University of Jyväskylä - Course TIEJ6003
intro2QC Summer2024: ex1 solved

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The exercises marked with * are considered important-to-solve; the remaining exercises are more advanced and meant to challenge you.

Exercise 1.1*: expectation

Suppose we prepare a quantum system in an **eigenstate** $|\psi\rangle$ of some observable M with corresponding eigenvalue m. What is the average observed value of M (that is, when repeatedly observing the same prepared state $|\psi\rangle$ by the operator M), and the standard deviation of this statistical process?

Solution

By the definition of expectation, we have that:

$$\langle M\rangle_{|\psi\rangle} = \langle\psi|M|\psi\rangle = \langle\psi|m|\psi\rangle = m\langle\psi|\psi\rangle = m$$

Where in the second equality we use that $|\psi\rangle$ is an eigenstate of M with eigenvalue m, and in the last equality we use that $|\psi\rangle$ is a normalized quantum state. Next, calculating $\langle M^2\rangle_{|\psi\rangle}$, we have:

$$\left\langle M^2\right\rangle_{|\psi\rangle} = \langle\psi|M^2|\psi\rangle = \langle\psi|MM|\psi\rangle = \langle\psi|Mm|\psi\rangle = m^2\langle\psi|\psi\rangle = m^2$$

Note that $M^\dagger = M$ and $m^* = m$ due to hermiticity.
Now calculating the standard deviation, we have:

$$\Delta(M) = \sqrt{\langle M^2\rangle - \langle M\rangle^2} = \sqrt{m^2 - (m)^2} = 0$$

Exercise 1.2*: the measurement postulate and cascades

Quantum measurements can be described by a collection $\{M_m\}$ of *measurement operators* (m refers to the measurement outcomes).
Given a quantum state $|\psi\rangle$, then the probability that result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle, \tag{1}$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}. \tag{2}$$

The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I. \tag{3}$$

Suppose $\{L_\ell\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by the measurement operators $\{L_\ell\}$ followed by a measurement defined by the measurement operators $\{M_m\}$ is physically equivalent to a single measurement defined by measurement operators $\{N_{lm}\}$ with the representation $N_{lm} \equiv M_m L_\ell$.

Solution

Suppose we have (normalized) initial quantum state $|\psi_0\rangle$. Then, the state after measurement of L_ℓ is given by definition to be:

$$|\psi_0\rangle \rightsquigarrow |\psi_1\rangle = \frac{L_\ell|\psi_0\rangle}{\sqrt{\langle\psi_0|L_\ell^\dagger L_\ell|\psi_0\rangle}}.$$

The state after measurement of M_m on $|\psi_1\rangle$ is then given to be:

$$\begin{aligned}
|\psi_1\rangle \rightsquigarrow |\psi_2\rangle &= \frac{M_m|\psi_1\rangle}{\sqrt{\langle\psi_1|M_m^\dagger M_m|\psi_1\rangle}} \\
&= \frac{M_m\left(\frac{L_\ell|\psi_0\rangle}{\sqrt{\langle\psi_0|L_\ell^\dagger L_\ell|\psi_0\rangle}}\right)}{\sqrt{\left(\frac{L_\ell^\dagger\langle\psi_0|}{\sqrt{\langle\psi_0|L_\ell^\dagger L_\ell|\psi_0\rangle}}\right) M_m^\dagger M_m \left(\frac{L_\ell|\psi_0\rangle}{\sqrt{\langle\psi_0|L_\ell^\dagger L_\ell|\psi_0\rangle}}\right)}} \\
&= \frac{M_m L_\ell|\psi_0\rangle}{\sqrt{\langle\psi_0|L_\ell^\dagger L_\ell|\psi_0\rangle}\sqrt{\langle\psi_0|L_\ell^\dagger M_m^\dagger M_m L_\ell|\psi_0\rangle}} \\
&= \frac{M_m L_\ell|\psi_0\rangle}{\sqrt{\langle\psi_0|L_\ell^\dagger L_\ell|\psi_0\rangle\langle\psi_0|L_\ell^\dagger M_m^\dagger M_m L_\ell|\psi_0\rangle}}.
\end{aligned}$$

Altogether, measuring $|\psi_0\rangle$ by $N_{lm} = M_m L_\ell$ yields:

$$|\psi_0\rangle \rightsquigarrow |\psi_3\rangle = \frac{M_m L_\ell|\psi_0\rangle}{\sqrt{\langle\psi_0|L_\ell^\dagger M_m^\dagger M_m L_\ell|\psi_0\rangle}}.$$

Clearly, $|\psi_2\rangle = |\psi_3\rangle$ (that is, the cascaded measurement produces the same result as the single measurement), proving the claim.

Exercise 1.3: operators

Consider a ket space spanned by the eigenkets $\{|a'\rangle\}$ of a Hermitian operator A. There is no degeneracy.

(a) Prove that

$$\prod_{a'}(A - a')$$

is a null operator – that is, applying it on any ket vector results in the zero vector.

(b) What is the significance of

$$\prod_{a'' \neq a'} \frac{(A - a'')}{a' - a''}?$$

(Note the double product notation, which is equivalent to $\prod_{a'} \prod_{a'' \neq a'}$

(c) Illustrate (a) and (b) by setting $A := S_z$ of a spin-$\frac{1}{2}$ system.

Solution

(a)

$$\begin{aligned}
\prod_{a'}(A - a')|\alpha\rangle &= \prod_{a'}(A - a') \sum_{a''} |a''\rangle \underbrace{\langle a''|\alpha\rangle}_{c_{a''}} \\
&= \sum_{a''} c_{a''} \prod_{a'}(A - a') \, |a''\rangle \\
&= \sum_{a''} c_{a''} \prod_{a'}(a'' - a') \, |a''\rangle \\
&= 0 \text{ //since } a^{''} \in \{ \, a^{'} \} \; .
\end{aligned}$$

(b)

$$\begin{aligned}
\left[\prod_{a'' \neq a'} \frac{(A - a'')}{a' - a''}\right] |\alpha\rangle &= \left[\prod_{a'' \neq a'} \frac{(A - a'')}{a' - a''}\right] \overbrace{\sum_{a'''} |a'''\rangle \langle a'''|}^{\mathbf{1}} |\alpha\rangle \\
&= \sum_{a'''} \langle a'''| \, |\alpha\rangle \prod_{a'' \neq a'} \frac{(a''' - a'')}{a' - a''} \, |a'''\rangle \\
&= \sum_{a'''} \langle a'''|\alpha\rangle \delta_{a'''a'} \, |a'''\rangle \\
&= \langle a'|\alpha\rangle \, |a'\rangle
\end{aligned}$$

So overall it projects to the eigenket $|a'\rangle$:

$$\left[\prod_{a'' \neq a'} \frac{(A - a'')}{a' - a''}\right] = |a'\rangle \langle a'| \equiv \Lambda_{a'}.$$

Solution

(c) The operator can be presented as $S_z \equiv \hbar/2(|+\rangle\langle+| - |-\rangle\langle-|)$.
It has eigenkets $|+\rangle$ and $|-\rangle$ with eigenvalues $\hbar/2$ and $-\hbar/2$ respectively. So

$$\begin{aligned}
\prod_{a'}(S_z - a'1) &= \prod_{a'}(S_z - a'1) \\
&= \left[\frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) - \frac{\hbar}{2}(|+\rangle\langle+| + |-\rangle\langle-|)\right] \\
&\quad \times \left[\frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) + \frac{\hbar}{2}(|+\rangle\langle+| + |-\rangle\langle-|)\right] \\
&= [-\hbar]|-\rangle\langle-|[\hbar]|+\rangle\langle+| = -\hbar^2|-\rangle\langle-|\langle+| = 0,
\end{aligned}$$

where we have used that $|+\rangle\langle+| + |-\rangle\langle-| = 1$.

We calculate for $a' = \hbar/2$ (3.1) and then for $a' = -\hbar/2$ (3.2):

$$\begin{aligned}
\prod_{a''\neq a'} \frac{(S_z - a''1)}{a' - a''} &= \prod_{a''\neq \hbar/2} \frac{(S_z - a''1)}{\hbar/2 - a''} = \frac{S_z + \hbar/2}{\hbar/2 + \hbar/2} \\
&= \frac{1}{\hbar}\left[\frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) + \frac{\hbar}{2}(|+\rangle\langle+| + |-\rangle\langle-|)\right] \\
&= \frac{1}{\hbar}\hbar|+\rangle\langle+| = |+\rangle\langle+|. \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\prod_{a''\neq a'} \frac{(S_z - a''1)}{a' - a''} &= \prod_{a''\neq -\hbar/2} \frac{(S_z - a''1)}{-\hbar/2 - a''} = \frac{S_z - \hbar/2}{-\hbar/2 - (-\hbar/2)} \\
&= \frac{1}{-\hbar}\left[\frac{\hbar}{2}(|+\rangle\langle+| - |-\rangle\langle-|) - \frac{\hbar}{2}(|+\rangle\langle+| + |-\rangle\langle-|)\right] \\
&= \frac{1}{-\hbar}[-\hbar]|-\rangle\langle-| = |-\rangle\langle-|. \qquad (3.2)
\end{aligned}$$