MATHEMATICAL MODELING FOR PHYSICAL SCIENCES

DIEGO HERRERA

ABSTRACT. Abstract goes here :) In the following, the introduction to each section contains the action plan, while the subsections are the plan applied to an example: the motion of a gravitational pendulum.

CONTENTS

1. PROBLEM BACKGROUND

A mathematical model of a physical phenomenon begins with the *problem background*, a little description or introduction to the subject. Specifically, it should contain:

- a *description* of the essential features of the physical process;
- an identification of the *objectives*, the key questions requiring answers.

A guide to write down the description is the answer to the questions what, who, where, how and why. Additionally, the inclusion of graphical illustrations aids not only in the description, but also in the definition of physical quantities and the establishment of hypotheses.

1.1. **Description.** Problem: analysis of a gravitational pendulum.

- What is a pendulum? A pendulum is a mechanical system consisting on a bob hanging by a rod attached, in turn, to a pivot.
- What is a gravitational pendulum? A gravitational pendulum is a pendulum object only to the action of gravitational interactions.
- How is the pendulum set into motion? There are basically three ways of setting a gravitational pendulum into motion:
  (1) by moving the bob an initial angle from its equilibrium position, θ_0, at time $t = 0$;
  (2) by applying a force that imprints an angular velocity to the bob, $\dot{\theta}_0$, at time $t = 0$;
  (3) or by both at the same time $t = 0$.
- What keeps the pendulum moving? Once the pendulum is swinging, gravitational action keeps it moving, since an interplay between kinetic energy and gravitational potential energy is established. Kinetic energy is impressed by the initial displacement or initial forces, while the gravitational potential acts as a restoring force that moves the bob to its equilibrium position.
- What forces act on it to damp its motion? The pendulum motion can be damped by friction on the pivot or by drag, when the pendulum is partially or totally submerged in a viscous fluid.

Date: September 30, 2013.
Key words and phrases. mathematical modeling.

1.2. **Objectives.** What we want to know, to model: A mathematical model of a pendulum seeks for a function, perhaps in a closed form, to predict how the pendulum amplitude varies with time; *i.e.*, a function of the form $\theta[t]$.

2. Problem formulation

Once the phenomenon under study is described, a *precise formulation* of it should be made. This formulation must be based on sound physical principles expressed expressed in mathematical terms.

The problem formulation aims to:

- identify key physical processes;
- interpret these processes mathematically;
- establish a mathematical model – governing equations and suitable initial conditions and boundary conditions;
- state clearly the assumptions.

A guide to the formulation is to define the physical framework (geometry, kinematics, dynamics, thermal transfer, and so on), define the physical quantities, constants, parameters, coefficients and their physical dimensions.

Usually, one begins with educated guessing, followed by dimensional analysis, order of magnitude analysis, analysis of extreme cases and ends with simplifications that can make the model less accurate – given a tolerance –, but less complex. Notice that order of magnitude analysis is preceded by dimensional analysis, since *always* the comparison of *dimensionless* quantities is meaningful! Dimensional quantities are relative.

2.1. **Physical processes.** In the case of a swinging gravitational pendulum, there are two main cases to study:

- free pendulum motion – where no frictional forces and no drag are taken into account and
- damp pendulum motion – where frictional forces or drag are considered.

In both cases, however, the interplay between kinetic energy and gravitational potential must be regarded, since it drives motion. It can be seen, finally, that the problem domain is that of dynamics.

2.2. **Mathematical interpretation.** The mathematical model wishes to find a function to predict the pendulum amplitude variation with time: $\theta = \theta[t]$.

First, one can hypothesize that the pendulum bob hangs by a massless, frictionless and inflexible rod. This implies that the system center of gravity will coincide with the bob center of gravity (massless rod), that no damping due to friction will happen and that the bob will trace a circular orbit of radius equal to the length of the rod (inflexible rod).

Next, let us list of the possible physical quantities that may influence the pendulum motion together with their symbols and dimensions:

- pendulum: amplitude, $\dim\theta = [1]$, initial amplitude, $\dim\theta_0 = \dim\theta[0] = [1]$;
- rod: length, $\dim l = [\mathsf{L}]$;
- bob: mass, $\dim m_{\text{bob}} = [\mathsf{M}]$, density, $\dim\rho_{\text{bob}} = [\mathsf{M}/\mathsf{L}^3]$, diameter, $\dim d_{\text{bob}} = [\mathsf{L}]$;
- fluid: density, $\dim\rho_{\text{fl}} = [\mathsf{M}/\mathsf{L}^3]$, dynamic viscosity, $\dim\mu = [\mathsf{M}/\mathsf{LT}]$;
- others: time, $\dim t = [\mathsf{T}]$, free fall acceleration, $\dim g = [\mathsf{L}/\mathsf{T}^2]$.

Since the problem belongs to dynamics, the chosen dimensional set was $\{\mathsf{L},\mathsf{M},\mathsf{T}\}$, with cardinality of three.

Because m_{bob}, ρ_{bob} and d_{bob} are related, we can keep two of the three quantities. We keep the density and the diameter. Then, there are 6 physical quantities and 3 independent dimensions. Thus, according to the Pi-theorem, the number of dimensionless quantities that can be formed is $9-3=6$. These quantities are

$$\Pi_1 = \theta\,, \quad \Pi_2 = \theta_0\,, \quad \Pi_3 = d/l\,, \quad \Pi_4 = \rho_{\text{fl}}/\rho_{\text{bob}}\,, \quad \Pi_5 = t\sqrt{g/l}\,, \quad \Pi_6 = \rho_{\text{fl}} d\sqrt{lg}/\mu\,.$$

Again, according to the Pi-theorem, the mathematical function we seek is of the *form*:

$$\theta = \psi\Big[\theta_0, d/l, \rho_{\text{fl}}/\rho_{\text{bob}}, t\sqrt{g/l}, \rho_{\text{fl}} d\sqrt{lg}/\mu\Big]\,.$$

Now, we use order of magnitude analysis to restrain the physical model and thus to simplify the mathematical model. We restrain the model by assuming:

- $l \gg d$, then $d/l \to 0$. The validity of this assumption depends on one's tolerance; *e.g.*, if the bob diameter is, say, 5 cm and the rod length 1 m, then the ratio $d/l = 0.05$, which might be enough for some purposes.
- $\rho_{\text{bob}} \gg \rho_{\text{fl}}$, then $\rho_{\text{fl}}/\rho_{\text{bob}} \to 0$. Say, for instance, that the bob is made of steel and swings through air at 15 °C and at sea level, then $\rho_{\text{air}}/\rho_{\text{steel}} = 1.225\,\text{kg/m}^3/7750\,\text{kg/m}^3 \sim 0.158 \times 10^{-4}$, which can be ignored. However, if the steel bob swings through oil with $\rho_{\text{oil}} = 850\,\text{kg/m}^3$, then $\rho_{\text{oil}}/\rho_{\text{steel}} = 850\,\text{kg/m}^3/7750\,\text{kg/m}^3 \sim 0.110$, which may be important for some applications.
- an inviscid fluid. An inviscid fluid is a fluid with no viscosity, then $\rho_{\text{fl}} d\sqrt{lg/\mu} = 0$. The validity of this assumption will also depend on the circumstances.

Finally, for mathematical purposes, we define a *simple gravitational pendulum* as a pendulum composed of a massive bob of mass m hanging by a massless, frictionless and inflexible rod of length l attached to a frictionless pivot. The pendulum swings through an inviscid fluid of negligible density.

The last definition leads to a very restricted physical model. However, this will allow us to find a closed form for the pendulum amplitude.

2.3. **Mathematical model.** Consider a simple pendulum composed of a bob of mass m and a rod of length l. Let ξ^θ be the amplitude of the pendulum for any time t. The pendulum begin moving by displacing the bob an initial angle ξ^θ_0 at time $t = 0$ and without applying any force. Motion is kept by gravitational action, with g being the free fall acceleration. Find the equation of motion for the pendulum.

Using ξ^θ

$$e_{\text{lag}} = \frac{1}{2} m\dot{\xi}^\theta \dot{\xi}^\theta + mgl\cos\left[\xi^\theta\right] .$$

Find next the generalized momentum and its temporal change

$$\partial_{\dot{\xi}^\theta} e_{\text{lag}} = ml^2\dot{\xi}^\theta \implies \mathrm{d}_t\partial_{\dot{\xi}^\theta} e_{\text{lag}} = ml^2\ddot{\xi}^\theta .$$

Calculate the generalized force:

$$\partial_{\xi^\theta} e_{\text{lag}} = -mgl\sin\left[\xi^\theta\right] .$$

Replace the generalized force and the temporal change of the generalized momentum in the Euler-Lagrange equation to find:

$$\Box_{\xi^\theta \dot{\xi}^\theta} e_{\text{lag}} = ml^2\ddot{\xi}^\theta + mgl\sin\left[\xi^\theta\right] = 0 .$$

Since the bob mass and the rod length are each not null, divide through ml^2 to have

$$\ddot{\xi}^\theta + \frac{g}{l}\sin\left[\xi^\theta\right] = 0 ,$$

object to the initial conditions $\xi^\theta[0] = \xi^\theta_0$ and $\dot{\xi}^\theta = 0$.

Note that we deduced the equation by means of Lagrangian mechanics. Writing down the Lagrangian was possible due to the fact that the system was assumed to be conservative; *i.e.*, neither friction nor drag were considered.

2.4. **Assumptions.** The equation of motion for the pendulum was found using the following assumptions:

3. Analysis

- Non-dimensionalize
- Analogies with other related problems
- Use analytic and numeric methods to obtain solutions/results.

In the case of analytic solutions, verify that they satisfy the model equations.

Verify assumptions, uncertainty analysis.

4. Results

Results/Conclusions/Discussion.

- Interpret results with respect to the original physical process and objectives.
- Identify limitations and extensions

Verify assumptions, confront with reference values and experimental data.