge with new notation. But what about the main goal of the present document, geometric algebra? Well we needed notation to better express vectors and operations with them. Now, it is time to look a bit closer to vectors and their properties.

In basic maths, mathematical objects – numbers, vectors, operators, functions, and so on – are either taken for granted or defined in isolation. For example, few times I have heard the term *vector space*. One only knows that they exist and learns how to operate with them. In advanced maths, on the other hand, mathematical objects are not taken for granted and not thought of as isolated entities. They are objects of mathematical structures, one of these structures is called *linear space*. We will thus begin our study with the formal definition of linear spaces.

Definition 4.1 (Linear Space): *Let V denote a non-empty set of objects, called* elements. *Let x, y and z be arbitrary elements in V and let a and b be arbitrary real numbers. The set V is called* a linear space *if it satisfies the following ten axioms listed in three groups.*

(1) Closure Axioms

Axiom 4.1 (Closure under addition): *There is a unique element in V called* the sum of x and y, *denoted by $x + y$.*

Axiom 4.2 (Closure by multiplication of real numbers): *There is a unique element in V called* the product of a and x, *denoted by ax.*

(2) Axioms For Addition

Axiom 4.3 (Commutative law): $x + y = y + x$.

Axiom 4.4 (Associative law): $(x + y) + z = x + (y + z)$.

Axiom 4.5 (Existence of zero element): *There is an element in V, denoted by 0, such that $x + 0 = x$.*

Axiom 4.6 (Existence of negatives): *The vector $(-1)x$ has the property $x + (-1)x = 0$, where $(-1) \in \mathbb{R}$.*

(3) Axioms For Multiplication By Numbers

Axiom 4.7 (Associative law): $a(bx) = (ab)x$.

Axiom 4.8 (Distributive law for addition in V): $a(x + y) = ax + ay$.

Axiom 4.9 (Distributive law for addition of numbers): $(a + b)x = ax + bx$.

Axiom 4.10 (Existence of identity): $1x = x$, *where* $1 \in \mathbb{R}$.

Herein, we will be concerned with vectors as depicted in geometry; *i.e.*, line segments from one point (tail) to another point (tip), where an arrow is placed into the vector tip. In such a context, if V is a set whose elements are *vectors* and if V satisfies the aforementioned axioms, then V will be called a *vector space*, instead of linear space. Additionally, the arbitrary real numbers will be referred to as *scalars*. Moreover, because we have used real numbers as scalars, a set V that satisfies the axioms of a linear space is called a *vector space over the real numbers*.

Note 1. *In the definition of a vector space, coordinates are nowhere to be found! Vectors and vector spaces are treated as pure mathematical entities on their own right;* i.e., *they are independent of any coordinate system, any choice of basis, any $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, and so on.*

NOTE 2. *The axioms do not tell us how to do computations, they tell us that the operations therein defined exist. For instance, let x and y in V. The axioms tell us that $x + y$ is defined, is a vector and is in V, but they do not tell us how to calculate its value(s).*

NOTATION 1. *For any two vectors x and y, the negative of x, $[(-1)x]$, is denoted by $-x$ and the difference $y - x$ is defined to be the sum $y + (-x)$.*

The definition of linear spaces is perhaps daunting, because it is abstract. However, instead of thinking of the axioms as a list of statements about abstract elements and abstract operations, it is better to regard the axioms as a *license* to do algebra with its elements.

The dot product is also called *inner product*. Moreover, the dot product of u and v is also called *scalar product*, since it returns scalars, w. That is to say, the dot product is a function that takes two vectors as arguments and returns a real number, as defined herein.

5. Notation, Notation, Notation!

Working with vectors as currently notated in engineering is exhausting: engineering notation is too verbose and inconsistent. Inconsistent, because, while the spatial axes are named x, y and z and while the components of $\vec{v}$ onto such axes are logically called v_x, v_y, and v_z (following the axis names), the unit vectors – supposed to represent the axes themselves – are called $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. Besides, the notation does not extend well to higher dimensions. What would happen with notation if we added another dimension to our studies? Let us say we wanted to form a 4-D vector, how would we note the fourth basis vector? $\hat{h}$, $\hat{l}$, $\hat{m}$, $\hat{n}$? What if we needed to work with vectors in 10-D? We need notation that eases working with vectors, extends nicely vector treatment to higher dimensions and saves some typing in the meantime.

Notation 2 (Undecorated variables). *From now on, we will generally* not *decorate variables;* e.g.*, a vector v will be denoted as v, not as $\vec{v}$, $\boldsymbol{v}$, $\hat{v}$, or alike. We will denote mathematical objects with italic font types. Scalars will be denoted with the first letters of the alphabet,* e.g.*, a, b, c; vectors with the last letters of the alphabet,* e.g.*, u, v, w; unit vectors with* one *Greek letter: γ. When needed, superscripts and subscripts will be added to variables. They will help to enforce this policy.*

Decorations look “cute”, but are troublesome. How would we tell the difference among scalars, vectors, unit vectors, *etc.*, in equations then? We found our rationale in Denker: (John Denker, Fundamental Notions of Vectors, Section 9.)

> [...] a name is not the same as an explanation. Do not expect the structure of a name or symbol to tell you everything you need to know. Most of what you need to know belongs in the legend. The name or symbol should allow you to look up the explanation in the legend.

Let us explain now what we mean by indexing variables. In Eq. 1, for instance, we noted the components of the vector v using subindices that indicate *space variables*; v_x, v_y and v_z. This is the piece of notation that does not extend nicely to higher dimensions. To overcome this limitation, let us agree that

Notation 3 (Index notation). *Numeric indices will replace space variable symbols in the following way:*

- *Basis vectors will be denoted with the Greek lowercase letter* gamma [3]*: γ.*
- *The* three spatial coordinates *will be replaced with* one indexed spatial coordinate *in the following way: $x \to x^1$, $y \to x^2$ and $z \to x^3$. For a spatial vector, say $x \in \mathbb{R}^3$, this replacement means*
$$x = \gamma_1 x^1 + \gamma_2 x^2 + \gamma_3 x^3.$$
- *For* non-spatial vectors*, we will index them as though they were spatial vectors. For a vector v in $\mathbb{R}^3$, this means $v_x \to v^1$, $v_y \to v^2$ and $v_z \to v^3$ or, more fully,*
$$v = \gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3.$$

It is to be noted that

(1) the number of indices represents the dimension of the variable: a scalar will have no indices, a vector will have one, a matrix will have two, and so on.
(2) the range of the indexing variables may be understood from context – a vector $v \in \mathbb{R}^3$ will have one index ranging from 1 to 3;

[3] This choice goes against the notation used in vector algebra and linear algebra, which use e instead. But, in geometric algebra, our ultimate goal, the usage of γ is the common choice.

(3) superscripts are indices rather than a exponents – "x^2" is therefore to be read "second component of v", instead of "x to the power of 2".
(4) the basis vectors, γ_1, γ_2, γ_3, follow the numbering of their respective vector component indices, but are *sub*scripts rather than *super*scripts. We will explain what it means *geometrically* in the following sections.

One immediate advantage of using these notational conventions is that vectors can be written in a rather compact way, for instance, by using the *sigma notation*. For our vector v in $\mathbb{R}^3$ defined in Eq. 1, we have

$$v = \sum_{i=1}^{3} \gamma_i v^i = \sum_{1 \leq i \leq 3} \gamma_i v^i. \tag{2}$$

or, *iff* [4] the range of i is understood,

$$v = \sum_{i} \gamma_i v^i,$$

instead of $\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{z}$.

Sweet! But, we could do even better. The problem is that, if you work quite a bit with vector components or summations, you could end up with many $\sum$'s. So many that they could amount the hundreds [5]. We would like to become proficient on vectors, not on the Greek alphabet. To overcome this, Einstein [6] came up with an idea that will cut down verbosity, while providing an elegant syntax:

NOTATION 4 (Einstein summation convention). *When an index variable appears* twice *in a single term, the convention implies summation of that term over all the values of the index. This means simply to eliminate the $\sum$ symbol and the summation limits, while keeping the summand. For instance, in Eq. 2, Einstein convention means*

$$v = \sum_{i=1}^{3} \gamma_i v^i \qquad \to \qquad v = \gamma_i v^i. \tag{3}$$

An index that is summed over is called a summation index*, in this case i. It is also called a* dummy index *since any symbol can replace i without changing the meaning of the expression, provided that it does not collide with index symbols in the same term. A free index appears* once and only once *within each additive term in an expression.*

An index that is not summed over is a free index *and should be found in each term of the equation or formula if it appears in any term. Compare dummy indices and free indices with free variables and bound variables.*

Free indices only appear as either super- or sub-script, never as both, and they must occur exactly once in every term.

Dummy indices occur once as super- and once as sub-script, so summation over these indices is implied. Dummy indices must never have the same label as a free index.

NOTE 3 (Summary of Einstein summation convention). *A brief summary on how index notation works in the form of "rules"(Primer on Index Notation, Massachusetts Institute of Technology Department of Physics, Physics 8.07, Fall 2004):*

Rule 1: *Repeated, doubled indices in quantities multiplied together are implicitly summed.*
Rule 2: *Indices that are not summed over (free indices) are allowed to take all possible values unless stated otherwise.*

[4] *iff* is the math lingo for "if and only if", not a typo.
[5] No kidding! Knuth... in their book... claimed to have used around 1000 $\sum$'s!
[6] Yep, *that* Einstein.

Rule 3: *It is illegal to use the same dummy index more than twice in a term unless its meaning is made explicit.*

NOTE 4 (Warning! Bad notation). *When establishing index notation or Einstein summation convention – like Eq. 3, some authors go from $\gamma_i v^i$ to simply v^i and say that "v^i are the components of v". They just drop the basis* [7]. *This notation is useless, so it will* not *be used here. It is useless, because the notation $\gamma_i v^i$ says* explicitly *where v is being* decomposed *onto. It gives the basis γ_i. Only when knowing the basis, phrases like "the components of v" make sense. Dropping the basis, like v^i, gives no information onto where the vector was decomposed.*

NOTE 5 (Warning! Source of errors). *We will see that when multiplying two vectors, say u and v, in index notation, we can be tempted to violate.*

Wow! Quite a journey from Eq. 1 to Eq. 3.

Sec. 6 is devoted to ... proficiency in using index notation. If you feel you need practice, check it out.

[7]For instance, inn "Tensor or Component Notation for Vectors", Leroy T. Kerth even says that we don't need them.

6. Exercises

Since the notations that we have just introduced are a bit too much to digest at once, let us dedicate this section to practice to which we have agreed. Note, while checking the examples, how the conventions ease working with vectors and how "good looking" [8] the resulting equations become.

Example 6.1. Rewrite the engineering vector of Eq. 1 using our agreed conventions.

Solution. Instead of doing that for 3-D, we will generalize the notation for vectors in n-D and get the solution as a special case [9].

Let v be an element of the $\mathbb{R}^n$ space and let $\{\gamma_i\,;\ i : 1, \ldots, n\}$ denote the space basis set. Then, v can be decomposed onto such a basis as

$$v = \gamma_1 v^1 + \gamma_2 v^2 + \cdots + \gamma_n v^n,$$

or, using sigma notation,

$$v = \sum_{i=1}^{3} \gamma_i v^i,$$

or, using index notation and Einstein summation convention,

$$v = \gamma_i v^i,$$

The engineering vector is found by setting n to 3; that is, $v = \gamma_i v^i$ where i runs from 1 to 3. Verbosely,

$$v = \gamma_i v^i = \gamma_1 v^1 + \gamma_2 v^2 + \gamma_3 v^3.$$

□

Each term in $\gamma_i v^i$ is composed of a scalar, v^i, and a vector, γ_i. By virtue of our definition of multiplication of vectors with scalars, $\gamma_i v^i$ can be rearranged to $v^i \gamma_i$ without changing the result [10]. Additionally, see that v^i indicate which component; while γ_i, which direction.

Example 6.2. Redefine vector addition.

Solution. Let u and v be two elements of $\mathbb{R}^n$. Then $w = u + v$, called *the addition of u and v*, is another vector in $\mathbb{R}^n$ defined as

$$\gamma_i w^i \equiv \gamma_i u^i + \gamma_i v^i.$$

□

Discussion. If u and v were in $\mathbb{R}^1$, *aka*, $\mathbb{R}$, the index i will be 1 – *i.e.*, we would be working in 1-D; if u and v were in $\mathbb{R}^2$, the index i will run from 1 to 2 – *i.e.*, we would be working in 2-D; if u and v were in $\mathbb{R}^3$, the index i will run from 1 to 3 – *i.e.*, we would be working in 3-D; and so on. For all of such cases, we get the correct results with the "improved" definition of vector addition. Nice! Although more abstract, our redefinition works nicely in *any* dimension. All that achieved with little typing and few explaining, at least not in the definition itself.

Example 6.3. Redefine multiplication of vectors with scalars.

Solution. Consider $a \in \mathbb{R}$ and $v \in \mathbb{R}^3$. Then $w = av$, called *the product of a and v*, is another vector in $\mathbb{R}^n$ defined [11] as

$$w \equiv a\gamma_i v^i = \gamma_i v^i a.$$

□

[8] Good looking equations form a part of what mathematicians call *mathematical beauty*.

[9] Phrases like this one are the origin of so many jokes about mathematicians.

[10] We will be forced to change this rule once we meet vectorial calculus.

[11] The symbol $\equiv$ is being used to denote definition. You probably have already figured it out.

So far we have been going from vector notation to index notation. What about going the other way around. Let's show it with a classic example. Do you remember that systems of linear equations can be expressed by means of vectors? Let us do some practicing with that.

EXAMPLE 6.4. Expand the following equations accordingly:

$$a_i = b_i + c_i,$$

where i runs from 1 to 3.

Solution.

$$a_1 = b_1 + c_1,$$
$$a_2 = b_2 + c_2,$$
$$a_3 = b_3 + c_3.$$

□

Wow, that was easy, right? We simply replaced the values of i one at a time. The key was to notice that i is a free index: a free index appears only once in every term and always means distinct equations, in this case three. Also notice that there is not repeated index, so no summation is implied.

Let's practice with double indices.

EXAMPLE 6.5. Expand a_{ii} in 3D.

Solution.

$$a_{ii} = a_{11} + a_{22} + a_{33}.$$

□

In this case, i is a dummy index. Dummy indices are the ones that imply summation.

What about matrices? If vectors can be expressed with indices, and matrices are composed out of vectors, then, maybe, matrices can be written using index notation. Yes, they can. But we need a bit more work to do. Since matrices have both rows and columns, we need to agree which index will represent what.

NOTATION 5 (Matrices in index notation). *A matrix M with n rows and m columns,* i.e., *a $n \times m$ matrix, can be expressed in index notation as*

$$M = M_{ij}.$$

where i runs from 1 to n and j from 1 to m.

Our first matrix to represent in index notation will be a classical one.

EXAMPLE 6.6. Represent the Kronecker delta in index notation.

Solution. Let δ be a $n \times m$ matrix whose elements are as follows

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where i runs from 1 to n and j from 1 to m. The matrix δ is also called *the Kronecker delta.* □

EXAMPLE 6.7. Represent the dot product in index notation and Einstein convention.

Solution. Let u and v be two elements of $\mathbb{R}^n$. Then $w = u \cdot v$, called the *dot product of u and v*, is defined as

$$w \equiv u^i v^i. \tag{4}$$

□

NOTE 6. *The last equation, Eq. 4, is sometimes taken as the "definition" of the dot product. It works, but it hides some things. Can you spot where the hidden bits lie and why?*

To see this, let us write the dot product as a bilinear, *i.e.*, distributive, operation between vectors rather than components in the following way:

- Accept the two vectors u and v as elements of $\mathbb{R}^n$.
- Express them using index notation: $\gamma_i u^i$ and $\gamma_j v^j$, with i and j both running from 1 to n.
- Calculate the dot product:

$$u \cdot v = \gamma_i u^i \cdot \gamma_j v^j = u^i v^j \gamma_i \cdot \gamma_j = u^i v^i.$$

Can you spot the bits now? Here is a list:

- Why i for u and j for v if both are 3-D vectors, and thus their indices run through the same values?
- What happened with the dot product of the basis vectors, $\gamma_i \cdot \gamma_j$? Was the dot product of the basis vectors set to one? To zero? To both?
- Why at the end the index of v was switched to i?

The answer for the first question is the Einstein summation convention rule 3, see Note 3. Thus, u and v need different indices for their expansion. In fact, every time that two or more vector expansions are joint together, every expansion needs a different index.

The answer for the rest of the questions are more elaborate and will be answered, but not now, because they require a bit more concepts to be developed. What we could say is that the dot product of basis "substitutes indices"; *e.g.*, in $u^i v^j \gamma_i \cdot \gamma_j$, the dot product takes and changes the index j of vi to i, leaving thus $u^i v^i$ as a result.