University of Jyväskylä - Course TIEJ6003
intro2QC Summer2024: ex2 solved

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Exercise 2.1: averaged measurements

Suppose we have qubit in the state $|0\rangle$, and we measure the observable X (Pauli's σ_x!).
What is the average value of X?
What is the standard deviation of X?

Solution

By the definition of expectation, we have

$$\langle X \rangle = \langle 0|X|0\rangle = \langle 0|1\rangle = 0$$

Next, calculating $\langle X^2 \rangle_{|0\rangle}$

$$\langle X^2 \rangle = \langle 0|XX|0\rangle = \langle 1|1\rangle = 1$$

Yielding altogether the standard deviation:

$$\Delta(X) = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{1-0} = 1$$

Exercise 2.2: Hadamard identities

Prove the following identities for the Hadamard gate H:

$$HZH = X; \quad HTH = R_x(\pi/4).$$

Solution

$$\begin{aligned}
HZH &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X
\end{aligned}$$

$$\begin{aligned}
HTH &= e^{-i\pi/8} H R_z(\frac{\pi}{4}) H \\
&= e^{-i\pi/8} H \left(\cos\left(\frac{\pi}{8}\right) I - i \sin\left(\frac{\pi}{8}\right) Z\right) H \\
&= e^{-i\pi/8} \left(\cos\left(\frac{\pi}{8}\right) I - i \sin\left(\frac{\pi}{8}\right) X\right) \\
&= e^{-i\pi/8} R_x(-\pi/4)
\end{aligned}$$

In the latter we used the fact that $T = R_z(-\pi/4)$ up to a global phase $e^{-i\pi/8}$.

Exercise 2.3: Hadamard via rotations

Express the Hadamard gate H as a product of R_x and R_z rotations,

$$H = R_z(\pi/2)R_x(\pi/2)R_z(\pi/2)$$

up to a global phase of $e^{-i\pi/2}$.

Solution

$$\begin{aligned}
R_z(\pi/2)R_x(\pi/2)R_z(\pi/2) &= \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \\
&= \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} e^{-i\pi/4} & -ie^{-i\pi/4} \\ -ie^{i\pi/4} & e^{i\pi/4} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\pi/2} & -i \\ -i & e^{i\pi/2} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\pi/2} & -e^{-i\pi/2} \\ -e^{i\pi/2} & e^{i\pi/2} \end{bmatrix} \\
&= \frac{e^{-i\pi/2}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= e^{-i\pi/2} H
\end{aligned}$$

Exercise 2.4: XY manipulations

Show that $XYX = -Y$ and use it to prove that $XR_y(\theta)X = R_y(-\theta)$.

Solution

For the first claim, we use that $XY = -YX$ as well as $X^2 = I$ – to obtain the following:

$$XYX = -YXX = -Y.$$

Next, by using this, we can show the main claim:

$$\begin{aligned}
XR_y(\theta)X &= X\left(\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)Y\right)X \\
&= \cos\left(\frac{\theta}{2}\right)XIX - i\sin\left(\frac{\theta}{2}\right)XYX \\
&= \cos\left(\frac{\theta}{2}\right)I + i\sin\left(\frac{\theta}{2}\right)Y \\
&= \cos\left(-\frac{\theta}{2}\right)I - i\sin\left(-\frac{\theta}{2}\right)Y \\
&= R_y(-\theta)
\end{aligned}$$

Exercise 2.5: $X_1 Z_2$

Show that the average value of the observable $X_1 Z_2$ for a 2-qubit system measured in the state $(|00\rangle + |11\rangle)/\sqrt{2}$ is zero.

Solution

Computing the expectation value of $X_1 Z_2$, we get:

$$
\begin{aligned}
\langle X_1 Z_2 \rangle &= \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) X_1 Z_2 \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) \\
&= \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) \left(\frac{X_1 Z_2 |00\rangle + X_1 Z_2 |11\rangle}{\sqrt{2}} \right) \\
&= \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) \left(\frac{|10\rangle - |01\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{2} \left(\langle 00|10\rangle - \langle 00|01\rangle + \langle 11|10\rangle - \langle 11|01\rangle \right) \\
&= \frac{1}{2} (0 + 0 + 0 + 0) \\
&= 0.
\end{aligned}
$$