Introduction to Data Processing and Representation (236201)

Spring 2022

Homework 4

Submitted by:
Shelly Golan 209290477
Ofer Nissim 312367576

June 27, 2022

Part I
Theory

1. Inverting the Second Derivative Operator

   (a) In this section we will provide an explicit formulation of the degradation of the matrix H and it's components. In this question we are given the following:

$$\varphi_{data,j} = -\frac{1}{12}\varphi_{j-2[M]} + \frac{4}{3}\varphi_{j-1[M]} - \frac{5}{2}\varphi_{j[M]} + \frac{4}{3}\varphi_{j+1[M]} - \frac{1}{12}\varphi_{j+2[M]}$$

Now we'll use φ_{data} to represent the rows of matrix H. The first row will be:

$$H_0 : \begin{pmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} & \frac{4}{3} \end{pmatrix}$$

as we denote: $j = 0$ so

$$\begin{gathered} j-2[M] = M-2 \\ j-1[M] = M-1 \\ j[M] = 0 \\ j+1[M] = 1 \\ j+2[M] = 2 \end{gathered}$$

We'll use the operator $\mathbb{J}$, the cyclic permutation matrix we saw in lecture 9, and write the matrix H as:

$$H = -\frac{5}{2}\mathbb{J}^M + \frac{4}{3}\mathbb{J}^{M-1} - \frac{1}{12}\mathbb{J}^{M-2} + \frac{4}{3}\mathbb{J}^1 - \frac{1}{12}\mathbb{J}^2$$

So, overall we can write the whole matrix H now:

$$H = \begin{pmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} & \frac{4}{3} \\ \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & \ddots & -\frac{1}{12} \\ -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 \\ 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & \ddots & \vdots \\ \vdots & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} & \frac{4}{3} & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & -\frac{1}{12} \\ -\frac{1}{12} & 0 & \cdots & 0 & \ddots & \ddots & \ddots & \frac{4}{3} \\ \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} & \frac{4}{3} & -\frac{5}{2} \end{pmatrix}$$

We can see the circulant proprety of the matrix H from it's definition above.

   (b) As we saw in the $10'th$ tutorial, the pseudo inverse filter $M = U\Sigma U^*$ under the assumption that H is diagonolized in a unitary basis, the same basis that M is diagonaziable with. Where,

$$\sigma_i = \begin{cases} \frac{1}{\lambda_i} & \lambda_i \neq 0 \\ 0 & \lambda_i = 0 \end{cases}$$

Since H is circulant it's diagonizable by the DFT^* matrix, which we'll denote as W^*. The eigenvalues of the matrix H are:

$$\lambda_i^H = \sum_{i=1}^{M-1} \lambda_0^i h_i$$

Where,

$$\lambda_k = e^{-\frac{2\pi ik}{M}} = w^{-k}$$

And $h_i's$ are determined according to the first row above:

$$H_0 : \begin{pmatrix} -\frac{5}{2} & \frac{4}{3} & -\frac{1}{12} & 0 & \cdots & 0 & -\frac{1}{12} & \frac{4}{3} \end{pmatrix}$$

Therefore, let k be a value in range $[0, M-1]$:

$$\lambda_k^H = \sum_{i=0}^{M-1} \lambda_k^i h_i = \sum_{i=0}^{M-1} w^{-ki} h_i =$$

$$= -\frac{5}{2} w^0 + \frac{4}{3} w^{-k} - \frac{1}{12} w^{-2k} + \frac{4}{3} w^{-(M-1)k} - \frac{1}{12} w^{-(M-2)k}$$

Now by using euler equation instead of the $w's$:

$$-\frac{5}{2}(cos0 - isin0) + \frac{4}{3}\left(cos\frac{2\pi k}{M} - isin\frac{2\pi k}{M}\right) - \frac{1}{12}\left(cos\frac{4\pi k}{M} - isin\frac{4\pi k}{M}\right) +$$

$$+\frac{4}{3}\left(cos\frac{2\pi k(M-1)}{M} - isin\frac{2\pi k(M-1)}{M}\right) - \frac{1}{12}\left(cos\frac{2\pi k(M-2)}{M} - isin\frac{2\pi k(M-2)}{M}\right) =$$

$$= -\frac{5}{2} + \frac{3}{8} cos\left(\frac{2\pi k}{M}\right) - \frac{1}{6} cos\left(\frac{4\pi k}{M}\right) =$$

$$= -\frac{5}{2} + \frac{3}{8} cos\left(\frac{2\pi k}{M}\right) - \frac{1}{6}\left(2cos^2\frac{2\pi k}{M} - 1\right) =$$

$$= -\frac{14}{6} + \frac{3}{8} cos\left(\frac{2\pi k}{M}\right) - \frac{1}{3} cos^2\left(\frac{2\pi k}{M}\right)$$

To build the pseudo inverse filter we need to find which eigenvalues equal to zero, since Σ has non zero values only when $\lambda_k \neq 0$.

$$-\frac{14}{6} + \frac{3}{8} cos\left(\frac{2\pi k}{M}\right) - \frac{1}{3} cos^2\left(\frac{2\pi k}{M}\right) = 0$$

$$cos\left(\frac{2\pi k}{M}\right) = \pm\frac{-16 + \sqrt{16^2 - 4 \cdot 2 \cdot 14}}{-4}$$

$$cos\left(\frac{2\pi k}{M}\right) = \begin{cases} 1 \\ 7 \end{cases}$$

$$\Rightarrow \frac{2\pi k}{M} = 0 + 2\pi n, \forall n \in \mathbb{N}$$

$$\Rightarrow \frac{k}{M} = n \in \mathbb{N}$$

but since $k \in [0, M-1]$, this is true only for $k = 0$.
so the pseudo inverse filter is:

$$M = W^* \cdot diag\left(0, \frac{1}{\sum_{i=1}^{n-1} \lambda_1^i h_i}, ..., \frac{1}{\sum_{i=1}^{n-1} \lambda_1^i h_i}\right) \cdot W$$

(c) Not any φ can be reconstructed by the filter we found above, since the first eigenvalue in the diagonal matrix is 0, which means that each signal that holds the condition: $\langle W_0^*, \varphi \rangle = 0$ cannot be reconstructed. Where W_0^* is the first column of the DFT^* matrix. Since those signals that hold this condition don't have a component in the direction of the first DFT^* column, so it's projection to the DFT^* space will be zero and we'll lose information in the reconstruction.

2. Let's Randomise

(a) We'll show that the random vector φ has a zero mean.
Let $i \in \mathbb{N}$. We'll denote the entries of the vector φ as φ_i.
Also, we know that $K \sim Uni\,[1, N]$. Let's compute $E\left(\varphi_i\right)$ and then generalize it to $E\left(\varphi\right)$.

$$E\left(\varphi_i\right) = E\left(E\left(\varphi_i \,|\, K=k\right)\right) = \sum_{k=1}^{N} E\left(\varphi_i \,|\, K=k\right) \cdot p\left(K=k\right) =$$

$$= \sum_{k=1}^{N} E\left(\varphi_i \,|\, K=k\right) \cdot \frac{1}{N} =$$

$$= \sum_{k=1 \neq i}^{N} E\left(\varphi_i \,|\, K=k\right) \cdot \frac{1}{N} + E\left(\varphi_i \,|\, K=i\right)$$

$$= \frac{N-1}{N} \cdot E\left(M\right) + \frac{1}{N} \cdot E\left(M+L\right) = E\left(M\right) + \frac{1}{N} \cdot E\left(L\right) \underbrace{=}_{*} E\left(M\right)$$

(*) We are given that both L_1, L_2 satisfy $E\left(L_1\right) = E\left(L_2\right) = 0$ so $E\left(L\right) = 0$
Also, we are given that $E\left(M\right) = 0$ so overall,

$$E\left(\varphi_i\right) = 0, \forall i$$

So $E\left(\varphi\right)$ is a vector of zeros only, and therefore has a mean of zero.

(b) We want to find the autocorrelation matrix. To do so, we'll find it's diagonal values first and then the rest of the values which we expect to be the same for all the entries which are not on the diagonal.
When $i = j$:

$$R_\varphi\left(i,i\right) = E\left(\varphi_i \varphi_i\right) = \sum_{k=1}^{N} E\left(\varphi_i^2 \,|\, K=k\right) \cdot \frac{1}{N} =$$

$$= \sum_{k=1\neq i}^{N} E\left(\varphi_i^2 \,|\, K=k\right) \cdot \frac{1}{N} + E\left(\varphi_i^2 \,|\, K=i\right) =$$

$$= \frac{N-1}{N} \cdot E\left(M^2\right) + \frac{1}{N} \cdot E\left(M+L\right)^2 =$$

$$= \frac{N-1}{N} \cdot E\left(M^2\right) + \frac{1}{N} E\left(M^2\right) + \frac{2}{N} E\left(ML\right) + \frac{1}{N} E\left(L^2\right) \underbrace{=}_{M,L\, independent}$$

$$= E\left(M^2\right) + \frac{2}{N} E\left(M\right) E\left(L\right) + \frac{1}{N} E\left(L^2\right) \underbrace{=}_{*}$$

$$= c + \frac{1}{N} E\left(L^2\right)$$

(*) We are given in the question that $E(M) = 0$ and $E(M^2) = c$.
So,

$$R_\varphi\left(i,i\right) = \begin{cases} c+a & K \leq \frac{N}{2} \\ c+b & otherwise \end{cases}$$

Now we'll calculate what happens if $i \neq j$:

$$R_\varphi\left(i,j\right) = E\left(\varphi_i \varphi_j\right) = \sum_{k=1}^{N} E\left(\varphi_i \varphi_j \,|\, K=k\right) \cdot \frac{1}{N} =$$

$$= \sum_{k=1\neq i,j}^{N} E\left(\varphi_i \varphi_j \,|\, K=k\right) \cdot \frac{1}{N} + E\left(\varphi_i \,|\, K=i\right) + E\left(\varphi_j \,|\, K=j\right) =$$

$$= \frac{N-2}{N} \cdot E\left(M^2\right) + \frac{2}{N} \cdot E\left(M\left(M+L\right)\right) =$$

$$= \frac{N-2}{N} \cdot E\left(M^2\right) + \frac{2}{N} E\left(M^2\right) + \frac{2}{N} E\left(ML\right) =$$

$$= E\left(M^2\right) + \frac{2}{N} E\left(M\right) E\left(L\right) \underbrace{=}_{E(M)=0} c$$

So, overall we get that

$$R_\varphi\left(i,j\right) = \begin{cases} c+a & K \leq \frac{N}{2}\, and\, i=j \\ c+b & K > \frac{N}{2}\, and\, i=j \\ c & otherwise \end{cases}$$

(c) The PCA matrix that is corresponding to the autocorrelation matrix R_φ, as we learnt in class, is the matrix containing the eigenvectors of R_φ in the order that corresponds to the sorted values of the eigenvalues in descending order.
If we demand that the DFT^* matrix is the PCA matrix, that means that the DFT^* columns are R_φ's eigenvectors. Which means, we want to demand that R_φ is diagonaziable using the DFT^* matrix, so R_φ is a circulant matrix. So, the most general condition on a, b, c we can demand in order that R_φ will be a circulant matrix is that $a = b$.

3. Let's Randomise Again!

(a) We'll calculate again just like we did in the previous question, for $i = j$ and $i \neq j$. We'll start from the case that $i = j$ or if $|i-j| = \frac{N}{2}$ meaning $(i-j)mod\frac{N}{2} = 0$:

$$R_\varphi(i,i) = E(\varphi_i\varphi_i) = \sum_{k=1}^{N} E\left(\varphi_i^2 \,|\, K=k\right) \cdot \frac{1}{N} =$$

$$= \sum_{k=1 \neq i, i-\frac{N}{2}}^{N} E\left(\varphi_i^2 \,|\, K=k\right) \cdot \frac{1}{N} + E\left(\varphi_i^2 \,|\, K=i\right) + E\left(\varphi_i^2 \,|\, K=i-\frac{N}{2}\right) =$$

$$= \frac{N-2}{N} \cdot E\left(M^2\right) + \frac{1}{N} \cdot E(M+L)^2 + \frac{1}{N} \cdot E(M+L)^2 =$$

$$= \frac{N-2}{N} \cdot E\left(M^2\right) + \frac{2}{N} E\left(M^2\right) + \frac{4}{N} E(ML) + \frac{2}{N} E\left(L^2\right) =$$

$$= E\left(M^2\right) + \frac{4}{N} E(M) E(L) + \frac{2}{N} E\left(L^2\right) =$$

$$= c + 0 + \frac{2}{N} \cdot \frac{N}{2} \cdot (1-c) = c + 1 - c = 1$$

(*) M, L are independent, $E(M^2) = c$, $E(L) = 0$, $E(L^2) = \frac{N}{2}(1-c)$
Now we'll calculate for $i \neq j$:

$$R_\varphi(i,j) = E(\varphi_i\varphi_j) = \sum_{k=1}^{N} E(\varphi_i\varphi_j \,|\, K=k) \cdot \frac{1}{N} =$$

$$= \sum_{k=1 \neq i,j,i-\frac{N}{2},j-\frac{N}{2}}^{N} E(\varphi_i\varphi_j \,|\, K=k) \cdot \frac{1}{N} + E(\varphi_i \,|\, K=i) + E(\varphi_j \,|\, K=j)$$

$$+E\left(\varphi_i\varphi_j \,|\, K=i-\frac{N}{2}\right) + E\left(\varphi_i\varphi_j \,|\, K=j-\frac{N}{2}\right) =$$

$$= \frac{N-4}{N} \cdot E\left(M^2\right) + \frac{4}{N} \cdot E(M(M+L)) =$$

$$= \frac{N-4}{N} \cdot E\left(M^2\right) + \frac{4}{N} E\left(M^2\right) + \frac{4}{N} E(ML) =$$

$$= E\left(M^2\right) + 0 = c$$

So overall,

$$R_\varphi(i,j) = \begin{cases} 1 & i = jmod\frac{N}{2} \\ c & otherwise \end{cases}$$

We can see that the form of R_φ is circulant by definition of circulant matrices.

(b) Since R_φ is a circulant matrix, we know it's eigenvalues from the last HW are just

$$\sqrt{N} DFT^* c_0^T$$

Where c_0 is the first row of the circulant matrix that defines the whole matrix.

4. Wiener Filter

(a) We'll compute the autocorrelation matrix of φ^*. We are given in the question a linear degradation operator $\mathcal{H}$, and an additive independent noise vector n with $E(n) = 0$. Also we are given the autocorrelation matrix of φ denoted as R_φ and the autocorrelation matrix of n as R_n. And $\varphi^* = \mathcal{H}\varphi + n$.
We'll denote $\mathcal{H}_i$ as the i'th row of $\mathcal{H}$. Now lets calculate the $R_{\varphi^*}(i,j)$:

$$R_{\varphi^*}(i,j) = E\left(\varphi_i^* \cdot \bar{\varphi_j^*}\right) = E\left((\mathcal{H}_i\varphi + n_i) \cdot \left(\mathcal{H}_j^*\bar{\varphi} + n_j^*\right)\right) =$$

$$= E\left(\mathcal{H}_i\varphi\mathcal{H}_j^*\bar{\varphi}\right) + E\left(\mathcal{H}_i\varphi\right)\underbrace{E\left(n_j^*\right)}_{=0} + \underbrace{E\left(n_i\right)}_{=0}E\left(\mathcal{H}_j^*\bar{\varphi}\right) + E\left(n_i n_j^*\right) =$$

$$= \mathcal{H}_i E\left(\varphi\bar{\varphi}\right)\mathcal{H}_j^* + R_n\left(i,j\right) =$$

$$= \mathcal{H}_i R_\varphi\left(i,j\right)\mathcal{H}_j^* + R_n\left(i,j\right)$$

So,

$$R_{\varphi^*} = \mathcal{H}R_\varphi\mathcal{H}^* + R_n$$

(b) We saw Weiner Filter in class, given as:

$$W = R_\varphi\mathcal{H}^*\left(\mathcal{H}R_\varphi\mathcal{H}^* + \sigma_n^2 I\right)^{-1}$$

We'll notice that if we use what we found in the previous question, that $R_{\varphi^*} = \mathcal{H}R_\varphi\mathcal{H}^* + R_n$ and the given information that $R_n = \sigma_n^2 I$, we'll simply get that

$$W = R_\varphi\mathcal{H}^* R_{\varphi^*}^{-1}$$

(c) We want to prove that if A is diagonaziable by the DFT^* matrix $\Rightarrow A$ is circulant. We'll denote $DFT^* = W^*$, Σ is the diagonal matrix of the eigenvalues. So

$$A = W^*\Sigma W$$

Let X be some circulant matrix of size NxN with same eigenvalues as of A. If X_0 is the first row of the matrix X, we can say that the eigenvalues of X are

$$\sqrt{N}WX_0^T$$

And also X is diagonaziable by W^*.
So we get that

$$X = W^*\Sigma W = A$$

And hence, $A = X$, so A is circulant.

(d) The Wiener Filter is not a shift invariant system. We'll show an example - if we define $R_\varphi = I$, and $\sigma^2 = 2$. So, we'll get that

$$W = R_\varphi \mathcal{H}^* \left(\mathcal{H} R_\varphi \mathcal{H}^* + \sigma_n^2 I\right)^{-1} = \mathcal{H}^* \left(\mathcal{H}\mathcal{H}^* + 2I\right)^{-1}$$

If we take $\mathcal{H}$ to be a diagonal matrix, we get that:

$$W = \mathcal{H}\left(\mathcal{H}^2 + 2I\right)^{-1}$$

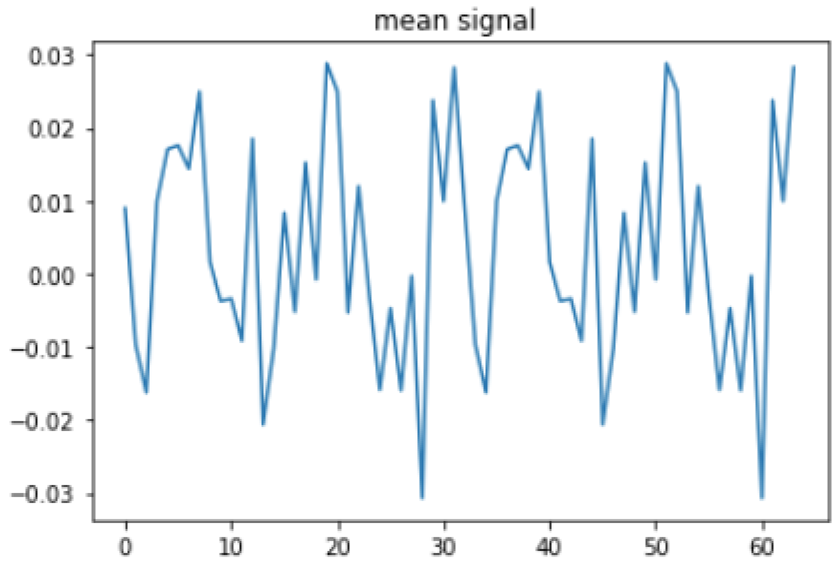
Which will be a diagonal matrix but with different values on the diagonal, as a multiplication and summation of diagonal matrices. A diagonal matrix of this kind is not a shift invariant system.

(e) We'll want to find conditions of R_φ,$\mathcal{H}$ and σ_n that W will be shift invariant.
If R_φ,$\mathcal{H}$ are both shift invariant they are diagonaziable by the DFT^* matrix, which means that also W is diagonaziable by the DFT^* matrix as a multipication and summation of those matrices. Which means that W is a shift invariant operator.

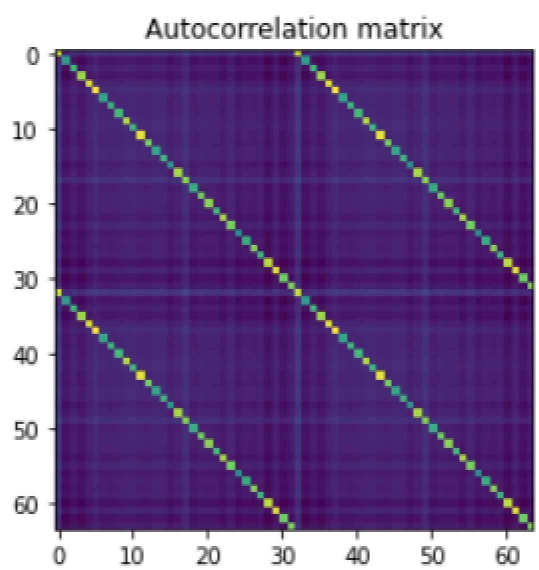
Part II

Implementation

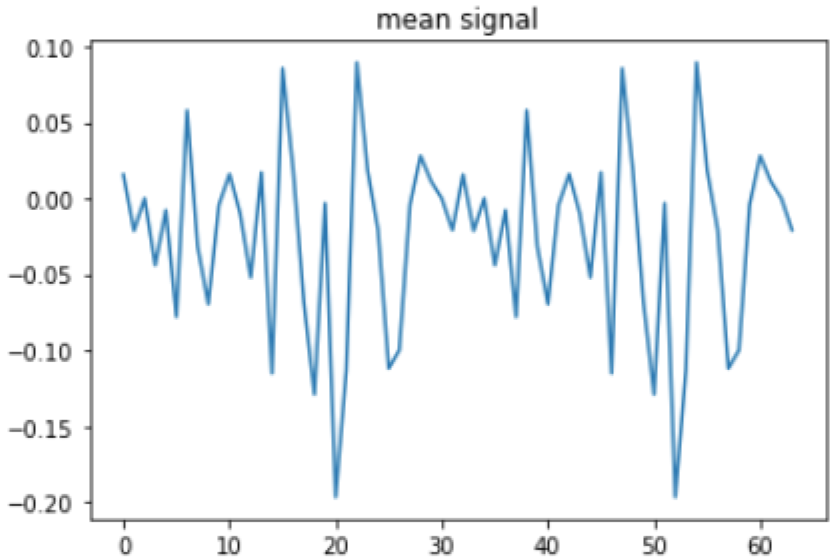
1. Reconsider the third exercise in the Theory part with some changes, $N = 64,\ M, L \sim N,\ c = 0.6$.
   (a) The empirical approximation of the mean signal:

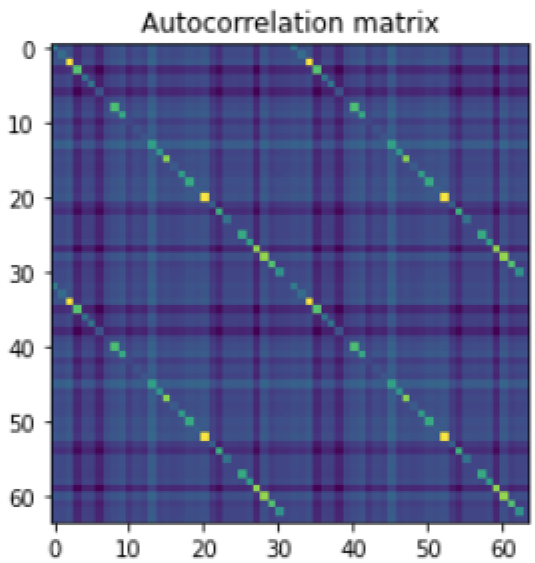


And the autocorrelation matrix:



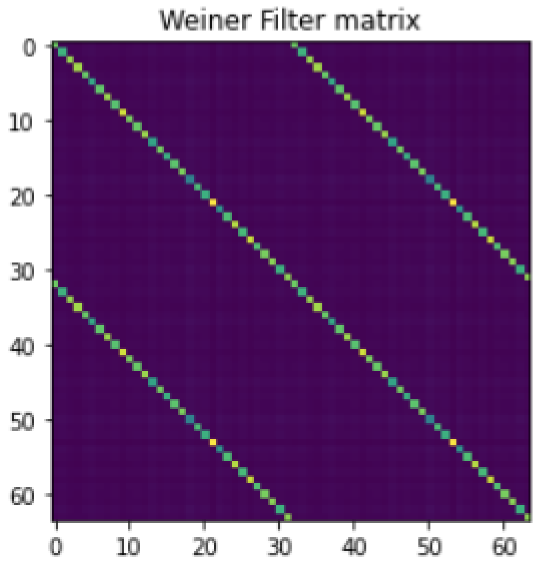
We notice that the autocorrelation matrix here in the plot is a good approximation of the one we computed by hand in Q3.
To see what is the number needed for a good realization, we ran more tests. The above plots are with $1,000$ realizations.
We also checked:
100 realizations:





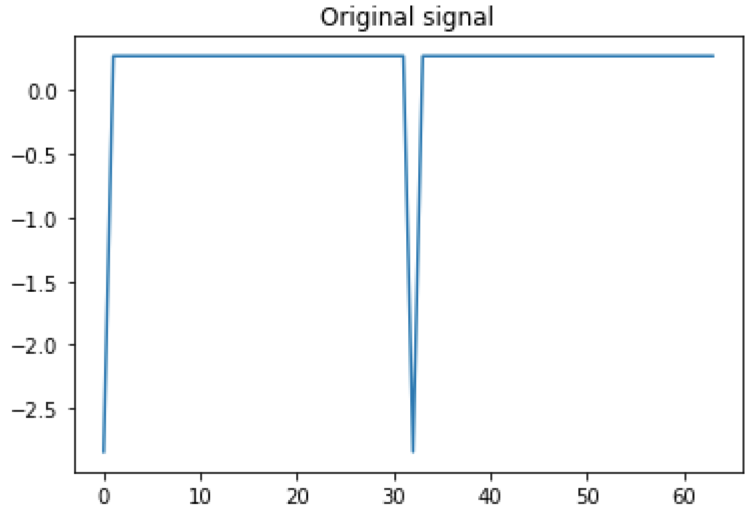
So as we can see we need at least $1,000$ for a good enough approximation.

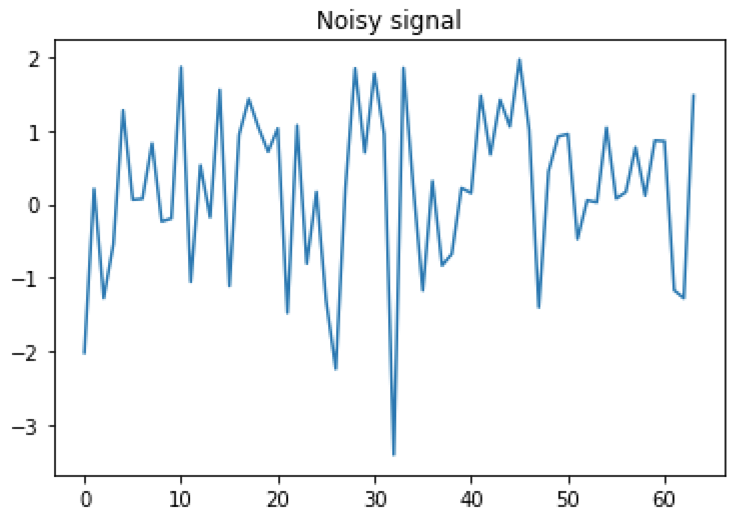
(b) The wiener filter matrix we generated:

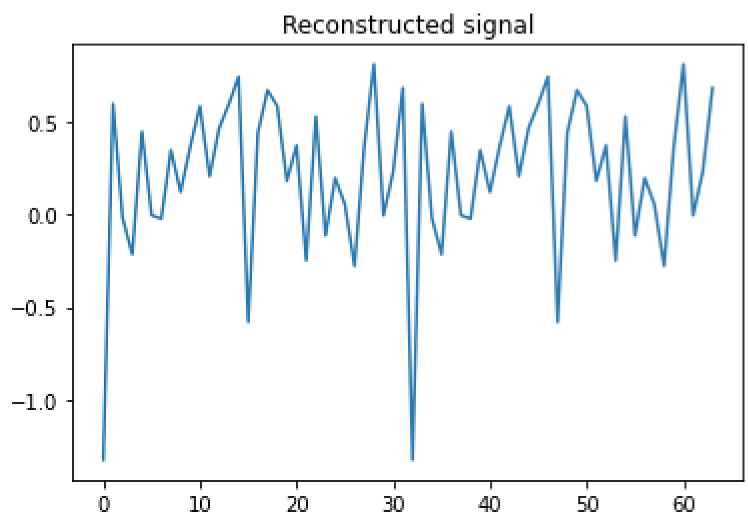


The filter's structure is circulant and very similiar to the autocorrelation matrix, intuitively.

Now we'll present the use of the wiener filter on a signal for different amounts of realizations:
For 10 realizations:

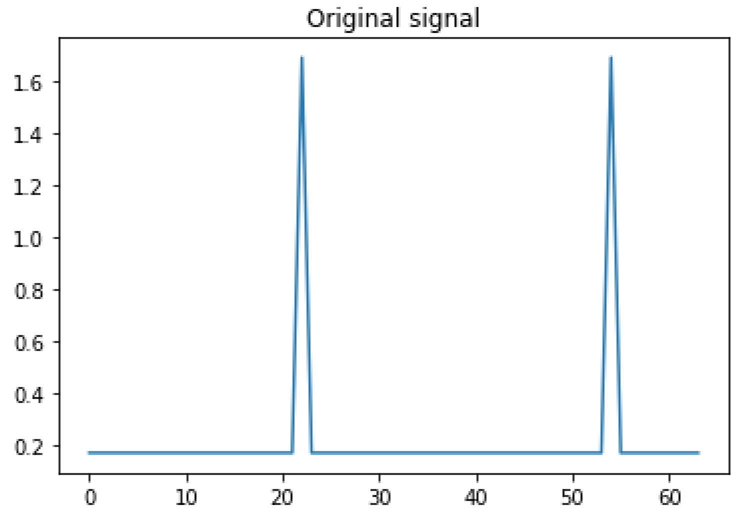


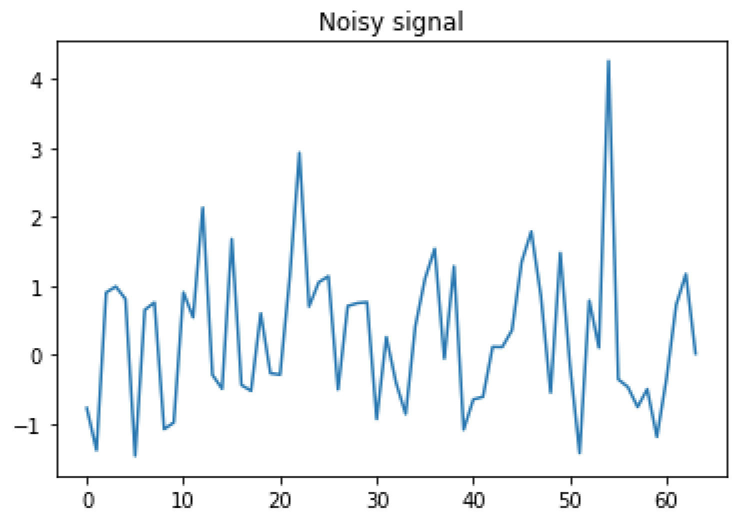


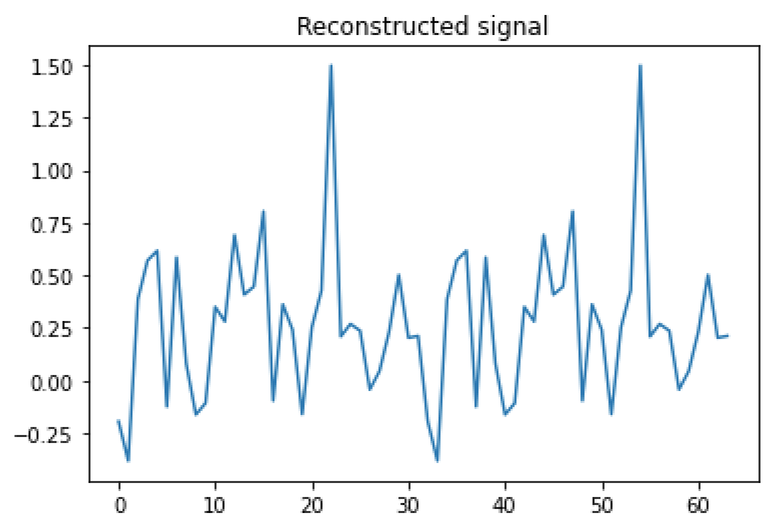



```
the mse for 10 realizations is: 0.18286726234650152
```

For 100 realizations:

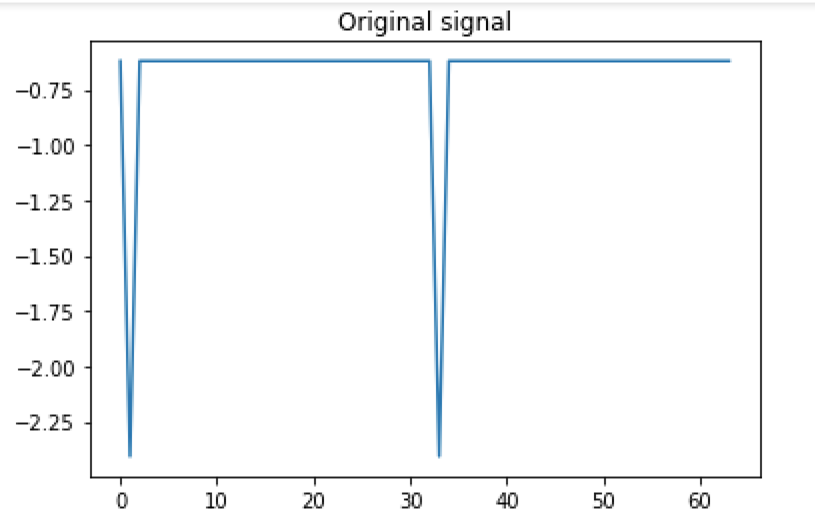


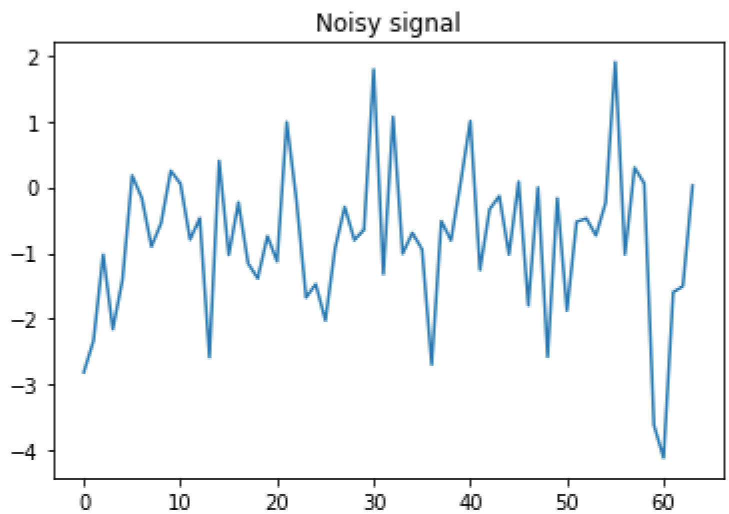


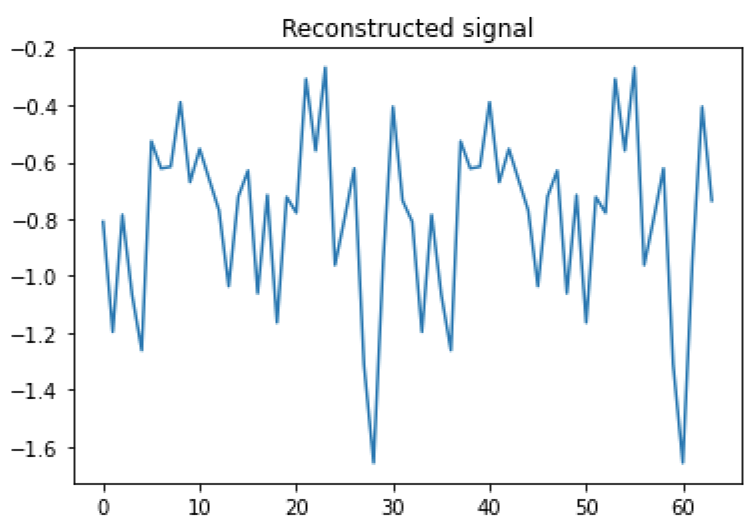



```
the mse for 100 realizations is: 0.08316462248497675
```

For 500 realizations:



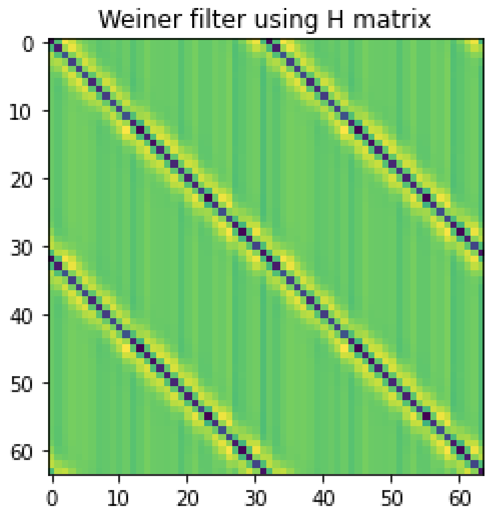




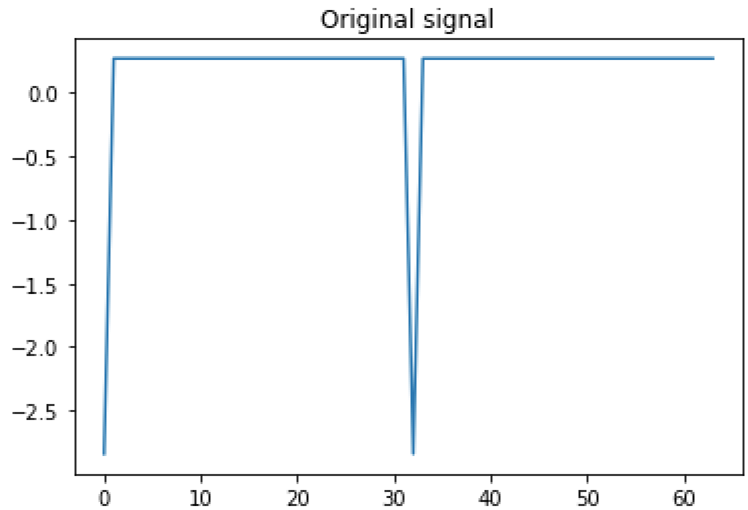

```
the mse for 500 realizations is: 0.15701853872613786
```

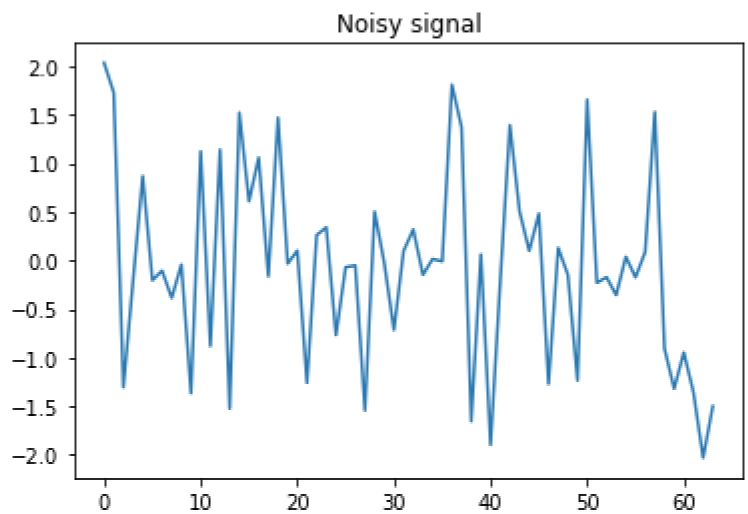
And the average MSE for all the realizations is: 0.2265.

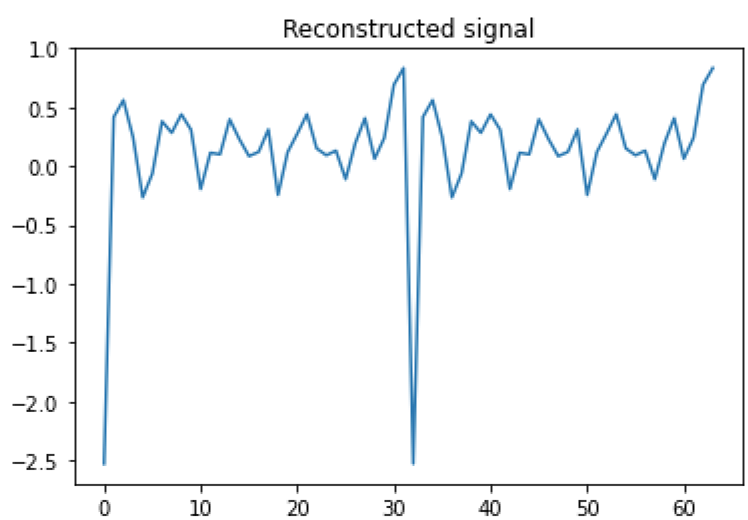
(c) The generated wiener filter using H matrix:



And the plots of the signals:
For 10 realizations:

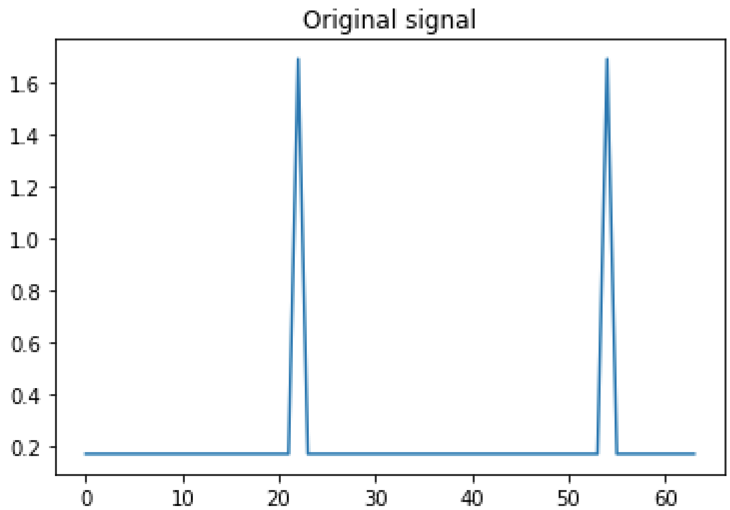


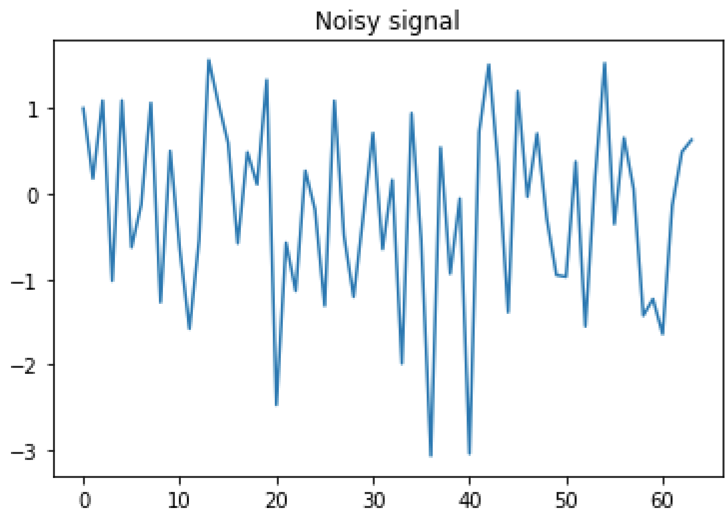


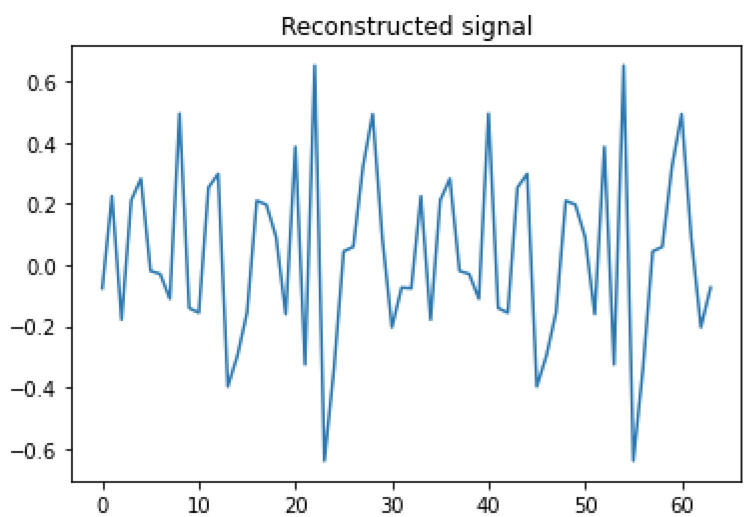



```
the mse for 10 realizations is: 0.06498643322511664
```

For 100 realizations:

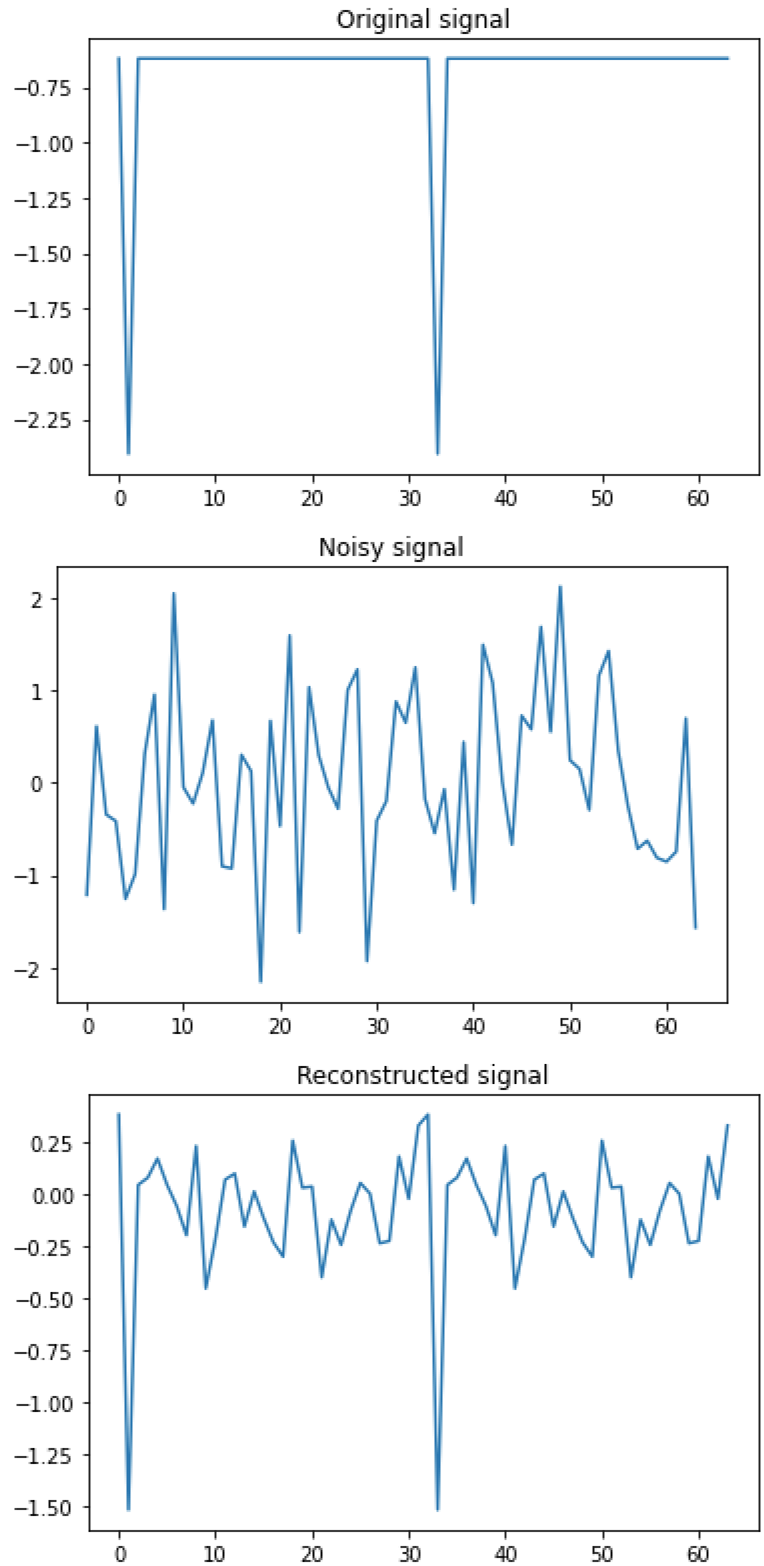






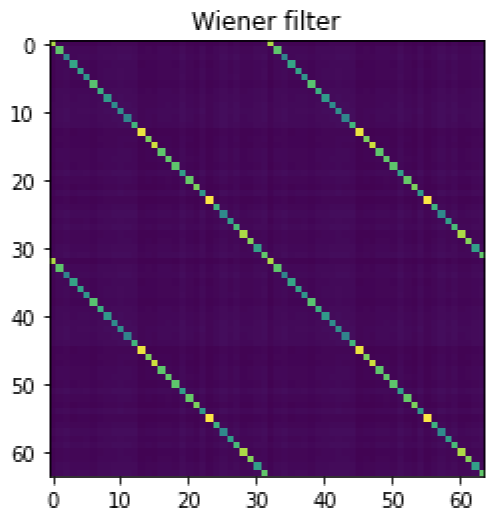

```
the mse for 100 realizations is: 0.1288326048221027
```

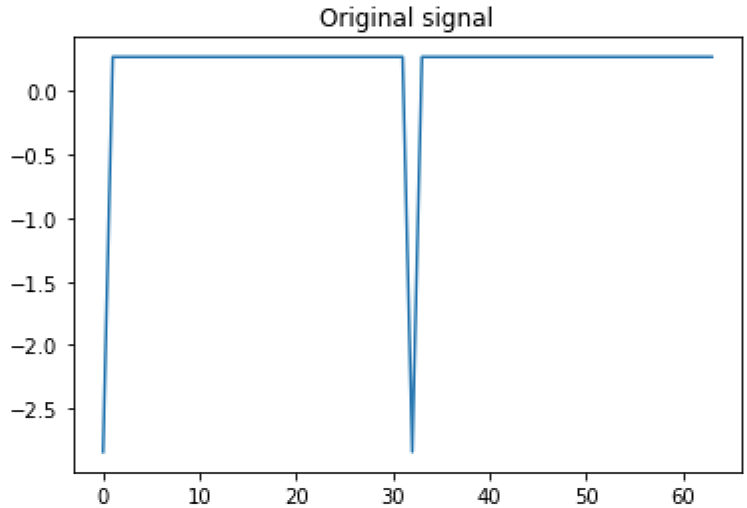
For 500 realizations:

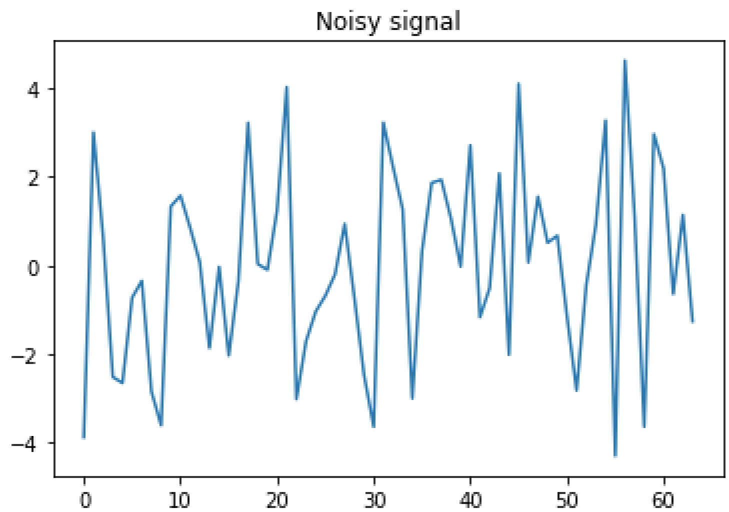


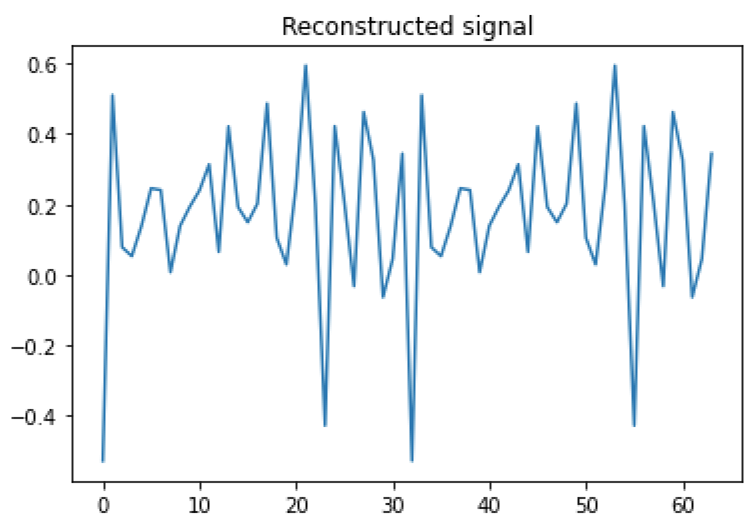

```
the mse for 500 realizations is: 0.3979305479103312
```

(d) We'll repeat the question with $\sigma_n^2 = 5$.

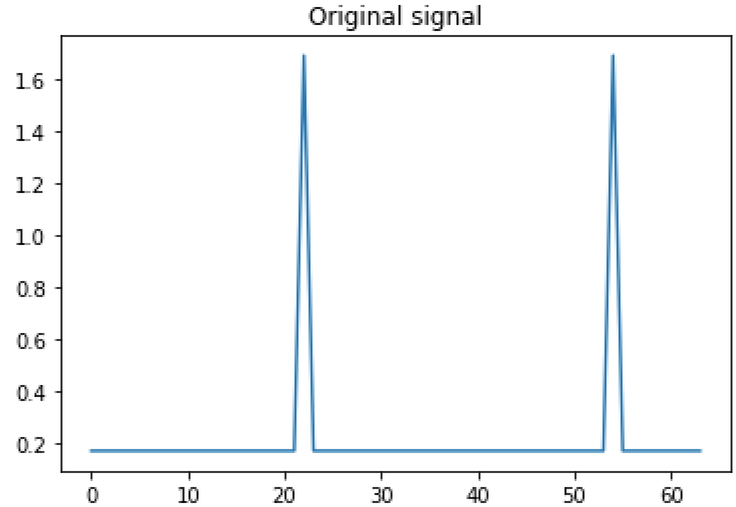

Wiener filter
0
10
20
30
40
50
60
0
10
20
30
40
50
60

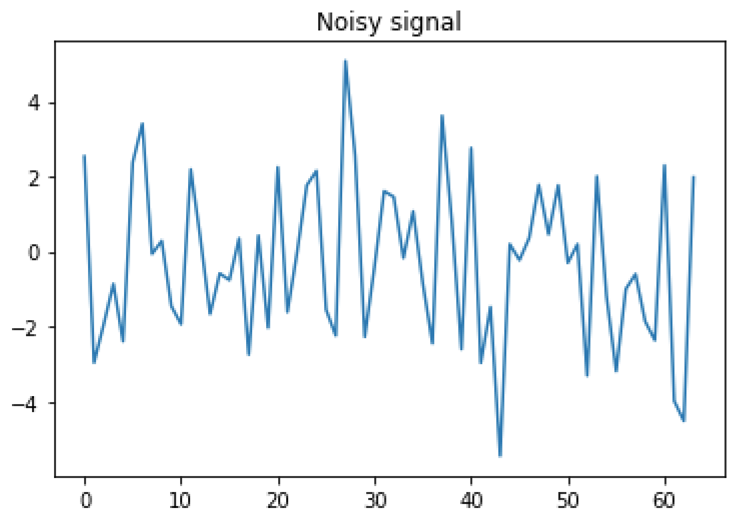


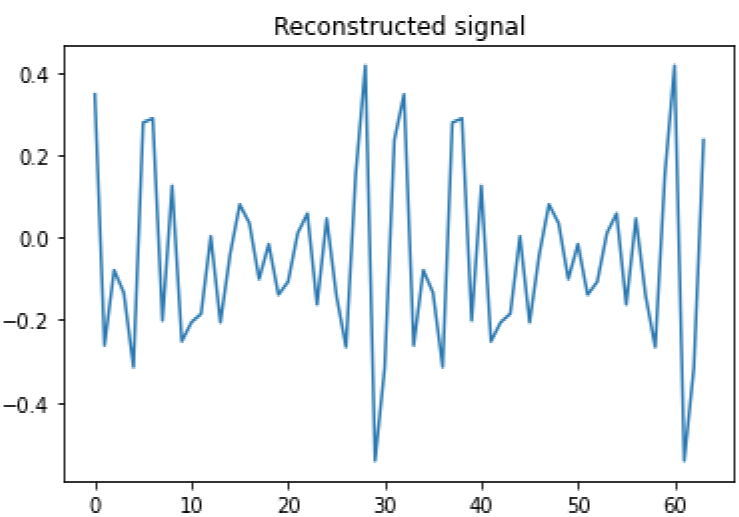




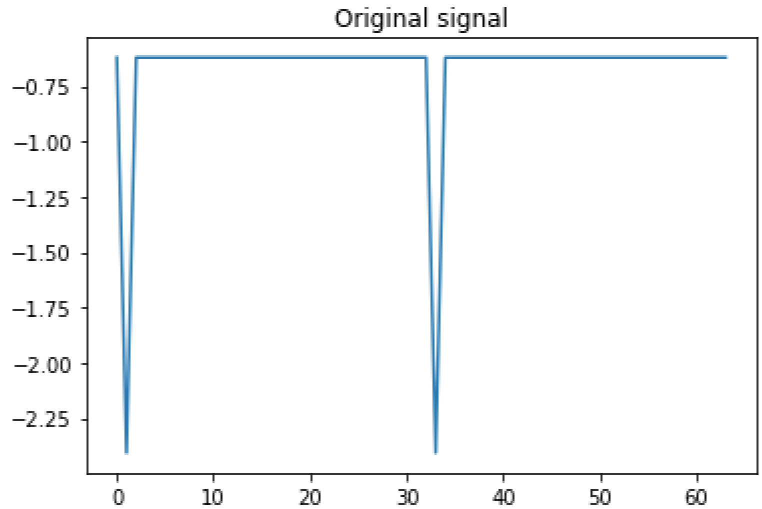

```
the mse for 10 realizations is: 0.21093497800409652
```

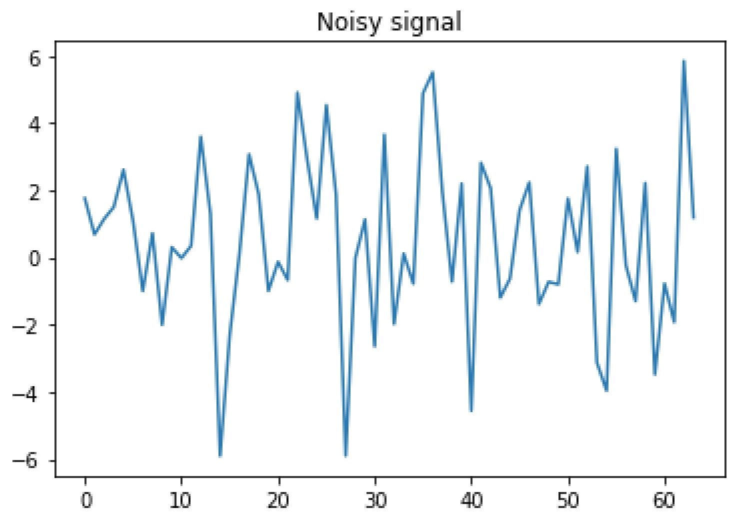


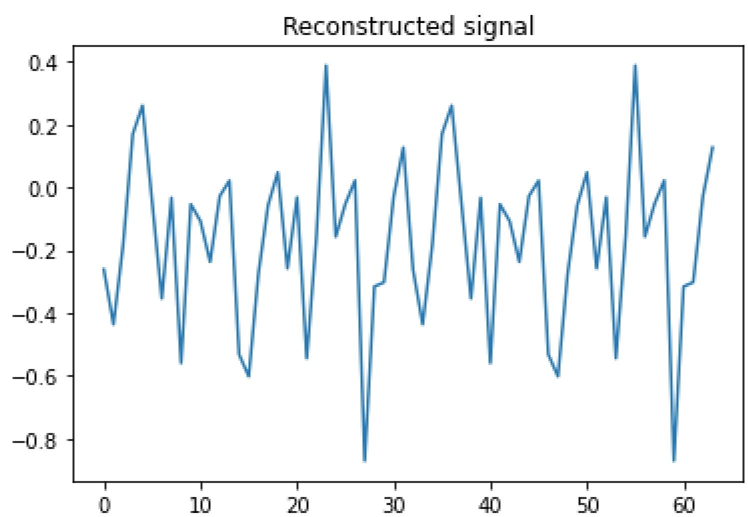




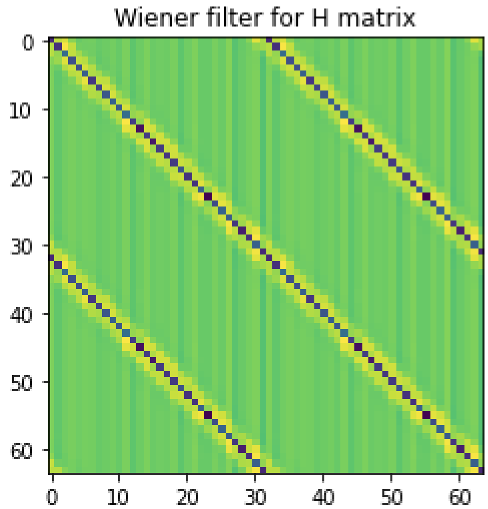

```
the mse for 100 realizations is: 0.17794777874392043
```



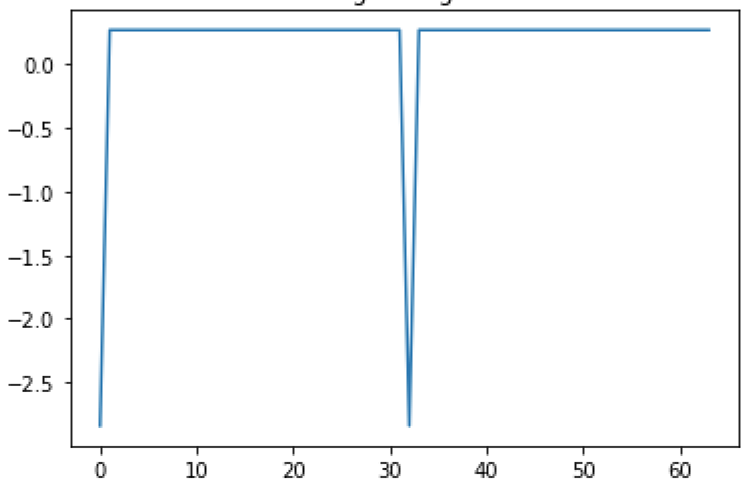




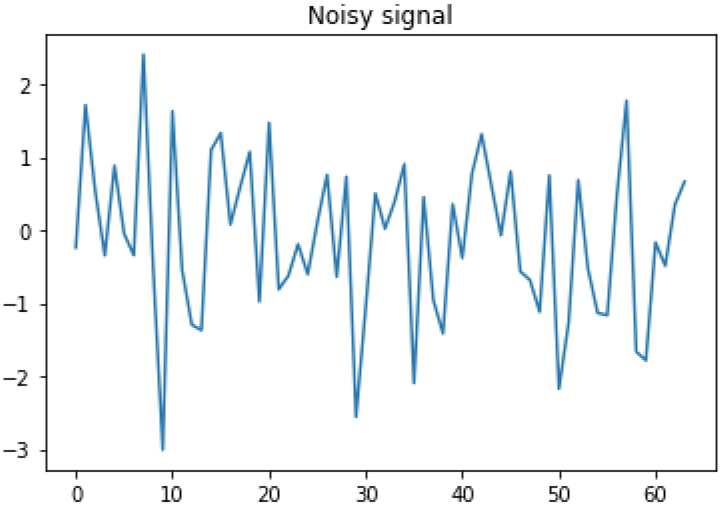

```
the mse for 500 realizations is: 0.39334178777769835
```


Wiener filter for H matrix
0
10
20
30
40
50
60
0
10
20
30
40
50
60

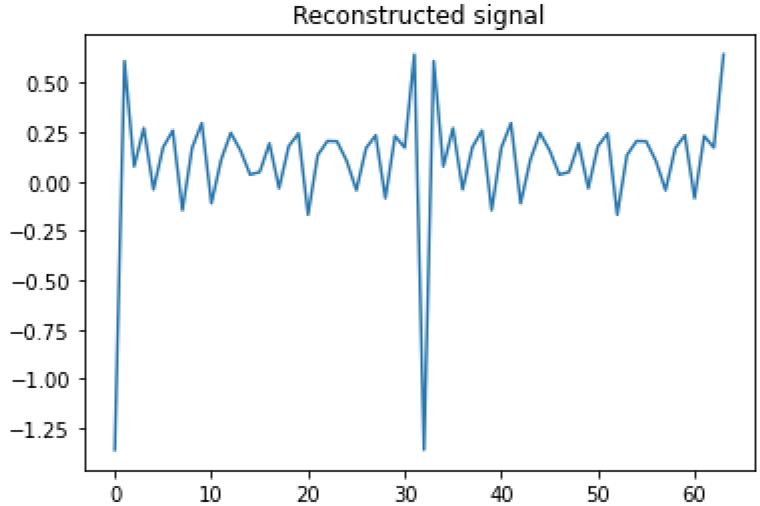
Original signal



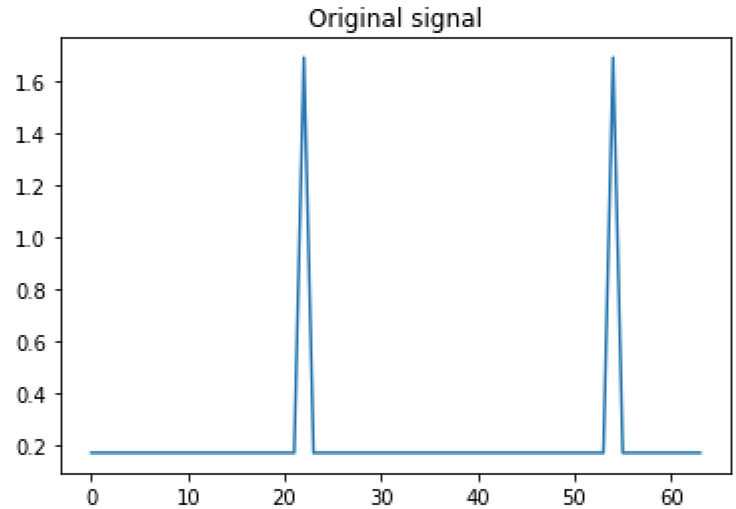
Noisy signal

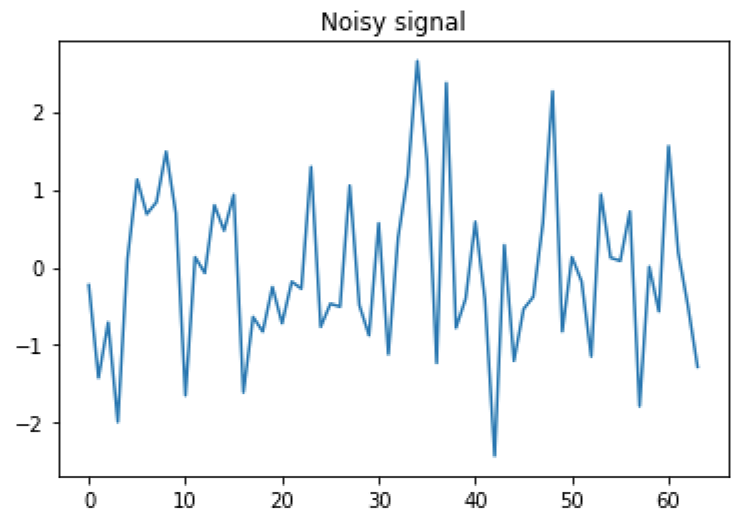


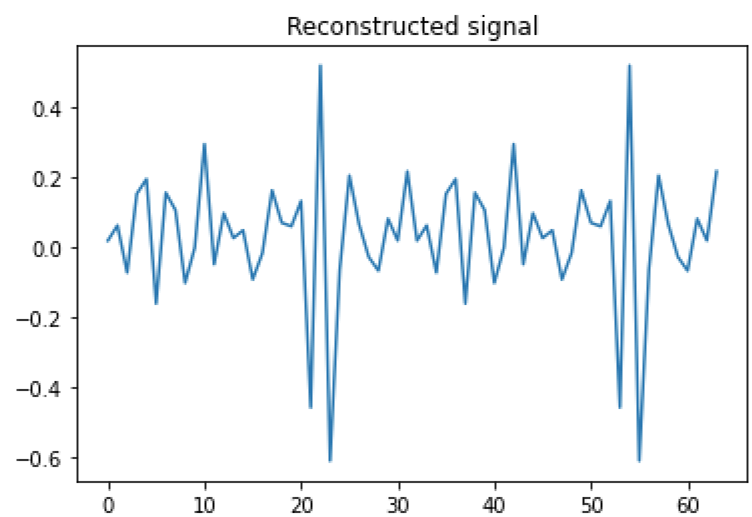
Reconstructed signal



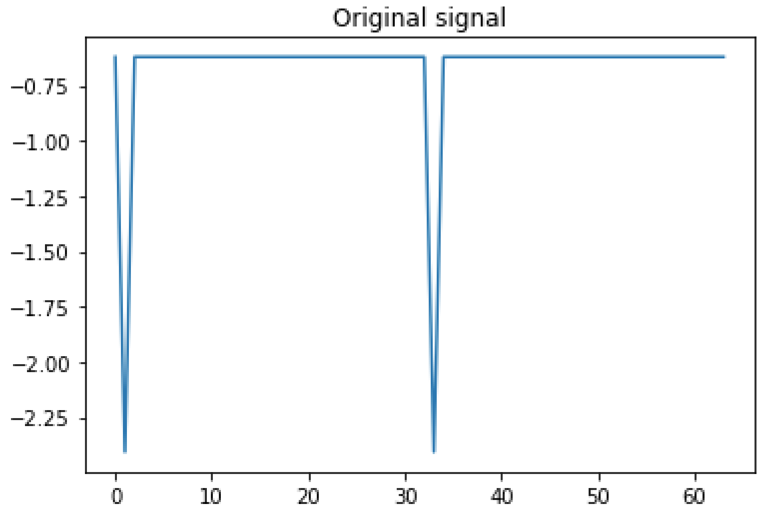
```
the mse for 10 realizations is: 0.11446837829836931
```

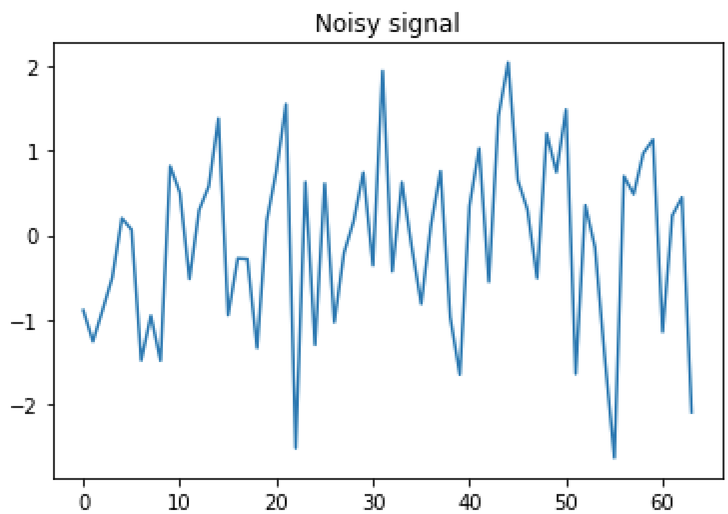


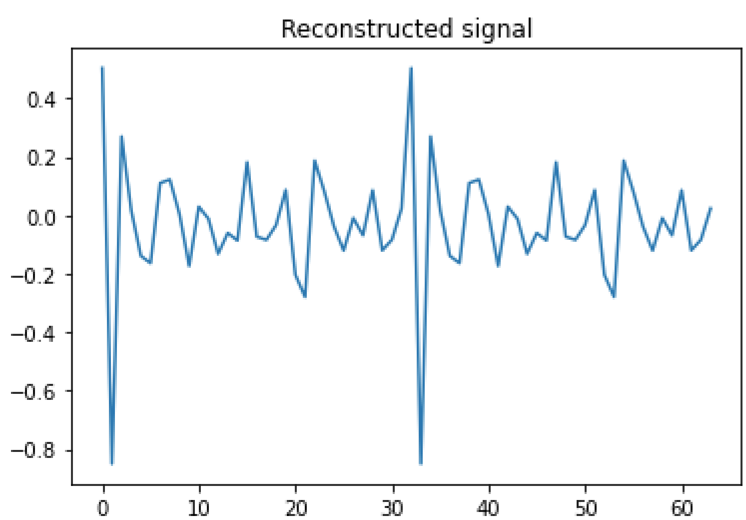





```
the mse for 100 realizations is: 0.0981663620111151
```

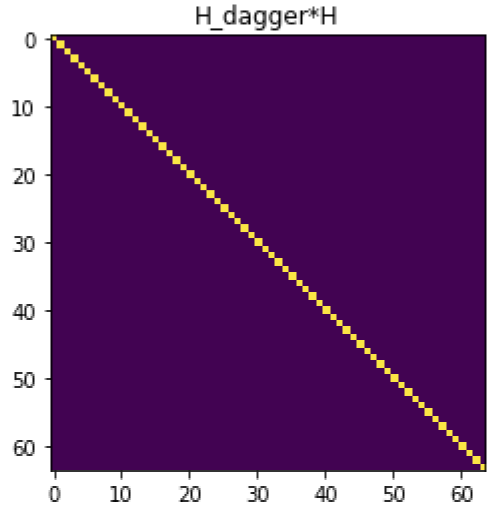


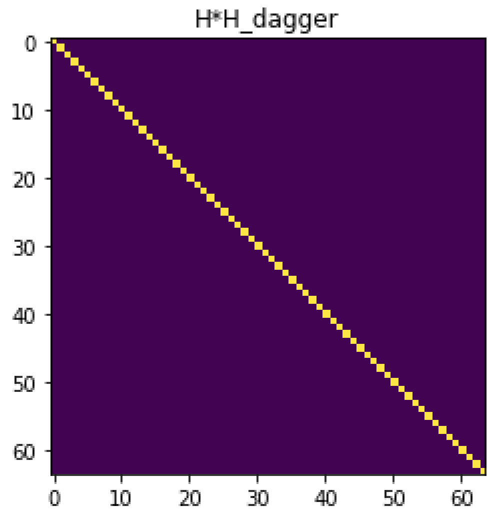





```
the mse for 500 realizations is: 0.46444457495654135
```

(e) We'll compute the pseudo inverse filter of H.
Plot of the result of $H^{\dagger}H$ and $HH^{\dagger}$:





The result is very similiar to the identity matrix except that the values on the main diagonal are not one's and the rest of the values are not zero. Other than that, both matrices are the same, which makes sense since H is a circulant matrix and so is it's pseudo inverse matrix, so the create permutations of the same sequences of multiplications, hence, they commute.

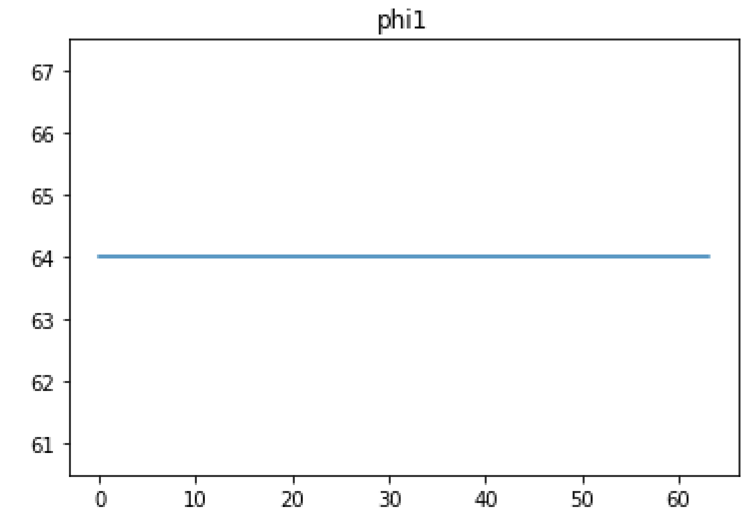
I'll choose 2 vectors, one of ones and another of zeros. To achieve the norm difference, I'll multiply the ones vector by 64, so we get that $\|\phi_2 - \phi_1\|_2 = 512$ which holds the condition. I chose vectors of this kind because they are contained in the nullity space of H, since 0 is the first eigenvalue and the eigenvector corresponds to it is a vector of ones, since both H and $H^\dagger$ are circulant it applies to them in the same way. That way, multipling H or $H^\dagger$ by ϕ_1 or ϕ_2 will be just zero. Hence, we satisfy the second condition that $H^\dagger\phi_1 = H^\dagger\phi_2 = 0$.

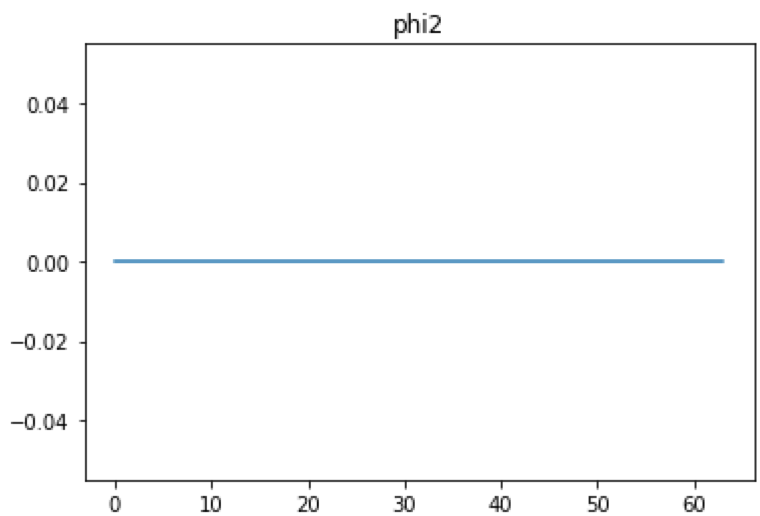
$$\|\phi_2 - \phi_1\|_2 = 512$$
$$\left\|H^\dagger\phi_2 - H^\dagger\phi_1\right\|_2 = 0$$

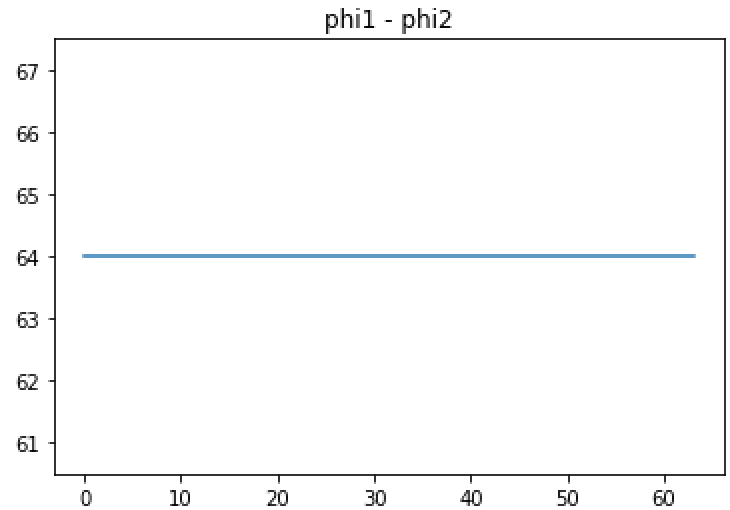
And in code we get:

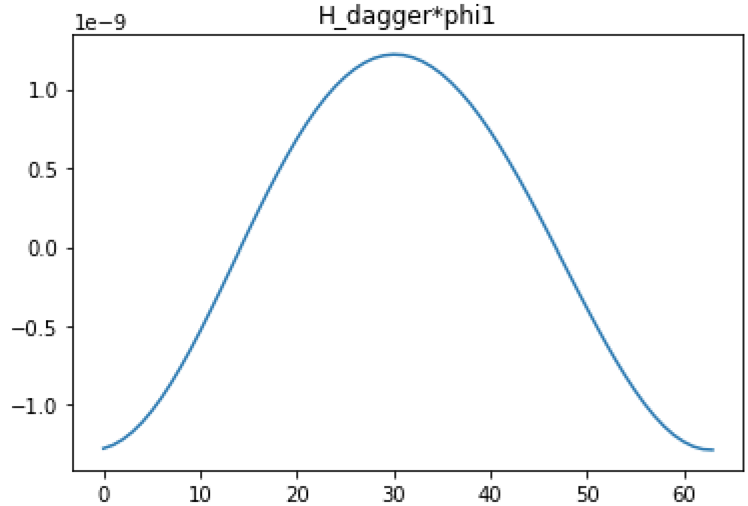
```
512.0
7.0966901994179025e-09
```

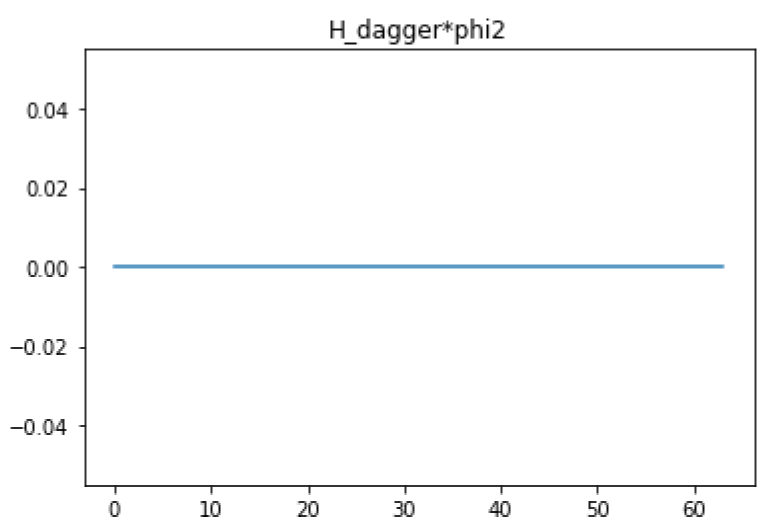
Plots:

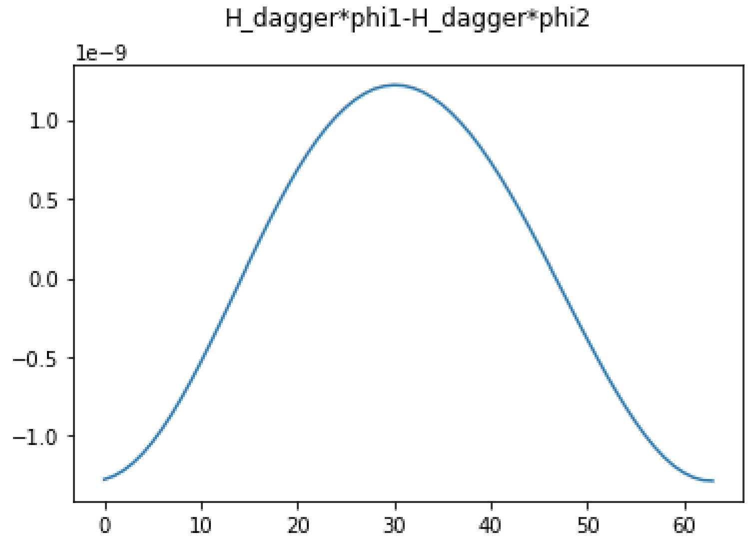





phi1 - phi2
67
66
65
64
63
62
61
0
10
20
30
40
50
60


1e−9
H_dagger*phi1
1.0
0.5
0.0
−0.5
−1.0
0
10
20
30
40
50
60


H_dagger*phi2
0.04
0.02
0.00
−0.02
−0.04
0
10
20
30
40
50
60



Since it's computed by code, there are ofcourse numerical errors, this is why we don't see exactly straight line of zero in some of the plots, but really small numbers that are really close to zero.