# Introduction to Data Processing and Representation (236201)

## Spring 2022

Homework 2

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# Part I
# Theory

1. In this question we consider the space of squared integrable functions $E = L^2(\mathbb{R}, \mathbb{C})$, to which we associate the natural Hermitian product. Let $f \in E$ and $F$ be a subspace of $E$ of finite dimension n.

   (a) Let $\beta_1, ..., \beta_n \in F$ be a finite family of functions such that $F = Vec(\beta_1, ..., \beta_n)$. Let $1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n$ be a set of $k$ increasing integers.

      i. We want to find the $k - term$ approximation of $f$ in $F$. As we saw in the tutorial we can write:

$$\left\|f - \hat{f}\right\|_2^2 = \left\|\underbrace{(f - proj_F f)}_{\perp F} + \underbrace{\left(proj_F f - \hat{f}\right)}_{\in F}\right\|_2^2 =$$

$$= \left\|\underbrace{(f - proj_F f)}_{constant}\right\|_2^2 + \left\|\left(proj_F f - \hat{f}\right)\right\|_2^2$$

Now using pythagoras we get that:

$$= \|f\|_2^2 - \|proj_F f\|_2^2$$

Where

$$\|proj_F\|_2^2 = \sum_i \left(\langle \beta_i, f \rangle\right)^2$$

So overall, we get that

$$\left\|f - \hat{f}\right\|_2^2 = \|f\|_2^2 - \sum_{i=1}^{n} \left(\langle \beta_i, f \rangle\right)^2$$

Now we'll calculate the SE according to the above, and the fact that $\hat{f} = \sum_{i \in \{i_1, ..., i_k\}} \langle \beta_i, f \rangle \beta_i$:

$$\left\|f - \hat{f}\right\|_2^2 = \left\|\left(f - \sum_i \langle \beta_i, f \rangle \beta_i\right)\right\|_2^2 + \left\|\left(\sum_i \langle \beta_i, f \rangle \beta_i - \sum_{j=1}^{k} \langle \beta_{i,j}, f \rangle \beta_{i,j}\right)\right\|_2^2 =$$

$$= \|f\|_2^2 - \sum_{i \in \{i_1, ..., i_k\}} \langle \beta_i, f \rangle^2$$

      ii. If we want to find the best $k$-approximation of $f$ in $F$ in the SE sense, we shall minimize the above

$$argmin\, \|f\|_2^2 - \sum_{i \in \{i_1, ..., i_k\}} \langle \beta_i, f \rangle^2$$

but since the norm of $f$ is a constant, the problem is basically maximazing $\sum_{i \in \{i_1, ..., i_k\}} \langle \beta_i, f \rangle^2$.

$$argmax \sum_{i \in \{i_1, ..., i_k\}} \langle \beta_i, f \rangle^2$$

The $k$ approximation might not be unique, because it depends on the inner products $\langle \beta_i, f \rangle$ that might be the same for $\beta_i$ and some other $\beta_j$ .

   (b) Let $\beta_1, ..., \beta_n \in F$ and $\tilde{\beta}_1, ..., \tilde{\beta}_n \in F$ be two different finite families of orthonormal functions.

      i. Both function families are different and independent thus both are basis for $F$. As written in the question above, $\hat{f} = \sum_{i=1}^{n} \langle \beta_i, f \rangle \beta_i$ and since $\left\{\tilde{\beta}_i\right\}_{i=1}^{n}$ is also an orthonormal basis, $\hat{f} = \sum_{i=1}^{n} \left\langle \tilde{\beta}_i, f \right\rangle \tilde{\beta}_i$. Therefore,

$$\left\|f - \hat{f}\right\|_2^2 = \|f\|_2^2 - \sum_{i=1}^{n} \left(\langle \beta_i, f \rangle\right)^2 = \|f\|_2^2 - \sum_{i=1}^{n} \left(\left\langle \tilde{\beta}_i, f \right\rangle\right)^2$$

Which means that the n-approximations of $f$, in the SE sense, using the $\beta$ family on one hand and the $\tilde{\beta}$ family on the other hand, are the same.

ii. Let $k \in \{1, ..., n-1\}$.
The SE for the $\beta$ functions family is:

$$\|f\|_2^2 - \sum_{i \in \{i_1, \ldots, i_k\}} \langle \beta_i, f \rangle^2$$

And for the $\tilde{\beta}$ functions family is:

$$\|f\|_2^2 - \sum_{i \in \{i_1, \ldots, i_k\}} \left\langle \tilde{\beta}_i, f \right\rangle^2$$

Since we don't have any other information about the basis $\{\beta_i\}_{i=1}^n$ and $\left\{\tilde{\beta}_i\right\}_{i=1}^n$ we can't say anything accurate about the k-term approximation other than what is said in the section above.

2. Haar matrix and Walsh-Hadamard matrix.
Given $t \in [0,1]$, consider the signal as $\phi(t) = a + bcos(2\pi t) + c \cdot cos^2(\pi t)$, where $a, b, c$ are constants.

(a) In this section, the $4x4$ Haar matrix is given by:

$$\mathbf{H}_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix}$$

i. We want to prove that the matrix above is unitary. We'll show that $\mathbf{H}_4^T \mathbf{H}_4 = \mathbf{H}_4 \mathbf{H}_4^T = I$.

$$\mathbf{H}_4 \mathbf{H}_4^T = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} =$$

$$= \frac{1}{4} \cdot \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I$$

$$\mathbf{H}_4^T \mathbf{H}_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} =$$

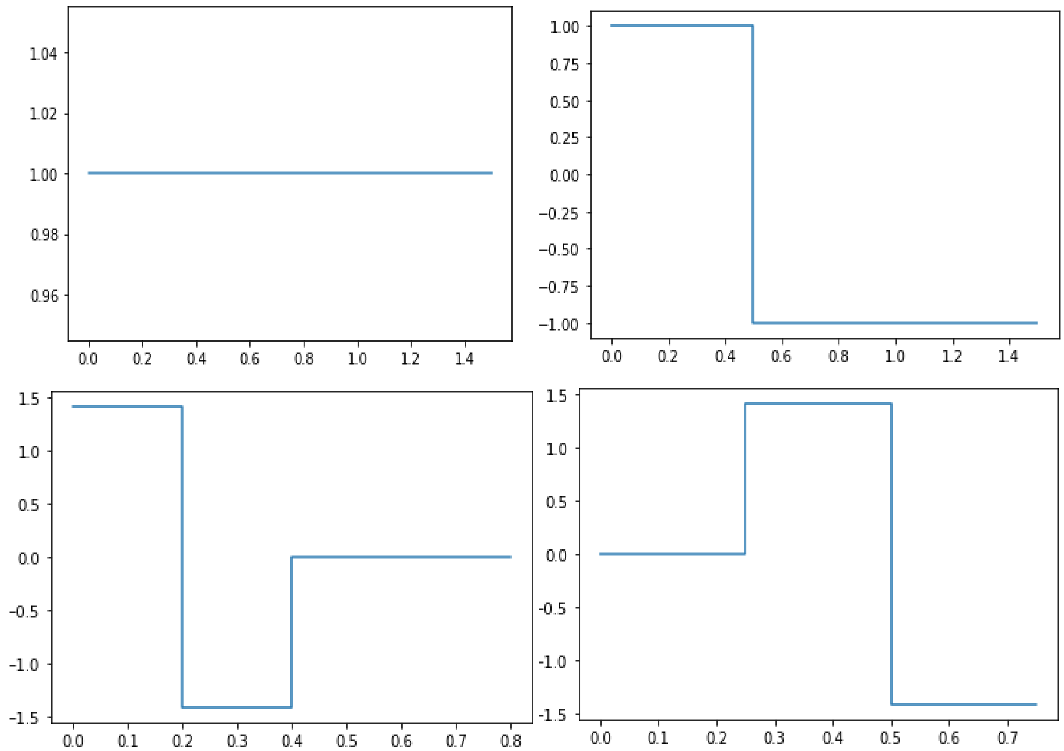
$$= \frac{1}{4} \cdot \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = I$$

Therefore,

$$\mathbf{H}_4^T \mathbf{H}_4 = \mathbf{H}_4 \mathbf{H}_4^T = I$$

So, $\mathbf{H}_4$ is a unitary matrix by definition.

ii. The wavelets are:




iii. The best approximation of $\phi$ using the Haar basis, as we saw in class,

$$\hat{\phi}_{optimal}(t) = \sum_{i=1}^{4} \langle \psi_i, \phi(t) \rangle \, \psi_i$$

Which is the projection of $\phi(t)$ onto the span of the coefficients $\{\psi_i\}_{i=1}^{n}$.
So we would like to find the exact coefficients by calculations to find the best approximation of $\phi$.
We saw in the lecture that $\psi_i$ can be written as:

$$\psi_i = \sqrt{N} \int_0^1 \phi(t) \mathbf{1}_{\Delta_i}(t) dt = \left\langle \sqrt{4} \mathbf{1}_{\Delta_i}(t), \phi(t) \right\rangle$$

And as for our case, we know that $\hat{\phi} = \sum_{i=1}^{4} \langle \psi_i, \phi(t) \rangle \, \psi_i = \sum_{i=1}^{4} \left\langle \sqrt{4} \mathbf{1}_{\Delta_i}(t), \phi(t) \right\rangle \sqrt{4} \mathbf{1}_{\Delta_i}(t)$.
Now we'll calculate each coefficient $\{\psi_i\}_{i=1}^{4}$.

$$\psi_1 = \left\langle \sqrt{4} \mathbf{1}_{\Delta_1}(t), \phi(t) \right\rangle = \sqrt{4} \int_0^1 \mathbf{1}_{\Delta_1} \cdot \left(a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)\right) dt =$$

$$= 2 \cdot \int_{\Delta_1} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) dt =$$

$$= 2 \cdot at|_0^{0.25} + 2b \cdot \frac{sin(2\pi t)}{2\pi}|_0^{0.25} + 2c \cdot \frac{2\pi t + sin(2\pi t)}{4\pi}|_0^{0.25} =$$

$$= \frac{a}{2} + \frac{b}{\pi} + \frac{2c + \pi c}{4\pi}$$

$$\psi_2 = \left\langle \sqrt{4} \mathbf{1}_{\Delta_2}(t), \phi(t) \right\rangle = \sqrt{4} \int_0^1 \mathbf{1}_{\Delta_2} \cdot \left(a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)\right) dt =$$

$$= 2 \cdot \int_{\Delta_2} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)dt =$$

$$= 2 \cdot at|_{0.25}^{0.5} + 2b \cdot \frac{sin(2\pi t)}{2\pi}|_{0.25}^{0.5} + 2c \cdot \frac{2\pi t + sin(2\pi t)}{4\pi}|_{0.25}^{0.5} =$$

$$= \frac{a}{2} - \frac{b}{\pi} + \frac{-2c + \pi c}{4\pi}$$

$$\psi_3 = \left\langle \sqrt{4}\mathbf{1}_{\Delta_3}(t), \phi(t) \right\rangle = \sqrt{4} \int_0^1 \mathbf{1}_{\Delta_3} \cdot \left(a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)\right) dt =$$

$$= 2 \cdot \int_{\Delta_3} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)dt =$$

$$= 2 \cdot at|_{0.5}^{0.75} + 2b \cdot \frac{sin(2\pi t)}{2\pi}|_{0.5}^{0.75} + 2c \cdot \frac{2\pi t + sin(2\pi t)}{4\pi}|_{0.5}^{0.75} =$$

$$= \frac{a}{2} - \frac{b}{\pi} + \frac{-2c + \pi c}{4\pi}$$

$$\psi_4 = \left\langle \sqrt{4}\mathbf{1}_{\Delta_4}(t), \phi(t) \right\rangle = \sqrt{4} \int_0^1 \mathbf{1}_{\Delta_4} \cdot \left(a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)\right) dt =$$

$$= 2 \cdot \int_{\Delta_4} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)dt =$$

$$= 2 \cdot at|_{0.75}^{1} + 2b \cdot \frac{sin(2\pi t)}{2\pi}|_{0.75}^{1} + 2c \cdot \frac{2\pi t + sin(2\pi t)}{4\pi}|_{0.75}^{1} =$$

$$= \frac{a}{2} + \frac{b}{\pi} + \frac{2c + \pi c}{4\pi}$$

Now we can plug in our calculations and compute the MSE as we learned in class:

$$\Psi\left(\phi - \hat{\phi}\right) = \int_0^1 (\phi(t))^2 \, dt - \sum_{i=1}^{4} (\psi_i)^2$$

iv. The best approximation for $i \in \{1, 2, 3, 4\}$:
We'll get the best approximation when we take the functions associated with the largest coefficiant $\psi_i$ .
We'll use the wavelest graph for this section to identify where the coefficiant is maximal.

$$\psi_1 = \frac{1}{2}\int_0^1 \phi(t)dt = \frac{1}{2}\int_0^1 a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) =$$

$$= \frac{1}{2}\left(a + \frac{1}{2}c\right)$$

$$\psi_2 = \frac{1}{2}\int_0^{0.5} \phi(t)dt - \frac{1}{2}\int_{0.5}^1 \phi(t)dt = 0$$

$$\psi_3 = \frac{\sqrt{2}}{2}\int_0^{0.25} \phi(t)dt - \frac{1}{2}\int_{0.25}^{0.5} \phi(t)dt =$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}b}{\pi} + \frac{\sqrt{2}c}{2\pi}\right)$$

$$\psi_4 = \frac{\sqrt{2}}{2}\int_{0.5}^{0.75} \phi(t)dt - \frac{1}{2}\int_{0.75}^1 \phi(t)dt =$$

$$= -\frac{1}{2}\left(\frac{\sqrt{2}b}{\pi} + \frac{\sqrt{2}c}{2\pi}\right)$$

Now that we have the coefficiants we can calculate the $k$-term approximation:

$$1\,term:\ \psi_1 \cdot \psi_1^H(t)$$

$$2\,term:\ \psi_3 \cdot \psi_3^H(t) + \psi_1 \cdot \psi_1^H$$

$$3\,term:\ \psi_4 \cdot \psi_4^H(t) + \psi_3 \cdot \psi_3^H(t) + \psi_1 \cdot \psi_1^H$$

$$4\,term:\ \psi_4 \cdot \psi_4^H(t) + \psi_3 \cdot \psi_3^H(t) + \psi_1 \cdot \psi_1^H$$

v. We are given now:

$$a = \frac{1}{\pi}, b = 1, c = \frac{3}{2}$$

we'll calculate the k-term approximations using those accurate numbers:

$$\psi_1 = \frac{1}{2}\left(\frac{1}{\pi} + \frac{3}{4}\right) = 0.53$$

$$\psi_2 = 0$$

$$\psi_3 = \frac{1}{\sqrt{2}\pi} + \frac{3}{4\sqrt{2}\pi} = 0.4$$

$$\psi_4 = -\frac{1}{\sqrt{2}\pi} - \frac{3}{4\sqrt{2}\pi} = -0.4$$

The largest coefficient is $\psi_1$ so the k-term approximation is the same as in the previous section.

(b) In this section we are given a $4x4$ Walsh-Hadamard matrix:

$$\mathbf{W}_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

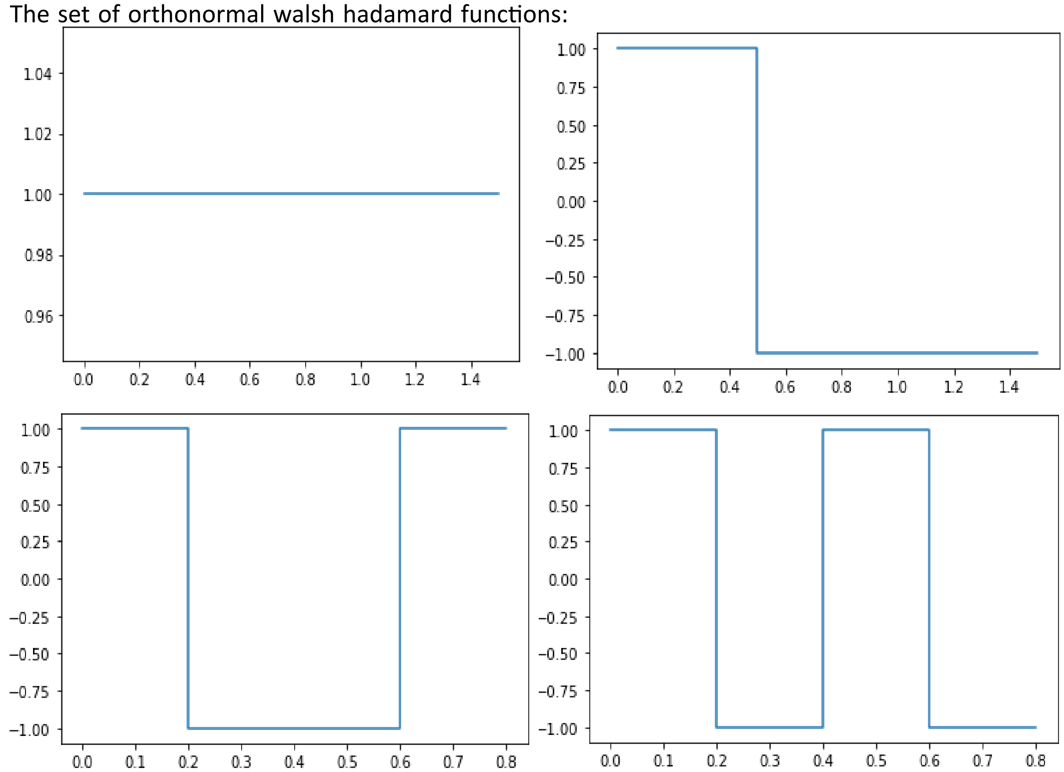
i. We'll show that the matrix above is unitary:

$$\mathbf{W}_4\mathbf{W}_4^T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = \mathbf{W}_4^T\mathbf{W}_4 = I$$

Therefore, by definition $\mathbf{W}_4$ is a unitary matrix.

ii. The set of orthonormal walsh hadamard functions:



iii. As we learnt in class, the best approximation of $\phi$ using the walsh hadamard basis is:

$$\hat{\phi}(t) = \sum_{i=1}^{4} \hat{\phi}_i \psi_i^W(t)$$

We saw in the lecture and tutorial that the best coefficiants are $\left\langle \psi_i^W(t), \phi(t) \right\rangle$ and the matching MSE is

$$\Psi\left(\phi - \hat{\phi}\right) = \int_0^1 (\phi(t))^2\, dt - \sum_{i=1}^{4} \left(\hat{\phi}_i\right)^2$$

iv. We'll once again calculate the coefficients like in the previous section:

$$\hat{\phi}_1 = \int_0^1 a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) dt =$$

$$= a + \frac{c}{2}$$

$$\hat{\phi}_2 = \int_0^{0.5} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) dt - \int_{0.5}^1 a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) dt =$$

$$= 0$$

$$\hat{\phi}_3 = \int_0^{0.25} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) dt - \int_{0.25}^{0.75} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) dt$$

$$+ \int_{0.75}^{1} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t) dt =$$

$$= \frac{(\pi + 2)\, c + 4b + 2\pi a}{8\pi} - \frac{(\pi - 2)\, c - 4b + 2\pi a}{4\pi} + \frac{\pi\,(c + 2a) + 4b + 2c}{8\pi} =$$

$$= \frac{8c + 16b}{8\pi} = \frac{1}{\pi}\,(c + 2b)$$

$$\hat{\phi}_4 = \int_0^{0.25} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)dt - \int_{0.25}^{0.5} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)dt$$
$$+ \int_{0.5}^{0.75} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)dt - \int_{0.75}^{1} a + b \cdot cos(2\pi t) + c \cdot cos^2(\pi t)dt =$$
$$== \frac{(\pi+2)\,c + 4b + 2\pi a}{8\pi} - \frac{(\pi-2)\,c - 4b + 2\pi a}{4\pi} - \frac{\pi\,(c+2a) + 4b + 2c}{8\pi} -$$
$$+ \frac{(\pi-2)\,c - 4b + 2\pi a}{4\pi} = 0$$

We know from the given conditions that - since, $a \geq b, c \geq 0$, $\hat{\phi}_3 \leq \hat{\phi}_1$ when plugging in these conditions. So overall we get that:

$$1\,term: \; \hat{\phi} = \hat{\phi}_1 \cdot \psi_1^W(t)$$
$$2\,term: \; \hat{\phi} = \hat{\phi}_1 \cdot \psi_1^W(t) + \hat{\phi}_3 \cdot \psi_3^W(t)$$
$$3\,term: \; \hat{\phi} = \hat{\phi}_1 \cdot \psi_1^W(t) + \hat{\phi}_2 \cdot \psi_2^W(t) + \hat{\phi}_3 \cdot \psi_3^W(t)$$
$$4\,term: \; \hat{\phi} = \hat{\phi}_4 \cdot \psi_4^W(t) + \hat{\phi}_1 \cdot \psi_1^W(t) + \hat{\phi}_2 \cdot \psi_2^W(t) + \hat{\phi}_3 \cdot \psi_3^W(t)$$

v. We'll plug in the given numbers

$$a = \frac{1}{\pi}, b = 1, c = \frac{3}{2}$$

and get that:

$$\hat{\phi}_1 = \frac{1}{\pi} + \frac{3}{4} = 1.068$$
$$\hat{\phi}_2 = 0$$
$$\hat{\phi}_3 = \frac{1}{\pi}\left(\frac{3}{2} + 2\right) = 1.114$$
$$\hat{\phi}_4 = 0$$

So since this time, $\hat{\phi}_3 \geq \hat{\phi}_1$, the k-term approximations are:

$$1\,term: \; \hat{\phi} = \hat{\phi}_3 \cdot \psi_3^W(t)$$
$$2\,term: \; \hat{\phi} = \hat{\phi}_1 \cdot \psi_1^W(t) + \hat{\phi}_3 \cdot \psi_3^W(t)$$
$$3\,term: \; \hat{\phi} = \hat{\phi}_1 \cdot \psi_1^W(t) + \hat{\phi}_2 \cdot \psi_2^W(t) + \hat{\phi}_3 \cdot \psi_3^W(t)$$
$$4\,term: \; \hat{\phi} = \hat{\phi}_4 \cdot \psi_4^W(t) + \hat{\phi}_1 \cdot \psi_1^W(t) + \hat{\phi}_2 \cdot \psi_2^W(t) + \hat{\phi}_3 \cdot \psi_3^W(t)$$

3. On Hadamard matrices.

(a) We want to prove that $\mathbf{H}_N$ is a symmetric, real and unitary matrix. Where $N = 2^n$, for some fixed $n \in \mathbb{N}^*$. Also we'll show that $\mathbf{H}_N = \lambda_N A$ for some constant $\lambda_N \in \mathbb{R}$, and $A$ is a matrix with only $\pm 1$ entries.
Since we want to prove the above for all $n \in \mathbb{N}^*$, we'll do that using induction.
Base:
The base case is for $n = 1$ so $N = 2^1 = 2$. The Hadamard matrix in this case is:

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{H}_2^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

And so the matrix is symmetric.
Unitarity:

$$\mathbf{H}_2\mathbf{H}_2^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

And because they are symmetric, $\mathbf{H}_2^T\mathbf{H}_2 = \mathbf{H}_2\mathbf{H}_2^T = I$. Therefore, the matrix is unitary by definition.
Also we can rewrite the matrix as:

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \lambda_2 \cdot A$$

where $\lambda_2$ is a constant and $A$ is a matrix with only $\pm 1$ entries.

(1) We assume the statements above are true for $n-1, N = 2^{n-1}$. Now we'll prove they hold for $n, N' = 2^n$.

$$\mathbf{H}_{N'} = \mathbf{H}_2 \bigotimes \mathbf{H}_N = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{H}_N & \mathbf{H}_N \\ \mathbf{H}_N & -\mathbf{H}_N \end{pmatrix}$$

$$\mathbf{H}_{N'}^T = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{H}_N & \mathbf{H}_N \\ \mathbf{H}_N & -\mathbf{H}_N \end{pmatrix}^T = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{H}_N^T & \mathbf{H}_N^T \\ \mathbf{H}_N^T & -\mathbf{H}_N^T \end{pmatrix} =$$

$$\underbrace{=}_{(1)}\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{H}_N & \mathbf{H}_N \\ \mathbf{H}_N & -\mathbf{H}_N \end{pmatrix} = \mathbf{H}_{N'}$$

So the matrix is symmetric.
Unitarity:

$$\mathbf{H}_{N'}\mathbf{H}_{N'}^T = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{H}_N & \mathbf{H}_N \\ \mathbf{H}_N & -\mathbf{H}_N \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{H}_N & \mathbf{H}_N \\ \mathbf{H}_N & -\mathbf{H}_N \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} \mathbf{H}_N\mathbf{H}_N + \mathbf{H}_N\mathbf{H}_N & \mathbf{H}_N\mathbf{H}_N - \mathbf{H}_N\mathbf{H}_N \\ \mathbf{H}_N\mathbf{H}_N - \mathbf{H}_N\mathbf{H}_N & \mathbf{H}_N\mathbf{H}_N + \mathbf{H}_N\mathbf{H}_N \end{pmatrix} =$$

$$\underbrace{=}_{(1)}\frac{1}{2}\begin{pmatrix} 2I & 0_{NxN} \\ 0_{NxN} & 2I \end{pmatrix} = I = \mathbf{H}_{N'}^T\mathbf{H}_{N'}$$

And if we denote for $\mathbf{H}_N = \lambda_N A_N$, we get that:

$$\mathbf{H}_{N'} = \frac{1}{\sqrt{2}}\begin{pmatrix} \lambda_N A_N & \lambda_N A_N \\ \lambda_N A_N & -\lambda_N A_N \end{pmatrix} = \frac{\lambda_N}{\sqrt{2}}\begin{pmatrix} A_N & A_N \\ A_N & -A_N \end{pmatrix}$$

We see that $A = \begin{pmatrix} A_N & A_N \\ A_N & -A_N \end{pmatrix}$ is a matrix with only $\pm 1$ entries, and $\lambda = \frac{\lambda_N}{\sqrt{2}}$.
Overall, we proved that $\mathbf{H}_N$ is a symmetric, real and unitary matrix, for all $n \in \mathbb{N}^*$.

(b) We denote $S(s)$ the number of changes of sign in s, a sequence of digit numbers taking the value $\pm 1$.

i. $s_1, s_2$ are squences of the same length. To calculate $S(s_1, s_2)$ we want to consider if there is a sign change between them when we concatinate the sequences. so:
If there is a sign change between $s_1, s_2$, meaning $s_1$ ends with the opposite sign to the start of $s_2$, we get that $S(s_1, s_2) = S(s_1) + S(s_2) + 1$.
If there is no sign change between $s_1, s_2$, meaning they end and start with the same sign, we get that $S(s_1, s_2) = S(s_1) + S(s_2)$.

ii. We want to prove that the number of changes of sign in the rows of $\mathbf{H}_N$ are the first N integers starting at 0 - $\{0, ..., N-1\}$.
Again because of the structure of the Hadamard matrix, we'll fix $n$ and prove the above using induction.
Base:
For $n = 1, N = 2^1$:

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

We have 1 change of sign in the second row only.
(1) We assume the statements above are true for $n-1, N = 2^{n-1}$.
using the assumption (1), each row has a value of sign changes between $0$ and $2^{n-1}$. The first $2^{n-1}$ rows of $H_{2n}$ are $r_i r_i$ and the last $2^{n-1}$ rows are $r_i(-r_i)$ for some $i \in \{1, ..., N\}$. So using the section above, we get that $S(s_i s_i) = 2 \cdot j \vee 2 \cdot j + 1$ where $j$ is the amount of sign changes in the row $r_i$.
So, overall we get that the largest amount of sign changes in $H_N$ is $2 \cdot (N-1) = 2^n - 1$ as needed.

4. On Haar matrices.

   (a) Is $H_{2N}$ symmetric? Answer: No.
   We'll give an example -

$$H_4 = \begin{pmatrix} H_2 \bigotimes & (1,1) \\ I_2 \bigotimes & (1,-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

   Which is not symmetric.

   (b) Is $H_{2N}$ orthogonal? Answer: No.
   We'll take the matrix above once again for the example:

$$H_4 H_4^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \neq I$$

   So it's not orthogonal by definition.

   (c) Is $H_{2N}$ unitary? Answer: No. Same example as in the previous section.

   (d) A recursive equation that produces the matrix $\tilde{H}_{2N}$:

$$\tilde{H}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\tilde{H}_{2N+2} = \begin{pmatrix} \tilde{H}_{2N} \bigotimes & (1,1) \\ I_{2N} \bigotimes & (1,-1) \end{pmatrix}$$

   (e) We want to prove that:

$$\left(A \bigotimes B\right)^T = A^T \bigotimes B^T$$

   By definition of kronecker product:

$$A \bigotimes B = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & ... & ... & ... \\ ... & ... & ... & ... \\ a_{n1}B & ... & ... & a_{nn}B \end{pmatrix}$$

   So

$$\left(A \bigotimes B\right)^T = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & ... & ... & ... \\ ... & ... & ... & ... \\ a_{n1}B & ... & ... & a_{nn}B \end{pmatrix}^T = \begin{pmatrix} a_{11}B^T & a_{21}B^T & ... & a_{n1}B^T \\ a_{12}B^T & ... & ... & ... \\ ... & ... & ... & ... \\ a_{1n}B^T & ... & ... & a_{nn}B^T \end{pmatrix} = A^T \bigotimes B^T$$

   As needed.

(f) The recursion equation between $\hat{H}$ matrices:

$$\hat{H}_2 = \tilde{H}_2^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Using the recursive definition from the sections above, we know that:

$$\tilde{H}_{2N+2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{H}_{2N} \bigotimes & (1,1) \\ I_{2N} \bigotimes & (1,-1) \end{pmatrix}$$

$$\Rightarrow \hat{H}_{2N+2} = \tilde{H}_{2N+2}^T = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{H}_{2N} \bigotimes & (1,1) \\ I_{2N} \bigotimes & (1,-1) \end{pmatrix}^T =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\tilde{H}_{2N} \bigotimes (1,1)\right)^T & (I_{2N} \bigotimes (1,-1))^T \end{pmatrix} \underbrace{=}_{previous\,question}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{H}_{2N}^T \bigotimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2N} \bigotimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{H}_{2N} \bigotimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2N} \bigotimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}$$

**Part II**

# Implementation

1. Considering the function $\phi(x,y) = Acos(2\pi\omega_x x)sin(2\pi\omega_y y)$ for $(x,y) \in [0,1]\times[0,1]$ and the given variables values: $A = 2500, \omega_x = 2$ and $\omega_y = 7$, we'll implement according to the following instructions:

   (a) For utilizing our calculations later, let us develop the following:

   **(*)** using trigonometric identity: $cos(2x) = 1 - 2sin^2 x$

$$\int_a^b sin^2(\alpha t)dt \overset{(*)}{=} \int_a^b \frac{1}{2}\left(1 - cos(2\alpha t)\right)dt = \frac{1}{2}\left(t - \frac{1}{2\alpha}sin(2\alpha t)\right)\Big|_a^b$$

   In addition, we'll remind one the results from the integral calculations in the previous part:

$$\int_a^b cos^2(\alpha t)dt = \frac{1}{2}\left(t + \frac{1}{2\alpha}sin(2\alpha t)\right)\Big|_a^b$$

   Now for the derivative energy, let's develop these:

$$\frac{\partial\phi(x,y)}{\partial x} = -2\pi\omega_x Asin(2\pi\omega_x x)sin(2\pi\omega_y y)$$
$$\frac{\partial\phi(x,y)}{\partial y} = 2\pi\omega_y Acos(2\pi\omega_x x)cos(2\pi\omega_y y)$$

   We notice that sin and cos functions are limited to the domain $[-1,1]$,and therefore our $\phi$ function is limited to $[-A,A]$, let's check it's range:

$$\phi\left(0, \frac{1}{4\omega_y}\right) = A = 2500$$
$$\phi\left(0, -\frac{1}{4\omega_y}\right) = A = -2500$$

We can conclude that $[-2500, 2500]$ is the function's actual range.
Let's compute the horizontal-derivative energy:

$$\int_0^1\int_0^1\left(\frac{\partial\phi(x,y)}{\partial x}\right)^2dxdy=\int_0^1\int_0^1\left(-2\pi\omega_x A sin(2\pi\omega_x x)sin(2\pi\omega_y y)\right)^2dxdy=$$

$$=4\pi^2\omega_x^2A^2\int_0^1 sin^2(2\pi\omega_x x)dx\int_0^1 sin^2(2\pi\omega_y y)dy=$$

$$=4\pi^2\omega_x^2A^2\left[\frac{x}{2}-\frac{sin(4\pi\omega_x x)}{8\pi\omega_x}\right]_0^1\left[\frac{y}{2}-\frac{sin(4\pi\omega_y y)}{8\pi\omega_y}\right]_0^1=$$

$$=\pi^2\omega_x^2A^2\cdot 1\cdot 1=\pi^2\omega_x^2A^2\approx\mathbf{2.47\times 10^8}$$

And at last, let's compute the vertical-derivative energy:

$$\int_0^1\int_0^1\left(\frac{\partial\phi(x,y)}{\partial y}\right)^2dxdy=\int_0^1\int_0^1\left(2\pi\omega_y A cos(2\pi\omega_x x)cos(2\pi\omega_y y)\right)^2dxdy$$

$$=4\pi^2\omega_y^2A^2\int_0^1 cos^2(2\pi\omega_x x)dx\int_0^1 cos^2(2\pi\omega_y y)dy=$$

$$4\pi^2\omega_y^2A^2\left[\frac{x}{2}+\frac{sin(4\pi\omega_x x)}{8\pi\omega_x}\right]_0^1\left[\frac{y}{2}+\frac{sin(4\pi\omega_y y)}{8\pi\omega_y}\right]_0^1=$$

$$==\pi^2\omega_y^2A^2\cdot 1\cdot 1=\pi^2\omega_y^2A^2\approx\mathbf{3.02\times 10^9}$$

(b) The approximated signal image (gray-level scaling), using a resolution of $512\times 512$:

(c) The approximation of the signal we made, used the renowned approximation of the derivative on a very small $h$ length interval,
and relied on the fact the given function is harmonic (periodic). Let's take a look in the obtained values we received:

```
Range is:  4997.8668752656495
Horizontal derivative energy is:  246563660.2512485
Vertical derivative energy is:  3018699452.86044
Horizontal derivative energy error is: 0.009183564 %
Vertical derivative energy error is: 0.112214553 %
```

The range we received was very close to the given value of $2A$, and also the derivatives energies were almost

identical:
Horizontal had a value of $2.465 \times 10^8$ and Vertical had a value of $3.018 \times 10^9$, both with an error rate of less than 0,12%, which is rather accurate.

(d) Implemented in code.

(e) Using numerical approximations, the obtained values are:

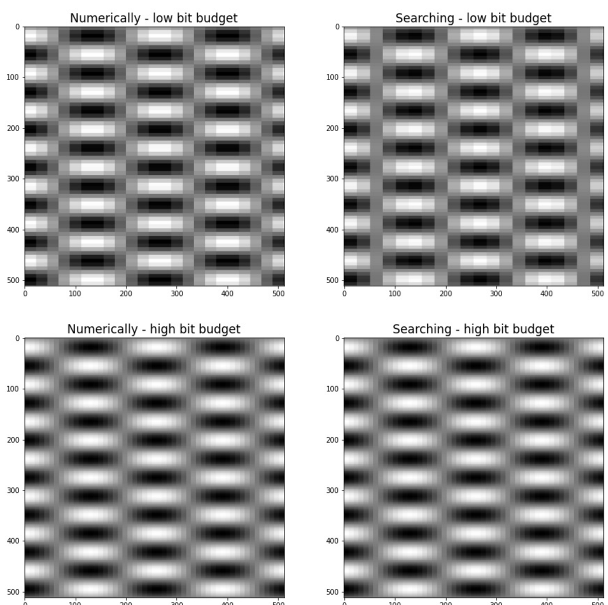
| | |
|---|---|
| $B = 5000$ | $B = 50000$ |
| $N_x = 24$ | $N_x = 54$ |
| $N_y = 69$ | $N_y = 185$ |
| $b = 3$ | $b = 5$ |

(f) Implemented in code.

(g) Using the implemented searching procedure, the obtained optimal values are:

| | |
|---|---|
| $B = 5000$ | $B = 50000$ |
| $N_x = 21$ | $N_x = 54$ |
| $N_y = 79$ | $N_y = 185$ |
| $b = 3$ | $b = 5$ |

These values are relatively similiar to the ones from question e, and paricularily "identical" (only when rounded) on the $B_{high} = 50000$.
We'll note that the values are rounded as seen in class, so there might be some differences stemming from the approximations done by the computer,
and also the methods used in the two procedures. The reconstructed images obtained in the experiments:




(h) Now we consider the following parameters: $A = 2500, \omega_x = 7, \omega_y = 2$ and repeat the analysis from the previous questions:

i. We notice that sin and cos functions are limited to the domain $[-1, 1]$,and therefore our $\phi$ function is limited to $[-A, A]$, let's check it's range:

$$\phi\left(0, \frac{1}{4\omega_y}\right) = A = 2500$$

$$\phi\left(0, -\frac{1}{4\omega_y}\right) = A = -2500$$

Similar to the previous range calculation (using the opposite parameters), $[-2500, 2500]$ is the function's

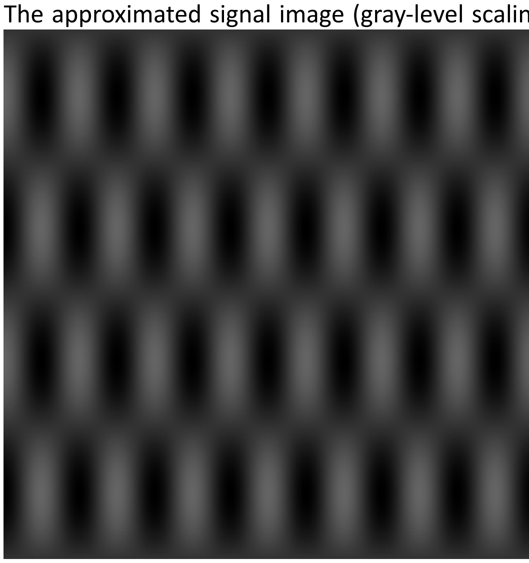
range.
Let's compute the horizontal-derivative energy:

$$\int_0^1 \int_0^1 \left(\frac{\partial\phi(x,y)}{\partial x}\right)^2 dxdy = \int_0^1 \int_0^1 \left(-2\pi\omega_x A sin(2\pi\omega_x x) sin(2\pi\omega_y y)\right)^2 dxdy =$$
$$= 4\pi^2\omega_x^2 A^2 \int_0^1 sin^2(2\pi\omega_x x)dx \int_0^1 sin^2(2\pi\omega_y y)dy =$$
$$= 4\pi^2\omega_x^2 A^2 \left[\frac{x}{2} - \frac{sin(4\pi\omega_x x)}{8\pi\omega_x}\right]_0^1 \left[\frac{y}{2} - \frac{sin(4\pi\omega_y y)}{8\pi\omega_y}\right]_0^1 =$$
$$= \pi^2\omega_x^2 A^2 \cdot 1 \cdot 1 = \pi^2\omega_x^2 A^2 \approx \mathbf{3.022 \times 10^9}$$

And at last, let's compute the vertical-derivative energy:

$$\int_0^1 \int_0^1 \left(\frac{\partial\phi(x,y)}{\partial y}\right)^2 dxdy = \int_0^1 \int_0^1 \left(2\pi\omega_y A cos(2\pi\omega_x x) cos(2\pi\omega_y y)\right)^2 dxdy$$
$$= 4\pi^2\omega_y^2 A^2 \int_0^1 cos^2(2\pi\omega_x x)dx \int_0^1 cos^2(2\pi\omega_y y)dy =$$
$$4\pi^2\omega_y^2 A^2 \left[\frac{x}{2} + \frac{sin(4\pi\omega_x x)}{8\pi\omega_x}\right]_0^1 \left[\frac{y}{2} + \frac{sin(4\pi\omega_y y)}{8\pi\omega_y}\right]_0^1 =$$
$$= \pi^2\omega_y^2 A^2 \cdot 1 \cdot 1 = \pi^2\omega_y^2 A^2 \approx \mathbf{2.467 \times 10^8}$$

ii. The approximated signal image (gray-level scaling), using a resolution of $512 \times 512$:



We note that the aformentioned image is a "90 degrees rotation" of the matching one with the opposite parameters,
which makes sense due to the flipping of frequencies that occured between the sin and the cos functions.

iii. The approximation of the signal we made, used the renowned approximation of the derivative on a very small $h$ length interval,
and relied on the fact the given function is harmonic (periodic). Let's take a look in the obtained values we received:

```
Range is:  4997.86687526565
Horizontal derivative energy is:  3018699452.8604403
Vertical derivative energy is:  246563660.25124842
Horizontal derivative energy error is: 0.000004306 %
Vertical derivative energy error is: 0.000017666 %
```

The range we received was again very close to the given value of $2A$.
also the derivatives energies were almost identical (and this time the horizontal and vertical values switched):
Horizontal had a value of $3.018 \times 10^9$ and Vertical had a value of $2.465 \times 10^8$, both with an error rate of less than $2 \times 10^{-5}$%,
which is even more accurate then before.

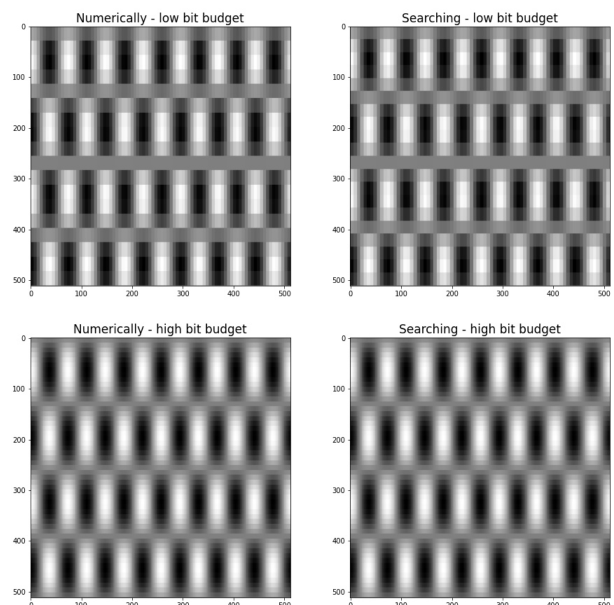
iv. Using <u>numerical approximations</u>, the obtained values are:

| | |
|---|---|
| $B = 5000$ | $B = 50000$ |
| $N_x = 87$ | $N_x = 185$ |
| $N_y = 19$ | $N_y = 54$ |
| $b = 3$ | $b = 5$ |

v. Using the implemented <u>searching procedure</u>, the obtained optimal values are:

| | |
|---|---|
| $B = 5000$ | $B = 50000$ |
| $N_x = 79$ | $N_x = 185$ |
| $N_y = 21$ | $N_y = 54$ |
| $b = 3$ | $b = 5$ |

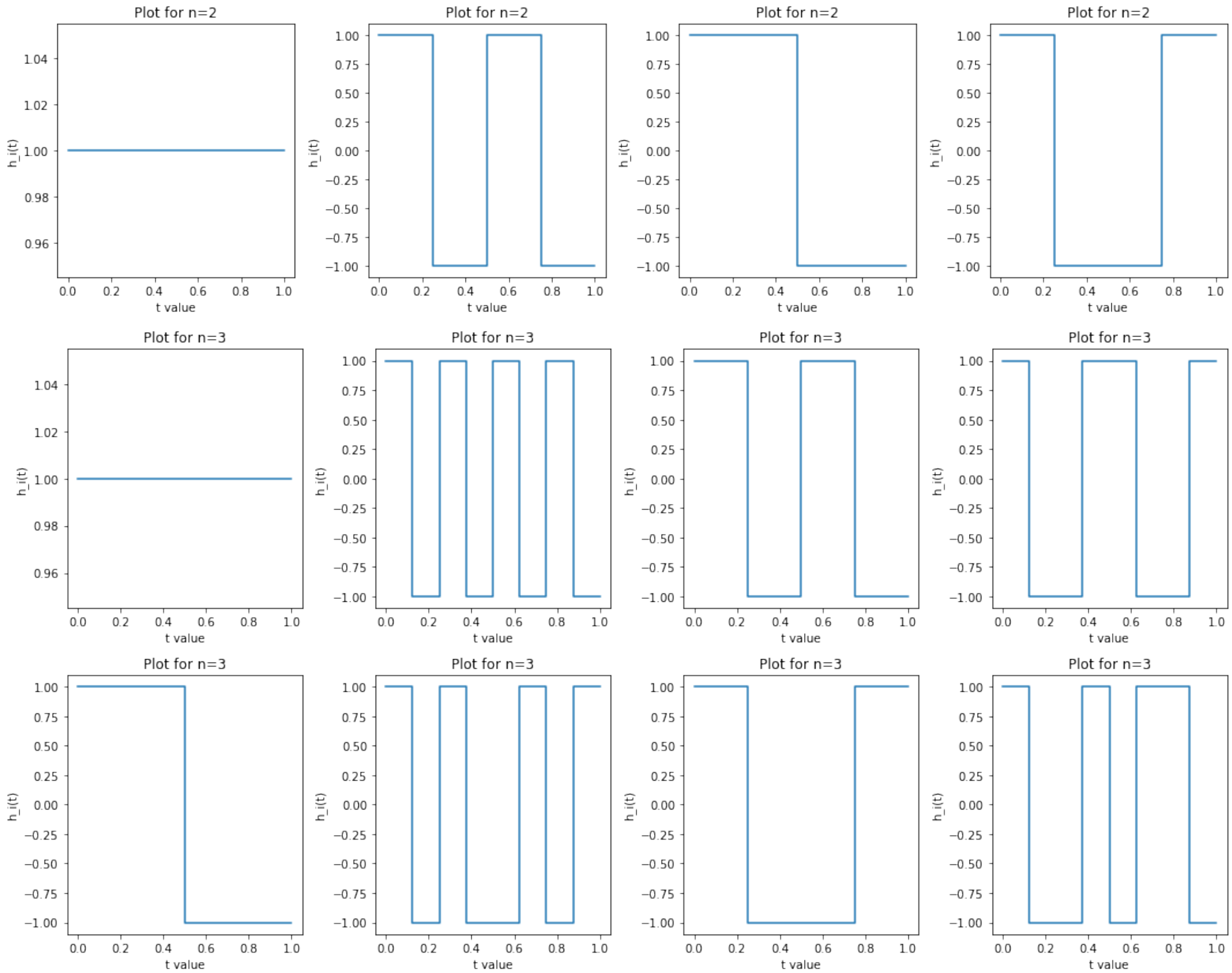
vi. These values are relatively similiar to the ones from obtained numerically, and again paricularily "identical" (only when rounded) on the $B_{high} = 50000$.
As seen in the previous clauses, the $N_x$ and $N_y$ values switched since the last comparison (question g), which is not surprising due to the fact that $\omega_x$and $\omega_y$ switched as well.
We'll note that these values are rounded as seen in class, so there might be some differences stemming from the approximations done by the computer,
and also the methods used in the two procedures. The reconstructed images obtained in the experiments:

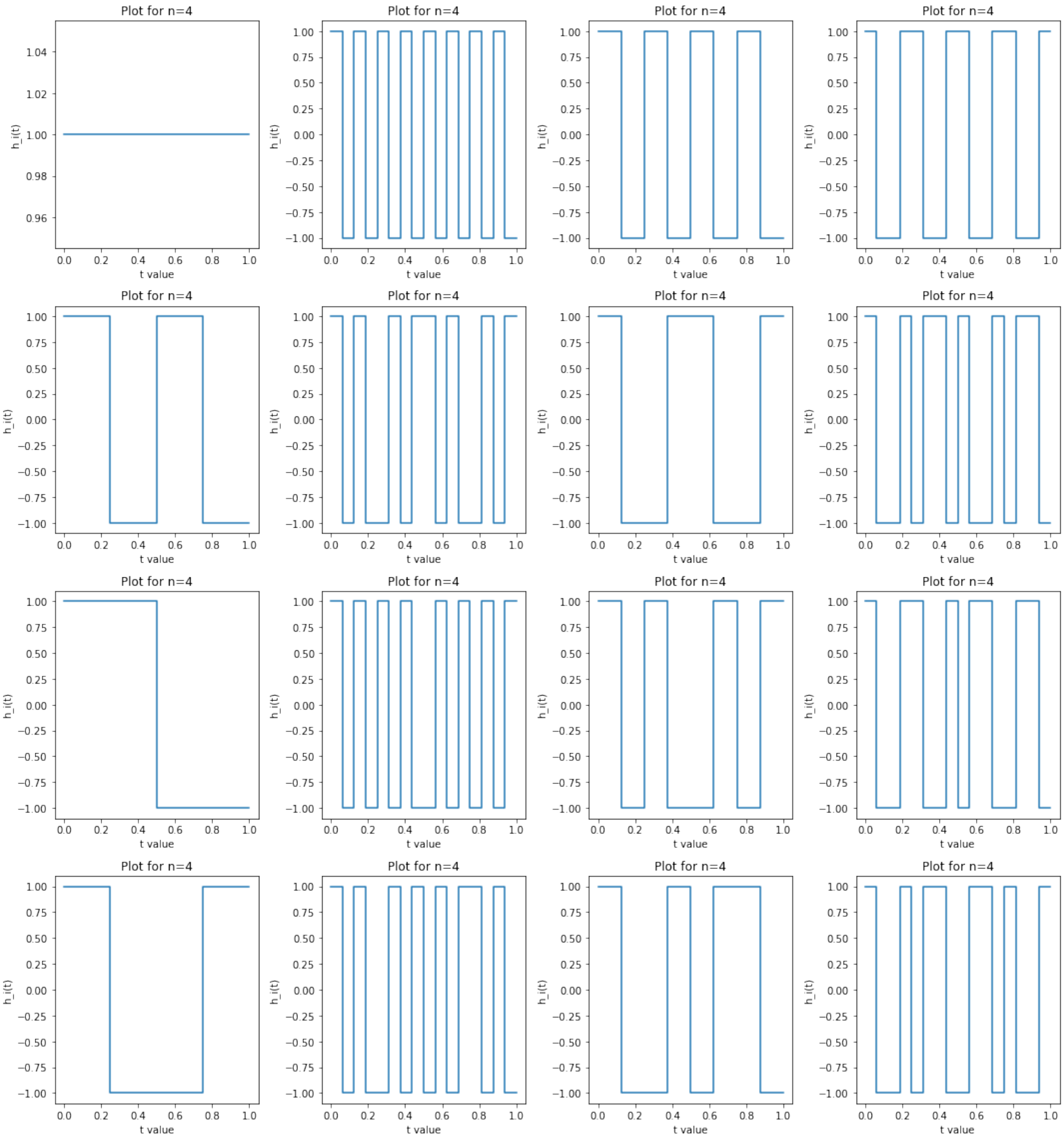



2. Hadamard, Hadamard-Walsh, and Haar matrices.

   (a) Implemented in code.

   (b) Attached plots for hadamard:


Plot for n=2
Plot for n=2
Plot for n=2
Plot for n=2
Plot for n=3
Plot for n=3
Plot for n=3
Plot for n=3
Plot for n=3
Plot for n=3
Plot for n=3
Plot for n=3
h_i(t)
t value


Plot for n=4
h_i(t)
t value
Plot for n=4
h_i(t)
t value
Plot for n=4
h_i(t)
t value
Plot for n=4
h_i(t)
t value
Plot for n=4
h_i(t)
t value
Plot for n=4
h_i(t)
t value
Plot for n=4
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Plot for n=4
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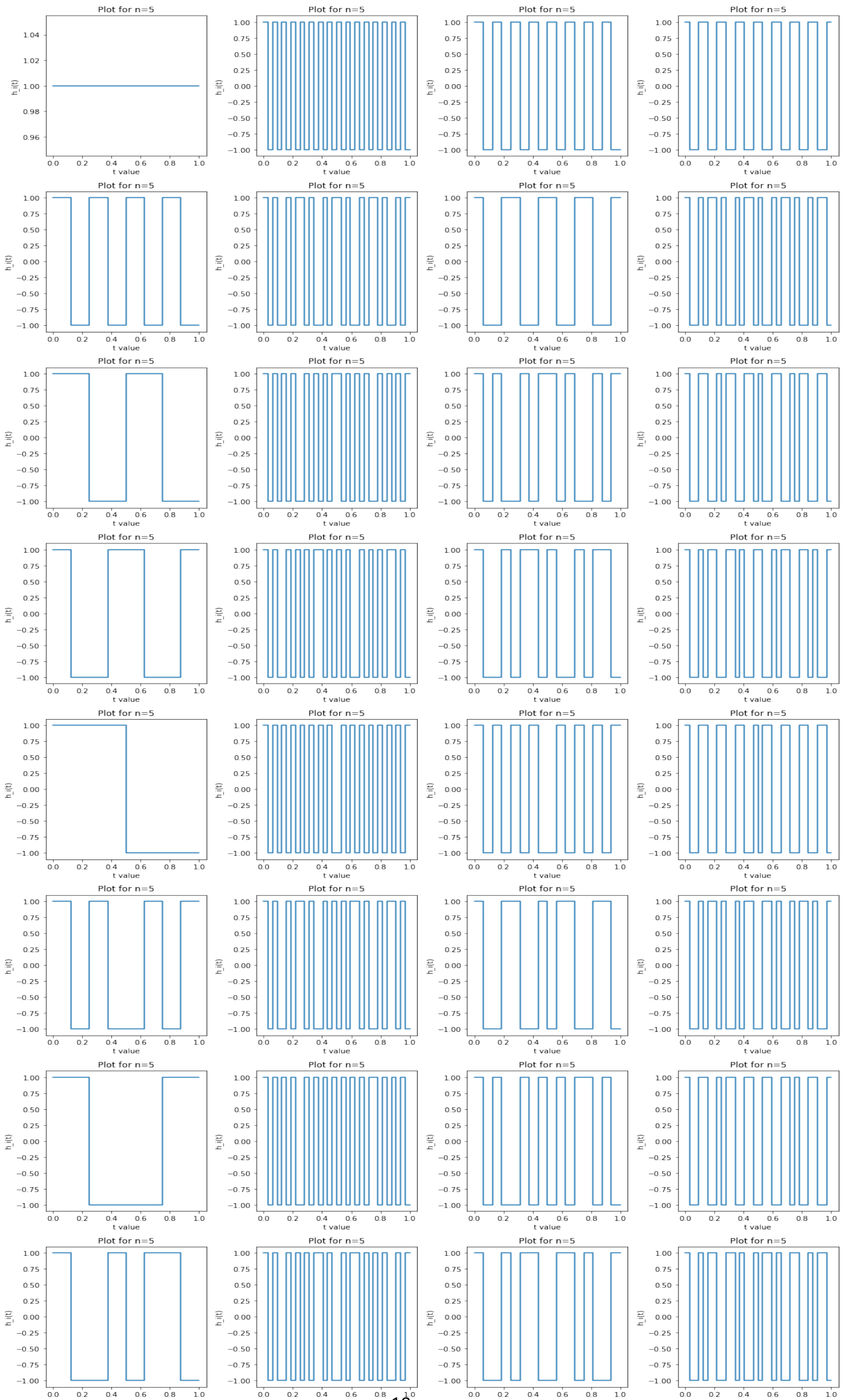
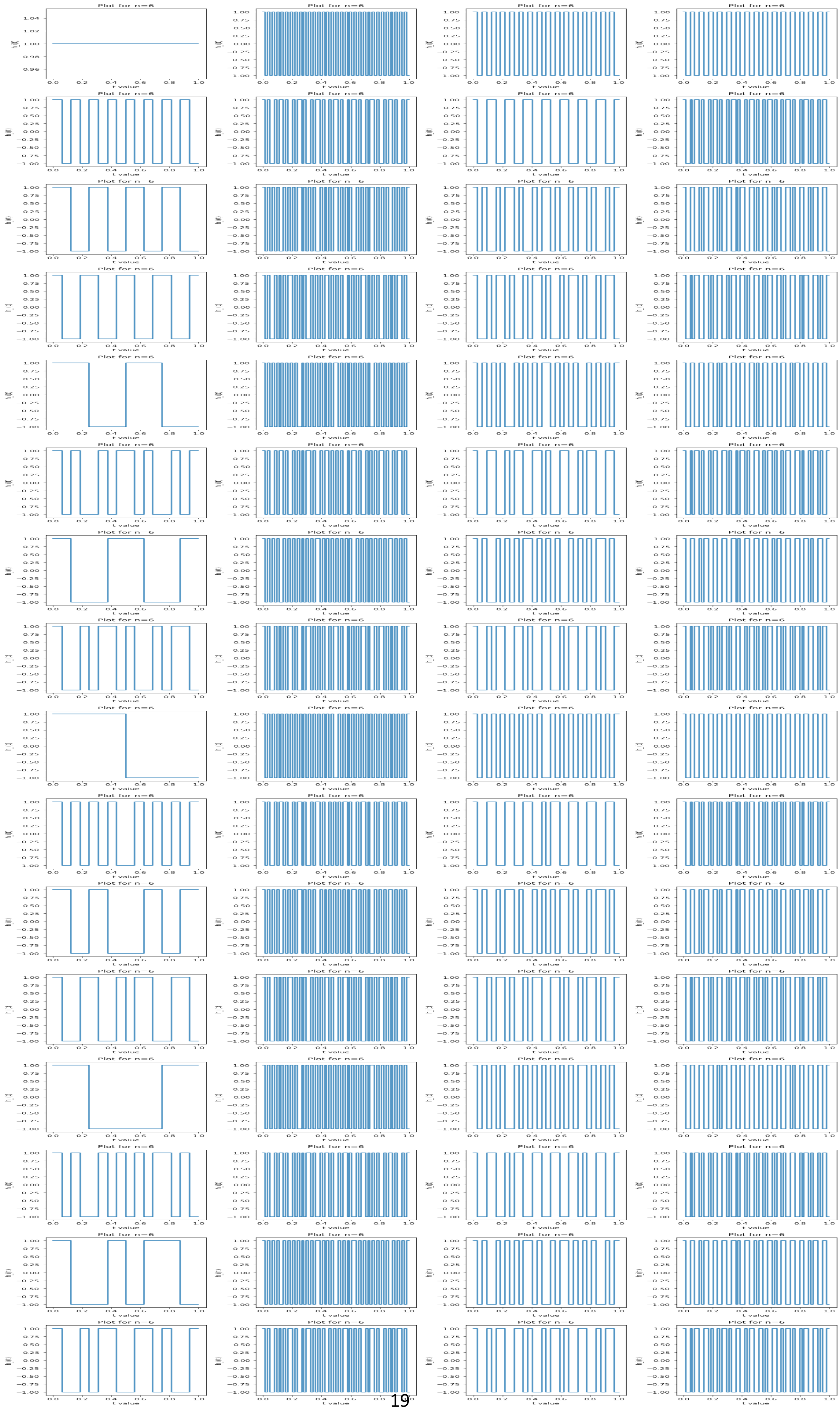
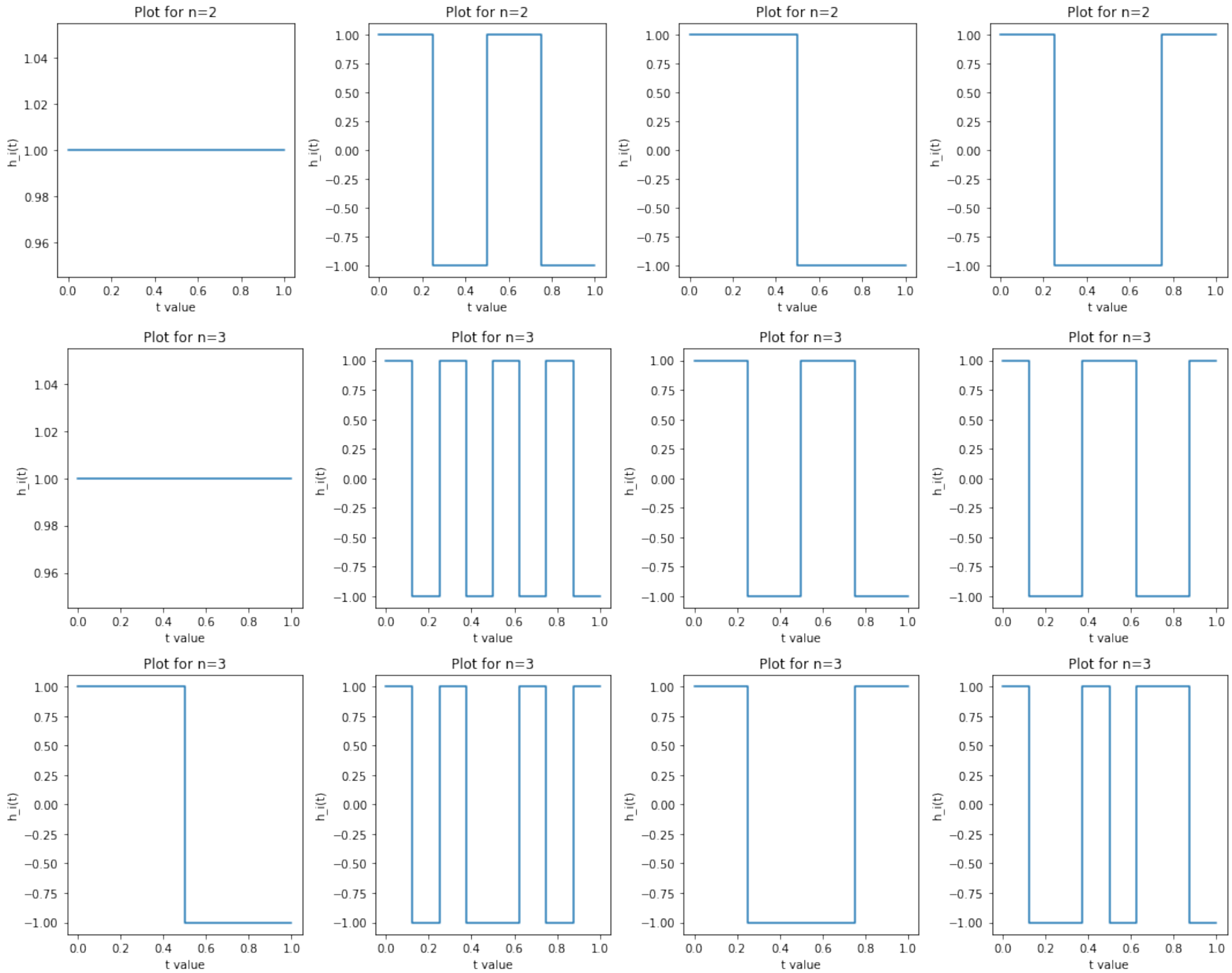
Plot for n=4
h_i(t)
t value
Plot for n=4
h_i(t)
t value

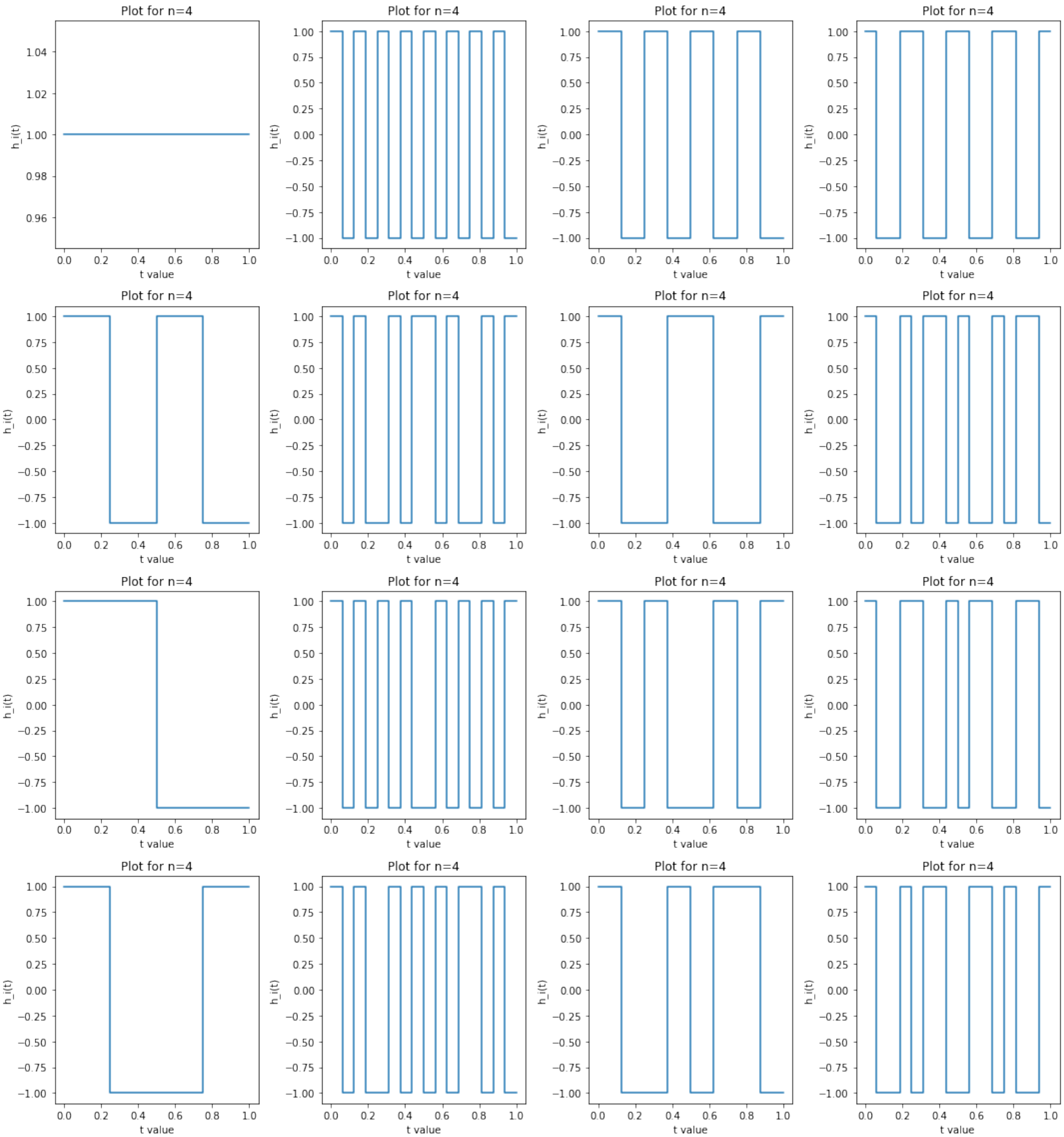

Plot for n=5
h_i(t)
t value

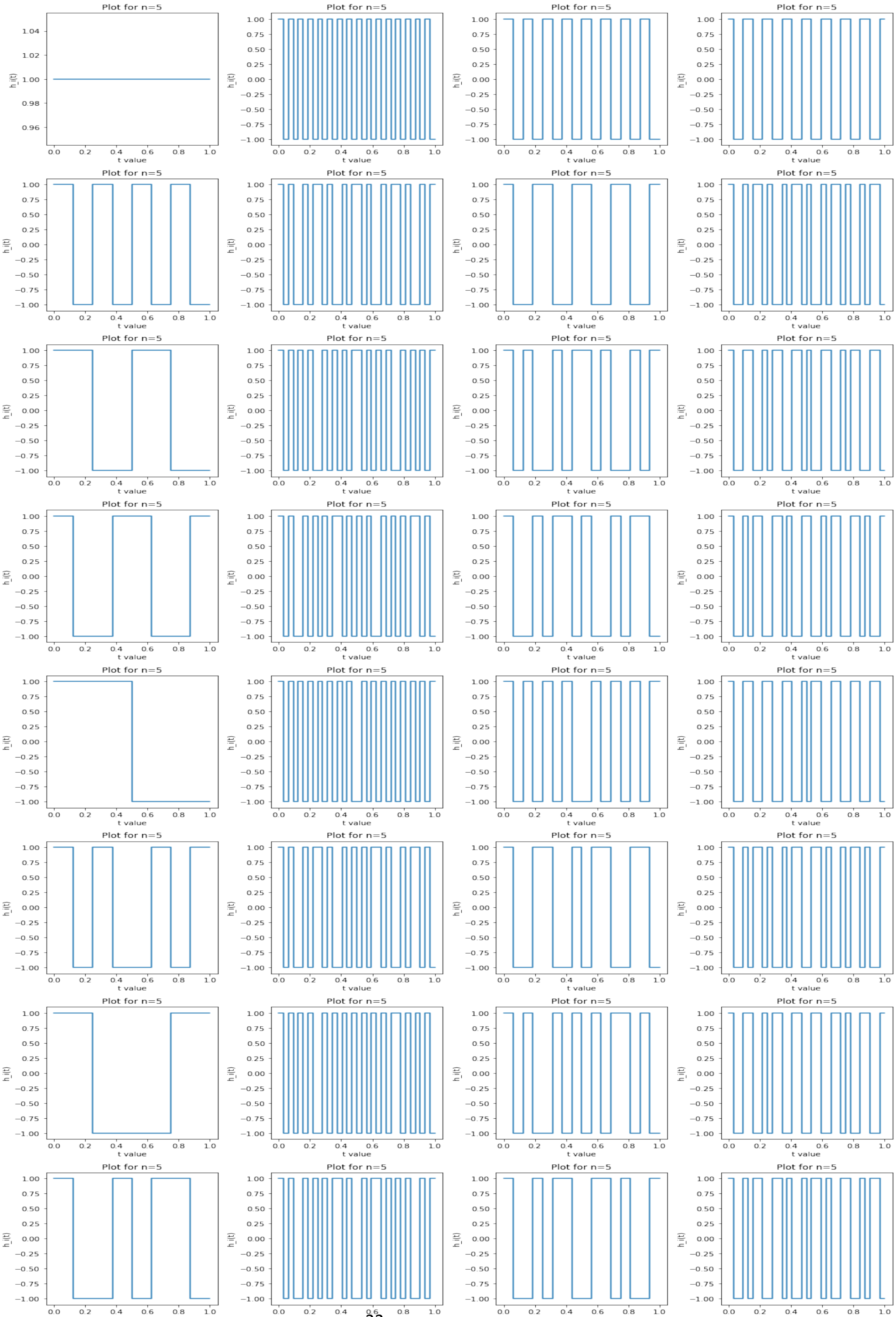

Plot for n=6
t value

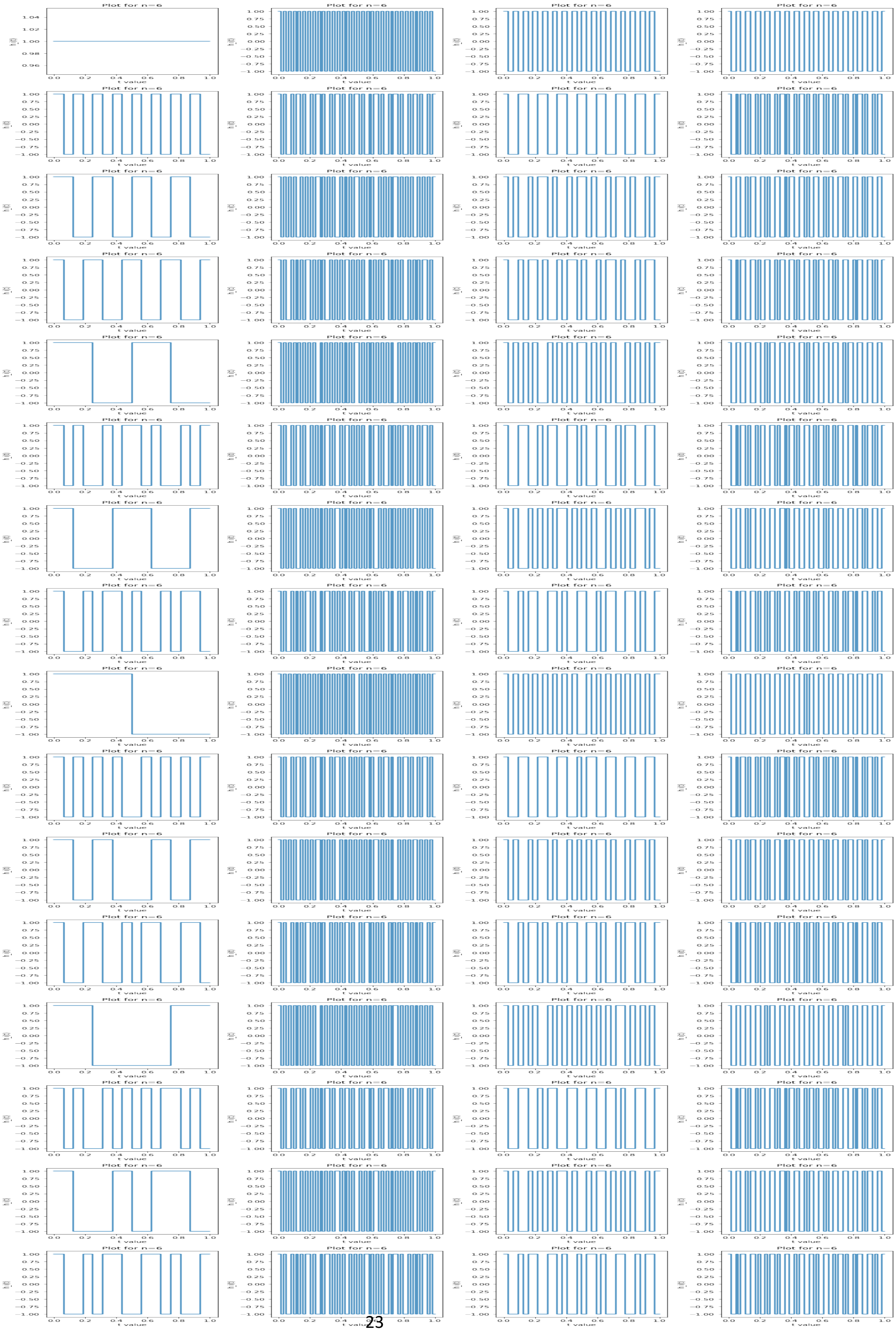
(c) Implemented.

(d) Attached plots for walsh:



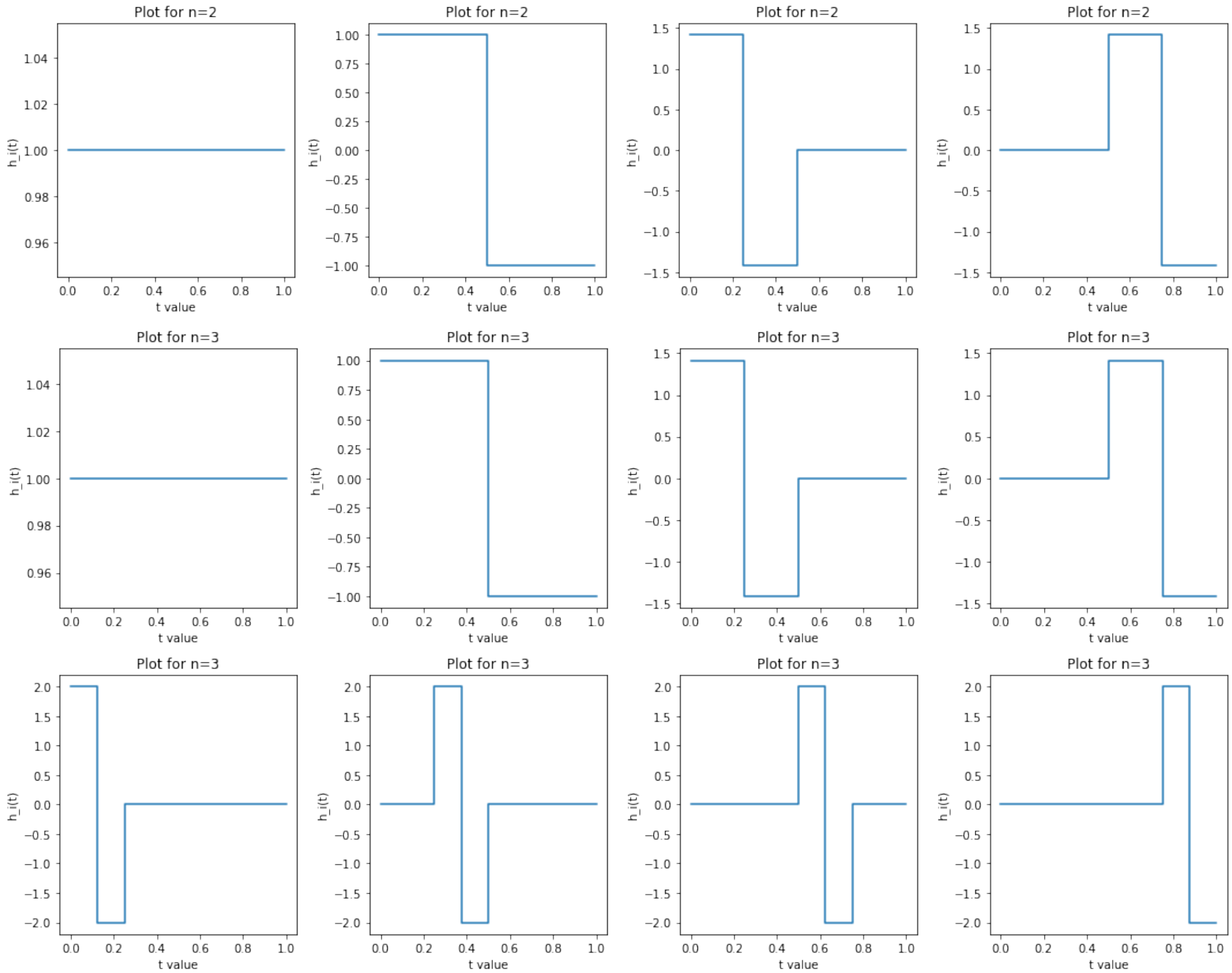

Plot for n=4
h_i(t)
t value

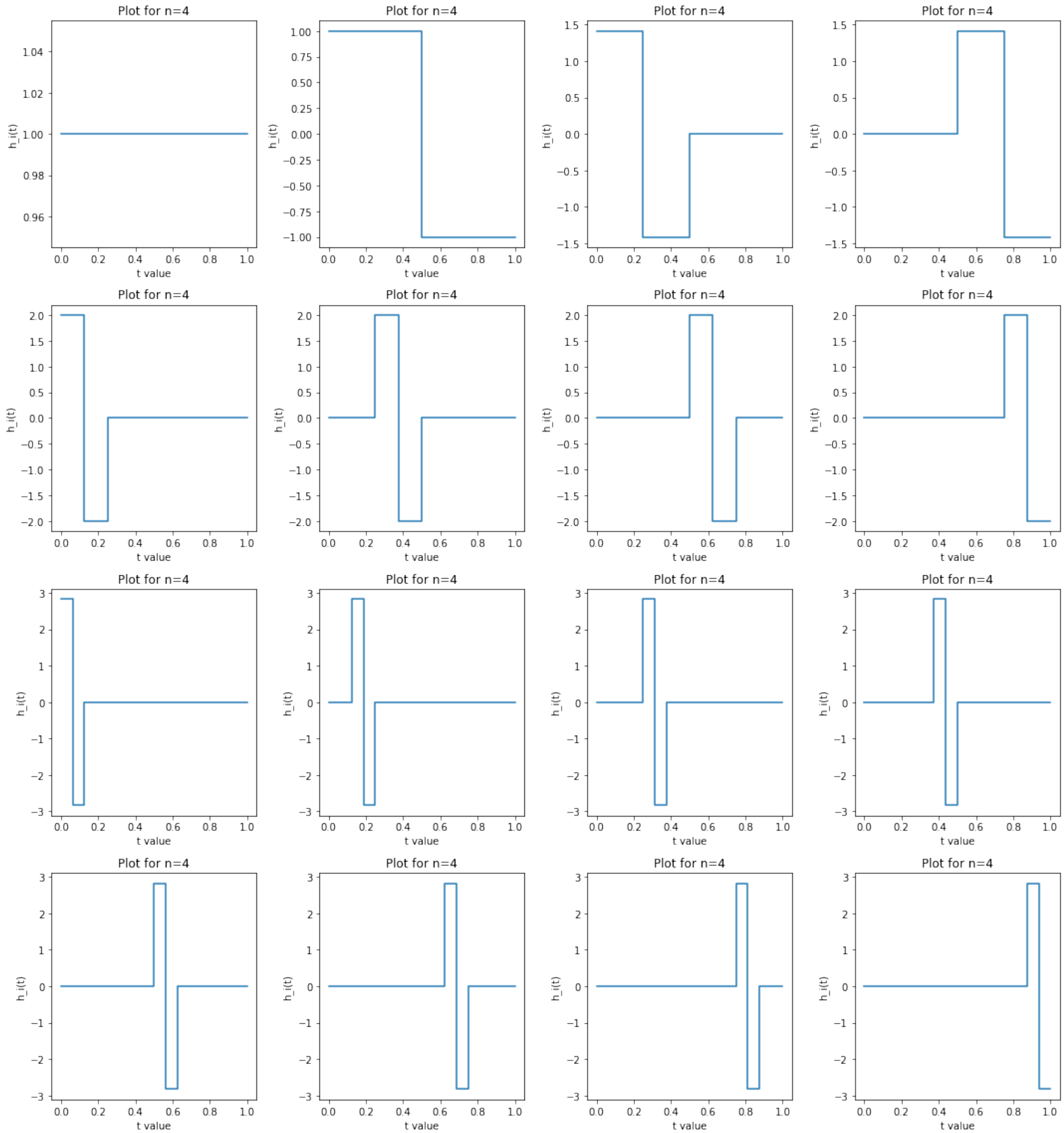

Plot for n=5
h_i(t)
t value

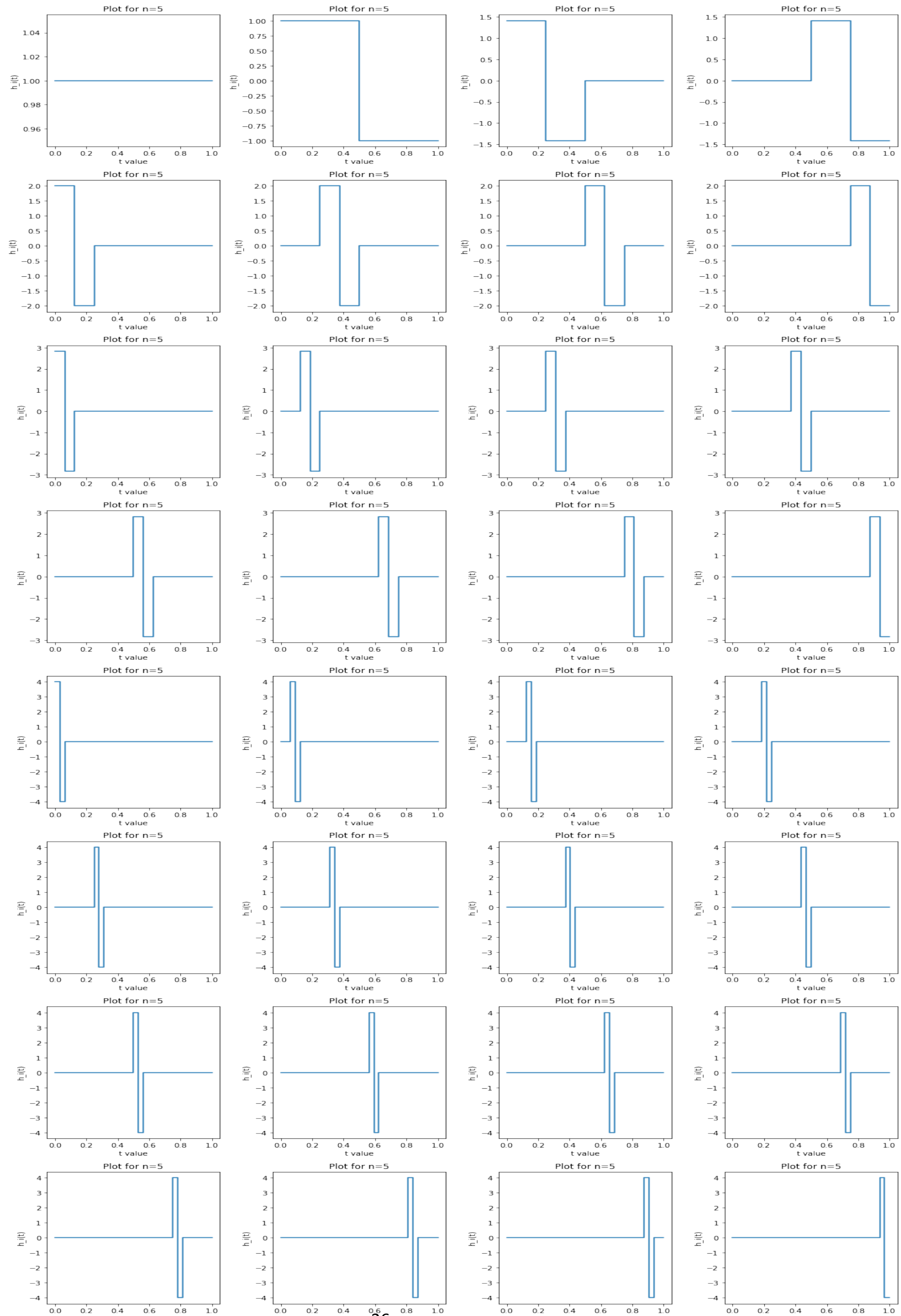

Plot for n=6
t value

(e) Implemented.

(f) Attached plots for haar:




Plot for n=4
h_i(t)
t value



Plot for n=5
h_i(t)
t value
Plot for n=5
h_i(t)
t value
Plot for n=5
h_i(t)
t value
Plot for n=5
h_i(t)
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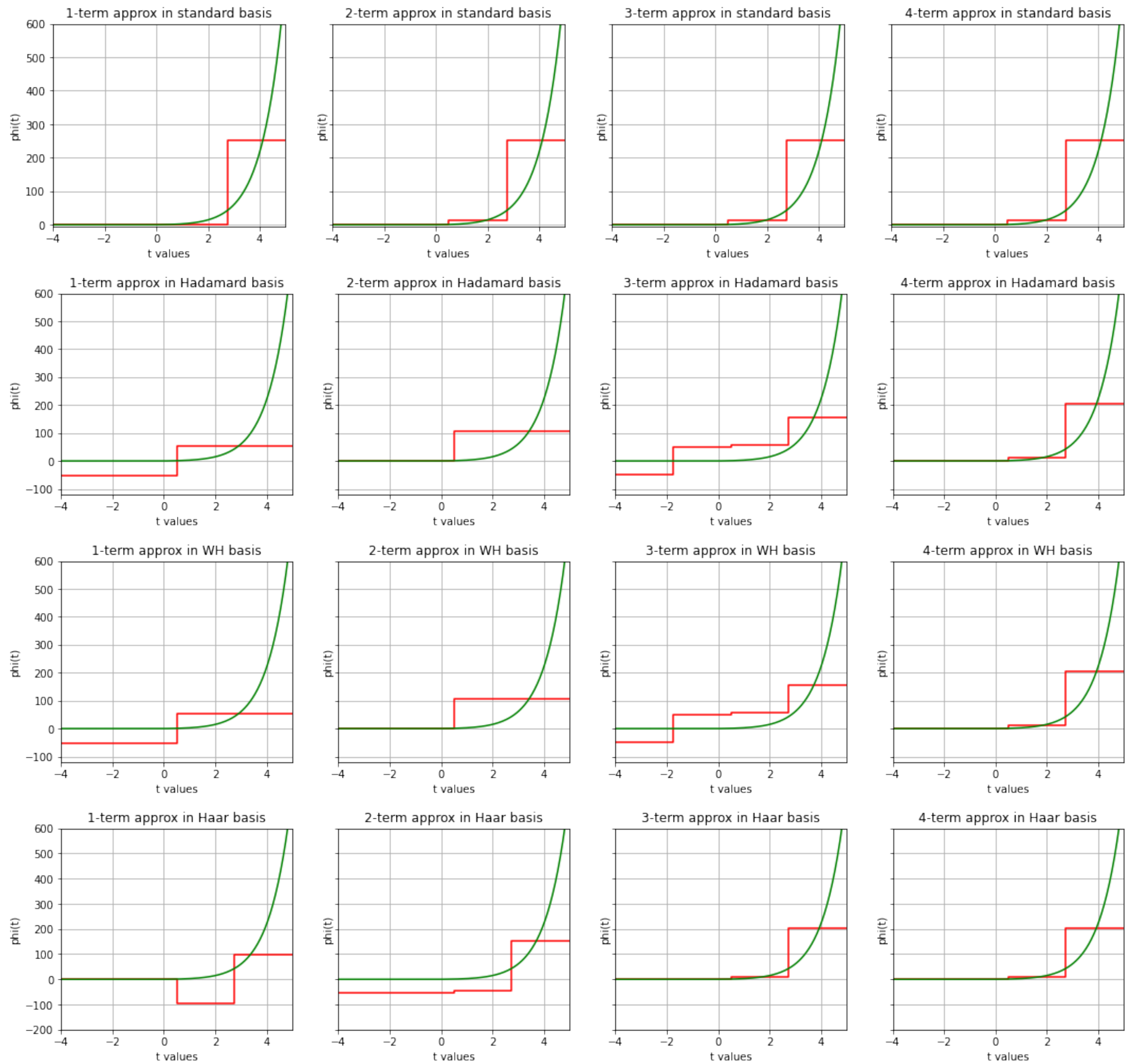
Plot for n=5
h_i(t)
t value
Plot for n=5
h_i(t)
t value
Plot for n=5
h_i(t)
t value


Plot for n=6
t value
h(t)

(g) Attached result of the best k-term approximations of $\phi(t)$ in each basis:




And also attached the MSE results:

```
MSE in 1 for standard basis-approx is 83250.91186720502
MSE in 2 for standard basis-approx is 82897.49777481775
MSE in 3 for standard basis-approx is 82897.44441927201
MSE in 4 for standard basis-approx is 82897.3780958027
MSE in 1 for Hadamard basis-approx is 193886.54961459708
MSE in 2 for Hadamard basis-approx is 162158.65230070206
MSE in 3 for Hadamard basis-approx is 136097.17849203415
MSE in 4 for Hadamard basis-approx is 110043.41773967282
```

```
MSE in 1 for WH basis-approx is 193886.54961459708
MSE in 2 for WH basis-approx is 162158.65230070206
MSE in 3 for WH basis-approx is 136097.17849203415
MSE in 4 for WH basis-approx is 110043.41773967282
MSE in 1 for Haar basis-approx is 173656.03668263083
MSE in 2 for Haar basis-approx is 141771.315338951
MSE in 3 for Haar basis-approx is 110043.41802505597
MSE in 4 for Haar basis-approx is 110043.41773967283
```