University of Jyväskylä - Course TIEJ6003
Introduction to Quantum Computing:
Final Assignment

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Deadline: Monday 19-08-2024 23:59

Solve the following five problems, which will be equally weighted for the grading.
Partial solutions will also contribute to the grade.
We will use the Finnish Universities' grading scale [0-5], where 1 is the lowest passing grade, and 5 is the best attainable.
Please submit your solution by email. **Good luck!**

1 Arbitrary Single-Qubit Operators

An arbitrary single qubit unitary operator can be written in the form

$$\mathcal{U} = \exp(i\alpha) R_{\hat{\mathbf{n}}}(\theta)$$

for some real numbers α and θ, and a real three-dimensional unit vector $\hat{\mathbf{n}}$.

1. Prove this claim.
2. Present the Hadamard gate H in this form (that is, specify the values of α, θ, and the vector $\hat{\mathbf{n}}$).
3. Present the Phase gate S in this form.

2 Expectation of an Observable

Show, using probability theory's basic principles, that the expectation value of any observable A in Quantum Mechanics per a given state $|\psi\rangle$ reads $\langle\psi| A |\psi\rangle$.

3 Three-Dimensional Hilbert Space

Consider the following operators on a three-dimensional Hilbert space:

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1. What are the possible values one can obtain if L_z is measured?
2. Take the state in which $L_z = 1$. In this state, what are $\langle L_x \rangle$, $\langle L_z \rangle$, and ΔL_x (standard deviation)?
3. Find the normalized eigenstates and the eigenvalues of L_x in the L_z basis.
4. If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities?

4 Grover's Subspaces

In Grover's search algorithm with M solutions, S was defined as the *solution set* and its complement as $S^{\perp} := \{0, 1, \ldots, N-1\} \setminus S$. We accordingly constructed during the lecture two vectors:

$$|\Psi_S\rangle := \frac{1}{\sqrt{M}} \sum_{x \in S} |x\rangle$$
$$|\Psi_{S^{\perp}}\rangle := \frac{1}{\sqrt{N-M}} \sum_{x \in S^{\perp}} |x\rangle .$$

Show that the uniform superposition state $|h\rangle := H^{\otimes n} |0\rangle^{\otimes n}$ can be expanded as:

$$|h\rangle = \sqrt{\frac{M}{N}} |\Psi_S\rangle + \sqrt{\frac{N-M}{N}} |\Psi_{S^{\perp}}\rangle .$$

5 Fourier for Factoring

Suppose $f(x+r) = f(x)$, and $0 \le x < N$, for N an integer multiple of r.
Compute the following sum:

$$\hat{f}(\ell) := \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) \exp\left(-2\pi i \ell x / N\right) \ .$$