Introduction to Data Processing and Representation
(236201)
Spring 2022

Homework 3

- **Published date: 25/05/2022**
- **Deadline date: 07/06/2022**

- Submission is in pairs only.
- Submit your entire solution (including the theoretical part, the Python part and the Python code) electronically via the course website. The file should be a zip file containing the your PDF submission and Python code.
- The submission should be in English and in a clear printed form (recommended) or a clear hand-writing.
- Rigorous mathematical proofs and reasoning are required for theoretical questions. Vague answers and unjustified claims will not be accepted.

I Theory

1. On Circulant Matrices

In this exercise, we use the normalised convention for the DFT matrix.

a. Consider the matrix $J = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$. For $k \in \mathbb{N}$, compute J^k.
In particular, what is J^n?

b. Compute the eigenvalues of J.

c. Do the full eigendecomposition of J. Is J diagonalisable? If yes, can it be diagonalised in a unitary basis?

d. Consider the general circulant matrix $H = \begin{pmatrix} h_0 & h_{n-1} & h_{n-2} & \dots & h_1 \\ h_1 & h_0 & h_{n-1} & \dots & h_2 \\ h_2 & h_1 & h_0 & \dots & h_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_{n-2} & h_{n-3} & \dots & h_0 \end{pmatrix}$. Show that H and J are linked by a polynomial expression, i.e. find a polynomial P such that $H = P(J)$.

e. Compute the full eigendecomposition of H. Is it diagonalisable and if so is it in a unitary basis?

f. Show that the diagonalisation basis matrix B can be chosen as the DFT^* matrix.

g. Prove that the eigenvalues, stacked in a column, equals to the product of B and the first row of H rewritten in column form up to a normalisation constant, i.e. if $\lambda_0, \dots, \lambda_{n-1}$ are the eigenvalues of H then:

$$\begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{n-1} \end{pmatrix} = \sqrt{n} B \begin{pmatrix} h_0 \\ h_{n-1} \\ \dots \\ h_1 \end{pmatrix}$$

.

h. Consider two circulant matrices H_1 and H_2. Show that they commute. Compute $H_1 H_2$, is this matrix circulant?

i. Compute DFT^k for $k \in \mathbb{N}$. What are the resulting matrices?

j. Prove that a convolution of n-dimensional signals can be computed by point-wise multiplication of the signals in the Fourier domain, up to normalisation. This means prove that if $z = x \otimes y$ where $\otimes$ the convolution operator, then $(DFT)z = \sqrt{n}(DFT)x \odot (DFT)y$ where $\odot$ is the Hadamard product.

2. Fourier Transform

a. Given two functions $f(t)$ and $g(t)$ and denote the convolution of the two functions by $h(t)$, that is

$$f(t) * g(t) = h(t).$$

What is $f(t-1) * g(t+1)$ in terms of $h(t)$?

b. Given two functions $f(t)$ and $g(t)$, show that the following condition holds

$$\int_{-\infty}^{\infty} f(t)g(-t)dt = \int_{-\infty}^{\infty} \mathcal{F}(u)\mathcal{G}(u)du,$$

where $\mathcal{F}(u)$ and $\mathcal{G}(u)$ are the Fourier transform of $f(t)$ and $g(t)$ respectively.

3. Discrete Fourier Transform

Denote a 1D signal with $2N$ elements as $\boldsymbol{\phi} \in \mathbb{R}^{2N}$ given by

$$\boldsymbol{\phi} = [1, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}]^\top$$

We use zero-based indexing in this exercise.

a. What is the DFT of $\boldsymbol{\phi}$?

b. Consider another 1D signal $\boldsymbol{\psi}$ with N elements and denote its DFT-domain representation by $\boldsymbol{\psi}^F$. Consider a new signal $\boldsymbol{\gamma}$ by inserting zeros between the elements of $\boldsymbol{\psi}$, i.e.,

$$\boldsymbol{\gamma} = [\boldsymbol{\psi}_0, 0, \boldsymbol{\psi}_1, 0, \boldsymbol{\psi}_2, 0, \cdots, \boldsymbol{\psi}_{N-1}, 0]^\top \in \mathbb{R}^{2N}$$

Find the DFT of $\boldsymbol{\gamma}$ in terms of $\boldsymbol{\psi}^F$.

c. Show that the convolution of $\boldsymbol{\gamma}$ and $\boldsymbol{\phi}$ is the linear interpolation of $\boldsymbol{\psi}$, that is,

$$\boldsymbol{h} = \boldsymbol{\gamma} * \boldsymbol{\phi} = [\boldsymbol{\psi}_0, \frac{\boldsymbol{\psi}_0 + \boldsymbol{\psi}_1}{2}, \boldsymbol{\psi}_1, \frac{\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2}{2}, \boldsymbol{\psi}_2, \cdots, \boldsymbol{\psi}_{N-1}, \frac{\boldsymbol{\psi}_{N-1} + \boldsymbol{\psi}_0}{2}]^\top$$

d. Find the DFT of $\boldsymbol{h}$ in terms of $\boldsymbol{\psi}^F$.

II Implementation

The purpose of this exercise is to get familiar with notch filtering. We will start indexing at 0.

Instruction:

- Figures should be titled in a appropriate font size.
- Submit the code and a report including the dry part and describing the results and your understanding of the exercise.

1. Periodic noise

a. Select your favourite gray-scale image I of size 256×256 pixels encoded in $[0, 1]$ and create three deteriorated versions of it by adding noise in the following way. For each $k \in \{1, 2\}$, denote $r_i^{(k)} \in \mathbb{R}^{256}$ a random harmonic noise vector with entries: $r_{i,j}^{(k)} = A_i^{(k)} \cos(2\pi f_k j + \varphi_i^{(k)})$, where the frequencies are fixed $f_1 = \frac{1}{8}$ and $f_2 = \frac{1}{32}$, but the amplitudes $A_i^{(k)}$ and phases $\varphi_i^{(k)}$ are random independently sampled per row $i \in \{1, \ldots, 256\}$ and per k, with $\varphi_i^{(k)} \sim \mathcal{U}([0, 2\pi])$ uniform and $A_i^{(k)} \sim \mathcal{N}(\mu, \sigma^2)$ a Gaussian of mean $\mu = \frac{1}{10}$ standard deviation $\sigma = \frac{1}{20}$. Degrade I by constructing $I^{(1)} = I + r^{(1)}$, $I^{(2)} = I + r^{(2)}$, and $I^{(12)} = I + \frac{r^{(1)}+r^{(2)}}{2}$. Plot I, $I^{(1)}$, $I^{(2)}$, and $I^{(12)}$.

b. Compute the DFT representation of each row of an image degraded by additive harmonic noise with fixed frequency but random iid amplitude and phase per row: $I_{i,j}^{noisy} = I + A_i \cos(2\pi f j + \varphi_i)$, where $\frac{1}{f}$ divides n the number of columns in the image. Provide a theoretical derivation.

c. Deduce the DFT representation of an image degraded by the weighted average of two different and independent harmonic noise vectors with different frequencies $\frac{1}{f_1}$ and $\frac{1}{f_2}$ both dividing n. Provide a theoretical derivation.

d. Compute empirically the DFT representations of I, $I^{(1)}$, $I^{(2)}$, and $I^{(12)}$. Compare with the theory. Note that you should implement yourselves the DFT transform.

e. Assuming the f_k are provided by an oracle, we can enhance the images by removing as best as possible the noise in the frequency domain. A common strategy is to zero out the frequencies contaminated with noise[1]. Apply this filtering strategy to enhance images $I^{(1)}$, $I^{(2)}$, and $I^{(12)}$. Compute the MSE of the reconstructions. Comment on the reconstruction results you obtain.

[1] Future material will give a theoretical explanation as to why this strategy is optimal in some sense.