TRANSPORT PHENOMENA

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1. Energy transport

> People goes to high place, water flows to low place
>
> — JUNPING SHI, Partial Differential Equations and Mathematical Biology

> All streams flow to the sea because it is lower than they are. Humility gives it its power.
>
> — LAO TZU, Tao Te Ching

> The flame that burns twice as bright burns half as long.
>
> — LAO TZU, Tao Te Ching

Be careful with notation:

Caution. We use the prime notation (Lagrange's notation for derivatives) to express time rate changes; *e.g.*, when expressing a body internal energy time rate change: i', or, equivalently, $i' = \partial_t i$. Note that $\dim i' = E/T$.

Under the same dimensional basis, we also use the prime notation to express energy flows; *e.g.*, when expressing thermal flow, *aka* thermal power, e'; since $\dim e' = E/T$. The same applies for other flows, like work change rate, *aka* mechanical power, w'.

Date: August 27, 2013.
Key words and phrases. math notation.

On the other hand, however, in order to preserve undecorated variables and similarly looking equations, we use an analogous notation for fluxes: the double prime notation; *e.g.*, when expressing thermal energy flux: e''. Note that now $\dim e'' = E/L^2T$, which is *not* the second derivative of thermal energy with respect to time; *i.e.*, $e'' \neq \ddot{e} = \mathrm{d}_t\mathrm{d}_t e$.

Therefore, in the case of any doubt, use the physicist's best allies: mathematical interpretation, physical reasoning and dimensional analysis.

1.1. **Relation of thermal energy transfer and thermodynamics.**

1.1.1. *Theory.* Consider a body of mass m and volume v being heated by an inflow of *thermal energy*, *thermal flow* or *thermal power*, e'. Due to heating, the body expands and performs *work* onto the surroundings at a rate w'. While both processes take place, on the other hand, the body *internal energy* i changes, or accumulates, at a rate i' given by the *energy conservation principle*:

$$i' = e' - w' ,$$

where the IUPAC sign convention for thermodynamics was used to find the signs for the different terms.

The body internal energy is related to the body *thermodynamic temperature* θ via the phenomenological expression

$$i = cm\theta ,$$

where c is a property of the body material called *specific heat capacity* [1], or the capacity that a body has to store or release thermal energy, $\dim c = E/M\Theta$, and θ is the body thermodynamic temperature.

If $p\,\mathrm{d}v$ is the only work that occurs, then,

$$e' = pv' + i' .$$

This last equation has two well-known special cases: when the body is allowed to expand freely, *constant pressure process*, and when expansion is constrained, *constant volume process*:

$$e' = \begin{cases} mc_v\theta' , & \text{for constant volume processes; } i.e., \ \mathrm{d}v = 0, \\ mc_p\theta' , & \text{for constant pressure processes; } i.e., \ \mathrm{d}p = 0, \end{cases}$$

where c_v is the *specific heat capacity at constant volume* and c_p the *specific heat capacity at constant pressure*.

When the body is made of an incompressible substance, then, for any pressure variation, $\mathrm{d}v = 0$ and thus the two specific capacities are equal:

$$c_v = c_p = c .$$

This approximation works well for solids and liquids. With this estimate, the energy conservation equation becomes

$$e' = i' = mc\theta' .$$

The work now is to solve the last equation to predict e. Finding the solution, however, is not possible at this stage, for i is not known *a priori*, so some principles must be added to complete the problem formulation. These principles are called *transport laws* or *constitutive equations* and are *not* a part of thermostatics. They include Fourier's law of thermal conduction, Newton's law of cooling and Stephan-Boltzmann's law of thermal radiation. Moreover, constitutive equations express physical quantities, in this case the internal energy i and thermal energy e, in terms of *state variables*, in this case temperature θ.

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[1] This property should really be called *specific energy capacity*, for "heat" is no more than energy in transit!

1.1.2. *Technical notes.* Some technical notes on energy transfer theory.

Technical note. The mathematical treatment of physical phenomena of systems composed of many particles depends on the size of the system; *i.e.*, on the number of particles composing it. To better describe such a dependence, we rely on the concepts of microscopic and macroscopic systems. Microscopic systems can be mathematically treated by statistical mechanics, while macroscopic systems by energy transfer theory.

Consider a system to be composed of n particles. A system is called *macroscopic* if

$$\frac{1}{\sqrt{n}} \ll 1\,,$$

which means that statistical arguments can be applied to reasonable accuracy. For instance, if we wish to keep the statistical error below one percent then a macroscopic system would have to contain more than about ten thousand particles. Any system containing less than this number of particles would be regarded as *microscopic*, and, hence, statistical arguments could not be applied to such a system without unacceptable error. Statistical quantum mechanics is required to analyze such systems.

Thermal energy transfer is based on the macroscopic description of physical phenomena; it is thus an approximation to the microscopic description of phenomena ultimately given by statistical quantum mechanics. However, thermal energy transfer can be approximated by classical statistical mechanics. This approximation is valid when the number of particles in the system.

Technical note. Thermal flow is also called *thermal energy transfer rate*.

Technical note. IUPAC sign convention: All net energy transfers *to* the system are positive. All net transfers *from* the system are negative. A useful mnemonic is the wording '+accumulation = +inflow - outflow + release - storage'.

Note. Statistical mechanics provides a microscopic explanation of *temperature*, based on macroscopic systems' being composed of many particles, such as molecules and ions of various species, the particles of a species being all alike. It explains macroscopic phenomena in terms of the mechanics of the molecules and ions, and statistical assessments of their joint adventures. In the statistical thermodynamic approach, by the *equipartition theorem* each classical degree of freedom that the particle has will have an average energy of $k_\mathrm{b}\theta/2$, where k_b is Boltzmann's constant. The *translational motion* of the particle has three degrees of freedom, one in every direction of space, so that, except at very low temperatures where quantum effects predominate, the average translational energy of a particle in an system with temperature θ will be $3k_\mathrm{b}\theta/2$.

On the molecular level, temperature is the result of the motion of the particles that constitute the material. Moving particles carry kinetic energy. Temperature increases as this motion and the kinetic energy increase. The motion may be the translational motion of particles or the energy of the particle due to molecular vibration or the excitation of an electron energy level.

In other words, since every degree of freedom that a particle has carries energy e that equals $k_\mathrm{b}\theta/2$ and since k_b is a *universal constant*, then energy is proportional to temperature: $e \propto \theta$. Therefore, one is entitled to think on temperature as energy, energy in "disguise".

Dim. Analysis. When analyzing thermal energy transfer, the most suitable *dimensional system* is the $ELT\Theta$ system – a *dimensionally independent system*. Within this framework, the dimensions of energy transfer, *aka* heat, of work and of internal energy are $\dim e = \dim w = \dim i = E$ and of energy flow, *aka* thermal power or thermal flow, of work rate, *aka* mechanical power, and of internal energy change rate, *aka* accumulation, are $\dim e' = \dim w' = \dim i' = E/T$.

Technical note. In giving the dimensions of the specific heat capacity c as $E/M\Theta$, we are abusing of the $ELT\Theta$ system by adding mass, M, to it. This addition extends the dimensionally *independent* $ELT\Theta$ system to the dimensionally *dependent*, redundant, $EMLT\Theta$ system. We do so consciously, so to stress physics instead of mathematical purity; for, in the "pure" $ELT\Theta$ system, the dimensions of mass are ET^2/L^2 and then the dimensions of the c are $L^2/T^2\Theta$; disguising therefore its physical meaning. In other words, we prefer to interpret specific heat capacity as the capacity that a body has to store energy at a given temperature per unit mass or $\dim c = E/M\Theta$, rather than the less informative $L^2/T^2\Theta$.

1.2. **Modes of energy transfer.** Thermal energy is transferred inside a body or among bodies by three means:

(1) thermal (heat) conduction;
(2) thermal (heat) convection and

(3) thermal radiation.

1.2.1. *Thermal conduction.* Consider a body, say a rod, being heated up to temperature θ_1 at one end while simultaneously being cooled down to θ_2 at the other end. Consider also the body to be thermally isolated everywhere but not at the ends. Then, it can be observed that, driven by the temperature difference, $\Delta\theta = \theta_1 - \theta_2$, the thermal flow happens from the hot end to the cold one. Since such a flow happens through the body surface, it is useful to define physical quantity, *thermal flux*, to better express the energy flow and surface relation.

Thermal flux, e'', is thus defined as thermal flow per unit area, $\dim e'' = E/L^2T$. Then, *thermal conduction*, often called heat conduction, through a body is based on experimental observation: *Fourier's law*.

Fourier's law states that

> the local thermal flux resulting from thermal conduction is proportional to the magnitude of the temperature gradient and opposite to it in sign. Mathematically,
> $$e'' = -k \operatorname{grad} \theta \,.$$
> The minus sign accounts for the second law of thermostatics: thermal flow happens in the direction of *falling* temperature.

The proportionality "constant" in Fourier's law, k, is a definite positive number describing a substance thermal property, *thermal conductivity*, $\dim k = E/TL\Theta$. Thermal conductivity, rather than a constant, is a coefficient, because its value depends on temperature, pressure and, in mixtures, on the composition. It is a scalar as long as the material is *isotropic*, which means that the ability of the material to conduct thermal energy depends on position within the material, but for a given position not on the direction.

Thermal conductivity values: Because of how molecules are arranged, solids will have generally higher thermal conductivity values than gases. Thus, the process of thermal energy transfer is more efficient in solids than in gases.

In a gas, k is proportional to the molecular speed and molar specific heat and inversely proportional to the cross-sectional area of molecules.

The values for k are experimentally found and presented in references as tables or figures.

Example. The front of a slab of lead ($k = 35\,\mathrm{W/m\,K}$) is kept at $110\,^\circ\mathrm{C}$ and the back is kept at $50\,^\circ\mathrm{C}$. If the area of the slab is $0.4\,\mathrm{m}^2$ and it is $0.03\,\mathrm{m}$ thick, then compute the thermal energy flux and the thermal energy transfer rate.

Solution. Use Fourier's law to find the energy flux:

$$e'' = -k \operatorname{grad} \theta \sim -k\Delta\theta/\Delta x = -(35)((110-50)+273.15)/(0.03) = 70\,\mathrm{kW/m^2}\,.$$

The thermal energy transfer rate is thus

$$e' = e''s = (70)(0.4) = 28\,\mathrm{kW}\,. \qquad \square$$

1.2.2. *Thermal convection.* Consider a typical convective cooling situation: cool gas flows past a warm body. The fluid immediately adjacent to the body forms a thin slowed-down region called *boundary layer*. Thermal energy is conducted into this layer, which sweeps it away and, farther downstream, mixes it into the stream. We call such a process of carrying thermal energy away from a body surface by a moving fluid *convection*. Isaac Newton considered the convective process and suggested that the cooling would be such that

$$\dot{\theta}_{\text{body}} \propto \theta_{\text{body}} - \theta_{\text{fluid}}\,,$$

wherein θ_{fluid} is the temperature of the incoming fluid. This statement suggests that energy is flowing from the body. But if energy is constantly replenished, then the body temperature need not change. Therefore, with $e' = mc\dot{\theta}$, we get

$$e' \propto \theta_{\text{body}} - \theta_{\text{fluid}} \implies e'' = \bar{f}\left(\theta_{\text{body}} - \theta_{\text{fluid}}\right)\,,$$

where $e'' = e'/s$, s is the surface area of the body and $\bar{f}$ is the *film coefficient* or *heat transfer coefficient*. The bar over $\bar{f}$ indicates that's an *average* over the surface of the body.

Without the bar, f denotes the *local* value of the film coefficient at a point on the surface. The dimensions of both coefficients are $\dim f = \dim \bar{f} = E/TL^2\Theta$.

It turns out that Newton oversimplified the process description, when he made his conjecture. Thermal convection is complicated and $\bar{f}$ can depend on the temperature difference $\theta_{\mathrm{body}} - \theta_{\mathrm{fluid}} = \Delta\theta$:

- f is really independent of $\Delta\theta$ when the fluid is forced past a body and $\Delta\theta$ is not too large. This is called *forced convection*.
- when fluid buoys up from a hot body or down from a cold one, f varies as some weak power of $\Delta\theta$ – typically as $\Delta\theta^{1/4}$ or $\Delta\theta^{1/3}$. This is called *free* or *natural convection*. If the body is hot enough to boil a liquid surrounding it, then $\bar{f}$ will typically vary as $\Delta\theta^2$.

Typical values of the film coefficient are presented in equations, tables or figures. However, they should be judiciously applied in actual designs.

Lumped-capacity solution – heat equation. The problem now is to predict the transient cooling of a convectively cooled object. Apply the familiar filst law statement to have

$$e' = i' \implies -\bar{f}s\left(\theta - \theta_{\mathrm{fluid}}\right) = \mathrm{d}_t\left(\rho c v\left(\theta - \theta_{\mathrm{ref}}\right)\right) ,$$

where s and v are the surface area and volume of the body, θ the temperature of the body, $\theta = \theta\,[t]$, and θ_{ref} is an arbitrary temperature at which the internal energy i is defined to equal zero.

The last equation can be solved by separating the variables θ and t. After solving the equation, after applying the initial condition $\theta\,[t=0] = \theta_{\mathrm{i}}$, wherein θ_{i} the body initial temperature, and after rearranging terms, one has

$$\frac{\theta - \theta_{\mathrm{fluid}}}{\theta_{\mathrm{i}} - \theta_{\mathrm{fluid}}} = \exp\left[-t/\tau\right] .$$

Note that all the physical parameters in the problem have now been “lumped” into the *time constant* τ. This time constant represents the time required for a body to cool to $1/e$ or 35% of its initial temperature difference above (or below) θ_{fluid}. The ratio t/τ can also be interpreted as

$$\frac{t}{\tau} = \frac{\bar{f}st}{\rho c v} = \frac{\text{capacity for convection from surface}}{\text{thermal capacity of the body}} .$$

Note that the thermal conductivity of the body is missing from the last equations. The reason is that we have assumed that the body temperature is nearly uniform and thus means that internal conduction is unimportant. If $L/k\bar{f} \ll 1$, then the body temperature θ is almost constant within the body at any time. Therefore,

$$\frac{\bar{f}L}{k} \ll 1 \implies \theta\,[\xi, t] \sim \theta\,[t] \sim \theta_{\mathrm{surface}}$$

and the thermal conductivity k becomes irrelevant to the cooling process. This condition must be satisfied if the lumped solution is to be accurate.

We call the group $\bar{f}L/k = \Pi_{\mathrm{bi}}$ *Biot number*. If Π_{bi} were large, then the situation would be reversed in this case, $\Pi_{\mathrm{bi}} \ll 1$, and then the convection process offers little resistance to the thermal transfer (conduction). We could solve the *thermal diffusion equation*, *aka* heat diffusion equation:

$$\alpha\partial_{xx}\theta = \partial_t\theta ,$$

object to the simple boundary condition $\theta\,[\xi, t] = \theta_{\mathrm{fluid}}$, when $x = L$ to determine the temperature in the body and its rate of cooling, in this case.

Biot number will be therefore the basis for determining what sort of problem we have to solve.

1.2.3. *Thermal radiation.* When thermal energy is applied to a body, it generates motion of the charged particles in the body matter. Then, the body radiates electromagnetic energy – *thermal radiation*. Equivalently, this means that all matter with temperature greater than the absolute zero emits thermal radiation, or, in other words, thermal radiation can be seen as the conversion of thermal energy into electromagnetic energy. Examples of thermal radiation are the visible light and infrared light emitted by an incandescent light bulb, the infrared radiation emitted by animals and detectable with an infrared camera.

The thermal radiation of real bodies is modeled after a hypothetical radiative body called a *black-body*. If a radiation-emitting object meets the physical characteristics of a black body in thermodynamic equilibrium, then the radiation is called *black-body radiation*. There are three laws that describe the physical properties of black-bodies: Planck's law describes the *spectrum* of black-body radiation, which depends only on the object's temperature; Wien's displacement law determines the most likely *frequency of the emitted radiation* and, finally, Stefan-Boltzmann law gives the black-body *radiant emissivity*.

Thermal transfer by thermal radiation: All bodies constantly emit energy by a process of electromagnetic radiation. The intensity of such energy flux depends upon the body temperature. Most of the thermal energy that reaches you when you sit in front of a fire is radiant energy. Radiant energy warms you when you walk under the sun.

Objects cooler than the fire or the sun emit much less energy, because the energy emission varies as the fourth power of absolute temperature. Very often, the emission of energy, or radiant energy transfer, from cooler bodies can be neglected in comparison with convection and conduction. (Approximate solutions and order of magnitude physics can be helpful here!) But energy transfer processes that occur at high temperatures, or with conduction or convection suppressed by evacuated insulators, usually involve a significant fraction of radiation.

The electromagnetic spectrum: Thermal radiation occurs in a range of the electromagnetic spectrum of energy emission. Accordingly, it inhabits the same wavelike properties as light or radio waves. Each quantum of radiant energy has a wavelength λ and a frequency μ associated with it.

The full electromagnetic spectrum includes an enormous range of energy-bearing waves, of which thermal energy is only a small part.

Tables show the various forms over a range of wavelengths that spams 17 orders of magnitude. Thermal radiation, whose main component is usually the spectrum of infrared radiation, passes through a three-order-of-magnitude window in λ or μ. This window ranges $0.1\,\mu\mathrm{m}$ to $1000\,\mu\mathrm{m}$, with a geometric mean of $10\,\mu\mathrm{m}$.

Black-bodies: The model for the perfect thermal radiator is the so-called *black-body*. This is a body that absorbs all energy that reaches it and reflects nothing. The term is a bit confusing, since they *emit* energy. Thus, under infrared vision, a black-body would glow with "color" appropriate to its temperature. Perfect radiators are "black" in the sense that they absorb all visible light (and all other radiation) that reaches them.

To model a black-body, a *Hohlraum* is used. Whet are the important features of a thermally black-body? First consider a distinction between thermal (heat) radiation and infrared radiation. *Infrared radiation* refers to a particular range of wavelengths, while thermal radiation refers to the whole range of radiant energy flowing from one body to another. Suppose that a radiant thermal flux e'' falls upon a translucent plate that's not black. A fraction α of the total incident energy, called the *absorptance*, is absorbed by the body; a fraction ρ, called the *reflectance*, is reflected by the body and a fraction τ, called the *transmittance*, passes through. Thus,

$$1 = \alpha + \rho + \tau\,.$$

This relation can also be written for the energy carried by each wavelength in the distribution of wavelengths that makes up thermal radiant energy from a source at any temperature.

$$1 = \alpha_\lambda + \rho_\lambda + \tau_\lambda\,.$$

All radiant energy incident on a black-body is absorbed, so that $\alpha_\mathrm{b} = \alpha_{\lambda_\mathrm{b}} = 1$ and $\rho_\mathrm{b} = \tau_\mathrm{b} = 0$. Furthermore, the energy emitted by a black-body reaches a theoretical maximum given by Stefan-Boltzmann law.

Stefan-Boltzmann law: The energy flux radiating from a body is designated by $e''[t]$, $\dim e'' = E/L^2T$. The symbol $\check{e}_{\lambda_\mathrm{body}}[\lambda,\theta]$ designates the distribution function of radiative flux in λ, or the *monochromatic emission power*:

$$\check{e}_{\lambda_\mathrm{body}}[\lambda,\theta] = \mathrm{d}_t e''[\lambda,\theta] \qquad \text{or} \qquad e''[\lambda,\theta] = \int_0^\lambda \check{e}_{\lambda_\mathrm{body}}[\lambda,\theta]\,\mathrm{d}\lambda\,.$$

Thus,
$$e''[\theta] = e''[\infty, \theta] = \int_0^\infty \check{e}_{\lambda_{\rm body}}[\lambda, \theta]\, {\rm d}\lambda\,.$$
The dependence of $e''[\theta]$ on temperature θ for a black-body, $e''_{\rm bb}$, was found experimentally by Stefan and proved theoretically by Boltzmann. Stefan-Boltzmann law states that
$$e''_{\rm bb}[\theta] = \sigma_{\rm sb}\theta^4\,,$$
where Stefan-Boltzmann constant $\sigma_{\rm sb} = 5.670\,373(21) \times 10^{-8}\,{\rm W/m^2\,K^4}$. In terms of other fundamental constants:
$$\sigma_{\rm sb} = \frac{2\pi^5}{15}\frac{k_{\rm b}^4}{h_{\rm p}^3 c_0^2} = \frac{2\pi^5}{15}\frac{r_{\rm g}^4}{h_{\rm p}^3 c_0^2 n_{\rm a}^2}\,,$$
where $k_{\rm b}$ is Boltzmann constant, $h_{\rm p}$ Planck constant, c_0 speed of light in vacuum, $r_{\rm g}$ ideal gas constant and $n_{\rm a}$ Avogadro's number.

[a lot missing!]

1.3. **A look ahead.** To solve actual problems, three tasks must be completed:

(1) thermal energy (heat) diffusion equation must be solved object to appropriate boundary and initial conditions;
(2) the film (convective thermal transfer) coefficient must be determined if convection is relevant and
(3) the factor F_{1-2} must be determined to calculate radiative thermal energy transfer.

Appendix A. Energy continuity equation

A.1. **Derivation.** Consider a body being heated from one side while being cooled from the other, opposite, side. Consider also that no work is performed. Experience shows that the body internal energy varies with time t and position ξ due to the external energy difference; *i.e.*, external energy flowing through the body external surface gets temporally and spatially distributed within the body. To model such a distribution, we will make use of the energy conservation principle.

Let $i = i[t, \xi]$ be the internal energy density field of a non-moving control volume ${\rm d}v$ inside the body of volume v. Then, the internal energy in the whole body is $\int_v i\,{\rm d}v$. Therefore, the body internal energy changes, accumulates, at a rate
$${\rm d}_t \int_v i\,{\rm d}v = \int_v \partial_t i\,{\rm d}v = \int_v i'\,{\rm d}v\,.$$
On the other hand, let ∂v be the body outer surface, boundary, and, accordingly, let $n\,{\rm d}s$ be the directed control surface. Then, the thermal energy flowing *out* of the control surface is $-e''\cdot n\,{\rm d}s$. And, thus, the total thermal energy flowing out of the body through ∂v equals
$$-\int_{\partial v} e''\cdot n\,{\rm d}s = -\int_v \nabla\cdot e''\,{\rm d}v = -\int_v {\rm div}\, e''\,{\rm d}v\,.$$
By the energy conservation principle, the internal energy change rate and the thermal flow must balance one another:
$$\int_v i'\,{\rm d}v = -\int_v {\rm div}\, e''\,{\rm d}v\,.$$
The last equation must be equal regardless of any control volume choice:
$$i' + {\rm div}\, e'' = 0\,.$$
The next step is to relate the internal energy density with the thermal flux. This relation comes from two phenomenological laws, *aka constitutive equations*:

(1) As mentioned in the above sections, a body internal energy is related to the body temperature θ by
$$i = \rho c\theta\,,$$
where ρ and c are two body (material) properties, *mass density*, mass per unit volume, and *specific heat capacity*, the capacity that the body material has to store energy per unit mass and per unit temperature. Thus,
$$i' = (\rho c\theta)' = \partial_t(\rho c\theta) = \partial_t \rho c\theta\,.$$

(2) Fourier's law of thermal energy conduction states that the thermal energy flux is proportional to the temperature gradient:

$$e'' = -k \operatorname{grad} \theta \,.$$

The minus sign in this equation accounts for the second law of thermodynamics: thermal energy flows in the direction of falling temperature. The constant of proportionality k is a body (material) property, *thermal conductivity*, $\dim k = E/LT\Theta$. It is dependent on both the temperature and pressure – in mixtures also on the composition.

Using the empirical laws, find that

$$\partial_t \rho c \theta = \operatorname{div} k \operatorname{grad} \theta \,.$$

Consider, finally, the body to be composed of an homogeneous, isotropic material. Then, the mass density, the specific heat capacity and the thermal conductivity are constant. Therefore, write the last equation as

$$\partial_t \theta = \alpha \operatorname{lap} \theta \qquad \text{or as} \qquad \theta' = \alpha \operatorname{lap} \theta \,,$$

where α is the *thermal diffusivity*, $\dim \alpha = L^2/T$, and lap Laplace operator. Refer to the last equation as the *thermal transfer equation*, the *thermal diffusion equation*, or, more commonly, as *heat equation*.

Note the definition of thermal diffusivity: $\alpha = k/\rho c$; *i.e.*, thermal diffusivity is the ratio of thermal energy conduction, $\dim k = E/LT\Theta$, to the volumetric thermal capacity, $\dim \rho c = E/L^3\Theta$. Thus, in a sense, thermal diffusivity is the measure of thermal inertia. In a substance with high thermal diffusivity, thermal energy moves rapidly through it because the substance conducts energy quickly relative to its volumetric thermal capacity or "thermal bulk".

A.2. **Technical notes.** Some explanations on the maths involved during the derivation.

Technical note. It is assumed that, during the thermal energy transfer, the body material is not affected by it; *i.e.*, the body material does not undergo thermal decomposition.

Math note. For modeling the phenomenon, energy density was assumed to be a smooth scalar field and the body to be bounded by a smooth outer surface.

Notation. The symbol d_t defined by $\mathrm{d}_t \doteq \mathrm{d}/\mathrm{d}t$ denotes the *ordinary* derivative with respect to time and $\partial_t \doteq \partial/\partial t$ denotes the *partial* derivative with respect to time.

Math note. In going from the surface integral to the volume integral,

$$-\int_{\partial v} e'' \cdot n \,\mathrm{d}s = -\int_v \nabla \cdot e'' \,\mathrm{d}v = -\int_v \operatorname{div} e'' \,\mathrm{d}v \,,$$

the divergence theorem was used.

Dim. Analysis. The accumulation in the body must balance energy outflow:

$$\int_v i' \,\mathrm{d}v = \int_v \partial_t i \,\mathrm{d}v = -\int_v \operatorname{div} e'' \,\mathrm{d}v \,.$$

This equation satisfies a fundamental physics law: the energy conservation principle. It also satisfies another fundamental physics principle: dimensional homogeneity principle for physics law.

The dimensions of the terms in the left-hand-side, LHS, of the last equation are $\dim \partial_t = 1/T$, $\dim i = E/L^3$, *i.e.*, internal energy *density*, $\dim \mathrm{d}v = L^3$. Then, the whole LHS integral has dimensions of energy change rate, or energy flow, E/T.

The dimensions of the terms in the right-hand-side, RHS, are $\dim \operatorname{div} = \dim \nabla\cdot = 1/L$, $\dim e'' = E/L^2T$, $\dim \mathrm{d}v = L^3$. Then, the whole RHS integral has dimensions of energy flow, E/T.

Since the LHS and the RHS of the equation have the same dimensions, thus, the principle of dimensional homogeneity is satisfied.

Math note. During the derivation, we went from

$$\int_v \left(i' + \operatorname{div} e''\right) \mathrm{d}v = 0 \qquad \text{to} \qquad i' + \operatorname{div} e'' = 0 \,;$$

or, in other words, we made the integrand to vanish. To explain this "math move", making the integrand equal to zero, we need to analyze how the derivation was done, mathematically – the math framework behind the deduction.

To find the energy distribution inside our given body, we have followed Riemann's ideas on integration. He would have said:

> To find the energy distribution within the body, follow the process:
>
> - first, divide the body volume v into n smaller control volumes $\mathrm{d}v_i$, where i runs from 1 to n. Note that the number and sizes of the control volumes are up to you. There is neither a fixed number nor a fixed size. In fact, they not even need be all of the same size! Nevertheless, do make sure that n is large enough to cover the whole body volume.
> - Then, find the energy distribution in each control volume and
> - finally, find the total energy distribution inside of the whole of the body by adding the energy distribution of the n control volumes together by means of integration.
>
> Note that the result has to be *independent* on the number of control volumes and on their sizes.
>
> To explain this better. Say, for instance, that you take n control volumes of equal size $\mathrm{d}v_1$ and, say, I take m smaller control volumes of size $\mathrm{d}v_2$; or, $n < m$ since $\mathrm{d}v_1 > \mathrm{d}v_2$. Say both choices cover the whole body. Then, you calculate the energy distribution of your control volumes and so do I. To find the total energy distribution, you next add the energy distribution of every of your control volumes and so do I. Finally, we compare results. Good maths requires both results, your total energy distribution and mine, be the same!

This explanation implies, mathematically, the vanishing integrand.

Notation. Another common form of writing the heat equation is by using the geometric derivative: $\partial_t \theta = \alpha \nabla^2 \theta$.

Technical note. As it stands, the heat equation, $\theta' = \alpha \operatorname{lap} \theta$, is written in vector notation; thus, it is valid in *any* coordinate system. However, once a particular coordinate system is selected, then, the appropriate form of Laplace operator must be found. For instance, in a Cartesian coordinate system, $[x, y, z]$, the heat equation becomes

$$\theta' = \partial_t \theta = \alpha \partial_k \partial_l g^{kl} \theta \qquad \text{or} \qquad \theta_{,t} = \alpha \theta_{,\xi\xi} = \alpha \left(\theta_{,xx} + \theta_{,yy} + \theta_{,zz}\right) ,$$

The second equation follows from two facts of Cartesian coordinate systems: the metric g does not depend on position, thus $\partial_k \partial_l g^{kl} \theta = g^{kl} \partial_k \partial_l \theta$ and $g = \operatorname{diag}[1, 1, 1]$.

For non-Cartesian coordinate systems, however, g does depends on position, so $\partial_k \partial_l g^{kl} \theta \neq g^{kl} \partial_k \partial_l \theta$ and therefore the derivatives of the metric with respect to the coordinates and the metric coefficients g^{kl} must be found.

Notation. Consider a smooth scalar field $\phi = \phi\,[x, y, z]$. Then, the comma derivative notation is defined as

$$\phi_{,x} = \frac{\partial \phi}{\partial x} , \quad \phi_{,y} = \frac{\partial \phi}{\partial y} , \quad \phi_{,z} = \frac{\partial \phi}{\partial z} , \quad \phi_{,xy} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \phi \quad \text{and so on} .$$

Geometry. Remember that, in vector calculus, divergence is an operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. Or, more technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

Then, recall the equation:

$$i' = -\operatorname{div} e'' .$$

This equation models the phenomenon of a body being heated up from one side while being cooled down from another side or the phenomenon of a body releasing energy from its center while it cools down. Take the second phenomenon. The LHS of the model equation represents the rate at which a body internal energy changes with respect to time: the rate at which energy is released from the center to the outer portions of the body and finally to the environment surrounding the body.

The RHS, on the other hand, because of the divergence, represents how the central energy diffuses from the center throughout the body to the environment. It has a minus sign because energy is being dissipated instead of being accumulated.

The equality holds, physically, because energy is conserved: if it is being released and not accumulated within the body, energy has no other place to go but to its surrounding environment.

Geometry. Recall that Laplace operator $\operatorname{lap} f\,[\xi]$ of a function f at a point ξ, up to a constant depending on the dimension, is the rate at which the average value of f, over spheres centered at ξ, deviates from $f\,[\xi]$ as the radius of the sphere grows; *i.e.*, Laplace operator of a function represents the difference between the value of the function at a point and the average of the values at surrounding points. Another way of looking at Laplace operator is by writing it in a Cartesian coordinate system $[x, y, z]$:

$$\operatorname{lap} = \nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}\,.$$

See that, then, this operator finds the *change in the change* of the function (if you make a graph, the change in the slope) in all directions from the point of interest. That may not seem very interesting, until you consider that acceleration is the change in the change of position with time or that the maxima and minima of functions (peaks and valleys) are regions in which the slope changes significantly.

Now, recall the thermal diffusion equation:

$$\theta' = -\alpha \operatorname{lap} \theta\,.$$

This equation models the phenomenon of a body being heated up from one side while being cooled down from another side or the phenomenon of a body releasing energy from its center while it cools down. Take the second phenomenon. The LHS of the model equation represents the rate at which a body temperature changes with respect to time: the rate at which temperature diffuses from the center to the outer portions of the body and finally to the environment surrounding the body. In an analogous way as internal energy in the $i' = -\operatorname{div} e''$ equation. Something unsurprising, since energy and temperature are related via the equipartition theorem.

The RHS, on the other hand, because of Laplace operator, represents how the central temperature diverges from the center to the surrounding points. It has a minus sign because temperature is being dissipated instead of being gathered. The coefficient α, thermal diffusion, is a body material property that acts as both a dimensional conversion factor between temporal temperature changes and spatial temperature changes ($\dim \theta' = \Theta/T$ and $\dim \operatorname{lap} \theta = \Theta/L^2$) or as representing how a material diffuses or infuses temperature (internal energy). Thermal conductors diffuse (infuse) temperature (internal energy) at a greater rate than thermal insulators.

A.3. **Another derivation.** [John H. Lienhard IV, John H. Lienhard V, a heat transfer book] In this reference, the energy continuity equation is called heat diffusion equation, *aka* heat conduction equation in other sources.

A.3.1. *Objective.* We must now develop some ideas that will be needed for the design of thermal (heat) exchangers. The most important of these is the notion of an overall *thermal energy transfer coefficient*, *aka* heat transfer coefficient. This is a measure of the general resistance of a thermal (heat) exchanger to the flow of energy (heat), and usually it must be built up from analyses of component resistances. Although we shall count radiation among these resistances, this overall energy transfer coefficient is most often dominated by *convection and conduction.*

We need to know values of the film coefficient $\bar{f}$ to handle convection. Calculating $\bar{f}$ becomes sufficiently complex that we defer it to later chapters. For the moment, we shall take the appropriate value of $\bar{f}$ as known information and concentrate upon its use in the overall heat transfer coefficient.

The thermal (heat) conduction component also becomes more complex than the planar analyses we did in earlier chapters. But its calculation is within our present scope. Therefore we devote this section to deriving the full thermal conduction equation, *aka* heat conduction or heat diffusion equation, solving it in some fairly straightforward cases, and using our results in the overall coefficient. We undertake that task next.

Consider the general temperature distribution in a three-dimensional body as depicted in Fig. 2.1 (a body being heated by a candle). For some reason, say heating from one side, the temperature of the body varies with time and space. This field $\theta = \theta\,[\xi, t] = \theta\,[x, y, z, t]$, defines instantaneous *isothermal surfaces*, θ_1, θ_2 and so on.

We next consider a very important vector associated with the scalar, θ. The vector that has both the magnitude and direction of the maximum increase of temperature at each point is called the *temperature gradient*, $\operatorname{grad} \theta$:

$$\operatorname{grad} \theta = \nabla\theta = \gamma^i \partial_i \theta = \hat{x}\partial_x\theta + \hat{y}\partial_y\theta + \hat{z}\partial_z\theta\,.$$

A.3.2. *Fourier's law.* "Experience" – that is, physical observation – suggests two things about the thermal flow that results from temperature nonuniformities in a body. These are:

(1) e'' and $\operatorname{grad}\theta$ are exactly opposite one another in direction [2]:

$$\frac{e''}{|e''|} \propto -\frac{\nabla\theta}{|\nabla\theta|}$$

and

(2) the magnitude of the thermal flux is directly proportional to the temperature gradient:

$$|e''| \propto |\nabla\theta| \,.$$

Notice that the thermal flux is now written as a quantity that has a specified direction as well as a specified magnitude. Fourier's law summarizes this physical experience succinctly as

$$e'' = -k\nabla\theta = -k \operatorname{grad}\theta \,.$$

The coefficient k – the thermal conductivity – also depends on position and temperature in the most general case:

$$k = k\left[\xi, \theta\left[\xi, t\right]\right] \,.$$

Fortunately, most materials (though not all of them) are very nearly *homogeneous*. Thus we can usually write $k = k\left[\theta\right]$. The assumption that we really want to make is that k is constant. Whether or not that is legitimate must be determined in each case. As is apparent from Fig. 2.2 and Fig. 2.3, k almost always varies with temperature. It always rises with θ in gases at low pressures, but it may rise or fall in metals or liquids. The problem is that of assessing whether or not k is approximately constant in the range of interest. We could safely take k to be a constant for iron between $0\,^\circ\mathrm{C}$ to $40\,^\circ\mathrm{C}$ (see Fig. 2.2), but we would incur error between $-100\,^\circ\mathrm{C}$ to $800\,^\circ\mathrm{C}$. Thus, if $\Delta\theta$ is not large, one can still make a reasonably accurate approximation using a constant average value of k.

We must now write the thermal energy conduction equation in three dimensions. We begin with the energy conservation principle – thermal energy flow equals energy accumulation within the body:

$$e' = i' \,.$$

This time we apply the last equation to a three-dimensional control volume, as shown in Fig. 2.4.1. The control volume is a finite region of a conducting body, which we set aside for analysis. The surface is denoted as s and the volume and the region as v; both are at rest. An element of the surface, $\mathrm{d}s$, is identified and two vectors are shown on $\mathrm{d}s$: one is the unit normal vector, $\hat{n}$ (with $|\hat{n}| = 1$), and the other is the thermal energy flux vector, $e'' = -k \operatorname{grad}\theta = -k\nabla\theta$, at that point on the surface.

We also allow the possibility that a volumetric thermal energy release flow equal to r', $\dim r' = E/TL^3$, is distributed through the region. This might be the result of chemical or nuclear reaction, of electrical resistance heating (Joule heating), of external radiation into the region or of still other causes.

With reference to Fig. 2.4, we can write the thermal energy flow conducted *out* of $\mathrm{d}s$, in dimensions of E/T, as

$$(-k\nabla\theta) \cdot (\hat{n}\,\mathrm{d}s) \,.$$

The thermal energy released (or accumulated) within the region v must be added to the total energy flow into s to get the overall rate of thermal energy addition to v:

$$e' = -\int_s (-k\nabla\theta) \cdot (\hat{n}\,\mathrm{d}s) + \int_v r'\,\mathrm{d}v \,.$$

The rate of energy increase (accumulation) of the region v is

$$i' = \int_v \left(\partial_t \rho c \theta\right)\,\mathrm{d}v \,,$$

where the derivative of θ is in partial form because θ is a function of both position, ξ, and time, t.

[2] This is a consequence of the second law of thermostatics: thermal energy flows in the direction of falling temperature.

Finally, we combine e' and i' using the energy conservation principle. After rearranging the terms, we obtain

$$\int_s (-k\nabla\theta)\cdot(\hat{n}\,\mathrm{d}s) = \int_v \left(\partial_t \rho c\theta + r'\right)\,\mathrm{d}v\,.$$

To get the left-hand side into a convenient form, we introduce Gauss's theorem, which converts a surface integral into a volume integral. This reduces the last equation into

$$\int_v \left(\nabla\cdot k\nabla\theta - \partial_t \rho c\theta + r'\right)\,\mathrm{d}v\,.$$

Next, since the region v is arbitrary, the integrand must vanish identically. We therefore get the *thermal energy diffusion equation* in three dimensions:

$$\nabla\cdot k\nabla\theta + r' = \partial_t \rho c\theta\,.$$

The limitations on this equation are:

- Incompressible medium. (This was implied when no expansion work term was included.)
- No convection. (The medium cannot undergo any relative motion. However, it can be a liquid or gas as long as it sits still.)

If the variation of k with θ is small and if the medium is homogeneous, then k and ρc can be factored out of the last equation to get:

$$\nabla^2\theta + \frac{1}{k}r' = \frac{1}{\alpha}\partial_t\theta\,,$$

where $\alpha = k/\rho c$ is the body thermal diffusivity. In a sense, thermal diffusivity is the measure of thermal inertia. In a substance with high thermal diffusivity, thermal energy moves rapidly through it because the substance conducts energy quickly relative to its volumetric thermal capacity or "thermal bulk".

Geometry. As seen in the thermal energy diffusion equation, *aka* heat equation, when there are no volumetric thermal energy releases, *i.e.*, $r' = 0$, then we have

$$\theta_{,t} = \alpha\theta_{,\xi\xi}\,.$$

Here thermal diffusivity can be geometrically interpreted as the ratio of the time derivative of temperature to its *curvature*, quantifying thus the rate at which temperature concavity is "smoothed out".

Technical note. Notice the reactive term r'/k. It has the dimensions of

$$\dim\frac{r'}{k} = \frac{E/TL^3}{E/TL\Theta} = \frac{\text{flow of chemical energy release}}{\text{thermal energy conduction}}\,.$$

Thus, if thermal conduction is high, then thermal energy coming from the reaction flows quickly through the body, being then released to the environment. Otherwise, then thermal energy coming from the reaction is stored within the body, this, in turns, enhances the reaction rate, since $r = f[\theta]$.

A.4. **Yet another derivation.** [Evans M. Harrell II and James V. Herod, Linear Methods of Applied Mathematics. `http://www.mathphysics.com/pde/HEderiv.html`]

Newton articulated some principles of thermal flow through solids, but it was Fourier who created the correct systematic theory. Inside a solid there is no convective transfer of thermal energy and little radiative transfer, so temperature changes only by conduction, as the energy we now recognize as molecular kinetic energy flows from hotter regions to cooler regions.

(1) The first basic principle of thermal energy is that

> the thermal energy contained in a material (internal energy) is proportional to the temperature, the density of the material and a physical characteristic of the material called the *specific thermal capacity*. In mathematical terms,
>
> $$i = \int_v \rho c\theta\,[t,\xi]\,\mathrm{d}v\,.$$

(2) For the other principles of thermal transfer, let us do some experiments with the following materials: a hot stove, some iron rods of different, relatively short lengths and various widths and various ceramic rods of different lengths and widths. Since these will be thought experiments only, it will be safe to use a finger as the probe. Putting your finger right on the stove will convince you that the energy transfer is proportional to the difference in temperature between your finger and the stove. Using, if necessary, a different, undamaged finger, you will also find that the rate of thermal transfer is inversely proportional to the length of an iron rod intervening between your finger to the stove (fixing the cross-sectional area). In other words, the rate of thermal flow from one region to another is proportional to the temperature gradient between the two regions. You will probably also agree that the rate of thermal flow will be proportional to the area of the contact; for example, a short pin with one end on a hot stove and the other touching your hand is preferable to putting the palm of your hand on a frying pan. Finally, a ceramic material on the stove being usually more pleasant to the touch than hot iron, we see that the rate of thermal transfer depends on the material, as measured with a physical constant known as the *thermal conductivity*. The second basic principle is thus that

> the thermal transfer (thermal flow, thermal power) through the boundary of a region is proportional to the thermal conductivity, to the gradient of the temperature across the region and to the area of contact, so if the boundary of the region v is written as ∂v, with outward normal vector n, then
> $$e' = \int_{\partial v} kn \cdot \operatorname{grad} \theta\,[t,\xi]\;\mathrm{d}s\,.$$

If we differentiate the internal energy equation with respect to time, applying the differentiation under the integral sign, and apply divergence theorem to the thermal transfer integral, then we find that

$$\int_v \partial_t \rho c \theta\,[t,\xi]\;\mathrm{d}v = \int_v \operatorname{div} k \operatorname{grad} \theta\,[t,\xi]\;\mathrm{d}v\,.$$

Since the region v can be an arbitrary piece of the material under study, the integrands must be equal at almost every point. If the material under study is made out of an homogeneous, isotropic substance and if temperature gradients are not so big, then ρ, c and k are independent of the position ξ. Thus, we obtain the thermal diffusion equation

$$\partial_t \theta\,[t,\xi] = \alpha \operatorname{lap} \theta\,[t,\xi]\;,$$

where $\alpha = k/\rho c$ is called *thermal diffusivity*. Ordinary substances have values of α ranging from about $5\,\mathrm{cm}^2/\mathrm{g}$ to $9000\,\mathrm{cm}^2/\mathrm{g}$.

The one-dimensional thermal diffusion equation in Cartesian coordinates

$$\theta_{,t} = \alpha \theta_{,xx}$$

would apply, for instance, to the case of a long, thin metal rod wrapped with insulation, since the temperature of any cross-section will be constant, due to the rapid equilibration to be expected over short distances.

A.5. **Solutions to the thermal diffusion equation.** We are now in position to calculate the temperature distribution or thermal flux in bodies with the help of the heat diffusion equation. In every case, we first calculate $\theta\,[\xi,t]$. Then, if we want the thermal flux as well, we differentiate θ to get e'' from Fourier's law.

The thermal diffusion equation is a partial differential equation (p.d.e.) and the task of solving it may seem difficult, but we can actually do a lot with fairly elementary mathematical tools. For one thing, in one- dimensional steady-state situations the heat diffusion equation becomes an ordinary differential equation (o.d.e.); for another, the equation is linear and therefore not too formidable, in any case. Our procedure can be laid out, step by step:

(1) Play with geometrical analysis – make sketches, pics and drawings, physical analysis, dimensional analysis and approximate methods. Guess the solution

beforehand. Even if you are proven wrong by the formal analysis, guessimations enhance physical intuition!

Note. In the analysis of thermal transfer, the dimensionaly independent system $\{E, L, T, \Theta\}$ is generally the most adequate. However, the SI system, based on $\{M, L, T, \Theta\}$, should not be overlooked.

(2) Pick the coordinate scheme (coordinate system) that best fits the problem and identify the independent variables that determine θ.
(3) Write the appropriate d.e., starting with one of the forms of the thermal diffusion equation.

Note. Non-dimensionalization of the d.e. prior to obtain its solution provides a deeper physical understanding of the phenomenon regardless if it results in an "easier" equation to solve. Use and interpret physically the resulting non-dim. equation as well as any characteristic physical quantities and dimensionless numbers.

Remark. If non-dimensionalization is considered, then do not forget to include all of the physical quantities: quantities that appear in the diffusion equation, in the initial conditions and in the boundary conditions.

(4) Obtain the general solution of the d.e. (This is usually the easiest step.)
(5) Write the "side conditions" on the d.e. – the initial and boundary conditions. This is the trickiest part and the one that most seriously tests our physical or "practical" understanding any heat conduction problem. Normally, we have to make two specifications of temperature on each position coordinate and one on the time coordinate to get rid of the constants of integration in the general solution.

Warning. Very Important: Never, never introduce inaccessible information in a boundary or initial condition. Always stop and ask yourself, "Would I have access to a numerical value of the temperature (or other data) that I specify at a given position or time?" If the answer is no, then your result will be useless.

(6) Substitute the general solution in the boundary and initial conditions and solve for the constants. This process gets very complicated in the transient and multidimensional cases. Numerical methods are often needed to solve the problem.
(7) Put the calculated constants back in the general solution to get the particular solution to the problem.

Note. Non-dimensionalization of the solution can also be helpful at this stage. Not only, non-dim. reduces the number of physical quantities to analyze, so does become useful when plotting equations.

(8) Play with the solution – look it over– see what it has to tell you. Make any checks you can think of to be sure it is correct. Again dimensional analysis is the first tool to apply! Approximate methods and the use of characteristic quantities are also valuable.
(9) If the temperature field is now correctly established, we can, if we wish, calculate the thermal flux at any point in the body by substituting $\theta\,[\xi, t]$ back into Fourier's law.

A.6. **Examples.** Some examples on using the proposed method for solving the thermal diffusion equation.

Example. A large, thin concrete slab of thickness l is "setting". Setting is an exothermic process that releases r', $\dim r' = E/TL^3$, $\mathrm{unit}\, r' = \mathrm{W/m^3}$. The outside surfaces are kept at the ambient temperature, so $\theta_{\mathrm{w}} = \theta_\infty$. What is the maximum internal temperature?

Guess. Thermal energy is being released from the slab center to its walls. If its assumed that the center temperature is greater than the wall (ambient) temperature, then temperature attains its maximum at the center and decreases to ambient temperature. Additionally, since the outer surfaces are kept at constant temperature, then the process is to be steady, but ranging spatially throughout the slab thickness. If one measures the spatially variation as x, then the slab temperature must satisfy $\theta = \theta\,[x]$.

Another important point is given by the symmetry of the problem. The center temperature is maximum at the center, $l/2$, and minimum at the walls, $x = 0$ and $x = l$, with a smooth decay. This gives a room to think about the temperature distribution inside the slab as a parabola with its vertex at $\theta_{\max}$ and with its directrix greater than the horizontal line formed by the wall temperatures, $\theta_{\rm w}$, at $x = 0$ and $x = l$.

Dim. Analysis. Place a Cartesian coordinate axis that runs from one wall to the other covering the slab thickness; let x measure distances within the thickness: $0 \le x \le l$. Since temperature distribution is independent on time (steady process), then $\theta = \theta\,[x]$ and thus x is the independent quantity and θ the dependent quantity. Finally, the slab geometry provides two parameters, x and l, and the slab material, concrete, provides the thermal property, k, thermal conductivity.

Choose the set $\{E, L, T, \Theta\}$ as the dimensionally independent set of dimensions. Within this system, the quantities dimensions are:

$$\dim x = \dim l = L\,, \quad \dim\theta = \dim\theta_{\rm w} = \Theta\,, \quad \dim r' = E/TL^3 \quad \text{and} \qquad \dim k = E/LT\Theta\,.$$

According to Buckingham's theorem, the model can be described by $6 - 4 = 2$ dimensionless quantities. The first dimensionless quantity, Π_1, must include, as an advice, the sought quantity, θ. Form this group by [3]

$$\Pi_1 = \frac{k\,(\theta - \theta_{\rm w})}{r' l^2} = \frac{\text{conduction energy flow gradient}}{\text{released energy flow gradient}} = \frac{\text{conduction gradient}}{\text{“production” gradient}}\,.$$

The second dimensionless quantity, Π_2, can be formed by the geometrical parameters:

$$\Pi_2 = x/l\,.$$

Then, according to the dimensionally homogeneity principle for physics laws, the model can be written as

$$f\,[\Pi_1, \Pi_2] = 0 \implies \Pi_1 = f\,[\Pi_1, \Pi_2] \implies k\,(\theta - \theta_{\rm w})\,/r' l^2 = f\,[x/l]\;.$$

The last equation is the final result of dimensional analysis. The actual form of the function f has to be determined by experiment or by analytic considerations.

Approx. Solution. After dimensional analysis was carried away, we arrived to

$$k\,(\theta - \theta_{\rm w})\,/r' l^2 = f\,[x/l]\;.$$

When guessing the solution, on the other hand, we established that a parabola can express the temperature dependence on the thickness distance. Accordingly, the function f can be then hypothesize to satisfy

$$k\,(\theta - \theta_{\rm w})\,/r' l^2 = a\,(x/l)^2 + b\,(x/l) + c\,,$$

where the coefficients $\{a, b, c\}$ must be determined.

These coefficients can be calculated using the theorem that states that three points uniquely determine a parabola. These points are

$$\begin{cases} \left[x/l = 0, k\,(\theta - \theta_{\rm w})\,/r' l^2 = 0\right]\;, \\ \left[x/l = 1/2, k\,(\theta - \theta_{\rm w})\,/r' l^2 = \Pi\right]\;, \\ \left[x/l = 1, k\,(\theta - \theta_{\rm w})\,/r' l^2 = 0\right]\;, \end{cases}$$

where Π is an unknown dimensionless quantity that, geometrically, shows the parabola height, its distance from the horizontal line formed by the x-axis to the parabola vertex and, physically, the maximum value of temperature, $\theta_{\max} = \theta\,[1/2]$, attained at $x/l = 1/2$, or,

$$\Pi = k\,(\theta_{\max} - \theta_{\rm w})\,/r' l^2\,.$$

Using the points in the hypothetical equation for f, one finds that

$$a = -4\Pi\,, \quad b = 4\Pi \quad \text{and} \quad c = 0\,,$$

and, thus, one arrives at

$$\frac{k\,(\theta - \theta_{\rm w})}{r' l^2} = 4\Pi\frac{x}{l}\left(1 - \left(\frac{x}{l}\right)\right)\,.$$

Doing honest physics (*i.e.*, solving the differential equation), one finds that $4\Pi = 1/2$ or $\Pi = 1/8$, thus the model becomes

$$\frac{k\,(\theta - \theta_{\rm w})}{r' l^2} = \frac{1}{2}\frac{x}{l}\left(1 - \left(\frac{x}{l}\right)\right)$$

and, finally, the maximum temperature, $\theta_{\max}$, is attained at

$$\Pi = \frac{k\,(\theta_{\max} - \theta_{\rm w})}{r' l^2} = \frac{1}{8}\,. \qquad \square$$

Solution. This solution is based in the general procedure to solve the thermal diffusion equation.

[3] A dimensionless quantity that includes θ is readily available: $\theta/\theta_{\rm w}$. However, in thermal transfer, temperature differences are more important than temperatures alone, since differences are the driving force behind the transfer.

Choose a Cartesian coordinate system with ξ varying only in one-dimension, so that

$$0 \leq x \leq l \qquad \text{and} \qquad \theta = \theta\,[x]\ .$$

Write the model equation for a one-dimensional, steady state case:

$$\theta_{,xx} = -r'/k\,.$$

Obtain the general solution to the model equation by integration to have

$$\theta = -\frac{r'}{k}x^2 + c_1 x + c_2\,,$$

where c_1 and c_2 are integration constants.

Apply, next, the boundary conditions

$$\theta\,[x=0] = \theta_{\mathrm{w}} \qquad \text{and} \qquad \theta\,[x=l] = \theta_{\mathrm{w}}\,.$$

Substitute the boundary conditions into the general solution to have

$$c_1 = \frac{r'l}{2k} \qquad \text{and} \qquad c_2 = \theta_{\mathrm{w}}\,.$$

Replace the values of the constants into the general solution to find the particular solution

$$\theta = -\frac{1}{2}\frac{r'}{k}x^2 + \frac{1}{2}\frac{r'}{k}x + \theta_{\mathrm{w}}\,,$$

which can be written in the dimensionless form:

$$\frac{k\,(\theta - \theta_{\mathrm{w}})}{r'l^2} = \frac{1}{2}\frac{x}{l}\left(1 - \left(\frac{x}{l}\right)\right)\,.$$

Finally, as a verifying step, note that the resulting temperature distribution is parabolic and, as expected, symmetrical. It satisfies the boundary conditions at the wall and maximizes in the center. By nondimensionalizing the result, a simple curve can represent all situations. That is highly desirable when the calculations are not simple, as they are here. (Even here θ actually depends on five different quantities and its solution is a single curve on a two-coordinate graph.)□

Appendix B. Coordinate systems and index notation

B.1. **Cartesian coordinate system.** Consider a region $\mathcal{V}$ to be a part of an n dimensional Euclidean space, $\mathcal{E}^n$. Consider, next, a Cartesian coordinate system; *i.e.*, a coordinate system equipped with a frame $\{\gamma_k\}$ whose elements satisfy

$$\gamma_k \cdot \gamma_l = g_{kl} = [k = l]_{\mathrm{Iv}}\ .$$

If frame elements satisfy the last equation, then they are *orthogonal* to each other and *normal* (have unit length); thus, the frame $\{\gamma_k\}$ is said to be *othornormal*.

Express now the position of any point ξ in $\mathcal{V}$ as an n-tuple

$$\xi = \left[\xi^1, \ldots, \xi^n\right]\ ,$$

or as a linear combination of the frame elements

$$\xi = \xi^k \gamma_k\,,$$

where the elements of $\left\{\xi^k\right\}$ are called the *components of* ξ *onto the* $\{\gamma_k\}$ *frame*. (Einstein summation convention in force!)

To find the components of ξ on $\{\gamma_k\}$, apply

$$\xi^k = \xi \cdot \gamma_k\,.$$

B.1.1. *Reciprocal Cartesian coordinate system.* Consider now a reciprocal Cartesian coordinate system $\left\{\gamma^k\right\}$ whose elements are the inverse [4] of the $\{\gamma_k\}$ elements:

$$\gamma^k = {\gamma_k}^{-1} = \gamma_k\,,$$

since the frame elements are orthonormal; *viz.*, ${\gamma_k}^{-1} = \gamma_k/\gamma_k^2$, but $\gamma_k^2 = 1$.

[4] The square of a vector a is defined as $a^2 = aa$, which, by axiom, $a^2 \in \mathcal{E}^1$. The inverse of a nonzero vector a is then defined as $a^{-1} = a/a^2$.

B.1.2. *Geometric derivative, gradient, divergence, Laplacian.* Define the geometric derivative (operator) ∇ by its components as [5]

$$\nabla = \gamma^k \partial_k \, .$$

This means, treat the geometric derivative as another vector.

Consider a scalar field $\phi = \phi\,[\xi]$. Then, define the gradient of ϕ by

$$\operatorname{grad}\phi = \nabla\phi = \gamma^k \partial_k \phi \, .$$

Consider next a vector field $\psi = \psi\,[\xi]$. Then, define the divergence of ψ by

$$\operatorname{div}\psi = \nabla \cdot \psi = \gamma^k \partial_k \cdot \gamma^l \psi_l = \partial_k g^{kl} \psi_l \, ,$$

where $g^{kl} = \gamma^k \cdot \gamma^l$.

Finally, define Laplace operator on a scalar field ϕ by

$$\operatorname{lap}\phi = \operatorname{div}\operatorname{grad}\phi = \nabla \cdot \nabla\phi = \gamma^k \partial_k \cdot \gamma^l \partial_l \phi = \partial_k \partial_l g^{kl} \phi \, .$$

B.2. **Alternative coordinate systems.** Consider next another, alternative, coordinate system $\{\gamma_{k'}\}$, related to the Cartesian coordinate by

$$\xi^k = f\left[\xi^{k'}\right] \, .$$

It is possible now to express the Cartesian frame $\{\gamma_k\}$ in the alternative coordinate system by using the *tangent vectors* $\{\gamma_{k'}\}$ *defined* by [6]

$$\gamma_{k'} = \partial_{k'} \xi \, .$$

These tangent vectors need *not* be orthogonal nor normal.

Onto this alternative frame $\{\gamma_{k'}\}$, any point position ξ can be expressed as a linear combination of the alternative frame elements:

$$\xi = \xi^{k'} \gamma_{k'} \, ,$$

where $\xi^{k'} = \xi \cdot \gamma_{k'}$.

The metric coefficients for the alternative frame are found by

$$g_{k'l'} = \gamma_{k'} \cdot \gamma_{l'} \, .$$

Appendix C. Notation

C.1. **Maths.**

C.1.1. *Sets.*

- set: set A, $\mathcal{A}$.
- n-dim set: nset nA, $\mathcal{A}^n$.
- n-dim. Euclidean space: espace n, $\mathcal{E}^n$.
- region: region A, $\mathcal{A}$.

C.1.2. *Functions.*

- function value at: vat x, $[x]$.

C.1.3. *Geometric objects.*

- boundary: bound, ∂.
- surface: surf, s.
- volume: vol, v.

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[5] In traditional notation,

$$\nabla = \gamma^k \frac{\partial}{\partial \xi^k} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \, .$$

[6] In traditional notation:

$$\gamma_{k'} = \partial_{k'} \xi = \frac{\partial \xi}{\partial \xi^{k'}} \, .$$

C.1.4. *Geometric algebra.*

- magnitude: magn u, $|u|$.
- inner product: iprod, $a \cdot b$.
- outer product: oprod, $a \wedge b$.
- inverse: inv a, a^{-1}.
- unit vector: uvec a, $\hat{a}$.

C.1.5. *Geometric calculus.*

- geometric derivative: gder, ∇.
- Laplace operator (derivative): lder, ∇^2.
- divergence: div, div.
- gradient: grad, grad.
- Laplace operator: lap, lap.

C.1.6. *Calculus.*

- differential operator: dx, d.
- difference operator: Dx, Δ.

C.1.7. *Derivatives.*

- dot derivative: dt a, $\dot{a}$.
- dot dot derivative: ddt a, $\ddot{a}$.
- expanded ordinary derivative: xod at, $\dfrac{\mathrm{d}a}{\mathrm{d}t}$.
- expanded partial derivative: xpd at, $\dfrac{\partial a}{\partial t}$.
- indexed ordinary derivative: iod t, d_t.
- indexed partial derivative: ipd t, ∂_t.
- indexed geometric derivative: igder k, ∂_k.
- comma derivative: cder Tt, $T_{,t}$.

C.1.8. *Index notation.*

- basis vector: bvec, γ.
- frame (basis) element: fbvec 1, γ_1.
- reciprocal frame (basis) element: rbvec 1, γ^1.
- frame: frm k, $\{\gamma_k\}$.
- reciprocal frame: rfrm k, $\left\{\gamma^k\right\}$.
- metric: met, g.
- metric in frame: fmet 12, g_{12}.
- metric in reciprocal frame: rmet 12, g^{12}.
- up indexed partial derivative: upipd 1, ∂^1.
- down indexed partial derivative: dnipd 1, ∂_1.
- frame (contravariant) components: fvec pos 1, ξ^1.
- reciprocal frame (covariant) components: rvec pos 1, ξ_1.

C.1.9. *Matrices.*

- diagonal: diag, diag.
- signature: sig, sig.

C.1.10. *Brackets.*

- Iverson brackets: iverson k, $[k]_{\mathrm{Iv}}$.

C.2. **Physics.**

C.2.1. *Mechanics.*

- position: pos, ξ.
- pressure: press, p.

C.2.2. *Energy transport.*

- time change rate: rate a, a'.
- flow: flow q, q'.
- flux: flux q, q''.
- thermal energy (heat): then, e.
- mechanical work: work, w.
- internal energy: ien, i.
- accumulation (*e.g.*, internal energy): accu ien, i'.
- thermodynamic temperature: temp, θ.
- specific heat capacity: kshcap, c.
- thermal diffusivity: kthdiff, α.
- thermal conductivity: kthcond, k.
- local film (heat transfer) coefficient (thermal convection): kthconv, f.
- global (average) film coefficient: kavthconv, $\bar{f}$.
- time constant (conduction-convection lumped solution): ktime, τ.
- thermal radiation absorbance: absorb, α.
- thermal radiation reflectance: reflect, ρ.
- thermal radiation transmittance: transm, τ.
- body energy flux distribution function: dbflux, $\check{e}_{\lambda_{\mathrm{body}}}$.
- black-body energy flux: bbflux, e''_{bb}.
- black-body energy flux distribution function: dbbflux, $\check{e}_{\lambda_{\mathrm{bb}}}$.

C.2.3. *Mass transport.*

- n (number of) particles: npart, n.
- mass: mass, m.
- mass density: dens, ρ.
- chemical reaction thermal energy: chthen, r.

C.2.4. *Waves.*

- wavelength: wlen, λ.
- wave frequency: wfreq, μ.

C.2.5. *Physical constants.*

- Boltzmann constant: kboltz, k_{b}.
- Stefan-Boltzmann constant: kstef, σ_{sb}.
- Planck constant: kplanck, h_{p}.
- speed of light in vacuum: klight, c_0.
- ideal gas constant: kgas, r_{g}.
- Avogadro's number: kavog, n_{a}.

C.3. **Dimensional analysis.**

- units of a physical quantity: unit q, $\operatorname{unit} q$.
- dimension of a physical quantity: dim q, $\dim q$.
- dimensionless quantity: kdim, Π.
- Biot number: kbiot, Π_{bi}.