Introduction to Data Processing and Representation (236201)

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Homework 3

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Part I

Theory

1. In this question, we use the normalised convention for the DFT matrix.

   (a) We consider the matrix $J = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$. We notice that J is cirulant, so first, for J^2 and J^3:

$$J^2 = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 \end{pmatrix}$$

$$J^3 = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{pmatrix}$$

Thus, for $k \in \mathbb{N}$, let's compute J^k:$J^k = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}^k = \begin{pmatrix} 0 & \dots & 0 & \overbrace{1}^{\textbf{(*)}} & \dots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & 0 & \dots & 0 & 0 & 1 \\ \underbrace{1}_{\textbf{(**)}} & 0 & \dots & \dots & 0 & 0 \\ \vdots & 1 & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 & 0 \end{pmatrix}$

(*) $(n-k+1)(mod\, n)\ column$
(**) $(1+k)(mod\, n)\ row$

And in particular:

$$J^n = \begin{pmatrix} 1 & \dots & \dots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} = I^{n \times n}$$

(b) The eigenvalues of J are the solutions of the renowned following equation:

$$J\gamma\underline{\mathrm{v}}=\underline{\mathrm{v}}$$

Let us solve:

$$\underline{\mathrm{v}} = J^n \underline{\mathrm{v}} = \gamma^n \underline{\mathrm{v}} \ \Rightarrow\ \gamma^n = 1$$

The obtained complex roots are:

$$\forall k \in \{0, 1, ..., n-1\} : \gamma_k = exp\{\frac{2\pi i k(n-1)}{n}\} = w^{-k}$$

And these are the n eigenvalues of J.

(c) We found n different eigenvalues to J and its dimensions are $n \times n$, so it's indeed diagonalisable. Going through the eigendecomposition:
We know that all circulant matrices (including our J) are unitarly diagonalisable by the DFT matrix (Theorem from Numerical Algorithms course).
thus the corresponding eigenvector to the k-th eigenvalue (where $k \in \{0, ..., n-1\}$) is:

$$\begin{pmatrix} w^{-0 \cdot k} \\ w^{-1 \cdot k} \\ \vdots \\ w^{-(n-1) \cdot k} \end{pmatrix}$$

So we conclude that the eigenvectors basis matrix is:

$$W = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w^{-1} & w^{-2} & \dots & w^{-(n-1)} \\ 1 & w^{-2} & w^{-4} & \dots & w^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{-(n-1)} & w^{-2(n-1)} & \dots & w^{-(n-1)^2} \end{pmatrix}$$

Hence the obtained eigendecomposition would be:

$$J = W \Lambda W^H$$

where $\Lambda = \begin{pmatrix} \gamma_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_{n-1} \end{pmatrix}$.

(d) Let us rewrite H as a sum of n circulant matrices:

$$H = \begin{pmatrix} h_0 & h_{n-1} & h_{n-2} & \dots & h_1 \\ h_1 & h_0 & h_{n-1} & \dots & h_2 \\ h_2 & h_1 & h_0 & \dots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_0 \end{pmatrix} = h_0 \cdot \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} +$$

$$h_1 \cdot \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} + ... + h_{n-1} \cdot \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$= h_0 \cdot J^0 + h_1 \cdot J^1 + ... + h_{n-1} \cdot J^{n-1} \Rightarrow H = P(J) = \sum_{i=0}^{n-1} h_i \cdot J^i$$

(e) H is a circulant matrix, which means that according to the theorem used in clause c, it's also diagonalisable by the DFT matrix.
That means it has the same eigenvector as J, and according to another theorem regarding eigenvalues we get that the if the eigenvalues
of J are $\{\gamma_0, \gamma_1, ..., \gamma_{n-1}\}$,then the eigenvalues of H (which we showed above that is a polinomal form of J) are: $\{P(\gamma_0), P(\gamma_1), ..., P(\gamma_{n-1})\}$.
Hence, H is diagonalisable by DFT^*, in the unitary basis of W (DFT and W are the same):

$$H = W\tilde{\Lambda}W^H$$

where $\tilde{\Lambda} = \begin{pmatrix} P(\gamma_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & P(\gamma_{n-1}) \end{pmatrix}$.

(f) We showed the requested in the previous clause as needed.

(g) We'll denote the eigenvalues of H: $\lambda_0, \dots, \lambda_{n-1}$.
Earlier we found that the diagonal matrix containing H's eigenvalues is:

$$\tilde{\Lambda} = \begin{pmatrix} \lambda_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{n-1} \end{pmatrix} = \begin{pmatrix} P(\gamma_0) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & P(\gamma_{n-1}) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{n-1} h_i \cdot \gamma_0^i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sum_{i=0}^{n-1} h_i \cdot \gamma_{n-1}^i \end{pmatrix} =$$

$$h_0 \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} + h_1 \begin{pmatrix} \gamma_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_{n-1} \end{pmatrix} + h_2 \begin{pmatrix} \gamma_0^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_{n-1}^2 \end{pmatrix} ... + h_{n-1} \begin{pmatrix} \gamma_0^{n-1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_{n-1}^{n-1} \end{pmatrix}$$

We showed that $\forall k \in \{0, 1, ..., n-1\} : \gamma_k = exp\{\frac{2\pi i k(n-1)}{n}\} = w^{-k}$, thus we can represent λ_k using elements of the DFT matrix rows (denoted by DFT_k):

$$\lambda_k = \sum_{j=0}^{n-1} h_j \cdot \gamma_k^j = \sum_{j=0}^{n-1} h_j \cdot w^{-jk} \overset{definition\ of\ DFT}{=} \sqrt{n} \cdot DFT_k \begin{pmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{pmatrix}$$

At last we get as requested (where we showed $B = DFT$):

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n} \cdot \begin{pmatrix} DFT_0 \\ DFT_1 \\ \vdots \\ DFT_{n-1} \end{pmatrix} \begin{pmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{pmatrix} = \sqrt{n} \cdot DFT \begin{pmatrix} h_0 \\ h_{n-1} \\ \vdots \\ h_1 \end{pmatrix}$$

(h) We can rewrite every circulant matrix (with $n \times n$ dimensions) as a sum of n basic circulant matrices, thus we denote:

$$H_1 = h_0^1 \cdot J^0 + h_1^1 \cdot J^1 + ... + h_{n-1}^1 \cdot J^{n-1}, \quad H_2 = h_0^2 \cdot J^0 + h_1^2 \cdot J^1 + ... + h_{n-1}^2 \cdot J^{n-1}$$

Let's compute $H_1 H_2$:

$$H_1 H_2 = \left(\sum_{i=0}^{n-1} h_i^1 \cdot J^i \right) \cdot \left(\sum_{k=0}^{n-1} h_k^2 \cdot J^k \right) =$$

$$\left(h_0^1 \cdot h_0^2\right) J^0 + \left(\sum_{\substack{i+j=1\,(mod\,n) \\ 0 \le i < j \le n-1}} h_i^1 \cdot h_j^2 \right) J^1 + \left(\sum_{\substack{i+j=2\,(mod\,n) \\ 0 \le i < j \le n-1}} h_i^1 \cdot h_j^2 \right) J^2 +$$

$$\cdots + \left(\sum_{\substack{i+j=n-1\,(mod\,n) \\ 0 \le i < j \le n-1}} h_i^1 \cdot h_j^2 \right) J^{n-1} = H_2 H_1$$

We see that H_1 and H_2 commute (stems also from the commutativity of the real numbers field), and $H_1 H_2$ can be represented as
a sum of n basic circulant matrices, thus it is circulant too.

(i) First, let's compute DFT^2:

$$DFT \cdot DFT = \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w^{-1} & w^{-2} & \dots & w^{-(n-1)} \\ 1 & w^{-2} & w^{-4} & \dots & w^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{-(n-1)} & w^{-2(n-1)} & \dots & w^{-(n-1)^2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w^{-1} & w^{-2} & \dots & w^{-(n-1)} \\ 1 & w^{-2} & w^{-4} & \dots & w^{-2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{-(n-1)} & w^{-2(n-1)} & \dots & w^{-(n-1)^2} \end{pmatrix} =$$

$$\frac{1}{n} \begin{pmatrix} \sum_{i=0}^{n-1} w^{-0 \cdot i - i \cdot 0} & \sum_{i=0}^{n-1} w^{-0 \cdot i - i \cdot 1} & \dots & \sum_{i=0}^{n-1} w^{-0 \cdot i - i \cdot (n-1)} \\ \sum_{i=0}^{n-1} w^{-1 \cdot i - i \cdot 0} & \sum_{i=0}^{n-1} w^{-1 \cdot i - i \cdot 1} & \dots & \sum_{i=0}^{n-1} w^{-1 \cdot i - i \cdot (n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{n-1} w^{-(n-1) \cdot i - i \cdot 0} & \sum_{i=0}^{n-1} w^{-(n-1) \cdot i - i \cdot 1} & \dots & \sum_{i=0}^{n-1} w^{-(n-1) \cdot i - i \cdot (n-1)} \end{pmatrix}$$

We get that a general element in this matrix is: $DFT_{k,j}^2 = \frac{1}{n} \cdot \sum_{i=0}^{n-1} w^{-i(k+j)}$, therefore we understand that all elements on the anti-diagonals (main or sub-anti-diagonals) hold the same value.
Let's observe the obtained element values:

$$DFT_{k,j}^2 = \frac{1}{n} \cdot \sum_{i=0}^{n-1} w^{-i(k+j)} \overset{(*)}{=} \frac{1}{n} \cdot \sum_{i=0}^{n-1} w^{-ik} \cdot w^{i(n-j)\,(mod\,n)} = \frac{1}{n} \langle DFT_k, DFT_{n-j} \rangle$$

(*) using the identity: $w^x = w^{x\,(mod\,n)}$.
Because we know DFT matrix is orthonormal, we'll get the following:

$$DFT_{k,j}^2 = \frac{1}{n} \langle DFT_k, DFT_{n-j} \rangle = \begin{cases} 1 & (k+j) \, = 0\,(mod\,n) \\ 0 & O.W \end{cases}$$

We conclude that only on the main anti-diagonal the condition $(k+j) = 0\,(mod\,n)$ holds,
so for the anti-diagonal all element have 1's, are 0's in all the rest of the matrix, hence we got: $DFT^2 = AC$, which is the anti-circulant matrix from class.
We know it's a permutation matrix, thus: $AC^2 = I$. Let $m \in \mathbb{N}$,we'll check all different cases:

$$for\,k = 4m = 0\,(mod\,4):\; DFT^k = DFT^{4m} = \left(DFT^4\right)^m = \left(AC^2\right)^m = I^m = I$$

$$for\,k = 4m+1 = 1\,(mod\,4):\; DFT^k = DFT^{4m+1} = \left(DFT^4\right)^m \cdot DFT = I \cdot DFT = DFT$$

$$for\,k = 4m+2 = 2\,(mod\,4):\; DFT^k = DFT^{4m+2} = \left(DFT^4\right)^m \cdot DFT^2 = I \cdot AC = AC$$

$$for\,k = 4m+3 = 3\,(mod\,4):\; DFT^k = DFT^{4m+3} = \left(DFT^4\right)^m \cdot DFT^3 = I \cdot DFT^3 = DFT^*$$

Last equality stems from: $DFT \cdot DFT^3 = DFT^4 = I \Rightarrow DFT^* = DFT^3$.
In conclusion:

$$DFT^k = \begin{cases} I & k = 0\,(mod\,4) \\ DFT & k = 1\,(mod\,4) \\ AC & k = 2\,(mod\,4) \\ DFT^* & k = 3\,(mod\,4) \end{cases}$$

(j) Let $x, y \in \mathbb{R}^n$. The requested convolution z is the following:

$$z = x \otimes y = \begin{pmatrix} x_0 & x_{n-1} & x_{n-2} & \dots & x_1 \\ x_1 & x_0 & x_{n-1} & \dots & x_2 \\ x_2 & x_1 & h_0 & \dots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & x_{n-3} & \dots & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} \overset{(\$)}{=} DFT^* \cdot \Lambda \cdot DFT \cdot y$$

($) the X convolution matrix is circulant, thus can be diagonlized by the DFT matrix: $X = DFT^* \cdot \Lambda \cdot DFT$,where $\Lambda = \begin{pmatrix} \lambda_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{n-1} \end{pmatrix}$
Now we multiply both sides by DFT matrix (from the left):

$$DFT \cdot z = \Lambda \cdot (DFT \cdot y) = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} \odot (DFT \cdot y)$$

The last equality stems from the fact that multiplying by a diagonal matrix from the left is as same as multiplying each row
by the value on the diagonal, which yields the same as the Hadamard product.
Now we use the claim in clause g and obtain as requested:

$$DFT \cdot z = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} \odot (DFT \cdot y) = \sqrt{n} \cdot (DFT \cdot x) \odot (DFT \cdot y)$$

2. Fourier Transform.

(a) We are given $f(t), g(t)$ two functions and their convolution $h(t) = f(t) * g(t)$. We want to calculate in this question what is

$$f(t-1) * g(t+1)$$

In order to do so, we'll denote it as $\hat{f}\left(t\right)*\hat{g}\left(t\right)$, where the functions $\hat{f},\hat{g}$ are shifted such that $\hat{f}\left(t\right)=f\left(t-1\right)$ and $\hat{g}\left(t\right)=g\left(t+1\right)$.
Now we can use the definition and do the needed calculation:

$$\hat{f}\left(t\right)*\hat{g}\left(t\right)=\int_{-\infty}^{\infty}\hat{f}\left(\tau\right)\hat{g}\left(t-\tau\right)d\tau=\int_{-\infty}^{\infty}f\left(\tau-1\right)g\left(t-\tau+1\right)d\tau$$

Now we'll shift again and denote $s=\tau-1$, since the integration borders are $-\infty,\infty$ our change is not relevant to it.

$$\hat{f}\left(t\right)*\hat{g}\left(t\right)=\int_{-\infty}^{\infty}f\left(s\right)g\left(t-s\right)ds=f\left(t\right)*g\left(t\right)=h\left(t\right)$$

So actually we got that

$$f\left(t-1\right)*g\left(t+1\right)=h\left(t\right)$$

(b) The definition of the fourier transform is

$$\hat{f}\left(\xi\right)=\int_{-\infty}^{\infty}f\left(x\right)e^{-2\pi i\xi x}dx,\ \forall\xi\in\mathbb{R}$$

So in our case,

$$\mathcal{F}\left(u\right)=\int_{-\infty}^{\infty}f\left(t\right)e^{-2\pi iut}dt$$

and

$$\mathcal{G}\left(u\right)=\int_{-\infty}^{\infty}g\left(t\right)e^{-2\pi iut}dt$$

We can use the above to calculate:

$$\int_{-\infty}^{\infty}\mathcal{F}\left(u\right)\mathcal{G}\left(u\right)du$$

We'll plug in the fourier trasfrom definition:

$$\int_{-\infty}^{\infty}\mathcal{F}\left(u\right)\mathcal{G}\left(u\right)du=\int_{-\infty}^{\infty}\left(\left(\int_{-\infty}^{\infty}f\left(s\right)e^{-2\pi ius}ds\right)\cdot\left(\int_{-\infty}^{\infty}g\left(t\right)e^{-2\pi iut}dt\right)\right)du$$

We want to get at the end of the formulations a minus sign in the function g, so we should do a manipulation on the varaiable
t to get that, so we'll change it to be $-t$ and also follows it the change of dt to be $-dt$ and the integral borders swap, so now we got the following

$$\int_{-\infty}^{\infty}\mathcal{F}\left(u\right)\mathcal{G}\left(u\right)du=\int_{-\infty}^{\infty}\left(\left(\int_{-\infty}^{\infty}f\left(s\right)e^{-2\pi ius}ds\right)\cdot\left(\int_{-\infty}^{\infty}g\left(t\right)e^{-2\pi iut}dt\right)\right)du=$$

$$=\int_{-\infty}^{\infty}\left(\left(\int_{-\infty}^{\infty}f\left(s\right)e^{-2\pi ius}ds\right)\cdot\left(\int_{\infty}^{-\infty}g\left(-t\right)e^{2\pi iut}-dt\right)\right)du=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(s\right)e^{-2\pi ius}g\left(-t\right)e^{2\pi iut}dsdudt=$$

$$=\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(s\right)g\left(-t\right)dsdt\right)\left(\int_{-\infty}^{\infty}e^{-2\pi ius}e^{2\pi iut}du\right)=$$

Since the fourier basis vectors are orthonormal we get that

$$\int_{-\infty}^{\infty}e^{-2\pi ius}e^{2\pi iut}du=\begin{cases}1 & s=t\\ 0 & s\neq t\end{cases}$$

So overall

$$= \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt$$

And the rest zeroes out.
Hence:

$$\int_{-\infty}^{\infty} \mathcal{F}(u)\, \mathcal{G}(u)\, du = \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt$$

3. We denote a 1-D signal with 2N elements as $\phi \in \mathbb{R}^{2N}$ given by: $\phi = [1, \frac{1}{2}, 0, ..., 0, \frac{1}{2}]^T$

   (a) The DFT matrix elements have the following form: $w_{j,k} = \frac{1}{\sqrt{N}} \cdot e^{\frac{-2\pi ijk}{N}}$, thus we can compute:

   $$DFT(\phi) = W\phi = \phi^F$$

   Which is a vector with $2N$ elements, and each element ϕ_j^F ($\forall j \in \{0, ..., 2N-1\}$) equals to:

   $$\phi_j^F = \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \phi_k \cdot e^{\frac{-2\pi ijk}{2N}} \overset{(*)}{=} \frac{1}{\sqrt{2N}} \left(1 \cdot e^0 + \frac{1}{2} \cdot e^{\frac{-2\pi ij}{2N}} + \frac{1}{2} \cdot e^{\frac{-2\pi ij(2N-1)}{2N}} \right) =$$

   $$\frac{1}{\sqrt{2N}} \left(1 + \frac{1}{2} \left(e^{\frac{-2\pi ij}{2N}} + e^{\frac{-2\pi ij(2N-1)}{2N}} \right) \right)$$

   (*) All the elements of ϕ are zero, except for the first two the last one.
   From the last term we conclude that the DFT of ϕ is a normalized linear combination of a vector of 1's, and the second + last columns of the DFT matrix:

   $$DFT(\phi) = \phi^F = \frac{1}{\sqrt{2N}} (\mathbf{1}_{2N\times 1} + \frac{1}{2} \cdot W_1 + \frac{1}{2} \cdot W_{2N-1})$$

   (b) The DFT of ψ would be the following term:

   $$DFT(\psi) = W\psi = \psi^F$$

   Where ψ^F is a vector with N elements, and each element ψ_j^F ($\forall j \in \{0, ..., N-1\}$) equals to:

   $$\psi_j^F = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \psi_k \cdot e^{\frac{-2\pi ijk}{N}}$$

   As before, let us compute the DFT of γ:

   $$DFT(\gamma) = W\gamma = \gamma^F$$

   Where γ^F is a vector with $2N$ elements, and each element γ_j^F ($\forall j \in \{0, ..., 2N-1\}$) equals to:

   (c)

   $$\gamma_j^F = \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \gamma_k \cdot e^{\frac{-2\pi ijk}{2N}} \overset{(**)}{=} \frac{1}{\sqrt{2N}} \sum_{k=0}^{N-1} \psi_k \cdot e^{\frac{-2\pi ij(2k)}{2N}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \psi_k \cdot e^{\frac{-2\pi ijk}{N}} \overset{(!)}{=} \frac{1}{\sqrt{2}} \cdot \psi^F_{j\,(mod\, N)}$$

   (**) All the elements of γ in the odd indices are zero, and the each element in even index $2k$ ($\forall k \in \{0, ..., N-1\}$) has the value ψ_k.
   (!) using the identity $w^x = w^{x\,(mod\, N)}$.

   From the last term we conclude that the DFT of γ is a normalized vector, with two equal blocks of ψ^F:

   $$DFT(\gamma) = \gamma^F = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^F \\ \psi^F \end{pmatrix}$$

(d) Using the definition of the convolution operation, we obtain $\forall k \in \{0, ..., 2N-1\}$:

$$h_k = \gamma * \phi = \sum_{j=0}^{2N-1} \gamma_{(k-j)\,(mod\,2N)} \cdot \phi_j = 1 \cdot \gamma_{(k)\,(mod\,2N)} + \frac{1}{2} \cdot \gamma_{(k-1)\,(mod\,2N)} + \frac{1}{2} \cdot \gamma_{(k-(2N-1))\,(mod\,2N)} =$$

$$\gamma_{(k)\,(mod\,2N)} + \frac{1}{2}(\gamma_{(k-1)\,(mod\,2N)} + \gamma_{(k-2N+1)\,(mod\,2N)})$$

For odd k, with the definition of γ from the previous clause:

$$h_k = 0 + \frac{1}{2}\left(\psi_{\frac{(k-1)}{2}\,(mod\,2N)} + \psi_{\frac{(k-2N+1)}{2}\,(mod\,2N)}\right)$$

And for even k we get:

$$h_k = 1 \cdot \psi_{\frac{k}{2}} + \frac{1}{2} \cdot 0 = \psi_{\frac{k}{2}}$$

Therefore, the vector of the operation result would be as requested:

$$h = \begin{pmatrix} \psi_0 & \frac{1}{2}(\psi_0 + \psi_1) & \psi_1 & \frac{1}{2}(\psi_0 + \psi_1) & \dots & \psi_{N-1} & \frac{1}{2}(\psi_{N-1} + \psi_N) \end{pmatrix}^T$$

(e) Using the claim we proved in question 1 (clause J) and the definition $h = \gamma * \phi$,
The DFT of h would be the following term (last operation is the Hadamard product):

$$DFT(h) = DFT(\gamma * \phi) = \sqrt{2N} DFT(\gamma) \odot DFT(\phi)$$

Thus $\forall k \in \{0, ..., 2N-1\}$:

$$h_k^F = \sqrt{2N} \gamma_k^F \cdot \phi_k^F = \sqrt{2N} \cdot \frac{1}{\sqrt{2N}} \left(1 + \frac{1}{2}\left(e^{\frac{-2\pi ij}{2N}} + e^{\frac{-2\pi ij(2N-1)}{2N}}\right)\right) \cdot \frac{1}{\sqrt{2}} \psi^F_{j\,(mod\,N)} =$$

$$\frac{1}{\sqrt{2}} \psi^F_{j\,(mod\,N)} \left(1 + \frac{1}{2}\left(e^{\frac{-2\pi ij}{2N}} + e^{\frac{-2\pi ij(2N-1)}{2N}}\right)\right)$$

And finally h^F is a vector with $2N$ elements, which generally can be written as:

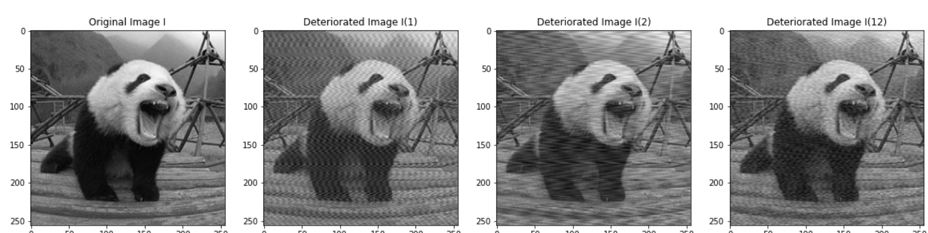
$$h^F = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi^F \\ \psi^F \end{pmatrix} \odot \frac{1}{\sqrt{2N}} (\mathbf{1}_{2N \times 1} + \frac{1}{2} \cdot W_1 + \frac{1}{2} \cdot W_{2N-1})$$

Part II

Implementation

1. Periodic noise

(a) After implementing the required, we received the following result plots:



(b) We denote: $DFT[I_k^{noisy}]$ - the DFT representation of the k-th row of a degraded image as described in the question.
Using the symmetry of the W matrix, Let us compute a general element represention:

$$DFT[I_{k,j}^{noisy}] = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j}^{noisy} = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (I_{k,j} + A_k cos(2\pi fj + \varphi_k)) =$$

$$\frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j} + \sum_{j=0}^{n-1} w^{-k\cdot n}\cdot A_k cos(2\pi fj + \varphi_k)\right) =$$

$$\frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j} + A_k\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot \frac{1}{2}\left(e^{i(2\pi fj+\varphi_k)} + e^{-i(2\pi fj+\varphi_k)}\right)\right) \overset{\textbf{(!)}}{=}$$

$$\frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j} + \frac{1}{2}A_k\cdot e^{i\varphi_k}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{\theta j}) + \frac{1}{2}A_k\cdot e^{-i\varphi_k}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{-\theta j})\right) =$$

$$\frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j} + \frac{1}{2}A_k\cdot e^{i\varphi_k}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{\theta j}) + \frac{1}{2}A_k\cdot e^{-i\varphi_k}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{-\theta j})\right)$$

(!) From the question we know that: $\frac{1}{f}\cdot\theta = n$, where $\theta\in\mathbb{N}$.
Let's look at the term: $\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{\theta j})$
When $k=\theta$:

$$\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{\theta j}) = \sum_{j=0}^{n-1} 1 = n$$

When $k\neq\theta$:

$$\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{\theta j}) = \sum_{j=0}^{n-1}\left(w^{(\theta-k)}\right)^j = \frac{\left(w^{(\theta-k)}\right)^n - 1}{w^{(\theta-k)}-1} = 0$$

Thus:

$$\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{\theta j}) = n\cdot\delta_{k,\theta} = \begin{cases}1 & k=\theta\\ 0 & O.W\end{cases}$$

And similarly (using the identity $w^x = w^{x\,(mod\,n)}$):

$$\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{-\theta j}) = n\cdot\delta_{k,n-\theta} = \begin{cases}1 & k=n-\theta\\ 0 & O.W\end{cases}$$

$$DFT[I_{k,j}^{noisy}] = \frac{1}{\sqrt{n}}\left(\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j} + \frac{1}{2}A_k\cdot e^{i\varphi_k}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{\theta j}) + \frac{1}{2}A_k\cdot e^{-i\varphi_k}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot (w^{-\theta j})\right) =$$

$$\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j} + \frac{\sqrt{n}}{2}A_k\left(e^{i\varphi_k}\cdot\delta_{k,\theta} + e^{-i\varphi_k}\cdot\delta_{k,n-\theta}\right) \overset{as\,seen\,in\,class}{=}$$

$$DFT[I_{k,j}] + \frac{\sqrt{n}}{2}A_k\left(e^{i\varphi_k}\cdot\delta_{k,\theta} + e^{-i\varphi_k}\cdot\delta_{k,n-\theta}\right)$$

Plugging in the given value $n=256$, we'll denote:

$$D_{k,j} = DFT[I_{k,j}] + \frac{\sqrt{256}}{2}A_k\left(e^{i\varphi_k}\cdot\delta_{k,\theta} + e^{-i\varphi_k}\cdot\delta_{k,n-\theta}\right) =$$

$$DFT[I_{k,j}] + 8A_k \left(e^{i\varphi_k} \cdot \delta_{k,\theta} + e^{-i\varphi_k} \cdot \delta_{k,n-\theta}\right)$$

In conclusion, the DFT representation of the k-th row of a degraded image (where $k \in \{0, 1, ..., n-1\}$) would be:

$$DFT[I_k^{noisy}] = [D_{k,0},\ D_{k,1},\ \ldots,\ D_{k,n-1}]$$

(c) At first, we'll denote the noisy signal element:

$$I_{k,j}^{noisy} = I_{k,j} + \omega_1 A_{1_k} cos(2\pi f_1 j + \varphi_{1_k}) + \omega_2 A_{2_k} cos(2\pi f_2 j + \varphi_{2_k})$$

Where ω_1, ω_2 are the normalized weights of the harmonic noise vectors, forming together the requested weighted average.
Now, similarly to the previous clause, we'll compute the representation of a general element:

$$DFT[I_{k,j}^{noisy}] = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot I_{k,j}^{noisy} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (I_{k,j} + \omega_1 A_{1_k} cos(2\pi f_1 j + \varphi_{1_k}) + \omega_2 A_{2_k} cos(2\pi f_2 j + \varphi_{2_k})) =$$

$$\frac{1}{\sqrt{n}} \left(\sum_{j=0}^{n-1} w^{-k\cdot j} \cdot I_{k,j} + \sum_{j=0}^{n-1} w^{-k\cdot n} \cdot \omega_1 A_{1_k} cos(2\pi f_1 j + \varphi_{1_k}) + \sum_{j=0}^{n-1} w^{-k\cdot n} \cdot \omega_2 A_{2_k} cos(2\pi f_2 j + \varphi_{2_k}) \right) =$$

$$\frac{1}{\sqrt{n}} (\sum_{j=0}^{n-1} w^{-k\cdot j} \cdot I_{k,j} + \omega_1 A_{1_k} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot \frac{1}{2} \left(e^{i(2\pi f_1 j + \varphi_{1_k})} + e^{-i(2\pi f_1 j + \varphi_{1_k})} \right) +$$

$$\omega_2 A_{2_k} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot \frac{1}{2} \left(e^{i(2\pi f_2 j + \varphi_{2_k})} + e^{-i(2\pi f_2 j + \varphi_{2_k})} \right)) \overset{\textbf{(!!)}}{=}$$

$$\frac{1}{\sqrt{n}} (\sum_{j=0}^{n-1} w^{-k\cdot j} \cdot I_{k,j} + \frac{1}{2} \omega_1 A_{1_k} \left(e^{i\varphi_{1_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{\theta_1 j}) + e^{i\varphi_{1_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{-\theta_1 j}) \right) +$$

$$\frac{1}{2} \omega_2 A_{2_k} \left(e^{i\varphi_{2_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{\theta_1 j}) + e^{i\varphi_{2_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{-\theta_1 j}) \right))$$

(!!) From the question we know that: $\frac{1}{f_1} \cdot \theta_1 = n$ and $\frac{1}{f_2} \cdot \theta_2 = n$, where $\theta_1, \theta_2 \in \mathbb{N}$.
We'll use the terms we found previously for $\theta \in \{\theta_1, \theta_2\}$ and obtain:

$$\sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{\theta j}) = n \cdot \delta_{k,\theta} = \begin{cases} 1 & k = \theta \\ 0 & O.W \end{cases}$$

And similarly (using the identity $w^x = w^{x \,(mod\, n)}$):

$$\sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{-\theta j}) = n \cdot \delta_{k,n-\theta} = \begin{cases} 1 & k = n - \theta \\ 0 & O.W \end{cases}$$

Hence:

$$DFT[I_{k,j}^{noisy}] = \frac{1}{\sqrt{n}} (\sum_{j=0}^{n-1} w^{-k\cdot j} \cdot I_{k,j} + \frac{1}{2} \omega_1 A_{1_k} \left(e^{i\varphi_{1_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{\theta_1 j}) + e^{i\varphi_{1_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{-\theta_1 j}) \right)$$

$$+ \frac{1}{2} \omega_2 A_{2_k} \left(e^{i\varphi_{2_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{\theta_1 j}) + e^{i\varphi_{2_k}} \sum_{j=0}^{n-1} w^{-k\cdot j} \cdot (w^{-\theta_1 j}) \right)) =$$

$$\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} w^{-k\cdot j}\cdot I_{k,j}+\frac{\sqrt{n}}{2}\omega_1 A_{1_k}\left(e^{i\varphi_{1_k}}\cdot\delta_{k,\theta_1}+e^{-i\varphi_{1_k}}\cdot\delta_{k,n-\theta_1}\right)+\frac{\sqrt{n}}{2}\omega_2 A_{2_k}\left(e^{i\varphi_{2_k}}\cdot\delta_{k,\theta_2}+e^{-i\varphi_{2_k}}\cdot\delta_{k,n-\theta_2}\right)\overset{as\ seen\ in\ class}{=}$$

$$DFT[I_{k,j}]+\frac{\sqrt{n}}{2}\omega_1 A_{1_k}\left(e^{i\varphi_{1_k}}\cdot\delta_{k,\theta_1}+e^{-i\varphi_{1_k}}\cdot\delta_{k,n-\theta_1}\right)+\frac{\sqrt{n}}{2}\omega_2 A_{2_k}\left(e^{i\varphi_{2_k}}\cdot\delta_{k,\theta_2}+e^{-i\varphi_{2_k}}\cdot\delta_{k,n-\theta_2}\right)$$

Plugging in the given value $n = 256$, we'll denote:

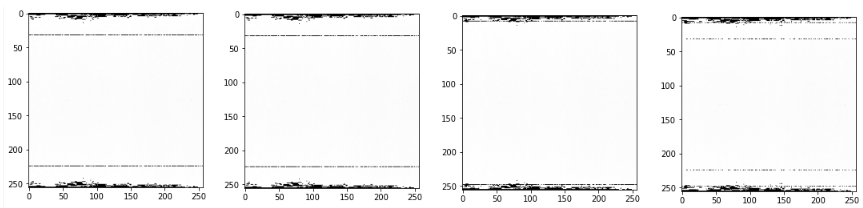
$$\tilde{D}_{k,j} = DFT[I_{k,j}]+\frac{\sqrt{256}}{2}\omega_1 A_{1_k}\left(e^{i\varphi_{1_k}}\cdot\delta_{k,\theta_1}+e^{-i\varphi_{1_k}}\cdot\delta_{k,n-\theta_1}\right)+\frac{\sqrt{256}}{2}\omega_2 A_{2_k}\left(e^{i\varphi_{2_k}}\cdot\delta_{k,\theta_2}+e^{-i\varphi_{2_k}}\cdot\delta_{k,n-\theta_2}\right)=$$

$$DFT[I_{k,j}]+8\omega_1 A_{1_k}\left(e^{i\varphi_{1_k}}\cdot\delta_{k,\theta_1}+e^{-i\varphi_{1_k}}\cdot\delta_{k,n-\theta_1}\right)+8\omega_2 A_{2_k}\left(e^{i\varphi_{2_k}}\cdot\delta_{k,\theta_2}+e^{-i\varphi_{2_k}}\cdot\delta_{k,n-\theta_2}\right)$$

In conclusion, the DFT representation of the k-th row of a degraded image (where $k \in \{0, 1, ..., n-1\}$) would be:

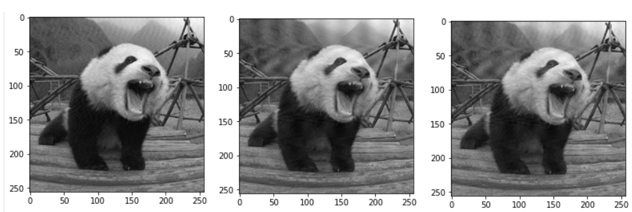
$$DFT[I_k^{noisy}] = \left[\tilde{D}_{k,0},\ \tilde{D}_{k,1},\ \ldots,\ \tilde{D}_{k,n-1}\right]$$

(d) Computing the required DFT representations empirically, we obtained the following results for $I, I^{(1)}, I^{(2)}, I^{(12)}$ (left to right):



After transforming to the frequency domain, we can see that the majority of the noise lies in the edges of the matrix, which goes well with
the result from the theory, where rows θ or $n-\theta$ provide $\delta_{k,\theta}$ or $\delta_{k,n-\theta}$ the value of 1 (respectively), while all other rows get 0.

(e) After applying the filter and reconstructing the image, we received the following results for $I^{(1)}, I^{(2)}, I^{(12)}$ (left to right):



We added the MSE values in the code, which are very small.
The filter managed to reduce most of the noise, but we can only see it's not perfect because of the characteristics of the "deletion" or information
in the noisy rows of the DFT representation. In our opinion, the best reconstruction using the filter was on $I^{(1)}$,which seems the smoothest.