University of Jyväskylä - Course TIEJ6003
intro2QC Summer2024: ex4 solved

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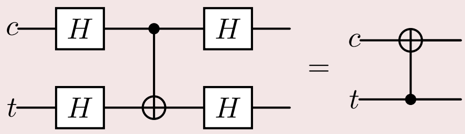
Exercise 4.1: circuits equivalence

Show this equivalence:



Solution

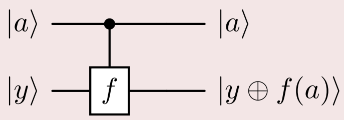
The 4×4 unitary matrix for the double-Hadamard is given by:

$$H_1 \otimes H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

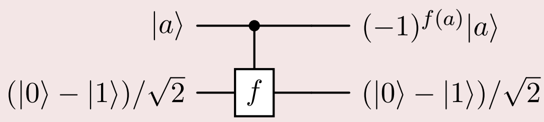
$$\begin{aligned}
(H_1 \otimes H_2)(CX_{1,2})(H_1 \otimes H_2) &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix} \\
&= CX_{2,1}
\end{aligned}$$

Exercise 4.2: controlled-f gate

Verify that given the following controlled-f gate,

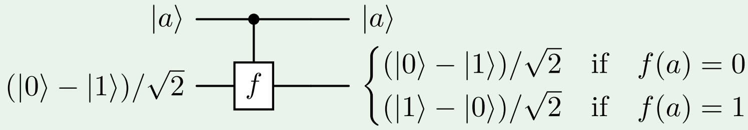


the following “trick” is valid:



Solution

By our definition of the controlled-f gate,



We are free to associate the phase factor $(-1)^{f(a)}$, obtained from the controlled-f, with the $|a\rangle$ register rather than with the “lower” qubit in state $(|0\rangle - |1\rangle)/\sqrt{2}$.

Exercise 4.3: identity

Let $H^{\otimes n}$ denote Hadamard gates applied individually to n qubits.
Let $P := 2\,|0^{\otimes n}\rangle\,\langle 0^{\otimes n}| - I^{\otimes n}$, where $|0^{\otimes n}\rangle\langle 0^{\otimes n}|$ is the projector onto the n-qubit zero state.

Prove that

$$H^{\otimes n} P H^{\otimes n} = 2\,|\psi_u\rangle\,\langle\psi_u| - I^{\otimes n}$$

where $|\psi_u\rangle$ is the uniform superposition over the computational basis states,

$$|\psi_u\rangle := \frac{1}{2^{n/2}} \sum_{j}^{2^n-1} |j\rangle$$

Solution

Recall that

$$H\,|0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, \quad H\,|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}, \quad HH^\dagger = I.$$

Since $H^{\otimes n}$ denotes Hadamard gates applied individually to each of the n qubits, it immediately follows that

$$H^{\otimes n} P H^{\otimes n} \equiv 2H^{\otimes n}\,|0^{\otimes n}\rangle\,\langle 0^{\otimes n}|\,(H^{\otimes n})^\dagger - H^{\otimes n} I^{\otimes n} (H^{\otimes n})^\dagger = 2\,|\psi_u\rangle\,\langle\psi_u| - I^{\otimes n},$$

as claimed.