CSCI6454: Advanced Algorithms

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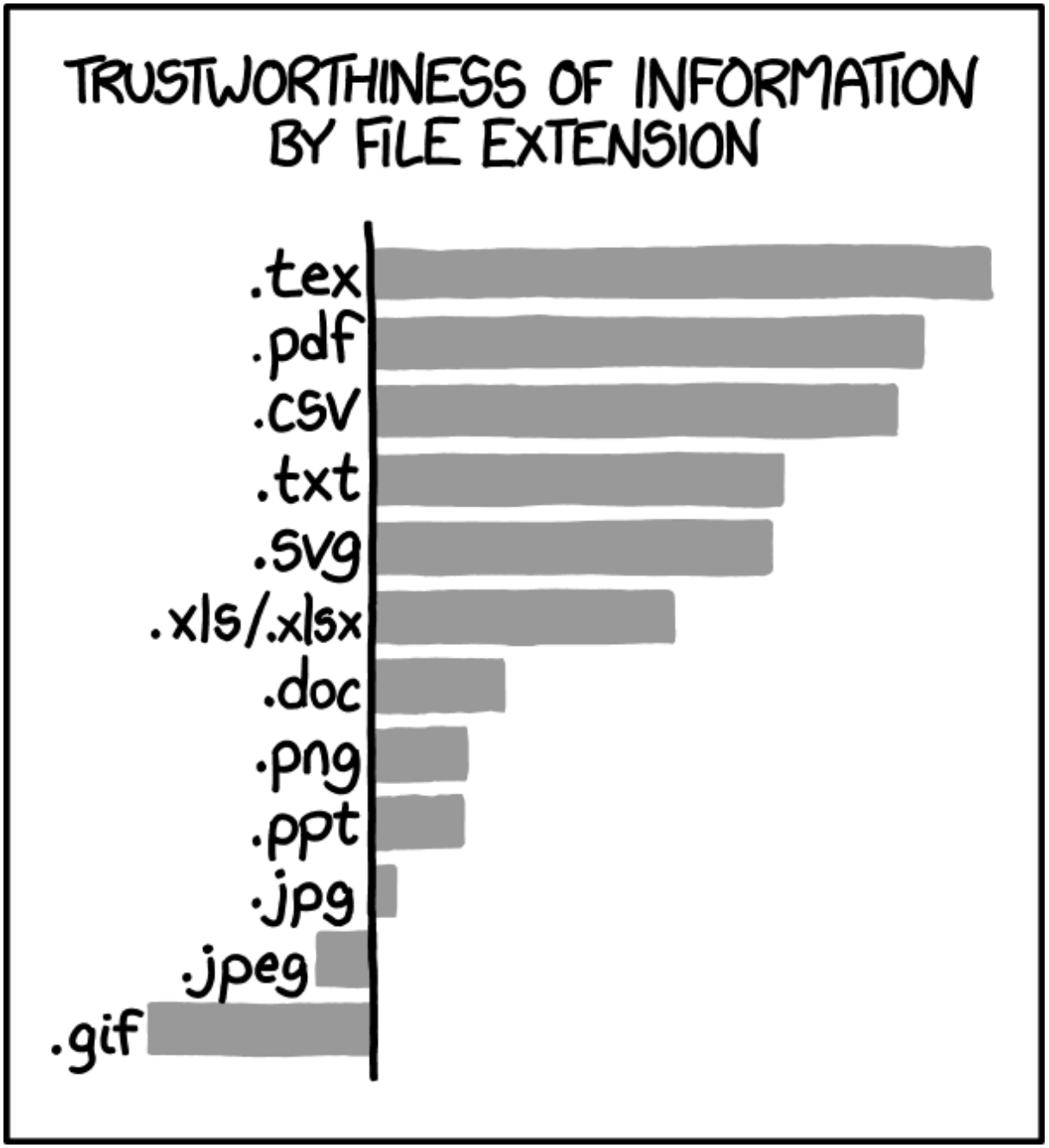
READINGS: VARIOUS

UNIVERSITY OF COLORADO BOULDER

EDITION 1



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Preface

To the interested reader,

This document is a compilation of lecture notes scribed during the Spring 2025 semester for CSCI6454: Advanced Algorithms at the University of Colorado Boulder. The content in these course notes is from lecture and several resources cited herein. The author only takes responsibility for the scribing of the content and lectures into the notes. This course was taught by Huck Bennett, Ph. D.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. If an error needs to be resolved, please contact Adithya Bhaskara at adithya@colorado.edu.

Best Regards,
Adithya Bhaskara

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Introduction to Fine-Grained Complexity

1.1 Lecture 1: January 14, 2025

1.1.1 Introduction to Fine-Grained Complexity

We hope to precisely understand the complexity of several algorithmic problems beyond the coarse information given by standard complexity classes, like P and NP. A central theme is proving lower bounds for the complexity of algorithms; this is a difficult task in general. We use reductions as an essential tool.

The high-level idea behind fine-grained complexity is to explain hardness, or the lack of algorithmic progress, on fundamental computational problems by giving reductions from well-studied problems. Recall the P v. NP problem. Loosely, problems in P are "easy," to solve, and problems in NP are "easy" to verify positive answers to. The hardest problems in NP, the NP-complete problems do not admit polynomial-time algorithms unless P $=$ NP. To show that a problem is NP-complete, we show that it is in NP, and exhibit a polynomial-time reduction from a known NP-complete problem.

The theory behind NP-completeness is robust and quite nice. But, we'd like finer results. Knowing whether a problem is in P or not doesn't really narrow down how efficiently we can solve it in practice. Even quadratic-time algorithms may be inefficient in production.

These ideas motivate fine-grained complexity. Here, we start by taking a well-studied problem, say L_1, and making a precise conjecture about the running time of optimal algorithms for L_1. Then, we give an efficient reduction from L_1 to a problem L_2, for which we are showing hardness. We start by providing some hypotheses corresponding to certain problems, and later, we'll show fairly precise running time bounds for problems in P.

Example 1. *Under certain assumptions, there is no $O(n^{2-\epsilon})$-time algorithm for the* Edit Distance *problem. This matches the $O(n^2)$ dynamic programming algorithm.*

Computational Problem 1 (k-SAT)**.**

- *Given a CNF formula φ with k literals in each clause,*
- *Decide: Does there exist a satisfying assignment?*

Hypothesis 1.1.1: Exponential Time Hypothesis (ETH)

The 3-SAT problem takes $2^{\Omega(n)}$ time.

Remark. *Note that* P $\neq$ NP *asserts that* 3-SAT *has no polynomial-time algorithm, whereas ETH asserts that* 3-SAT *has no subexponential-time algorithm. ETH implies* P $\neq$ NP*.*

Hypothesis 1.1.2: Strong Exponential Time Hypothesis (SETH)

For every $\epsilon > 0$, there exists $k \in \mathbb{Z}^+$ such that there is no $O\left(2^{(1-\epsilon)n}\right) = O((2-\epsilon)^n)$-time algorithm for k-SAT.

Remark. *SETH, at a high level, claims that 2^n-time is essentially optimal for k-SAT for large k.*

Computational Problem 2 (k-SUM)**.**

- *Given arrays $A_1, \dots, A_k$ each with n integers in $[-n^c, n^c]$,*
- *Decide: Does there exist $a_1 \in A_1, \dots, a_k \in A_k$ such that $a_1 + \cdots + a_k = 0$?*

Some variants include assuming that $A_1 = \cdots = A_k$, or wanting to find $a_1 \in A_1, \dots, a_k \in A_k$ where $a_1 + \cdots + a_{k-1} = a_k$.

Naively, we can solve k-SUM by trying all possible choices a_i and checking the sum. This can be done in $O(n^k)$ time. We can do better. For now, let $k = 3$. We start by computing all possible sums $a_1 + a_2$. Then, define $S = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$. Sort A_3. For each $s \in S$, search in A_3 for $-s$. If $-s$ is found, there exists a solution. Otherwise, there is no solution. This provides an $O(n^2 \log n)$ algorithm; computing the sums takes $O(n^2)$ time, sorting takes $O(n \log n)$ time, and the search procedure takes $O(n^2 \log n)$ time.

Hypothesis 1.1.3: No $O(n^{2-\epsilon})$-Time Algorithm for 3-SUM

For every $\epsilon > 0$, there is no $O\left(n^{2-\epsilon}\right)$-time algorithm for 3-SUM.

1.2 Lecture 2: January 16, 2025

1.2.1 Fine-Grained Complexity II

Now, we introduce a new problem.

Computational Problem 3 (All Pairs Shortest Path (APSP))**.**

- *Given a weighted graph $G = (V, E, w)$ on n vertices,*
- *Find: The lengths of the shortest paths $d(v_i, v_j)$ between every $v_i, v_j \in V$.*

One way to solve APSP is to run Dijkstra's algorithm on each vertex. Recall that Dijkstra's algorithm finds a single-source shortest path tree. We could also similarly use the Floyd-Warshall dynamic programming algorithm.

Hypothesis 1.2.1: No $O(n^{3-\epsilon})$-Time Algorithm for APSP

For every $\epsilon > 0$, there exists no $O(n^{3-\epsilon})$-time algorithm for APSP.

We now introduce the Floyd-Warshall dynamic programming algorithm in detail. Let $D_k[i, j]$ store the length of the shortest path from v_i to v_j, with the restriction that the path only passes through $v_1, \ldots, v_k$.

Example 2. *To illustrate our notion, consider the below figure.*

[AB: Insert Huck's example, see lecture notes.]

The idea is to build D_k from D_{k-1} comes from noting that either

1. the shortest path from v_i to v_j only using $v_1, \ldots, v_k$ uses v_k, or
2. it does not.

[AB: Insert figure here; see lecture notes.] From this basic idea, we get a simple algorithm.

Algorithm 1.2.1 Floyd-Warshall

Require: $G = (V, E, w)$.

procedure Floyd_Warshall(E)

$$D_0 \leftarrow \begin{cases} 0 & i = j \\ w_{i,j} & i \neq j, (v_i, v_j) \in E \\ \infty & (v_i, v_j) \notin E \end{cases}$$

for $k \in \{1, \ldots, n\}$ **do**
  for $i \in \{1, \ldots, n\}$ **do**
    for $j \in \{1, \ldots, n\}$ **do**
      $D_k[i, j] = \min\{D_{k-1}[i, k] + D_{k-1}[k, j], D_{k-1}[i, j]\}$
return D_n ▷ $D_n[i, j] = d(v_i, v_j)$

Consider the following problem.

Computational Problem 4 (Orthogonal Vectors (OV))**.**

- *Given two sets* $V = \{v_1, \dots, v_n\} \subseteq [0,1]^d$ *and* $W = \{w_1, \dots, w_n\} \subseteq [0,1]^d$,
- *Decide: Does there exist* $v_i \in W$ *and* $w_j \in W$ *such that* $\langle v_i, w_j \rangle = \sum_{k=1}^{d} v_i[k] w_j[k] = 0$*?*

The naive algorithm is to compare every pair, compute the inner products, and compare to 0. We have n^2 inner products to compute, and each takes d time, so the naive algorithm takes $O(dn^2)$ time. Think of $d = \log^k(n)$ for some k. In this case, $O(dn^2) = O(n^2 \log^k(n)) = \tilde{O}(n^2)$.

Hypothesis 1.2.2: No $O(n^{2-\epsilon})$-Time Algorithm for OV

For every $\epsilon > 0$, there exists no $O(n^{2-\epsilon})$-time algorithm for OV with $d = \text{polylog}(n)$.

So far, we have 5 hypotheses; what are the relations herein? We'll attack this question after a brief interlude on reductions.

We write $L_1 \leq L_2$ if problem L_1 reduces to problem L_2; that is, L_1 is no harder than L_2; and if L_1 is hard, then L_2 is hard. Equivalently, if L_2 is easy, then L_1 is easy. Consider the following definition, making precise the notion of a "fine-grained reduction."

Definition 1.2.1: (Vassilevska Williams) Fine-Grained Reductions

Suppose L_1 and L_2 are computational problems. Let $\ell_1 = \ell_1(n)$ and $\ell_2 = \ell_2(n)$ be associated runtime bounds, say conjectured optimal bounds. Then, L_1 (ℓ_1, ℓ_2)-reduces to L_2 if for every $\epsilon > 0$, there exists $\delta > 0$ and an algorithm R running in time $O(\ell_1(n)^{1-\delta})$ and making q calls to an oracle for L_2 on inputs of size $n_1, \dots, n_q$ where

$$\sum_{i=1}^{q} \ell_2(n_i)^{1-\epsilon} \leq O(\ell_1(n)^{1-\delta}).$$

Remark. *To understand Definition 1.2.1, consider the below figure. [AB: Add Figure; see Lecture Notes]*

Theorem 1.2.1: (Williams, 2004): SETH $\implies$ OV

Hypothesis 1.1.2 implies Hypothesis 1.2.2.

Proof. Using Definition 1.2.1, we can restate this theorem as follows: for every $k \in \mathbb{Z}^+$, k-SAT $(2^n, n^2)$-reduces to OV with $d = \text{polylog}(n)$. We exhibit such a reduction, which will be of exponential-time.

Let $\varphi(x_1, \dots, x_n) = \bigwedge_{j=1}^{m} C_j$ be our k-SAT instance, with each C_j a k-clause. Partition $X = \{x_i\}_{i=1}^{n}$ into $X_1 = \{x_1, \dots, x_{\frac{n}{2}}\}$ and $X_1 = \{x_{\frac{n}{2}+1}, \dots, x_n\}$. Consider the $N = 2^{\frac{n}{2}}$ assignments to each of X_1 and X_2 independently. Then, let $V_1, V_2 \subseteq \{0,1\}^m$. For an assignment, a_1 to literals in X_1, $v_{a_1} \in V_1$ is defined by

$$v_{a_1}[j] = \begin{cases} 0 & a_1 \text{ satisfies } C_j \\ 1 & \text{otherwise} \end{cases}.$$

Define $v_{a_2} \in V_2$ similarly corresponding to a_2 with literals in X_2. We claim that $\langle v_{a_i}, v_{a_2} \rangle = 0$ if and only if (a_1, a_2) is a satisfying assignment. Note that $m = O(n^k) = O((2\log_2 N)^k) = O(\log^k N)$, where $N = 2^{\frac{n}{2}}$. A $O\left(N^{2-\epsilon}\right)$-time algorithm for OV would then imply an $O\left(N^{2-\epsilon}\right) = O\left(\left(2^{\frac{n}{2}}\right)^{2-\epsilon}\right) = O\left(2^{\left(1-\frac{\epsilon}{2}\right)n}\right)$-time algorithm for k-SAT. □

Example 3. *Consider $n = 6$, $m = 3$, and $k = 3$ with*

$$\varphi(x_1, \dots, x_6) = ([AB : Finish!])$$

1.3 Lecture 3: January 21, 2025

1.3.1 Graph Diameter and Smarter k-SAT

Consider the following problem.

Computational Problem 5 (Graph Diameter)**.**

- *Given a graph* $G = (V, E, w)$*,*
- *Find:* $\max_{u,v \in V} d(u, v)$ *where* $d(u, v)$ *is the shortest path distance between* $u, v \in V$*.*

An easy algorithm is to run Floyd-Warshall and output the largest distance found. This takes $O(n^3)$ time.

Theorem 1.3.1: (Roddity-Vassilevska Williams) OV Reduction to GraphDiameter

The OV problem (n^2, n^2) reduces to GraphDiameter on undirected, unweighted graphs with $O(n)$ vertices and $\tilde{O}(n)$ edges.

Proof. Let $v = [v'_1, \ldots, v'_n] \in \{0, 1\}^d$ and $w = [w'_1, \ldots, w'_n] \in \{0, 1\}^d$ □

[AB: Finish this! I was very tired during lecture...]

1.4 Lecture 4: January 23, 2025

1.4.1 Smarter k-SAT, Part I

[Monien-Speckenmeyer, 1985]

We've shown a $O(1.913^n)$ algorithm for 3-SAT. Can we do better? Yes, we can! Consider a clause $C = x_1 \vee x_2 \vee x_2$. Note that

$$C = x_1 \vee x_2 \vee x_2 \iff x_1 \vee (\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3).$$

We can think of the above formula as either setting the truth of one, two, or three literals. This gives rise to a modification of the previous recursive algorithm, making 3 recursive calls. This gives us the recurrence $T(n) = T(n-1) + T(n-2) + T(n-3) + \text{poly}(n) = O^*(\alpha^n)$ where α is the unique real root to $\alpha^n - \alpha^{n-1} - \alpha^{n-2} - \alpha^{n-3} = 0$, or equivalently, $\alpha^3 - \alpha^2 - \alpha - 1$. Here, $\alpha \leq 1.84$. So, $T(n) = (1.84^n)$.

We now show how to reduce k-SAT to $k-1$-SAT where $k \geq 4$. Take the k-SAT formula

$$\varphi = \bigwedge_{j=1}^{m} C_j.$$

We will map each clause C_j in φ to the conjunction of a $k-1$-clause, and a 3-clause. We'll introduce a new variable y each time. Say

$$C_j = (x_1^j, \ldots, x_k^j) \mapsto (x_1^j, \ldots, x_{k-2}^j \vee y) \wedge (\neg y \vee x_{k-1}^j \vee x_k^j).$$

Now, we consider local search algorithms for 3-SAT [Spielman's Notes, [Sch99]]. Here, we harness the power of randomness.

Algorithm 1.4.2 Schöning SAT

```
1: procedure SCHÖNING_SAT_FRAMEWORK(φ, β_n)
2:     Guess a uniformly random assignment to a⃗ ∈ {0,1}^n to φ.
3:     for j ∈ {1, ..., β_n} do
4:         if φ is satisfied by a⃗ then
5:             return a⃗
6:         else
7:             Choose an unsatisfied clause C in φ, and choose a variable x_i in C uniformly at random.
8:             Flip the assignment a_i to x_i in a⃗.
9:     return a⃗
```

Example 4. *Say $C = (x_1, x_2, x_3)$ with $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. Since C is unsatisfied, we can flip one of the literals uniformly at random to satisfy C.*

Importantly, how do we choose β_n? Suppose that the initial assignment $\vec{a}$ agrees with the satisfying assignment $\vec{a}^*$ on a "decent" fraction of coordinates with "decent" probability. The steps in the loop can be thought of as doing a random walk on $\{0,1\}^n$. At each step, $\vec{a}$ gets closer to $\vec{a}^*$ with probability at least $\frac{1}{3}$ since at least one variable in each unsatisfied clause differs between $\vec{a}$ and $\vec{a}^*$.

For "Easy"-Schöning, take $\beta_n = n$. Say u is the number of coordinates in which $\vec{a}$ and $\vec{a}^*$ disagree.

Then,

$$\begin{aligned}
\Pr[\text{success}] &\geq \sum_{i=0}^{n} \Pr[u = i] \Pr[\text{first } i \text{ choices match}] \\
&= \sum_{i=0}^{n} \frac{1}{2^n} \binom{n}{i} \Pr[\text{first } i \text{ choices match}] \\
&\geq \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{3^i} \\
&= \frac{1}{2^n} \left(1 + \frac{1}{3}\right)^n \\
&= \left(\frac{2}{3}\right)^n .
\end{aligned}$$

Intuitively, we sample a random assignment $\vec{a}$ which disagrees in i coordinates from $\vec{a}^*$. We want to keep flipping assigments so that $\vec{a} = \vec{a}^*$. The algorithm succeeds with probability at least $\left(\frac{2}{3}\right)^n$. So, we can repeat Algorithm 1.4.2 roughly $\left(\frac{3}{2}\right)^n$ with $\beta = n$ times to find a solution with high probability. [AB: FINISH THIS! (and undertsnad this fully!)]

We now present another result: the sparsification lemma. Morally, we can reduce solving a k-SAT instance with n variables to solving several k-SAT instances each with n variables $O(n)$ clauses. Informally, "every k-SAT formula has $O(n)$ clauses."

Lemma 1.4.1: The Sparsification Lemma

Let $\epsilon > 0$ and $k \geq 3$. Then, there is a $2^{\epsilon n}\,\mathrm{poly}(n)$-time algorithm that takes a k-SAT formula φ on n variables as input and produces $2^{\epsilon n}$ k-SAT formulas $\varphi_1, \ldots, \varphi_{2^{\epsilon n}}$, each of which has n variables and $n\left(\frac{k}{\epsilon}\right)^{O(k)} = O(n)$ clauses. Furthermore, φ is satisfiable if and only if $\bigvee_i \varphi_i$ is satisfied.

With the help of Lemma 1.4.1, we can show that the "strong" exponential time hypothesis indeed implies the exponential time hypothesis. [AB: TODO! HW.]

1.5 Lecture 5: January 28, 2025

1.5.1 Smarter k-SAT, Part II

Now, what if we apply the framework behind Algorithm 1.4.2 with $\beta_n = 3n$? Let's analyze the probability that

1. the initial assignment $\vec{a}$ disagrees with $\vec{a}^*$ on exactly i coordinates, and
2. the first $3i$ steps of the walk include at least $2i$ "good" choices.

For (1), the analysis is very similar to "Easy"-Schöning. First, recall that for $i \geq 2$,

$$\binom{3i}{i} \geq \frac{1}{\sqrt{5i}} \frac{3^{3i}}{2^{2i}}$$

by Stirling's approximation. Then,

$$\begin{aligned}\Pr[\text{at least } 2i \text{ of the first } 3i \text{ choices are good}] &\geq \binom{3i}{i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \\ &\geq \frac{1}{\sqrt{3i}} \frac{3^{3i}}{2^{2i}} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i}\end{aligned}$$

Then,

$$\begin{aligned}\Pr[\text{success}] &\geq \sum_{i=0}^{n} \Pr[u = i] \Pr[\text{at least } 2i \text{ of the first } 3i \text{ choices are good}] \\ &\geq \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \binom{3i}{i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \\ &\geq \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{\sqrt{3i}} \frac{3^{3i}}{2^{2i}} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \\ &\geq \frac{1}{2^n \sqrt{5n}} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{\sqrt{3i}} \frac{3^{3i}}{2^{2i}} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \\ &= \frac{1}{2^n \sqrt{5n}} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i} \\ &= \frac{1}{2^n \sqrt{5n}} \left(1 + \frac{1}{2}\right)^{n} \\ &= \frac{1}{\sqrt{5n}} \left(\frac{3}{4}\right)^{n}.\end{aligned}$$

Proceeding similarly, we can run Algorithm 1.4.2 roughly $O^*\left(\frac{4}{3}\right)^n$ times with $\beta = 3n$ times to find a solution with high probability.

Question. *Is this optimal?*

Answer. *No! But, we'll end here.*

1.6 Lecture 6: January 30, 2025

1.6.1 Nontrivial Nondeterministic Algorithms: SETH and Other Assumptions

Consider the following definition.

Definition 1.6.1: NTIME

A language $L \subseteq \{0,1\}^n$ is in NTIME(f_n) if there exists an algorithm M such that for $x \in \{0,1\}^*$ with $|x| = n$, M runs in time $f(n)$ and

1. if $x \in L$, then there exists $w \in \{0,1\}^*$ with $M(x,w)$ accepting, and
2. if $x \notin L$, for all $w \in \{0,1\}^*$, $M(x,w)$ is rejecting.

Remark. *The familiar complexity class* NP *is given by*

$$\mathsf{NP} = \bigcup_{k \in \mathbb{N}} \mathsf{NTIME}(n^k).$$

Herein, we reference [CGI+16]. Recall that k-SAT $\in$ NP.

Question. *Can we say anything about verifying that a* k-SAT *formula is unsatisfiable?*

Hypothesis 1.6.1: NSETH

For every $\epsilon > 0$, there exists k with $\overline{k\text{-SAT}} \notin \mathsf{NTIME}\left(2^{(1-\epsilon)n}\right)$.

Remark. *NSETH essentially hypothesizes that* k-SAT *has no nontrivial verification algorithm.*

The key takeways of [CGI+16] are that

1. assuming NSETH, $\overline{k\text{-SAT}}$ does not have nontrivial nondeterministic algorithms,
2. $\overline{\text{3-SUM}}$ and $\overline{\text{APSP}}$ do have nontrivial nondeterministic algorithms, and
3. assuming (1), (2) and NSETH, there is no fine-grained reduction from k-SAT to 3-SUM or APSP, as opposed to the k-SAT to OV reduction.

Consider the following theorem.

Theorem 1.6.1: [CGI+16] Nondeterministically Solving $\overline{\text{3-Sum}}$ in $\tilde{O}(n^{\frac{3}{2}})$ Time

There is a $\tilde{O}\left(n^{\frac{3}{2}}\right)$-time nondeterministic algorithm for $\overline{\text{3-SUM}}$.

Proof. Take the variant of 3-SUM where we are given a single list $A = [a_1, \dots, a_n]$ where $a_i \in [-n^c, n^c]$, and we are to decide if there exist $a_i, a_j, a_k \in A$ with $a_i + a_j + a_k = 0$. So, for $\overline{\text{3-SUM}}$, we must verify that for all $i, j, k \in \{1, \dots, n\}$, $a_i + a_j + a_k \neq 0$. The witness is a triple (p, t, S) where

1. p is a prime which is at most the $n^{1.5\text{TH}}$ prime number,
2. $S = \{(i, j, k) : a_i + a_j + a_j \equiv 0 \pmod{p}\}$, and
3. $t \in \mathbb{Z} \cap [0, 3cn^{1.5} \log n]$ such that $t = |S|$.

We wish to show that such a witness exists. Let R be the set of all pairs $((i, j, k), p)$ where p is a prime which is at most the $n^{1.5\text{TH}}$ prime number and $a_i + a_j + a_k \equiv 0 \pmod{p}$. Then, $|R| \leq n^3 \log(3n^c) \leq 3cn^3 \log n$. There are n^3 triples (i, j, k), and $3n^c$ gives an upper bound on the magnitude of $a_i + a_j + a_k$, and the logarithm is an upper bound on the number of prime factors of this quantity. Now, by an averaging argument, there must exist p_0 which is at most the $n^{1.5\text{TH}}$ prime number such that the number of pairs $((i, j, k), p_0)$ is at most $\frac{|R|}{n^{1.5}} \leq 3cn^{1.5} \log n$. We now show verification given input A. First, check that for all $r \in \{1, \dots, t\}$,

$$a_{ir} + a_{jr} + a_{kr} \equiv 0 \pmod{p}, \quad a_{ir} + a_{jr} + a_{kr} \neq 0.$$

Then compute the number of triples summing to 0 modulo p compare this with t. To compute all triple sums, map

$$A = [a_1, \dots, a_n] \mapsto q_A(x) = \sum_i x^{a_i \bmod p}$$

and use the fast Fourier transform to compute $(q_A(x))^3$. If b_j is the coefficient of x^j in $(q_A(x))^3$, b_3 is the number of triples in $\{a_1 \bmod p, \dots, a_n \bmod p\}$ summing to j. Now, since $\deg(q_A(x))^3 \leq 3(p-1) < 3p$, we can just check

$$b_0 + b_p + b_{2p} = t,$$

and accept if we have equality, and reject otherwise.

For complexity, the first check for $r \in \{1, \dots, t\}$ takes time $O(t) = O\left(n^{\frac{3}{2}} \log n\right)$. The fast Fourier transform computation takes time $O(p \log p) = \tilde{O}\left(n^{\frac{3}{2}}\right)$ where $p = O\left(n^{\frac{3}{2}} \log n\right)$. In short, our procedure takes $O\left(n^{\frac{3}{2}} \log n\right)$, as desired. □

2
Matrix Multiplication

2.1 Lecture 6: January 30, 2025

2.1.1 Introduction to Matrix Multiplication

Consider the following problem.

Computational Problem 6 (Matrix Multiplication)**.**

- *Given matrices* $A, B \in \mathbb{R}^{n \times n}$,
- *Find:* $C = AB$, *where* $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Naively, we immediately have an $O(n^3)$ algorithm. We compute n^2 inner products, each taking time $O(n)$.

Question. *Can we do better?*

Answer. *Yes, we can!*

Definition 2.1.1: Matrix Multiplication Exponent

The Matrix Multiplication exponent ω is

$$\omega = \inf \left\{ \omega' > 0 : \forall \epsilon, \exists \text{ an algorithm to solve Matrix Multiplication in } O\left(n^{\omega'+\epsilon}\right) \text{ time} \right\}.$$

Remark. *We know* $\omega \in [2, 3]$. *It is a conjecture that* $\omega = 2$. *The best known upper bound is* $\omega < 2.371539$ *due to [ADW+24].*

We show how to improve ω by exploring Strassen's algorithm, using the divide and conquer paradigm. The key idea is to write A and B in 2×2 block form, and compute the product blockwise. That is,

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

2.2 Lecture 7: February 4, 2025

2.2.1 Strassen's Algorithm & Tensor Rank Techniques

Naively, a divide and conquer approach for matrix multiplication, following the outline of the previous section, would give the recurrence $8T\left(\frac{n}{2}\right) + O(n^2)$ since we'd have 8 multiplications, each taking $O(n^2)$ time. But, $T(n) = O(n^{\log_2 8}) = O(n^3)$. It turns out that if we're clever, we can reduce 8 multiplications to 7. In this section, we explore this approach, [Str69]. Those familiar with Karatsuba's algorithm for multiplication should notice parallels.

Algorithm 2.2.1 Strassen

procedure $\textsc{Strassen}(A, B)$
  $M_1 \leftarrow \textsc{Strassen}(A_{11} + A_{22}, B_{11} + B_{22})$
  $M_2 \leftarrow \textsc{Strassen}(A_{21} + A_{22}, B_{11})$
  $M_3 \leftarrow \textsc{Strassen}(A_{11}, B_{12} - B_{22})$
  $M_4 \leftarrow \textsc{Strassen}(A_{22}, B_{21} - B_{11})$
  $M_5 \leftarrow \textsc{Strassen}(A_{11} + A_{12}, B_{22})$
  $M_6 \leftarrow \textsc{Strassen}(A_{21} - A_{11}, B_{11} + B_{12})$
  $M_7 \leftarrow \textsc{Strassen}(A_{12} - A_{22}, B_{21} + B_{22})$
  return $\begin{bmatrix} M_1 + M_4 - M_3 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{bmatrix}$

To analyze runtime for Strassen's algorithm, we have that $7T\left(\frac{n}{2}\right) + O(n^2) = O\left(n^{\log_2 7}\right) = O\left(n^{2.808}\right)$. Now, we seek to generalize Strassen's method. We follow [Alm21].

Definition 2.2.1: Order 3 Tensors

Let $\mathbb{F}$ be a field. An order 3 tensor $T \in \mathbb{F}^{a \times b \times c}$ over $\mathbb{F}$ is a bilinear map

$$T : \mathbb{F}^a \times \mathbb{F}^b \to \mathbb{F}^c.$$

Remark. *Equivalently, we can think of order* 3 *tensors as*

1. 3*-dimensional array structures over* $\mathbb{F}$*,*
2. *a trilinear map* $T : \mathbb{F}^a \times \mathbb{F}^b \times \mathbb{F}^c \to \mathbb{F}$*, or*
3. *a trilinear polynomial*

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} T[i, j, k] x_i y_j z_k.$$

We can think of $n \times n$ square matrix multiplication as a bilinear map $\mathbb{F}^{n^2} \times \mathbb{F}^{n^2} \to \mathbb{F}^{n^2}$.

Definition 2.2.2: Matrix Multiplication Tensor

Let $\mathbb{F}$ be a field. The $a \times b \times c$ matrix multiplication tensor $\langle a, b, c \rangle$ is given by

$$\sum_{i=1}^{a}\sum_{k=1}^{b}\sum_{j=1}^{c} x_{i,k}y_{k,j}z_{i,j} = \sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j}\left(\sum_{k=1}^{b} x_{i,k}y_{k,j}\right),$$

corresponding to a bilinear map $\mathbb{F}^{ac} \times \mathbb{F}^{cb} \to \mathbb{F}^{ab}$.

Remark. *If we substitute in the entries of x and y from the matrices, the coefficient of $z_{i,j}$ corresponds to the i, j entry in the matrix product.*

Definition 2.2.3: Tensor Rank

Let T be a tensor over finite sets of variables X, Y, and Z. A tensor over $\mathbb{F}$ has rank 1 if there exist coefficients $\alpha_x, \beta_y, \gamma_z \in \mathbb{F}$ with

$$T = \left(\sum_{x \in X} \alpha_x X\right)\left(\sum_{y \in Y} \beta_y y\right)\left(\sum_{z \in Z} \gamma_z z\right).$$

Then, the rank, $\text{rank}(T)$, of a general tensor T is the minimum of rank 1 tensors that sum up to T.

We now detail an intimate connection between tensor rank and fast matrix multiplication.

Theorem 2.2.1: Upper Bounding Matrix Multiplication Tensor Rank → Fast Algorithms

If $\text{rank}(\langle q, q, q \rangle) \leq r$, then there exists a $O\left(n^{\log_q r}\right)$-time matrix multiplication algorithm.

Proof. By the definition of tensor rank, we have that

$$\begin{aligned}\langle q, q, q \rangle &= \sum_{i=1}^{q}\sum_{j=1}^{q}\sum_{k=1}^{q} x_{i,k}y_{k,j}z_{i,j} \\ &= \sum_{\ell=1}^{r}\left(\sum_{i=1}^{q}\sum_{k=1}^{q}\alpha_{i,k}^{(\ell)}x_{i,k}\right)\left(\sum_{k=1}^{q}\sum_{j=1}^{q}\beta_{k,j}^{(\ell)}y_{k,j}\right)\left(\sum_{i=1}^{q}\sum_{j=1}^{q}\gamma_{i,j}^{(\ell)}y_{i,j}\right)\end{aligned}$$

for coefficients $\alpha_{i,k}^{(\ell)}, \beta_{k,j}^{(\ell)}, \gamma_{i,j}^{(\ell)} \in \mathbb{F}$. Assume $q|n$ without loss of generality, and form $q \times q$ blocks of $X, Y \in \mathbb{F}^{n \times n}$. Then, each block $X_{i,k}, Y_{k,j} \in \mathbb{F}^{\frac{n}{q} \times \frac{n}{q}}$. Then, substitute in the blocks into the right-hand side of the above equation, and compute the coefficient of each $z_{i,j}$ to compute $Z_{i,j}$. Recurse to compute products of the $\frac{n}{q} \times \frac{n}{q}$ matrices. Each term corresponding to a rank 1 tensor indexed by ℓ takes $3q^2$ additions and one multiplication of $\frac{n}{q} \times \frac{n}{q}$ matrices. This gives us the recurrence

$$T(n) = rT\left(\frac{n}{q}\right) + O(n^2) = O\left(n^{\log_q r}\right),$$

as desired. □

Applying Theorem 2.2.1 from above, we have the following results.

Corollary 2.2.1: [Str69]'s Bound

Note $\text{rank}(\langle 2,2,2\rangle) \leq 7$, so we have an $O\left(n^{\log_2 7}\right)$-time matrix multiplication algorithm.

Corollary 2.2.2: [Pan78]'s Bound

Note $\text{rank}(\langle 70,70,70\rangle) \leq 143640$, so we have an $O\left(n^{\log_{70} 143640}\right)$-time matrix multiplication algorithm.

Definition 2.2.4: Rectangular Matrix Multiplication Exponent

The RECTANGULAR MATRIX MULTIPLICATION exponent ω is

$$\omega(a,b,c) = \inf\left\{\omega' > 0 : \forall \epsilon > 0, \exists \text{ an algorithm for multiplying } A \in \mathbb{R}^{n^a \times n^b} \text{ and } B \in \mathbb{R}^{n^b \times n^c} \text{ in } O\left(n^{\omega'+\epsilon} \text{ time}\right)\right\}.$$

Definition 2.2.5: Dual Matrix Multiplication Exponent

The dual MATRIX MULTIPLICATION exponent α is

$$\alpha = \sup\{\alpha' > 0 : \omega(1,1,\alpha') = 2\}.$$

Remark. *By [WXXZ23], $\alpha \geq 0.321334$. Also, $\alpha = 1$ if and only if $\omega = 2$.*

2.3 Lecture 8: February 6, 2025

2.3.1 Matrix Multiplication Verification

Consider the following problem.

Computational Problem 7 (Matrix Multiplication Verification)**.**

- *Given matrices* $A, B, C \in \mathbb{R}^{n \times n}$,
- *Decide:* $AB = C$*?*

Naively, we immediately have an $O(n^\omega)$ algorithm by an easy reduction to Matrix Multiplication.

Question. *Can we do better?*

Answer. *Yes, we can!*

We present a natural randomized algorithm, due to [Fre79].

Algorithm 2.3.2 Freivalds' Matrix Multiplication Verification

```
1: procedure FREIVALDS_MMV(A, B, C)
2:     Sample x ~ {0,1}^n uniformly at random.
3:     return ABx = Cx                          ▷ Compute ABx as A(Bx)
```

For correctness, let $D = AB - C$ and observe that $Dx = 0$ if and only if $ABx = Cx$. If $D = 0$, then $Dx = 0$ with probability 1. If $D \neq 0$, D has a nonzero row d. Then,

$$\Pr_x[Dx = 0] \leq \Pr_x[\langle d, x \rangle = 0] \leq \frac{1}{2}.$$

Algorithm 2.3.2 runs in $O(n^2)$ time. Note that Freivalds' procedure uses n random bits. Naturally, we ask how to partially derandomize while still retaining $o(n^\omega)$ time.

2.3.2 Sparse Matrix Multiplication

Another natural question is how we can do better in the case of sparse matrices. Here, we follow [YZ05].

Computational Problem 8 (Sparse Matrix Multiplication)**.**

- *Given matrices* $A, B \in \mathbb{R}^{n \times n}$*, each with at most* m *nonzero entries,*
- *Find* $C = AB$*.*

We have two critical observations that we rely on. First, let $A, B \in \mathbb{R}^{n \times n}$ where A has columns a_i and B has rows b_i. Then, $AB = \sum_{i=1}^n a_i b_i$. Also, for any permutation π, $\sum_{i=1}^n a_i b_i = \sum_{i=1}^n a_{\pi(i)} b_{\pi(i)}$.

Appendices

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