University of Jyväskylä - Course TIEJ6003
intro2QC Summer2024: ex3 solved

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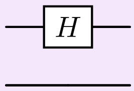
Exercise 3.1: scaling-up

What is the 4×4 unitary matrix for this circuit?



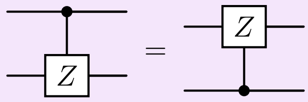
Solution

The unitary matrix for the circuit is given by:

$$H_1 \otimes I_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Exercise 3.2: equivalence

Show that



Solution

It suffices to verify that the two gates have the same effect on the 2-qubit computational basis states (as it will then follow by linearity that they will have the same effect on any such superposition of the basis states). Checking the 8 necessary cases, we then have that:

$$
\begin{aligned}
&CZ_{1,2}(|0\rangle_1 \otimes |0\rangle_2) = |0\rangle_1 \otimes |0\rangle_2 \\
&CZ_{2,1}(|0\rangle_1 \otimes |0\rangle_2) = |0\rangle_1 \otimes |0\rangle_2 \\
&CZ_{1,2}(|1\rangle_1 \otimes |0\rangle_2) = |1\rangle_1 \otimes Z(|0\rangle_2) = |1\rangle_1 \otimes |0\rangle_2 \\
&CZ_{2,1}(|1\rangle_1 \otimes |0\rangle_2) = |1\rangle_1 \otimes |0\rangle_2 \\
&CZ_{1,2}(|0\rangle_1 \otimes |1\rangle_2) = |0\rangle_1 \otimes |1\rangle_2 \\
&CZ_{2,1}(|0\rangle_1 \otimes |1\rangle_2) = Z(|0\rangle_1) \otimes |1\rangle_2 = |0\rangle_1 \otimes |1\rangle_2 \\
&CZ_{1,2}(|1\rangle_1 \otimes |1\rangle_2) = |1\rangle_1 \otimes Z(|1\rangle_2) = |1\rangle_1 \otimes -|1\rangle_2 = -(|1\rangle_1 \otimes |1\rangle_2) \\
&CZ_{2,1}(|1\rangle_1 \otimes |1\rangle_2) = Z(|1\rangle_1) \otimes |1\rangle_2 = -|1\rangle_1 \otimes |1\rangle_2 = -(|1\rangle_1 \otimes |1\rangle_2)
\end{aligned}
$$

from which we observe equality for each. The claim follows.
Remark: More compactly, we have $CZ_{1,2}\,|b_1b_2\rangle = |b_1\rangle \otimes Z^{b_1}\,|b_2\rangle = (-1)^{b_1b_2}\,|b_1b_2\rangle$ for computational basis states $b_1, b_2 \in \{0,1\}$. Using this form we can write

$$
\begin{aligned}
CZ_{1,2}\,|b_1b_2\rangle &= (-1)^{b_1b_2}\,|b_1b_2\rangle \\
&= (-1)^{b_2b_1}\,|b_1b_2\rangle \\
&= Z^{b_2}\,|b_1\rangle \otimes |b_2\rangle \\
&=: CZ_{2,1}\,|b_1b_2\rangle\,.
\end{aligned}
$$

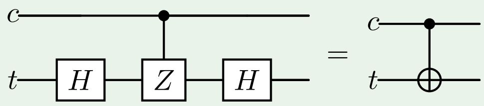
Exercise 3.3: CNOT from controlled-Z gates

Construct a CNOT gate from one controlled-Z gate, that is, the gate whose action in the computational basis is specified by the unitary matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Solution

We previously showed that $HZH = X$. Hence, to obtain a CNOT gate from a single controlled Z gate, we can conjugate the target qubit with Hadamard gates:



We can verify this via matrix multiplication, relying on previous results:

$$\begin{aligned}
(I_1 \otimes H_2)(CZ_{1,2})(I_1 \otimes H_2) &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} = CX_{1,2}
\end{aligned}$$

Exercise 3.4: Bell states are orthonormal basis

Verify that the Bell states form an *orthonormal basis* for the *2-qubit state space*.

Solution

$$\langle B_{00}|B_{00}\rangle = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle + \langle 11|11\rangle\right) = 1$$
$$\langle B_{00}|B_{01}\rangle = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| + \langle 10|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 00|01\rangle + \langle 00|10\rangle + \langle 11|01\rangle + \langle 11|10\rangle\right) = 0$$
$$\langle B_{00}|B_{10}\rangle = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 00|01\rangle - \langle 00|10\rangle + \langle 11|01\rangle - \langle 11|10\rangle\right) = 0$$
$$\langle B_{00}|B_{11}\rangle = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| - \langle 11|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 00|00\rangle - \langle 00|11\rangle + \langle 11|00\rangle - \langle 11|11\rangle\right) = 1$$
$$\langle B_{01}|B_{01}\rangle = \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| + \langle 10|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 01|01\rangle + \langle 01|10\rangle + \langle 10|01\rangle + \langle 10|10\rangle\right) = 1$$
$$\langle B_{01}|B_{10}\rangle = \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 01|01\rangle - \langle 01|10\rangle + \langle 10|01\rangle - \langle 10|10\rangle\right) = 0$$
$$\langle B_{01}|B_{11}\rangle = \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| - \langle 11|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 01|01\rangle - \langle 01|10\rangle - \langle 10|01\rangle + \langle 10|10\rangle\right) = 0$$
$$\langle B_{10}|B_{10}\rangle = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 01|01\rangle - \langle 01|10\rangle - \langle 10|01\rangle + \langle 10|10\rangle\right) = 1$$
$$\langle B_{10}|B_{11}\rangle = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| - \langle 11|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 01|01\rangle - \langle 01|10\rangle - \langle 10|01\rangle + \langle 10|10\rangle\right) = 0$$
$$\langle B_{11}|B_{11}\rangle = \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| - \langle 11|}{\sqrt{2}}\right) = \frac{1}{2}\left(\langle 00|00\rangle - \langle 00|11\rangle - \langle 11|00\rangle + \langle 11|11\rangle\right) = 1$$

Orthonormality is thus verified, next we show that it is a basis. We may write any vector $|\psi\rangle$ in the 2 qubit state space as:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.
We then observe that this is equivalent to:

$$\left(\frac{\alpha+\delta}{\sqrt{2}}\right)|B_{00}\rangle + \left(\frac{\alpha-\delta}{\sqrt{2}}\right)|B_{01}\rangle + \left(\frac{\beta+\gamma}{\sqrt{2}}\right)|B_{10}\rangle + \left(\frac{\beta-\gamma}{\sqrt{2}}\right)|B_{11}\rangle \quad (*)$$

as:

$$\begin{aligned}
\left(\frac{\alpha+\delta}{\sqrt{2}}\right)\frac{|00\rangle + |11\rangle}{\sqrt{2}} + \left(\frac{\alpha-\delta}{\sqrt{2}}\right)\frac{|00\rangle - |11\rangle}{\sqrt{2}} + \left(\frac{\beta+\gamma}{\sqrt{2}}\right)\frac{|01\rangle + |10\rangle}{\sqrt{2}} + \left(\frac{\beta-\gamma}{\sqrt{2}}\right)\frac{|01\rangle - |10\rangle}{\sqrt{2}}& \\
= \left(\frac{\alpha}{2} + \frac{\delta}{2} + \frac{\alpha}{2} - \frac{\delta}{2}\right)|00\rangle + \left(\frac{\beta}{2} + \frac{\gamma}{2} + \frac{\beta}{2} - \frac{\gamma}{2}\right)|01\rangle& \\
+ \left(\frac{\beta}{2} + \frac{\gamma}{2} - \frac{\beta}{2} + \frac{\gamma}{2}\right)|10\rangle + \left(\frac{\alpha}{2} + \frac{\delta}{2} - \frac{\alpha}{2} + \frac{\delta}{2}\right)|11\rangle& \\
= \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle = |\psi\rangle&
\end{aligned}$$