Scars of the Transverse Field Ising Model on Discrete Geometries (Polyhedra)

September 18, 2025

Introduction

We are studying scars of the simple Ising model on discrete geometries (polyhedra). Here, *scars* are identified as special, sparser eigenstates of the Hamiltonian which are simultaneously eigenstates of the Ising term and of the transverse-field (TF) term separately; in addition, each such state is annihilated by exactly one of the two terms. In all of the following examples, the Hamiltonian is

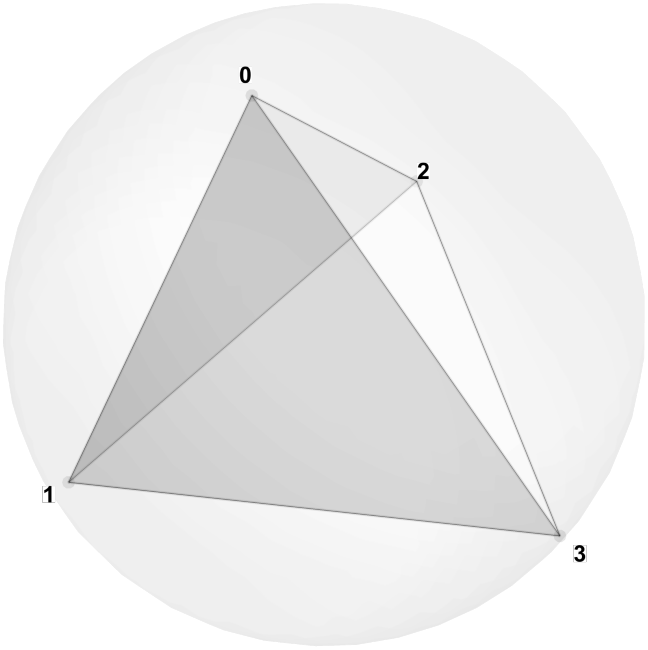
$$H = H_{\text{Ising}} + H_{\text{TF}}, \quad H_{\text{Ising}} = J \sum_{\langle i,j \rangle} \sigma_i^x \sigma_j^x, \quad H_{\text{TF}} = -h \sum_i \sigma_i^z \tag{1}$$

where $J = 1, h = 3$ (antiferromagnetic, non critical).

Platonic Solids

Tetrahedron

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4,\ F = 3$ (equilateral triangles), $E = 6$
- **Solid point group:** T_d
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

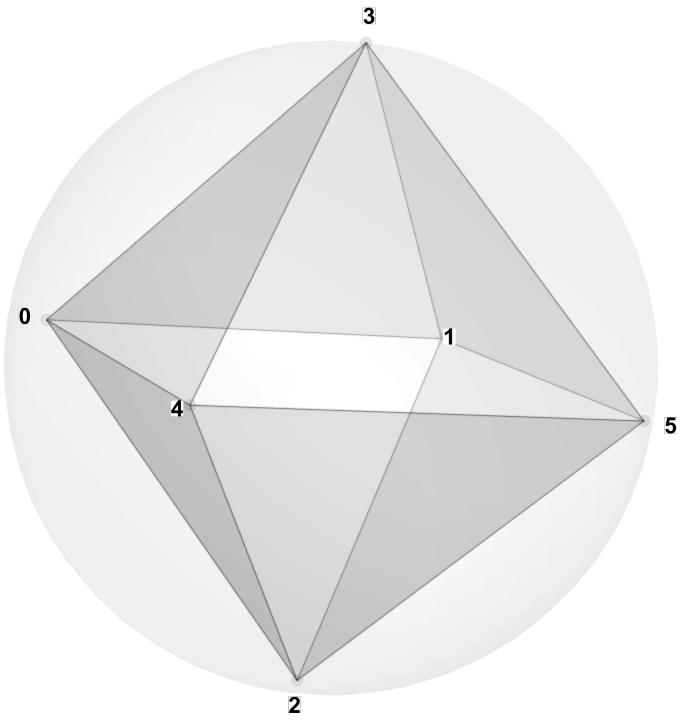
Multiplet label	**Energy** E	**Degeneracy**	**Annihilated by**	**Non-zero components (vs.** $2^4 = 16$**)**
S_1	-2	2	H_{TF}	4,6

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 1$ sites, with $n < V/2, (V = 4)$
- **Compactness criterion:** NA, the single point 0 was chosen
- **Diagnostics:** 1-site RDMs for both scars are full rank (system size $V = 4$ is too small)

Octahedron

Overview and data



- **Duality / paired solid:** cube
- **Vertices (V), Faces (F), Edges (E):** $V = 6,\ F = 8$ (equilateral triangles), $E = 12$
- **Solid point group:** O_h
  Vertex stabilizer subgroup: C_4 for rotations only, C_{4v} for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^6 = 64$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-18.80, 19.67]$

Scar structure: sets and multiplets

- **Number of scar sets:** 3

  For each scar set S_k:

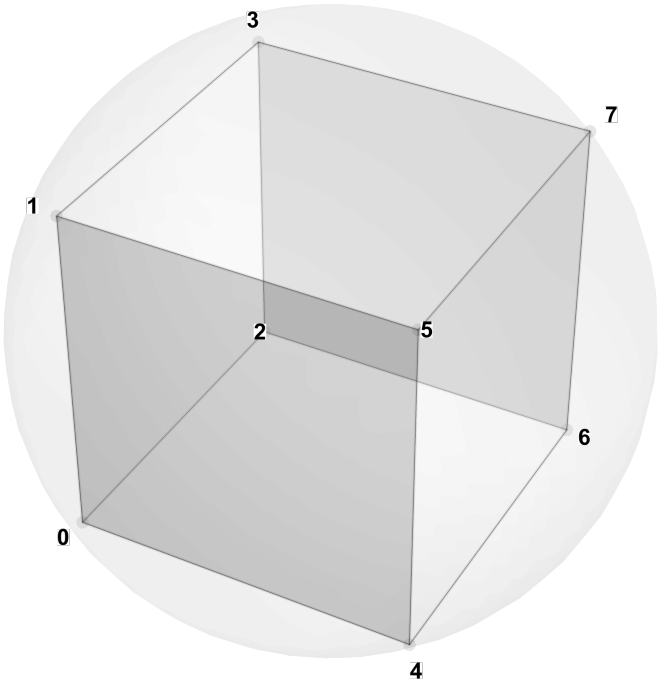
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^6 = 64$)
S_1	-6	3	H_{Ising}	12
S_2	-4	1	H_{TF}	12
S_3	6	3	H_{Ising}	12

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2$ sites, with $n < V/2, (V = 6)$
- **Compactness criterion:** nearest-neighbor, for example $[0, 1]$
- **Diagnostics:** 2-sites RDMs for all 7 scars are full rank (system size $V = 6$ is too small)

Cube

Overview and data



- **Duality / paired solid:** octahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 8,\ F = 6$ (squares), $E = 12$
- **Solid point group:** O_h
  Vertex stabilizer subgroup: C_3 for rotations only, C_{3v} for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^8 = 256$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-25.11, 25.11]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

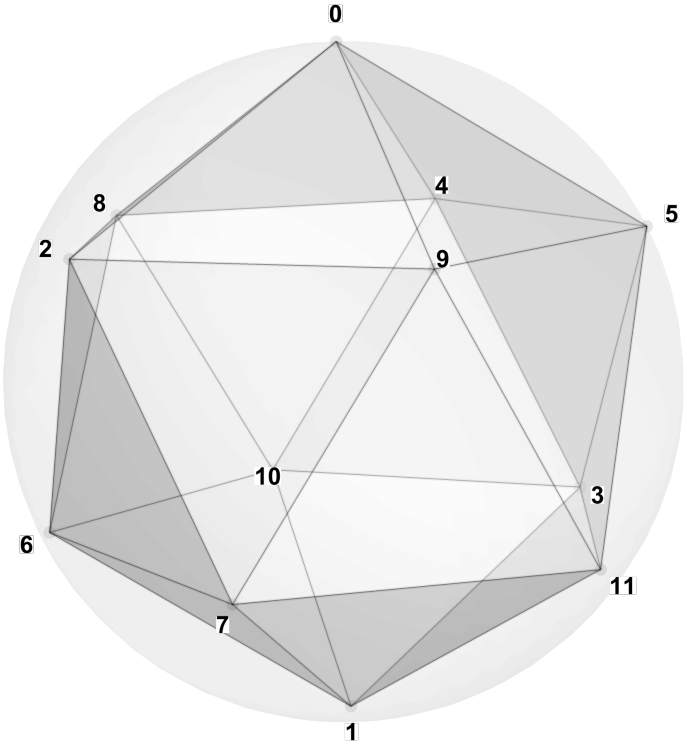
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. $2^8 = 256$)
S_1	-2	3	H_{TF}	48
S_2	2	3	H_{TF}	48

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3$ sites, with $n < V/2, (V = 8)$
- **Compactness criterion:** nearest-neighbor + most compact, for example $[0, 1], [0, 1, 2]$
- **Diagnostics:** 2-sites RDMs for all 6 scars are full rank (system size $V = 8$ is too small)

Icosahedron

Overview and data



- **Duality / paired solid:** dodecahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 12,\ F = 20$ (equilateral triangles), $E = 30$
- **Solid point group:** I_h
  Vertex stabilizer subgroup: C_5 for rotations only, D_5 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^{12} = 4096$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-37.95, 41.29]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

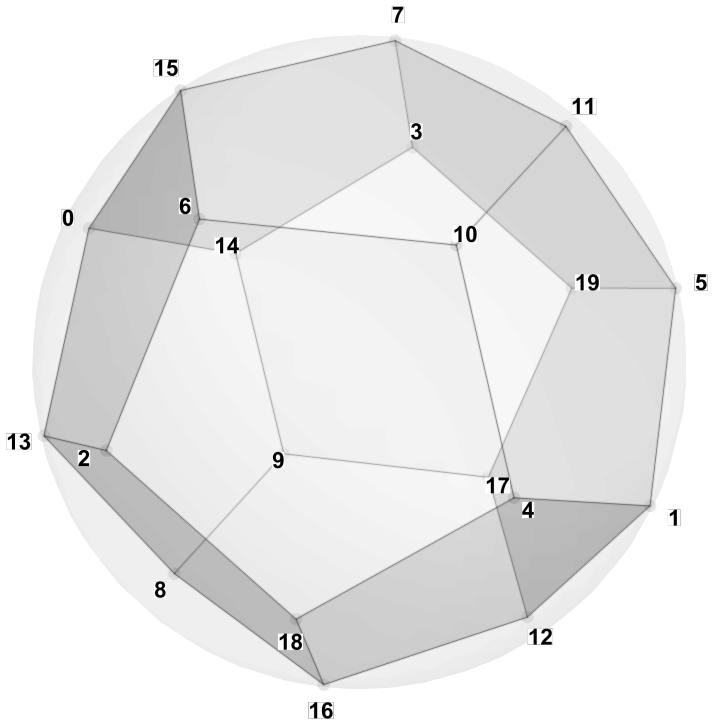
Multiplet label	**Energy** E	**Degeneracy**	**Annihilated by**	**Non-zero components (vs.** $2^{12} = 4096$**)**
S_1	-6	5	H_{TF}	280

Local properties (RDMs)

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5$ sites, with $n < V/2, (V = 12)$.
- **Compactness criterion:** `<how subsets chosen>`.
- **Diagnostics:**
  - 2/3-sites RDMs for all 5 scars are full rank
  - 4-sites RDMs for all 5 scars have reduced rank of 11 = 16 - 5
  - 5-sites RDMs for all 5 scars have reduced rank of 18 = 32 - 14

Dodecahedron

Overview and data



- **Duality / paired solid:** icosahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 20,\ F = 12$ (equilateral triangles), $E = 30$

- **Solid point group:** I_h
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^{20} = 1,048,576$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 1

  For each scar set S_k:

Multiplet label	**Energy** E	**Degeneracy**	**Annihilated by**	**Non-zero components (vs.** $2^{20} = 1,048,576$**)**
S_1	$\pm$ `<int>`	`<deg>`	H_{Ising} `/` H_{TF} `/ both`	`<# non-zero> /` $2^{\texttt{<V>}}$

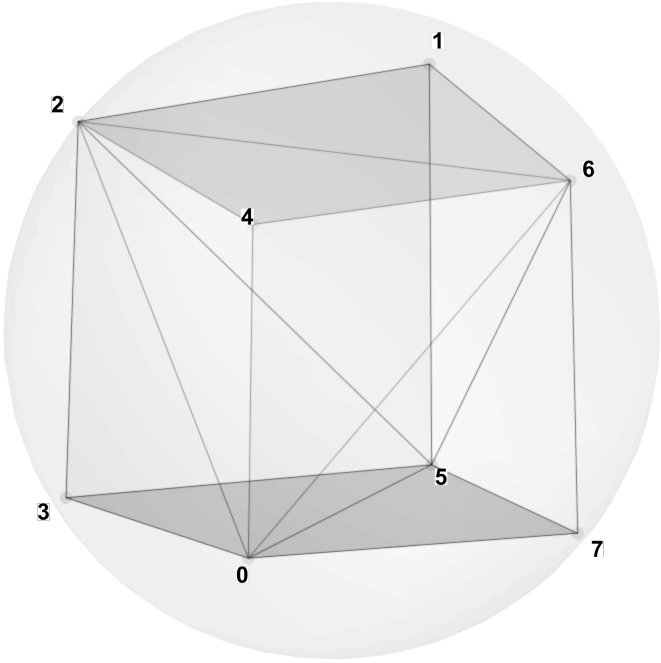
Local properties (RDMs).

- **Local RDM definition:** $\rho_A = \text{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6, 7, 8, 9$ sites, with $n < V/2, (V = 20)$.
- **Compactness criterion:** `<how subsets chosen>`.
- **Diagnostics:** `<observables, entropies, purity, etc.>`.

Catalan Solids

Triakis Tetrahedron

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4,\ F = 3$ (equilateral triangles), $E = 6$
- **Solid point group:** $T_d \cong S_4$
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

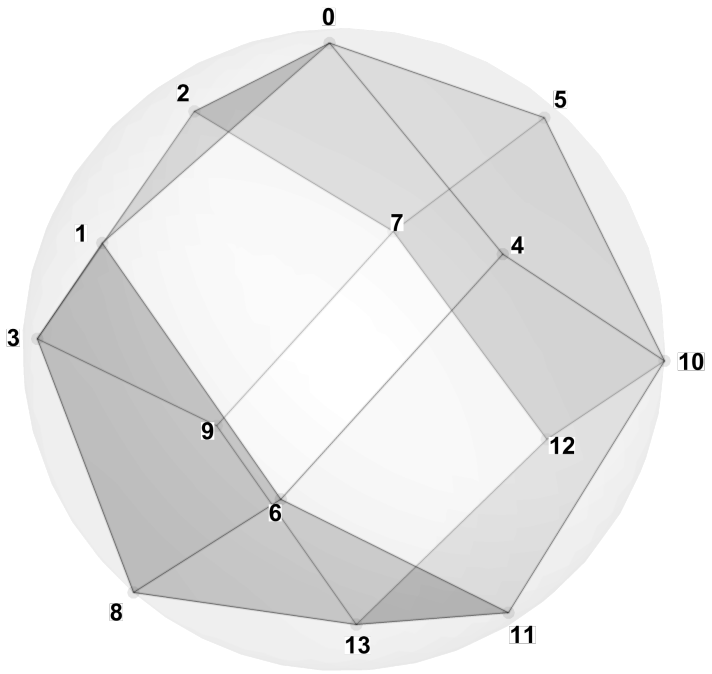
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. 2^V)
S_1	$\pm$ `<int>`	`<deg>`	H_{Ising} / H_{TF} / `both`	`<# non-zero>` / $2^{\texttt{<V>}}$
S_2	$\pm$ `<int>`	`<deg>`	H_{Ising} / H_{TF} / `both`	`<# non-zero>` / $2^{\texttt{<V>}}$

Local properties (RDMs).

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2$.
- **Compactness criterion:** `<how subsets chosen>`.
- **Diagnostics:** `<observables, entropies, purity, etc.>`.

Rhombic Dodecahedron

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4,\ F = 3$ (equilateral triangles), $E = 6$
- **Solid point group:** $T_d \cong S_4$
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

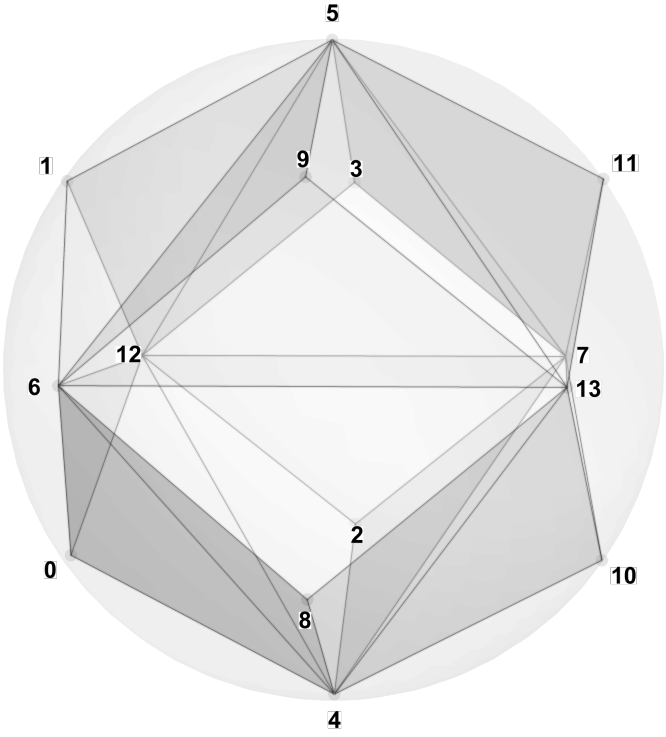
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. 2^V)
S_1	$\pm$ <int>	<deg>	H_{Ising} / H_{TF} / both	<# non-zero> / $2^{\text{<V>}}$
S_2	$\pm$ <int>	<deg>	H_{Ising} / H_{TF} / both	<# non-zero> / $2^{\text{<V>}}$

Local properties (RDMs).

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2$.
- **Compactness criterion:** <how subsets chosen>.
- **Diagnostics:** <observables, entropies, purity, etc.>.

Triakis Octahedron

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4$, $F = 3$ (equilateral triangles), $E = 6$
- **Solid point group:** $T_d \cong S_4$
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. 2^V)
S_1	$\pm$ <int>	<deg>	H_{Ising} / H_{TF} / both	<# non-zero> / $2^{\text{<V>}}$
S_2	$\pm$ <int>	<deg>	H_{Ising} / H_{TF} / both	<# non-zero> / $2^{\text{<V>}}$

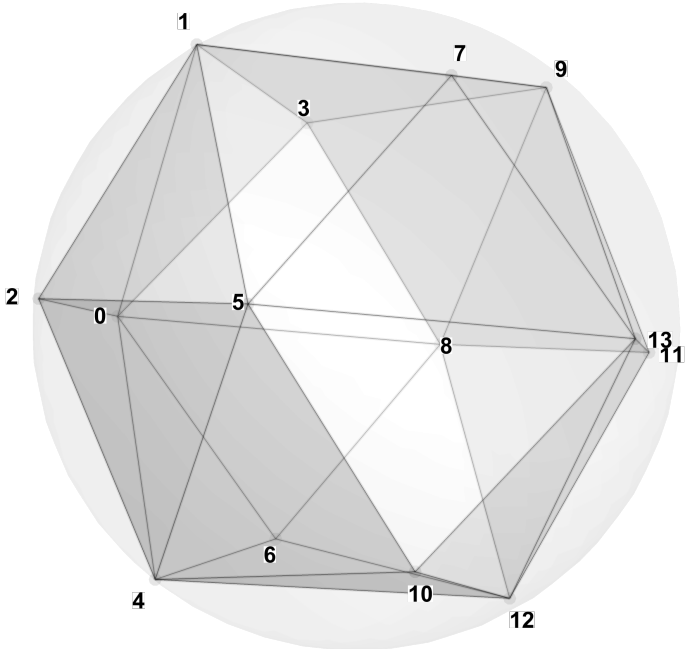
Local properties (RDMs).

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2$.
- **Compactness criterion:** <how subsets chosen>.
- **Diagnostics:** <observables, entropies, purity, etc.>.

Tetrakis Hexahedron

Overview and data.

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4$, $F = 3$ (equilateral triangles), $E = 6$

- **Solid point group:** $T_d \cong S_4$
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

Multiplet label	**Energy** E	**Degeneracy**	**Annihilated by**	**Non-zero components (vs.** 2^V**)**
S_1	$\pm$ `<int>`	`<deg>`	H_{Ising} / H_{TF} / `both`	`<# non-zero>` / $2^{\texttt{<V>}}$
S_2	$\pm$ `<int>`	`<deg>`	H_{Ising} / H_{TF} / `both`	`<# non-zero>` / $2^{\texttt{<V>}}$

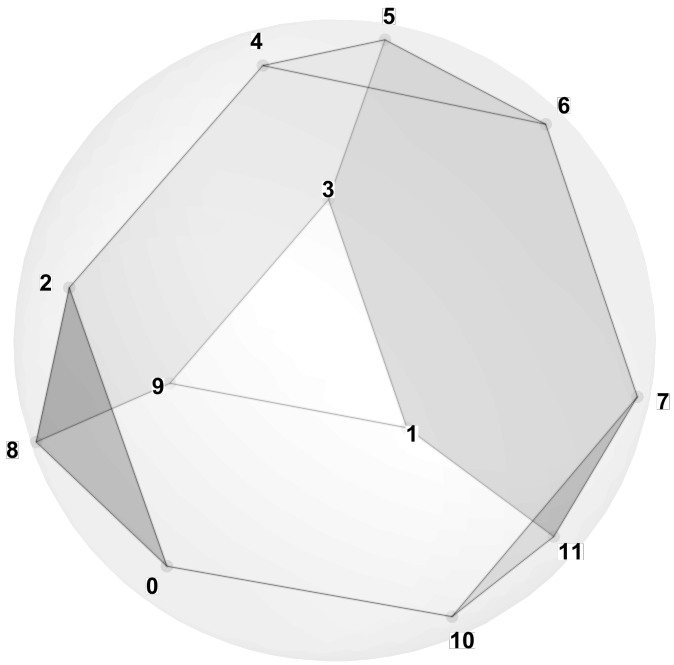
Local properties (RDMs).

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2$.
- **Compactness criterion:** `<how subsets chosen>`.
- **Diagnostics:** `<observables, entropies, purity, etc.>`.

Archimedean Solids

Truncated Tetrahedron

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4,\ F = 3$ (equilateral triangles), $E = 6$
- **Solid point group:** $T_d \cong S_4$
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

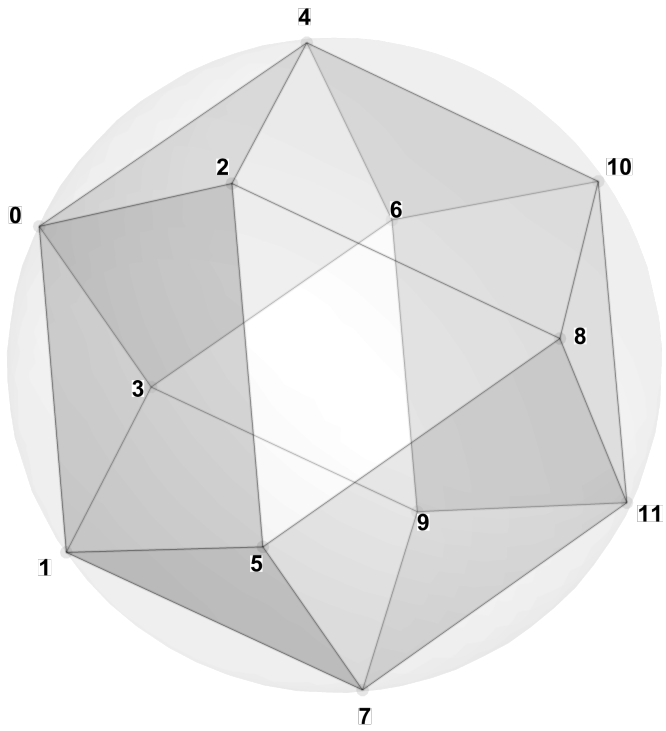
Multiplet label	Energy E	Degeneracy	Annihilated by	Non-zero components (vs. 2^V)
S_1	$\pm$ `<int>`	`<deg>`	H_{Ising} / H_{TF} / `both`	`<# non-zero>` / $2^{\texttt{<V>}}$
S_2	$\pm$ `<int>`	`<deg>`	H_{Ising} / H_{TF} / `both`	`<# non-zero>` / $2^{\texttt{<V>}}$

Local properties (RDMs).

- **Local RDM definition:** $\rho_A = \mathrm{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2$.
- **Compactness criterion:** `<how subsets chosen>`.
- **Diagnostics:** `<observables, entropies, purity, etc.>`.

Cuboctahedron

Overview and data



- **Duality / paired solid:** self-dual, tetrahedron
- **Vertices (V), Faces (F), Edges (E):** $V = 4$, $F = 3$ (equilateral triangles), $E = 6$
- **Solid point group:** $T_d \cong S_4$
  Vertex stabilizer subgroup: C_3 for rotations only, D_3 for full symmetry group
- **Hilbert space:** $\dim \mathcal{H} = 2^4 = 16$ (spin-$\frac{1}{2}$ on each vertex)
- **Eigenvalue range:** $[-12.37, 12.71]$

Scar structure: sets and multiplets

- **Number of scar sets:** 2

  For each scar set S_k:

Multiplet label	**Energy E**	**Degeneracy**	**Annihilated by**	**Non-zero components (vs. 2^V)**
S_1	$\pm$ <int>	<deg>	H_{Ising} / H_{TF} / both	<# non-zero> / $2^{\text{<V>}}$
S_2	$\pm$ <int>	<deg>	H_{Ising} / H_{TF} / both	<# non-zero> / $2^{\text{<V>}}$

Local properties (RDMs).

- **Local RDM definition:** $\rho_A = \text{Tr}_{\bar{A}}(|\psi\rangle\langle\psi|)$ on compact subsets of $n = 2, 3, 4, 5, 6$ sites, with $n < V/2$.
- **Compactness criterion:** <how subsets chosen>.
- **Diagnostics:** <observables, entropies, purity, etc.>.

Conclusions The eigenstates corresponding to eigenvalue 0, which are annihilated by both H_{Ising} and H_{TF}, are probably not proper scars, since they are not really sparse.