Scars of the Transverse Field Ising Model on Discrete Geometries (Polyhedra)

November 19, 2025

Introduction

We are studying scars of the simple Ising model on discrete geometries (polyhedra). Here, *scars* are identified as special, sparser eigenstates of the Hamiltonian which are simultaneously eigenstates of the Ising term and of the transverse-field (TF) term separately; in addition, each such state is annihilated by exactly one of the two terms. In all of the following examples, the Hamiltonian is

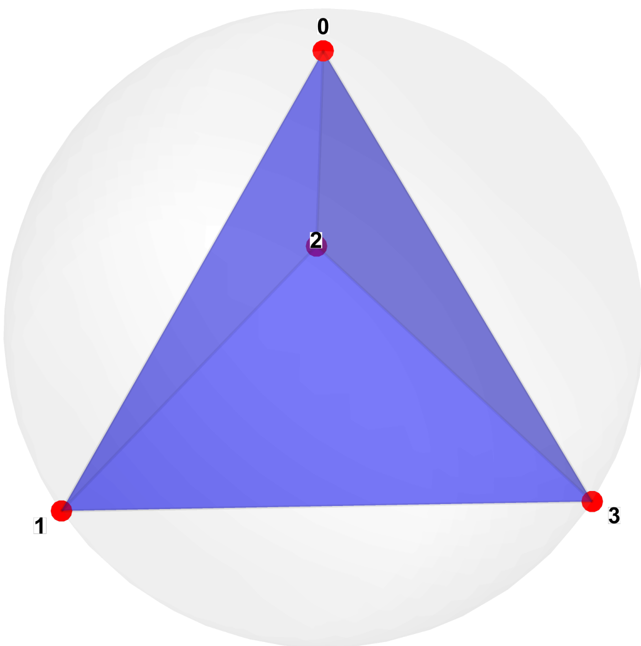
$$H = H_{\text{Ising}} + H_{\text{TF}}, \quad H_{\text{Ising}} = J \sum_{\langle i,j \rangle} \sigma_i^x \sigma_j^x, \quad H_{\text{TF}} = -h \sum_i \sigma_i^z \tag{1}$$

where $J = 1, h = 3$ (antiferromagnetic, non critical).

Platonic Solids

Tetrahedron

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4$, $F = 3$ (equilateral triangles), $E = 6$
- **Point group:** T_d
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

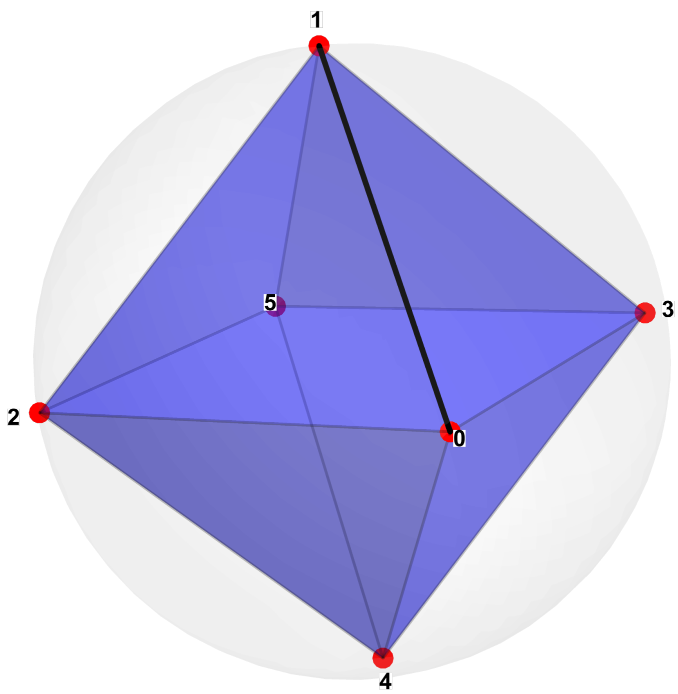
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^4 = 16$)
S_1	-2	2	H_{TF}	4,6

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 1$ sites, with $n < V/2, (V = 4)$
- **Compactness criterion:** NA, the single point 0 was chosen
- **Diagnostics:** 1-site RDMs for both scars have full rank (system size $V = 4$ is too small)

Octahedron

Overview and data



- **Duality / paired solid:** cube
- **Vertices (V), Faces (F), Edges (E):** $V = 6,\ \ F = 8$ (equilateral triangles), $\ E = 12$
- **Point group:** O_h
- **Hilbert space:** $\dim \mathcal{H} = 2^6 = 64$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-18.80, 19.67]$

Scar structure: sets and multiplets

- **Number of scar sets:** 3

  For each scar set S_k:

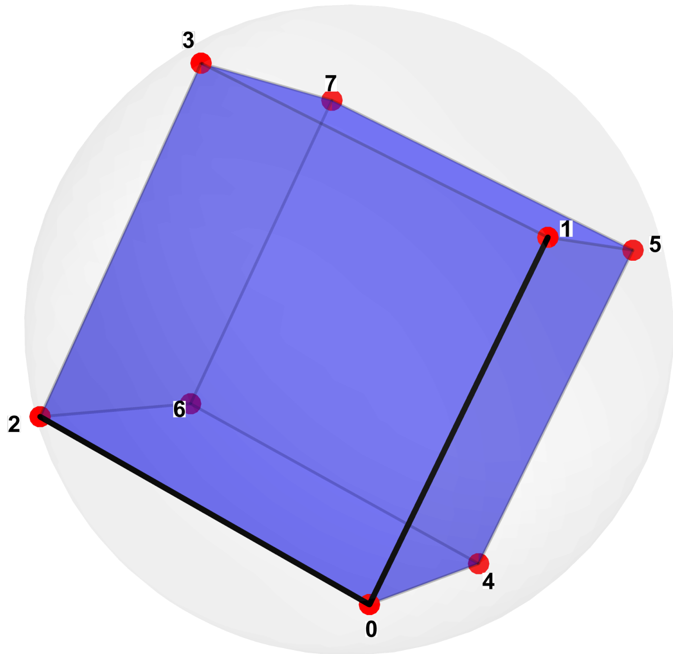
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^6 = 64$)
S_1	-6	3	H_{Ising}	12
S_2	-4	1	H_{TF}	12
S_3	6	3	H_{Ising}	12

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \text{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2$ sites, with $n < V/2, (V = 6)$
- **Compactness criterion:** nearest-neighbor, for example $[0, 1]$ (see highlighted edges in figure)
- **Diagnostics:** 2-sites RDMs for all 7 scars have full rank (system size $V = 6$ is too small)

Cube

Overview and data



- **Duality / paired solid:** octahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 8,\ \ F = 6$ (squares), $\ E = 12$
- **Point group:** O_h
- **Hilbert space:** $\dim \mathcal{H} = 2^8 = 256$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-25.11, 25.11]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^8 = 256$)
S_1	-2	3	H_{TF}	48
S_2	2	3	H_{TF}	48

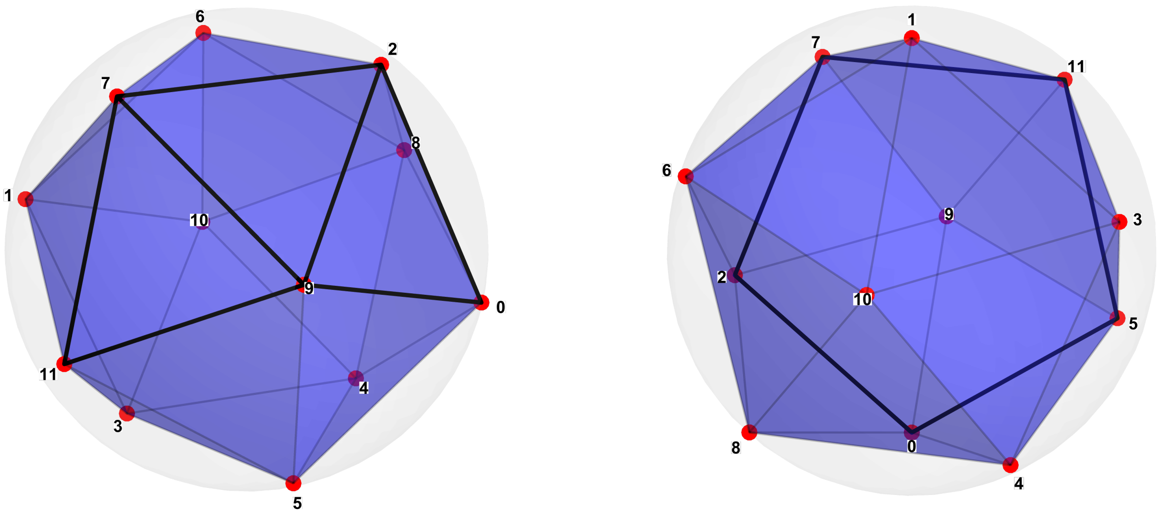
Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3$ sites, with $n < V/2, (V = 8)$

- **Compactness criterion:** nearest-neighbor + most compact, for example $[0, 1](n = 2)$, $[0, 1, 2](n = 3)$ (see highlighted edges in figure)
- **Diagnostics:** 2/3-sites RDMs for all 6 scars have full rank (system size $V = 8$ is too small)

Icosahedron

Overview and data



- **Duality / paired solid:** dodecahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 12$, $F = 20$ (equilateral triangles), $E = 30$
- **Point group:** I_h
- **Hilbert space:** $\dim \mathcal{H} = 2^{12} = 4096$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-37.95, 41.29]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^{12} = 4096$)
S_1	-6	5	H_{TF}	280

*We have also observed two additional, non-degenerate “scarred” states (let’s call them “pseudoscars”) at integer energies $E = 0, -4$, which are less sparse than the five-fold scarred multiplet, but are still sparser than the other eigenstates. These two pseudoscars have the peculiarity that they are not annihilated by either $H_{\mathrm{Ising}}, H_{\mathrm{TF}}$, and they are not even eigenstates of them (like it happens for the other scars), but instead their expectation value of H_{TF} is zero (for the pseudoscar at $E = 0$, the expectation value of H_{Ising} is also zero). If $|s\rangle$ is the scarred eigenstate of H corresponding to either $E = 0, -4$, we have that:

$$\langle s|H_{\mathrm{TF}}|s\rangle = 0 \tag{2}$$

In other words, H_{TF} projects $|s\rangle$ onto its orthogonal complement, causing the expectation value to be zero.

Multiplet label	**Energy E**	**Degeneracy**	**Annihilated by**	**Non-zero components (vs. $2^{12} = 4096$)**
S'_1	-4	1	neither	720
S'_2	0	1	neither	600

There are additional sparser eigenvectors which are not integer:

- 480 non-zero components: E= -8.64, -5.85, 3.85, 6.64
- 840 non-zero components: E= -16.87, -12.87, -8.27, -4.27, 0.27, 4.27, 8.87, 12.87

Local properties (RDMs)

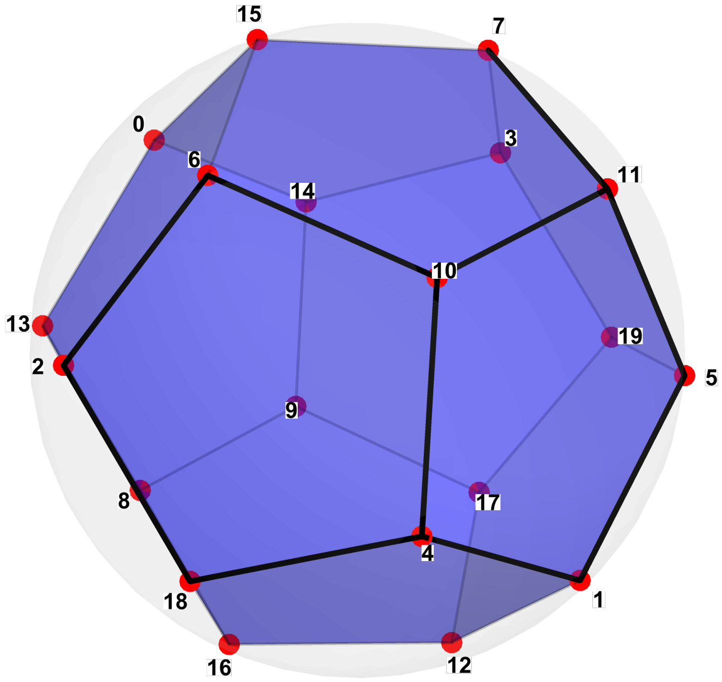
- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5$ sites, with $n < V/2, (V = 12)$.
- **Compactness criterion:** nearest-neighbor + most compact, for example $[0, 2]\,(n = 2)$, $[0, 2, 9]\,(n = 3), [0, 2, 7, 9]\,(n = 4), [0, 2, 7, 9, 11]\,(n = 5)$; for the pseudoscars, for n=5 we looked at the pentagon [0,2,5,7,11] (see highlighted edges in figure; first n=5 [0,2,7,9,11] is highlighted in the figure on the left, while n=5 [0,2,5,7,11] is highlighted in the figure on the right).
- **Diagnostics:**
  - 2/3-sites RDMs for all 5 scars + 2 pseudoscars have full rank
  - 4-sites RDMs for all 5 scars have reduced rank of 11 = 16 - 5; 2 pseudoscars have full rank
  - 5-sites ([0,2,7,9,11]) RDMs for all 5 scars have reduced rank of 18 = 32 - 14, while the 2 pseudoscars have full rank; for the pentagon 5-sites [0,2,5,7,11], the 2 pseudoscars (together with the other non-integer sparser eigenvectors found above) have both reduced rank of 32 - 8 = 24 (the 5 scars have also reduced rank of 32 - 6 = 26)

Observation on pseudoscars

The two pseudoscars at integer energies $E = 0, -4$ do not have sub-volume entanglement entropy (like for the scars studied in literature so far), but still show reduced rank for appropriately chosen local RDMs, such as the 5-sites pentagon one. This suggests that the two pseudoscars - if they are of any interest - are better characterized in terms of the FRH (Full Rank Hypothesis) rather than the ETH (Eigenstate Thermalization Hypothesis).

Dodecahedron

Overview and data



- **Duality / paired solid:** icosahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 20,\;\; F = 12$ (pentagons), $\;E = 30$
- **Point group:** I_h
- **Hilbert space:** $\dim \mathcal{H} = 2^{20} = 1,048,576$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-62.51, 62.61]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^{20} = 1,048,576$)
S_1			H_{Ising}	

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \text{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6, 7, 8, 9$ sites, with $n < V/2, (V = 20)$

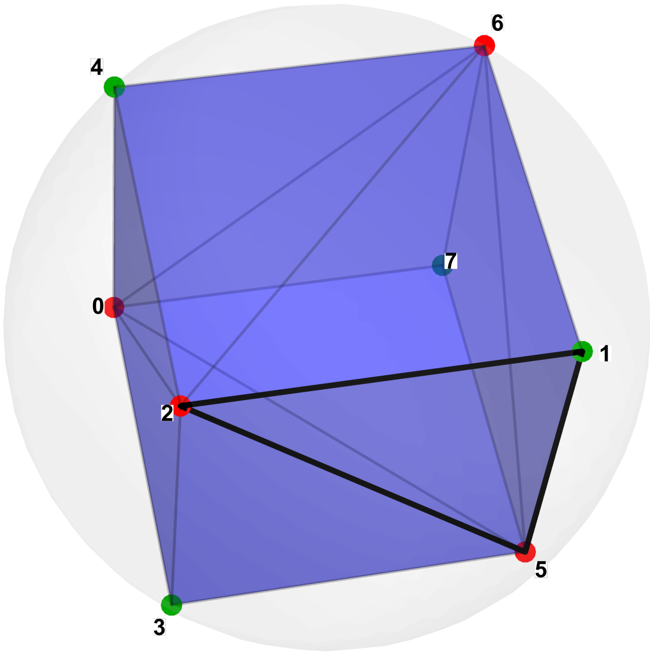
- **Compactness criterion:** nearest-neighbor + most compact, for example $[4, 10]$ $(n = 2)$, $[4, 6, 10]$ $(n = 3)$, $[2, 4, 6, 10]$ $(n = 4)$, $[2, 4, 6, 10, 18]$ $(n = 5)$, $[1, 2, 4, 6, 10, 18]$ $(n = 6)$, $[1, 2, 4, 5, 6, 10, 18]$ $(n = 7)$, $[1, 2, 4, 5, 6, 10, 11, 18]$ $(n = 8)$, $[1, 2, 4, 5, 6, 7, 10, 11, 18]$ $(n = 9)$ (see highlighted edges in figure)

- **Diagnostics:**
  - 2/3-sites RDMs for all scars have full rank
  - 4-sites RDMs for all scars have reduced rank
  - 5-sites RDMs for all scars have reduced rank
  - 6-sites RDMs for all scars have reduced rank
  - 7-sites RDMs for all scars have reduced rank
  - 8-sites RDMs for all scars have reduced rank
  - 9-sites RDMs for all scars have reduced rank

Catalan Solids

Triakis Tetrahedron

Overview and data



- **Duality / paired solid:** truncated tetrahedron (archimedean)
- **Vertices (V), Faces (F), Edges (E):** $V = 8$ (4 on the primary tetrahedron, 4 at the tip of the 4 pyramids built on top of each face of the tetrahedron), $F = 12$ (isosceles triangles), $E = 18$
- **Point group:** T_d
- **Hilbert space:** $\dim \mathcal{H} = 2^8 = 256$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-25.10, 27.07]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

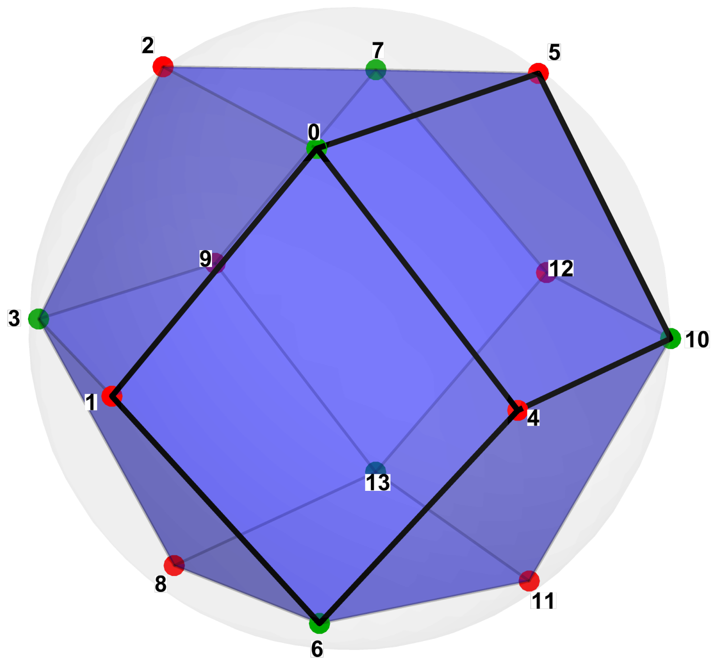
Multiplet label	**Energy** E	**Degeneracy**	**Annihilated by**	**Non-zero components (vs.** $2^8 = 256$**)**
S_1	-2	3	H_{TF}	36

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3$ sites, with $n < V/2, (V = 8)$
- **Compactness criterion:** nearest-neighbor + most compact, for example $[1, 2](n = 2), [1, 2, 5](n = 3)$ (see highlighted edges in figure)
- **Diagnostics:** 2/3-sites RDMs for all 3 scars have full rank (system size $V = 8$ is too small)

Rhombic Dodecahedron

Overview and data



- **Duality / paired solid:** cuboctahedron (archimedean)
- **Vertices (V), Faces (F), Edges (E):** $V = 14$ (8 acute vertices where 3 rhombi meet, 6 obtuse vertices where 4 rhombi meet), $F = 12$ (rhombi), $E = 24$
- **Point group:** O_h

- **Hilbert space:** $\dim \mathcal{H} = 2^{14} = 16,384$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-44.28, 44.28]$

Scar structure: sets and multiplets

- **Number of scar sets:** 4

  For each scar set S_k:

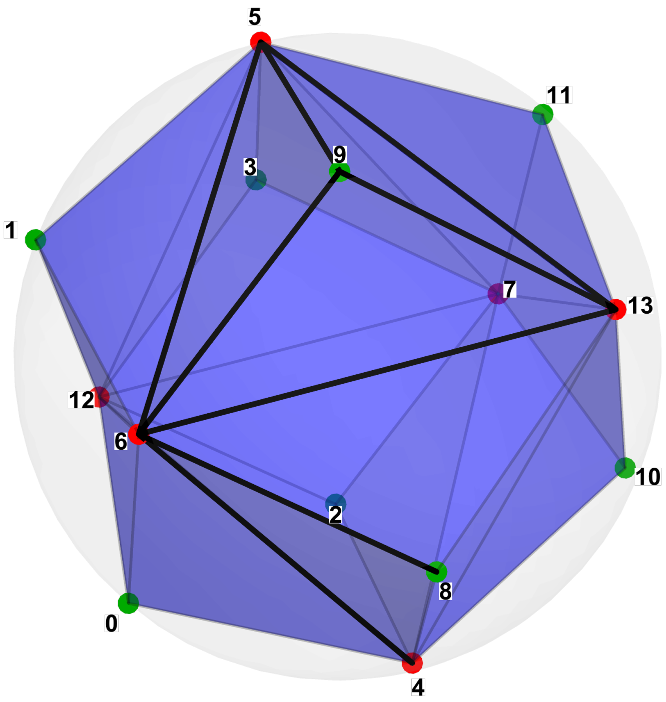
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^{14} = 16,384$)
S_1	-12	3	H_{Ising}	432
S_2	-6	18	H_{Ising}	432
S_3	6	18	H_{Ising}	432
S_4	12	3	H_{Ising}	432

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \text{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2, (V = 14)$
- **Compactness criterion:** nearest-neighbor + most compact, for example $[0, 1]$ $(n = 2)$, $[0, 1, 4]$ $(n = 3)$, $[0, 1, 4, 6]$ $(n = 4)$, $[0, 1, 5, 6, 10]$ $(n = 5)$, $[0, 1, 4, 5, 6, 10]$ $(n = 6)$ (see highlighted edges in figure)
- **Diagnostics:**
  - 2/3-sites RDMs for all scars have full rank
  - 4-sites RDMs for the S_1, S_4 scars have reduced rank of 12, and for S_2, S_3 have full rank
  - 5-sites RDMs for the S_1, S_4 scars have reduced rank of 16, and for S_2, S_3 have reduced rank of 28
  - 6-sites RDMs for the S_1, S_4 scars have reduced rank of 20, and for S_2, S_3 have reduced rank of 50

Triakis Octahedron

Overview and data



- **Duality / paired solid:** truncated cube (archimedean)
- **Vertices (V), Faces (F), Edges (E):** $V = 14$ (6 on the primary octahedron, 8 at the tip of the 8 pyramids built on top of each face of the octahedron), $F = 24$ (isosceles triangles), $E = 36$
- **Point group:** O_h
- **Hilbert space:** $\dim \mathcal{H} = 2^{14} = 16,384$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-44.21, 50.01]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^{14} = 16,384$)
S_1	-4	1	H_{TF}	192
S_2	0	4	both[1]	368

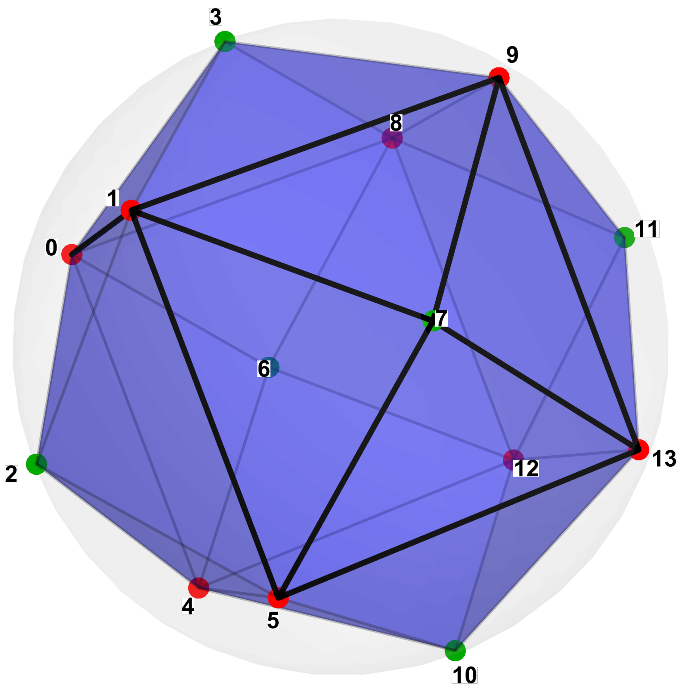
[1] The eigenvalues are not exactly zero for H_{Ising}, but very small, of the order of $10^{-13/14}$.

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2, (V = 14)$
- **Compactness criterion:** nearest-neighbor + most compact, for example $[5, 6]$ $(n = 2)$, $[5, 6, 9]$ $(n = 3)$, $[5, 6, 9, 13]$ $(n = 4)$, $[4, 5, 6, 9, 13]$ $(n = 5)$, $[4, 5, 6, 8, 9, 13]$ $(n = 6)$ (see highlighted edges in figure)
- **Diagnostics:**
  - 2/3-sites RDMs for all scars have full rank
  - 4-sites RDMs for the S_1 scar have reduced rank of 8, and for S_2 have full rank
  - 5-sites RDMs for the S_1 scar have reduced rank of 8, and for S_2 have reduced rank of 16
  - 6-sites RDMs for the S_1 scar have reduced rank of 16, and for S_2 have reduced rank of 20

Tetrakis Hexahedron

Overview and data



- **Duality / paired solid:** truncated octahedron (archimedean)
- **Vertices (V), Faces (F), Edges (E):** $V = 14$ (8 on the primary cube/hexahedron, 6 at the tip of the 8 pyramids built on top of each face of the cube/hexahedron), $F = 24$ (isosceles triangles), $F = 24$ (isosceles triangles), $E = 36$
- **Point group:** O_h
- **Hilbert space:** $\dim \mathcal{H} = 2^{14} = 16,384$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-44.39, 48.90]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^{14} = 16,384$)
S_1	-4	12	H_{TF}	1172
S_2	0	1	both[2]	192

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \text{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2, (V = 14)$
- **Compactness criterion:** nearest-neighbor + most compact, for example $[1, 5]$ $(n = 2)$, $[1, 5, 7]$ $(n = 3)$, $[1, 5, 9, 13]$ $(n = 4)$, $[1, 5, 7, 9, 13]$ $(n = 5)$, $[0, 1, 5, 7, 9, 13]$ $(n = 6)$ (see highlighted edges in figure)
- **Diagnostics:**
  - 2/3-sites RDMs for all scars have full rank
  - 4-sites RDMs for the S_1 scars have full rank, and for S_2 have reduced rank of 6
  - 5-sites RDMs for the S_1 scar have reduced rank of 26, and for S_2 have reduced rank of 12
  - 6-sites RDMs for the S_1 scar have reduced rank of 52, and for S_2 have reduced rank of 24

NB: in the triakis octahedron and tetrakis hexahedron there is a clear hierarchy of the vertices - the (red) vertices of the primary solids, the octahedron and the cube, are prioritized over the (green) vertices at the tip of the pyramids when computing reduced sites RDMs. This explains why the 6th vertices of 4 (for the triakis octahedron) and of 0 (for the tetrakis hexahedron) were respectively added to the 6-sites subset. Also, the vertices hierarchy forces to pick 4 primary vertices + 3 pyramid vertices when computing the bipartite entanglement entropy.

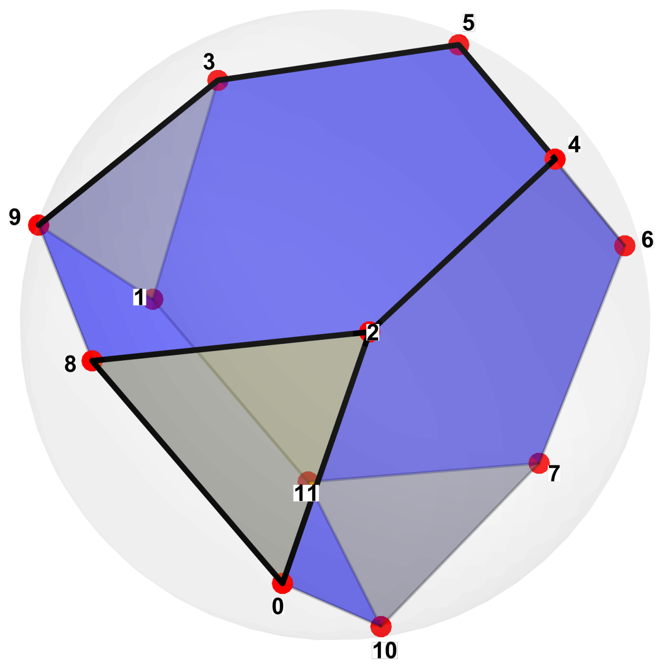
Archimedean Solids

In this case, the hyerarchy is for the faces only. I am not even sure if this is a real hyerarchy - i.e. there are two types of faces, but is one type of face more important than the other? In the case of the truncated tetrahedron, which is the only archimedean solid studied with a scar, the bipartite entanglement entropy seems to be lower when prioritizing the traingular faces over the hexagonal ones. However, when computing 3/4/5-sites RDMs, none of the vertices combination, including the ones prioritizing the vertices on the triangular faces, seems to give a reduced rank RDM.

[2]The eigenvalue is not exactly zero for H_{Ising}, but very small, of the order of 10^{-14}.

Truncated Tetrahedron

Overview and data



- **Duality / paired solid:** triakis tetrahedron (catalan)
- **Vertices (V), Faces (F), Edges (E):** $V = 12,\ \ F = 8$ (4 equilateral triangles, 4 hexagons), $E = 18$
- **Point group:** T_d
- **Hilbert space:** $\dim \mathcal{H} = 2^{12} = 4,096$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-37.36, 37.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

Multiplet label	**Energy E**	**Degeneracy**	**Annihilated by**	**Non-zero components (vs. $2^{12} = 4,096$)**
S_1	0	1	both[3]	48

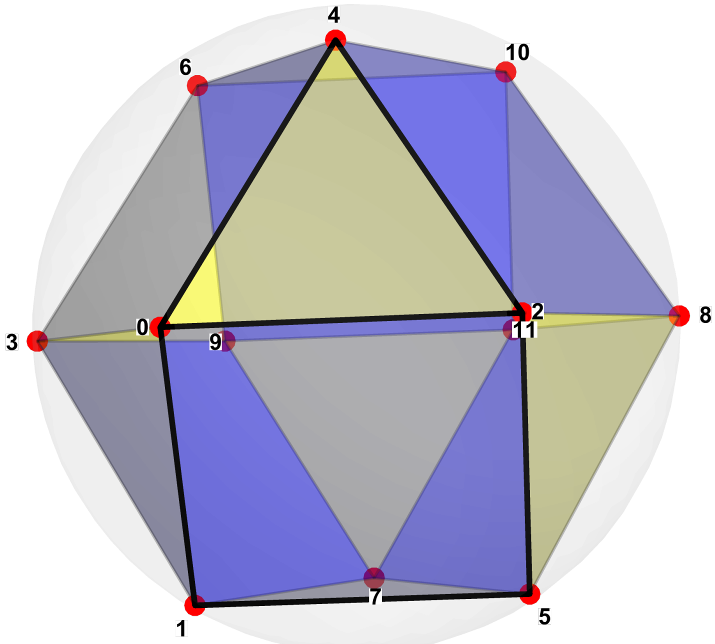
[3]The eigenvalue is not exactly zero for H_{Ising}, but very small, of the order of 10^{-14}.

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5$ sites, with $n < V/2, (V = 12)$
- **Compactness criterion:** nearest-neighbor + most compact, for example $[0, 2]$ $(n = 2)$, $[0, 2, 8]$ $(n = 3)$, $[2, 3, 4, 5]$ $(n = 4)$, $[2, 3, 4, 5, 9]$ $(n = 5)$ (see highlighted edges in figure)
- **Diagnostics:** all RDMs for the scar have full rank

Cuboctahedron

Overview and data



- **Duality / paired solid:** rhombic dodecahedron (catalan)
- **Vertices (V), Faces (F), Edges (E):** $V = 12,\;\; F = 14$ (8 equilateral triangles, 6 squares), $E = 24$
- **Point group:** O_h
- **Hilbert space:** $\dim \mathcal{H} = 2^{12} = 4,096$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-37.73, 38.81]$

Scar structure: sets and multiplets

- **Number of scar sets:** no scars observed

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5$ sites, with $n < V/2, (V = 12)$

- **Compactness criterion:** nearest-neighbor + most compact, for example $[0, 2]$ ($n = 2$), $[0, 2, 4]$ ($n = 3$), $[0, 1, 2, 5]$ ($n = 4$), $[0, 1, 2, 4, 5]$ ($n = 5$) (see highlighted edges in figure)
- **Diagnostics:** all RDMs have full rank because there are no scars

Observations/Conclusions

- The number of scars depends on whether the solid is Platonic, Catalan or Archimedean, and possibly it corresponds to the vertex orbits number of each solid class: Platonic and Archimedean solids have only 1 vertex orbit, since they are vertex-transitive, and so they have only 1 type of scar - up to algebraic sign (exception that I still do not understand: cuboctahedron doesn't seem to have any scar); Catalan solids have 2 vertex orbits, because they have two types of vertices, and they have two sets of scars, again up to an algebraic sign.

- The degeneracy of each scarred multiplet depends on the point group T_d, O_h, I_h: For Platonic solids, the degeneracies are always 2 for T_d, 3 for O_h, and 5 for I_h. I am still not sure where these numbers come from, but my guess would be that they are related to the ‘'rotational" symmetry subgroup of each point group. For Catalan and Archimedean solids, the description seems to be more complicated, because we have 2 types of vertices and faces, respectively, but it should follow a similar, more convoluted reasoning.
- In some cases, the scarred multiplets come in symmetric pairs at $\pm n$, where $n \in \mathbb{N}$. I would like to think this is related to frustration - this seems to happen when the faces of each solid have an even number of vertices, and in this case the scars are usually annihilated by H_{Ising} and not by H_{TF}. There are however exception to this observation: in primis the octahedron, which shouldn't have symmetric scars since it has triangular faces, but it does; the cube, which despite having square faces, it does have a pair of symmetric scars, but these are annihilated by H_{TF} and not H_{Ising}. This aspect is also confusing for Archimedean solids, since they have both even and odd-number of vertices faces.