CSCI6454: Advanced Algorithms

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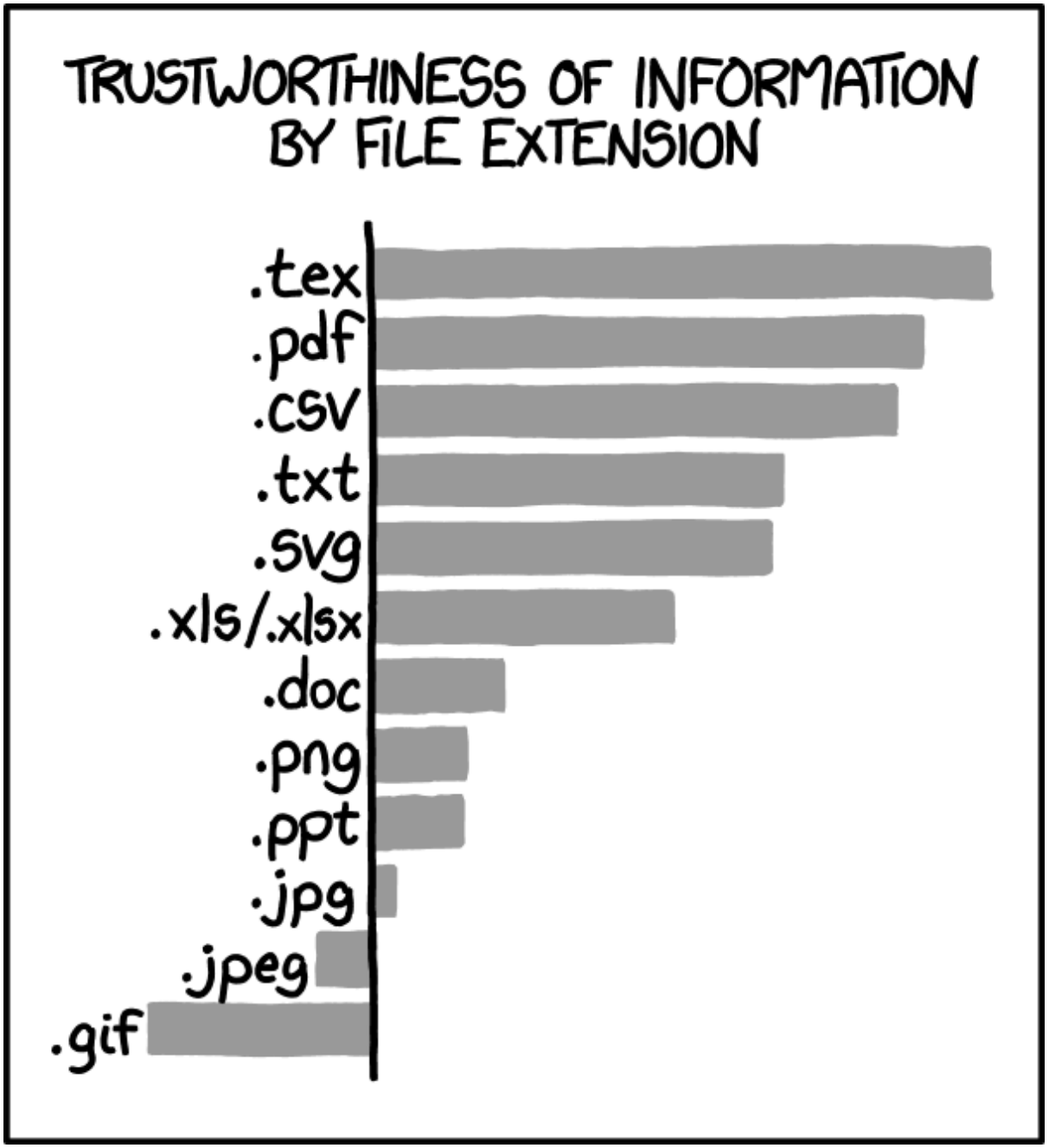
UNIVERSITY OF COLORADO BOULDER

EDITION 1



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Preface

To the interested reader,

This document is a compilation of lecture notes scribed during the Spring 2025 semester for CSCI6454: Advanced Algorithms at the University of Colorado Boulder. The content of these course notes is from lecture and several resources cited herein. I only take credit for the scribing of the content and lectures into the notes. This course was taught by Huck Bennett, Ph. D. and the notes would not be possible without Huck's great lectures.

I have taken liberties, deviating from the lecture notes, when it comes to notation and ordering of content between chapters.

This course is designed as a graduate third course in algorithms. Hence, we assume as prerequisites the topics covered in undergraduate algorithms and introductory graduate courses. At CU Boulder, the prerequisites correspond to CSCI 3104 and CSCI 5454.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. I take full responsibility for all remaining errors. If an error needs to be resolved, please contact me at adithya@colorado.edu.

Best Regards,
Adithya Bhaskara

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1 Introduction to Fine-Grained Complexity

1.1 Lecture 1: January 14, 2025

1.1.1 Introduction to Fine-Grained Complexity

We hope to precisely understand the complexity of several algorithmic problems beyond the coarse information given by standard complexity classes like **P** and **NP**. A central theme is proving lower bounds for the complexity of algorithms; this is a difficult task in general. Here, reductions are an essential tool.

The high-level idea behind fine-grained complexity is to explain hardness, or the lack of algorithmic progress, on fundamental computational problems by giving reductions from well-studied problems. Recall the **P** v. **NP** problem. Loosely, problems in **P** are "easy," to solve, and problems in **NP** are "easy" to verify positive answers to. The hardest problems in **NP**, the **NP**-complete problems, do not admit polynomial-time algorithms unless $\mathbf{P} = \mathbf{NP}$. To show that a problem is **NP**-complete, we show that it is in **NP**, and exhibit a polynomial-time reduction from a known **NP**-complete problem. We assume familiarity with the technical definitions and implications, and defer the reader to [HMU01, Sip12] for details.

The theory behind **NP**-completeness is robust and quite nice. But, we'd like finer results. Knowing whether a problem is in **P** or not doesn't really narrow down how efficiently we can solve it in practice. Even quadratic-time algorithms may be inefficient in production.

These ideas motivate fine-grained complexity. Here, we start by taking a well-studied problem, say L_1, and making a precise conjecture about the running time of optimal algorithms for L_1. Then, we give an efficient reduction from L_1 to a problem L_2, for which we are showing hardness. We start by providing some hypotheses corresponding to certain problems, and later, we'll show fairly precise running time bounds for problems in **P**.

Example 1. *Under certain assumptions, there is no $O(n^{2-\epsilon})$-time algorithm for the* Edit Distance *problem. This matches the $O(n^2)$ dynamic programming algorithm.*

Computational Problem 1 (k-SAT)**.**

- *Given a CNF formula φ with k literals in each clause,*
- *Decide: Does there exist a satisfying assignment?*

Hypothesis 1.1.1: [IP01] Exponential Time Hypothesis (ETH)

The 3-SAT problem takes $2^{\Omega(n)}$ time.

Remark. *Note that* **P** $\neq$ **NP** *asserts that* 3-SAT *has no polynomial-time algorithm, whereas ETH asserts that* 3-SAT *has no subexponential-time algorithm. Hence, ETH implies* **P** $\neq$ **NP**.

Hypothesis 1.1.2: [IP01, IPZ01] Strong Exponential Time Hypothesis (SETH)

For every $\epsilon > 0$, there exists $k \in \mathbb{Z}^+$ such that there is no $O\left(2^{(1-\epsilon)n}\right) = O((2-\epsilon)^n)$-time algorithm for3-SAT

Remark. *SETH, at a high level, claims that 2^n-time is essentially optimal for k-SAT with large k.*

Remark. *In [IP01, IPZ01], ETH and SETH are instead stated as follows. Let*

$$s_k = \inf\left\{\delta > 0 : \exists \text{ an algorithm to solve } k\text{-SAT in } O^*\left(2^{\delta n}\right) \text{ time}\right\}.$$

ETH is the statement that for $k \geq 3$, $s_k > 0$. Assuming ETH, s_k is increasing infinitely often, [IPZ01]. Let $s_\infty = \lim_{k\to\infty} s_k$. Then, SETH is the statement that $s_\infty = 1$.

Computational Problem 2 (k-SUM)**.**

- *Given arrays $A_1, \ldots, A_k$ each with n integers in $[-n^c, n^c]$,*
- *Decide: Does there exist $a_1 \in A_1, \ldots, a_k \in A_k$ such that $a_1 + \cdots + a_k = 0$?*

Some variants include assuming that $A_1 = \cdots = A_k$, or wanting to find $a_1 \in A_1, \ldots, a_k \in A_k$ where $a_1 + \cdots + a_{k-1} = a_k$.

A naive algorithm for k-SUM is to try all possible choices a_i and verify the sum. This gives rise to an $O(n^k)$-time algorithm.

Question. *Can we do better?*

Answer. *Yes, we can!*

For now, let $k = 3$. Consider the following, slightly better, algorithm.

Algorithm 1.1.1 Better 3-SUM

```
1: procedure 3-SUM-BETTER(A_1, A_2, A_3)
2:     S ← {a_1 + a_2 : a_1 ∈ A, a_2 ∈ A_2}          ▷ O(n^2)
3:     Sort A_3.                                      ▷ O(n log n)
4:     for s ∈ S do                                   ▷ O(n^2 log n)
5:         if −s ∈ A_3 then return TRUE
6:     return FALSE
```

Algorithm 1.1.1 takes $O(n^2 \log n)$ time, using binary search on the sorted array A_3.

Question. *Can we do better?*

Answer. *Yes, we can!*

Our first example of a helpful reduction is as follows. Given a 3-SUM instance with arrays of length n, we can construct n 2-SUM instances. This procedure gives us an $O(n^2)$ algorithm for 3-SUM, which we describe in Algorithm 1.1.2.

Algorithm 1.1.2 Best 3-SUM

```
1: procedure 3-SUM-BETTER(A_1, A_2, A_3)
2:     Sort A_1 and A_2.
3:     for a_3 ∈ A_3 do
4:         A_2 + a_3 ← [a_{2,1} + a_3, ..., a_{2,n} + a_3]
5:         if 2-SUM(A_1, A_2 + a_3) then return TRUE
6:     return FALSE
```

Sorting takes $O(n \log n)$ time. Then, we can n calls of 2-SUM that each take $O(n)$ time since the arrays are sorted. Therefore, Algorithm 1.1.2 runs in $O(n^2)$ time. We have the following hypothesis.

Hypothesis 1.1.3: No $O(n^{2-\epsilon})$-Time Algorithm for 3-Sum

For every $\epsilon > 0$, there is no $O\left(n^{2-\epsilon}\right)$-time algorithm for 3-SUM.

1.2 Lecture 2: January 16, 2025

1.2.1 Fine-Grained Complexity II

Now, we introduce a new problem.

Computational Problem 3 (All Pairs Shortest Path (APSP))**.**

- *Given a weighted graph $G = (V, E, w)$ on n vertices,*
- *Find: The lengths of the shortest paths $d(v_i, v_j)$ between every $v_i, v_j \in V$.*

One way to solve APSP is to run Dijkstra's algorithm on each vertex. Recall that Dijkstra's algorithm finds a single-source shortest path tree. We could also similarly use the Floyd-Warshall dynamic programming algorithm which we introduce below.

Let $D_k[i, j]$ store the length of the shortest path from v_i to v_j, with the restriction that the path only passes through $v_1, \ldots, v_k$.

Example 2. *To illustrate this notion, consider the below graph.*

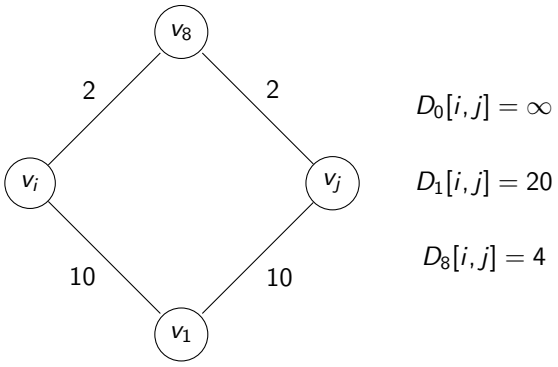

Figure 1.1: Example of $D_k[i, j]$

Illustrated in Figure 1.2, the idea behind Floyd-Warshall is to build D_k from D_{k-1}, noting that either

1. the shortest path from v_i to v_j only using $v_1, \dots, v_k$ uses v_k, or
2. it does not.

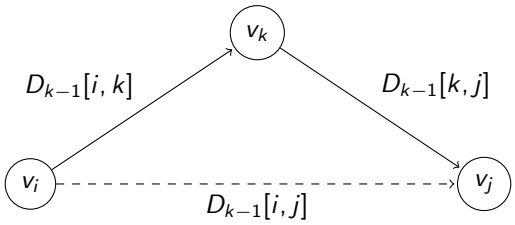

Figure 1.2: Update of $D_k[i,j]$

From this basic idea, we get a simple algorithm.

Algorithm 1.2.3 Floyd-Warshall

Require: $G = (V, E, w)$.

procedure FLOYD_WARSHALL(E)

$$D_0 \leftarrow \begin{cases} 0 & i = j \\ w_{i,j} & i \neq j, (v_i, v_j) \in E \\ \infty & (v_i, v_j) \notin E \end{cases}$$

for $k \in \{1, \dots, n\}$ **do**
  for $i \in \{1, \dots, n\}$ **do**
    for $j \in \{1, \dots, n\}$ **do**
      $D_k[i,j] = \min\{D_{k-1}[i,k] + D_{k-1}[k,j], D_{k-1}[i,j]\}$
return D_n ▷ $D_n[i,j] = d(v_i, v_j)$

It turns out that as far as we know, Algorithm 1.2.3 with complexity $O(n^3)$ is the best we can do.

Hypothesis 1.2.1: No $O(n^{3-\epsilon})$-Time Algorithm for APSP

For every $\epsilon > 0$, there exists no $O(n^{3-\epsilon})$-time algorithm for APSP.

We now turn to a new problem.

Computational Problem 4 (Orthogonal Vectors (OV))**.**

- *Given two sets* $V = \{v_1, \dots, v_n\} \subseteq \{0,1\}^d$ *and* $W = \{w_1, \dots, w_n\} \subseteq \{0,1\}^d$*,*
- *Decide: Does there exist* $v_i \in W$ *and* $w_j \in W$ *such that* $\langle v_i, w_j \rangle = \sum_{k=1}^{d} v_i[k] w_j[k] = 0$*?*

The naive algorithm is to compare every pair, compute the inner products, and compare to 0. We have n^2 inner products to compute, and each takes d time, so the naive algorithm takes $O(dn^2)$ time. Think of $d = \log^k(n)$ for some k. In this case, $O(dn^2) = O(n^2 \log^k(n)) = \tilde{O}(n^2)$.

Hypothesis 1.2.2: No $O(n^{2-\epsilon})$-Time Algorithm for OV

For every $\epsilon > 0$, there exists no $O(n^{2-\epsilon})$-time algorithm for OV with $d = \text{polylog}(n)$.

So far, we have 5 hypotheses; what are the relations herein? We'll attack this question after a brief interlude on reductions.

We write $L_1 \leq L_2$ if problem L_1 reduces to problem L_2; that is, L_1 is no harder than L_2; and if L_1 is hard, then L_2 is hard. Equivalently, if L_2 is easy, then L_1 is easy. Consider the following definition, making precise the notion of a "fine-grained reduction."

Definition 1.2.1: [VW19] Fine-Grained Reductions

Suppose L_1 and L_2 are computational problems. Let $\ell_1 = \ell_1(n)$ and $\ell_2 = \ell_2(n)$ be associated conjectured optimal runtime bounds. Then, L_1 (ℓ_1, ℓ_2)-reduces to L_2 if for every $\epsilon > 0$, there exists $\delta > 0$ and an algorithm R for L_1 running in time $O(\ell_1(n)^{1-\delta})$ and making q calls to an oracle for L_2 on inputs of size $n_1, \dots, n_q$ where

$$\sum_{i=1}^{q} \ell_2(n_i)^{1-\epsilon} \leq O(\ell_1(n)^{1-\delta}).$$

We write $L_1 \leq_{\ell_1,\ell_2} L_2$. If both $L_1 \leq_{\ell_1,\ell_2} L_2$ and $L_2 \leq_{\ell_1,\ell_2} L_1$, we say $L_1 =_{\ell_1,\ell_2} L_2$.

Theorem 1.2.1: [Wil04] SETH $\implies$ OV

Hypothesis 1.1.2 implies Hypothesis 1.2.2.

Proof. Using Definition 1.2.1, we can restate this theorem as follows: for every $k \in \mathbb{Z}^+$, k-SAT $(2^n, n^2)$-reduces to OV with $d = \text{polylog}(n)$. We exhibit such a reduction, which will be of exponential-time.

Let $\varphi(x_1, \dots, x_n) = \bigwedge_{j=1}^{m} C_j$ be our k-SAT instance, with each C_j a k-clause. Partition $X = \{x_i\}_{i=1}^{n}$ into $X_1 = \{x_1, \dots, x_{\frac{n}{2}}\}$ and $X_2 = \{x_{\frac{n}{2}+1}, \dots, x_n\}$. Consider the $N = 2^{\frac{n}{2}}$ assignments to each of X_1 and X_2 independently. Then, let $V_1, V_2 \subseteq \{0,1\}^m$. For an assignment a_1 to literals in X_1, $v_{a_1} \in V_1$ is defined by

$$v_{a_1}[j] = \begin{cases} 0 & a_1 \text{ satisfies } C_j \\ 1 & \text{otherwise} \end{cases}.$$

Define $v_{a_2} \in V_2$ similarly corresponding to a_2 with literals in X_2. We claim that $\langle v_{a_i}, v_{a_2} \rangle = 0$ if and only if $[a_1, a_2]$ is a satisfying assignment. Note that $m = O(n^k) = O((2\log_2 N)^k) = O(\log^k N)$, where $N = 2^{\frac{n}{2}}$. A $O\left(N^{2-\epsilon}\right)$-time algorithm for OV would then imply an $O\left(N^{2-\epsilon}\right) = O\left(\left(2^{\frac{n}{2}}\right)^{2-\epsilon}\right) = O\left(2^{\left(1-\frac{\epsilon}{2}\right)n}\right)$-time algorithm for k-SAT. ■

Example 3. *To illustrate the reduction in the proof of Theorem 1.2.1, consider k-SAT with $n = 6$, $m = 3$, and $k = 3$ and formula*

$$\varphi(x_1, \dots, x_6) = (x_1 \vee \neg x_2 \vee x_6) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_4 \vee \neg x_3 \vee x_6).$$

For $a_1 = [0,1,0]$, we have $v_{a_1} = [1,0,0] \in V_1$ and for $a_2 = [0,1,1]$, we have $v_{a_2} = [0,1,0] \in V_2$. Indeed $\varphi(a_1, a_2) = 1$ and $\langle v_{a_1}, v_{a_2} \rangle = 0$.

1.3 Lecture 3: January 21, 2025

1.3.1 Graph Diameter

Consider the following problem.

Computational Problem 5 (Graph Diameter)**.**

- *Given a graph* $G = (V, E, w)$*,*
- *Find:* $\max_{u,v \in V} d(u, v)$ *where* $d(u, v)$ *is the shortest path distance between* $u, v \in V$*.*

An easy algorithm is to run Floyd-Warshall and output the largest distance found. This takes $O(n^3)$ time.

Theorem 1.3.1: [RVW13] OV Reduction to GraphDiameter

The OV problem (n^2, n^2)-reduces to GraphDiameter on undirected, unweighted graphs with $O(n)$ vertices and $\tilde{O}(n)$ edges.

Proof. Let $V = [v'_1, \ldots, v'_n] \subseteq \{0,1\}^d$ and $W = [w'_1, \ldots, w'_n] \subseteq \{0,1\}^d$ ■

Remark. *Theorem 1.3.1 immediately shows that assuming Hypothesis 1.2.2 or SETH, there exists no* $O\left(n^{2-\epsilon}\right)$*-time algorithm to* $\left(\frac{3}{2} - \delta\right)$*-approximate* GraphDiameter *for all* $\epsilon, \delta > 0$*.*

[AB: Finish this! I was very tired during lecture...]

1.4 Lecture 4: January 23, 2025

1.4.1 Smarter k-SAT, Part I

In this section, we detail faster algorithms for solving 3-SAT. But first, we recall some standard facts; 2-SAT is in **P** and k-SAT is **NP**-complete for $k \geq 3$. The brute force algorithm takes $O(2^n)$ time. Now, we show how to convert φ, a k-SAT formula with n variables and m clauses, into a 3-SAT formula. Write

$$\begin{aligned}\varphi &= \bigwedge_{i=1}^{m} (x_i^1 \vee \cdots \vee x_i^k) \\ &\iff \bigwedge_{i=1}^{m} \left[(x_i^1 \vee x_i^2 \vee y_i^1) \wedge \left(\overline{y_i^1} \vee x_i^3 \vee y_i^2\right) \wedge \cdots \wedge \left(\overline{y_i^{k-2}} \vee x_{k-2} \vee y_i^{k-3}\right) \wedge \left(\overline{y_i^{k-3}} \vee x_{k-1} \vee x_k\right)\right]\end{aligned}$$

Our new 3-SAT formula has $n + m(k-3)$ literals and $m(k-3)$ clauses, which is apparent from the line above.

We also present the useful sparsification lemma. Morally, we can reduce solving a k-SAT instance with n variables to solving several k-SAT instances each with n variables $O(n)$ clauses. Informally, "every k-SAT formula has $O(n)$ clauses."

Theorem 1.4.1: The Sparsification Lemma

Let $\epsilon > 0$ and $k \geq 3$. Then, there is a $2^{\epsilon n}\,\mathrm{poly}(n)$-time algorithm that takes a k-SAT formula φ on n variables as input and produces $2^{\epsilon n}$ k-SAT formulas $\varphi_1, \ldots, \varphi_{2^{\epsilon n}}$, each of which has n variables and $n\left(\frac{k}{\epsilon}\right)^{O(k)} = O(n)$ clauses. Furthermore, φ is satisfiable if and only if $\bigvee_i \varphi_i$ is satisfied.

With the help of Theorem 1.4.1, we can show that the "strong" exponential time hypothesis indeed implies the exponential time hypothesis. The proof follows by contraposition, first applying the sparsification lemma to the k-SAT formula, and then converting each φ_i into a 3-SAT formula. Indeed, SETH implies ETH.

Now, we tackle 3-SAT itself. A very simple, but clever optimization gives us a $O(1.913^n)$-time algorithm. Let $\varphi; \mathbf{a}$ denote a partial assignment to φ according to $\mathbf{a}$.

Algorithm 1.4.4 3-SAT Easy Recursive Algorithm; $O(1.913^n)$

```
1: procedure 3-SAT-Easy-Recursive(φ)
2:     if φ only has 1 and 2-clauses then return 2-SAT(φ)        ▷ in poly(n) time
3:     C ← ℓ1 ∨ ℓ2 ∨ ℓ3
4:     for each of 7 partial satisfying partial assignments a_C to C do
5:         return 3-SAT-Easy-Recursive(φ; a_C)
6:     return FALSE
```

Algorithm 1.4.4 takes time

$$\begin{aligned}T(n) &= 7T(n-3) + \mathrm{poly}(n) \\ &= 7^{\frac{n}{3}}\,\mathrm{poly}(n) = 2^{\frac{\log_2 7}{3}}\,\mathrm{poly}(n) = O(1.913),\end{aligned}$$

as desired.

Question. *Can we do better?*

Answer. *Yes, we can!*

To improve, we take the approach of [MS85]. Consider a clause $C = x_1 \vee x_2 \vee x_2$. Note that

$$C = x_1 \vee x_2 \vee x_2 \iff x_1 \vee (\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3).$$

We can think of the above formula as either setting the truth of one, two, or three literals. This gives rise to a modification of Algorithm 1.4.4, making 3 recursive calls. This gives us the recurrence

$$T(n) = T(n-1) + T(n-2) + T(n-3) + \text{poly}(n) = O^*(\alpha^n)$$

where α is the unique real root to $\alpha^n - \alpha^{n-1} - \alpha^{n-2} - \alpha^{n-3} = 0$, or equivalently, $\alpha^3 - \alpha^2 - \alpha - 1$. Here, $\alpha \leq 1.84$. So, $T(n) = (1.84^n)$.

Question. *Can we do better?*

Answer. *Yes, we can!*

Local search algorithms for 3-SAT turn out to be very powerful. We proceed, following [Sch99, Spi07]. Here, we harness the power of randomness.

Algorithm 1.4.5 Schöning SAT

```
1: procedure SCHÖNING_SAT_FRAMEWORK(φ, β_n)
2:     Guess a uniformly random assignment to a ∈ {0,1}^n to φ.
3:     for j ∈ {1, ..., β_n} do
4:         if φ is satisfied by a then return a
5:         else
6:             Choose an unsatisfied clause C in φ, and choose a variable x_i in C uniformly at random.
7:             Flip the assignment a_i to x_i in a.
8:     return a
```

Example 4. *Say $C = (x_1, x_2, x_3)$ with $a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. Since C is unsatisfied, we can flip one of the literals uniformly at random to satisfy C.*

Importantly, how do we choose β_n? Suppose that the initial assignment $\mathbf{a}$ agrees with the satisfying assignment $\mathbf{a}^*$ on a "decent" fraction of coordinates with "decent" probability. The steps in the loop can be thought of as doing a random walk on $\{0,1\}^n$. At each step, $\mathbf{a}$ gets closer to $\mathbf{a}^*$ with probability at least $\frac{1}{3}$ since at least one variable in each unsatisfied clause differs between $\mathbf{a}$ and $\mathbf{a}^*$.

For "Easy"-Schöning, take $\beta_n = n$. Say u is the number of coordinates in which $\mathbf{a}$ and $\mathbf{a}^*$ are different. Then, the probability of Algorithm 1.4.5 finding a satisfying assignment is given by

$$\begin{aligned}
\Pr[\text{satisfying assignment found}] &\geq \sum_{i=0}^{n} \Pr[u = i] \Pr[\text{first } i \text{ choices match}] \\
&= \sum_{i=0}^{n} \frac{1}{2^n} \binom{n}{i} \Pr[\text{first } i \text{ choices match}] \\
&\geq \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{3^i} \\
&= \frac{1}{2^n} \left(1 + \frac{1}{3}\right)^n \\
&= \left(\frac{2}{3}\right)^n .
\end{aligned}$$

Intuitively, we sample a random assignment $\mathbf{a}$ which differs from $\mathbf{a}^*$ in i coordinates. We want to keep flipping assigments so that $\mathbf{a} = \mathbf{a}^*$. The above procedure succeeds with probability at least $\left(\frac{2}{3}\right)^n$. So, we can repeat Algorithm 1.4.5 roughly $\left(\frac{3}{2}\right)^n$ times with $\beta_n = n$ to find a solution with high probability.

1.5 Lecture 5: January 28, 2025

1.5.1 Smarter k-SAT, Part II

To get an even better runtime, we can refine the procedure described in the previous section. What if we apply Algorithm 1.4.5 with $\beta_n = 3n$? Let's analyze the probability that

1. the initial assignment $\mathbf{a}$ differs with $\mathbf{a}^*$ on exactly i coordinates, and
2. the first $3i$ steps of the walk include at least $2i$ choices satisfying all clauses.

For (1), the analysis is very similar to "Easy"-Schöning. First, recall that for $i \geq 2$,

$$\binom{3i}{i} \geq \frac{1}{\sqrt{5i}} \frac{3^{3i}}{2^{2i}}$$

by Stirling's approximation. Then,

$$\Pr[\text{at least } 2i \text{ of the first } 3i \text{ choices satisfy all clauses}] \geq \binom{3i}{i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \geq \frac{1}{\sqrt{5i}} \frac{3^{3i}}{2^{2i}} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i}.$$

Then,

$$\begin{aligned}
\Pr[\text{satisfying assignment found}] &\geq \sum_{i=0}^{n} \Pr[u = i] \Pr[\text{at least } 2i \text{ of the first } 3i \text{ choices satisfy all clauses}] \\
&\geq \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \binom{3i}{i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \\
&\geq \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{\sqrt{5i}} \frac{3^{3i}}{2^{2i}} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \\
&\geq \frac{1}{2^n \sqrt{5n}} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{\sqrt{5i}} \frac{3^{3i}}{2^{2i}} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{i} \\
&= \frac{1}{2^n \sqrt{5n}} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i} \\
&= \frac{1}{2^n \sqrt{5n}} \left(1 + \frac{1}{2}\right)^{n} \\
&= \frac{1}{\sqrt{5n}} \left(\frac{3}{4}\right)^{n}.
\end{aligned}$$

Proceeding similarly, we can run Algorithm 1.4.5 roughly $O^*\left(\frac{4}{3}\right)^n$ times with $\beta = 3n$ to find a solution with high probability.

Question. *Is this optimal?*

Answer. *No! But, we'll end here.*

We present a concluding remark.

Remark. *A similar approach to [Sch99] shows how* 2-SAT *can be solved in polynomial time; this approach is due to [Pap91] and these methods are known as "WalkSAT."*

1.6 Lecture 6: January 30, 2025

1.6.1 Nontrivial Nondeterministic Algorithms: SETH and Other Assumptions

Consider the following definition.

Definition 1.6.1: NTIME

A language $L \subseteq \{0,1\}^*$ is in NTIME($f(n)$) if there exists an algorithm M such that for $x \in \{0,1\}^*$ with $|x| = n$, M runs in time $O(f(n))$ and

1. if $x \in L$, then there exists $w \in \{0,1\}^*$ with $M(x,w)$ accepting, and
2. if $x \notin L$, for all $w \in \{0,1\}^*$, $M(x,w)$ is rejecting.

Remark. *The familiar complexity class* **NP** *is given by*

$$\mathbf{NP} = \bigcup_{k \in \mathbb{N}} \text{NTIME}(n^k).$$

Herein, we reference [CGI+16]. Recall that k-SAT $\in$ **NP**.

Question. *Can we say anything about verifying that a k-SAT formula is unsatisfiable?*

Hypothesis 1.6.1: NSETH

For every $\epsilon > 0$, there exists k with $\overline{k\text{-SAT}} \notin \text{NTIME}\left(2^{(1-\epsilon)n}\right)$.

Remark. *NSETH essentially hypothesizes that k-SAT has no nontrivial verification algorithm.*

The key takeways of [CGI+16] are that

1. assuming NSETH, $\overline{k\text{-SAT}}$ does not have nontrivial nondeterministic algorithms,
2. $\overline{\text{3-SUM}}$ and $\overline{\text{APSP}}$ do have nontrivial nondeterministic algorithms, and
3. assuming (1), (2) and NSETH, there is no fine-grained reduction from k-SAT to 3-SUM or APSP, as opposed to the k-SAT to OV reduction.

Consider the following theorem.

Theorem 1.6.1: [CGI+16] Nondeterministically Solving $\overline{\text{3-Sum}}$ in $\tilde{O}(n^{\frac{3}{2}})$ Time

There is a $\tilde{O}\left(n^{\frac{3}{2}}\right)$-time nondeterministic algorithm for $\overline{\text{3-SUM}}$.

Proof. Take the variant of 3-SUM where we are given a single list $A = [a_1, \ldots, a_n]$ where $a_i \in [-n^c, n^c]$, and we are to decide if there exist $a_i, a_j, a_k \in A$ with $a_i + a_j + a_k = 0$. So, for $\overline{\text{3-SUM}}$, we must verify that for all $i, j, k \in \{1, \ldots, n\}$, $a_i + a_j + a_k \neq 0$. The witness is a triple (p, t, S) where

1. p is a prime which is at most the $n^{1.5\text{TH}}$ prime number,
2. $S = \{(i, j, k) : a_i + a_j + a_j \equiv 0 \pmod{p}\}$, and
3. $t \in \mathbb{Z} \cap [0, 3cn^{1.5} \log n]$ such that $t = |S|$.

We wish to show that such a witness exists. Let R be the set of all pairs $((i, j, k), p)$ where p is a prime which is at most the $n^{1.5\text{TH}}$ prime number and $a_i + a_j + a_k \equiv 0 \pmod{p}$. Then, $|R| \leq n^3 \log(3n^c) \leq 3cn^3 \log n$. There are n^3 triples (i, j, k), and $3n^c$ gives an upper bound on the magnitude of $a_i + a_j + a_k$. The logarithm is an upper bound on the number of prime factors of this quantity. Now, by an averaging argument, there must exist p_0 which is at most the $n^{1.5\text{TH}}$ prime number such that the number of pairs $((i, j, k), p_0)$ is at most $\frac{|R|}{n^{1.5}} \leq 3cn^{1.5} \log n$. We now show verification given input A. First, check that for all $r \in \{1, \ldots, t\}$,

$$a_{ir} + a_{jr} + a_{kr} \equiv 0 \pmod{p}, \quad a_{ir} + a_{jr} + a_{kr} \neq 0.$$

Then compute the number of triples summing to 0 modulo p compare this with t. To compute all triple sums, map

$$A = [a_1, \ldots, a_n] \mapsto q_A(x) = \sum_i x^{a_i \bmod p}$$

and use the fast Fourier transform to compute $(q_A(x))^3$. If b_j is the coefficient of x^j in $(q_A(x))^3$, b_3 is the number of triples in $\{a_1 \bmod p, \ldots, a_n \bmod p\}$ summing to j. Now, since $\deg(q_A(x))^3 \leq 3(p-1) < 3p$, and hence is nonzero for $j \in \{0, \ldots, 3(p-1)\}$, we can just check

$$b_0 + b_p + b_{2p} = t,$$

and accept if we have equality, and reject otherwise.

For complexity, the first check for $r \in \{1, \ldots, t\}$ takes time $O(t) = O\left(n^{\frac{3}{2}} \log n\right)$. The fast Fourier transform computation takes time $O(p \log p) = \tilde{O}\left(n^{\frac{3}{2}}\right)$ where $p = O\left(n^{\frac{3}{2}} \log n\right)$. In short, our procedure takes $O\left(n^{\frac{3}{2}} \log n\right)$, as desired. ■

2
Matrix Multiplication

2.1 Lecture 6: January 30, 2025

2.1.1 Introduction to Matrix Multiplication

Consider the following problem.

Computational Problem 6 (Matrix Multiplication)**.**

- *Given matrices* $A, B \in \mathbb{R}^{n \times n}$,
- *Find:* $C = AB$, *where* $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Naively, we immediately have an $O(n^3)$ algorithm. We compute n^2 inner products, each taking time $O(n)$.

Question. *Can we do better?*

Answer. *Yes, we can!*

Definition 2.1.1: Matrix Multiplication Exponent

The Matrix Multiplication exponent ω is

$$\omega = \inf \left\{ \omega' > 0 : \forall \epsilon, \exists \text{ an algorithm to solve Matrix Multiplication in } O\left(n^{\omega'+\epsilon}\right) \text{ time} \right\}.$$

Remark. *We know* $\omega \in [2, 3]$. *It is a conjecture that* $\omega = 2$. *The best known upper bound is* $\omega < 2.371539$ *due to [ADW+24].*

We show how to improve ω by exploring Strassen's algorithm, using the divide and conquer paradigm. The key idea is to write A and B in 2×2 block form, and compute the product blockwise. That is,

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

2.2 Lecture 7: February 4, 2025

2.2.1 Strassen's Algorithm & Tensor Rank Techniques

Naively, a divide and conquer approach for matrix multiplication, following the outline of the previous section, would give the recurrence $8T\left(\frac{n}{2}\right) + O(n^2)$ since we'd have 8 multiplications, each taking $O(n^2)$ time. But, $T(n) = O(n^{\log_2 8}) = O(n^3)$. It turns out that if we're clever, we can reduce 8 multiplications to 7. In this section, we explore this approach, [Str69]. Those familiar with Karatsuba's algorithm for multiplication should notice parallels.

Algorithm 2.2.1 Strassen

procedure $\textsc{Strassen}(A, B)$
  $M_1 \leftarrow \textsc{Strassen}(A_{11} + A_{22}, B_{11} + B_{22})$
  $M_2 \leftarrow \textsc{Strassen}(A_{21} + A_{22}, B_{11})$
  $M_3 \leftarrow \textsc{Strassen}(A_{11}, B_{12} - B_{22})$
  $M_4 \leftarrow \textsc{Strassen}(A_{22}, B_{21} - B_{11})$
  $M_5 \leftarrow \textsc{Strassen}(A_{11} + A_{12}, B_{22})$
  $M_6 \leftarrow \textsc{Strassen}(A_{21} - A_{11}, B_{11} + B_{12})$
  $M_7 \leftarrow \textsc{Strassen}(A_{12} - A_{22}, B_{21} + B_{22})$
  return $\begin{bmatrix} M_1 + M_4 - M_3 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{bmatrix}$

To analyze runtime for Strassen's algorithm, we have that $7T\left(\frac{n}{2}\right) + O(n^2) = O\left(n^{\log_2 7}\right) = O\left(n^{2.808}\right)$. Now, we seek to generalize Strassen's method. We follow [Alm21].

Definition 2.2.1: Order 3 Tensors

Let $\mathbb{F}$ be a field. An order 3 tensor $T \in \mathbb{F}^{a \times b \times c}$ over $\mathbb{F}$ is a bilinear map

$$T : \mathbb{F}^a \times \mathbb{F}^b \to \mathbb{F}^c.$$

Remark. *Equivalently, we can think of order* 3 *tensors as*

1. 3*-dimensional array structures over* $\mathbb{F}$*,*
2. *a trilinear map* $T : \mathbb{F}^a \times \mathbb{F}^b \times \mathbb{F}^c \to \mathbb{F}$*, or*
3. *a trilinear polynomial*

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} T[i, j, k] x_i y_j z_k.$$

We can think of $n \times n$ square matrix multiplication as a bilinear map $\mathbb{F}^{n^2} \times \mathbb{F}^{n^2} \to \mathbb{F}^{n^2}$.

Definition 2.2.2: Matrix Multiplication Tensor

Let $\mathbb{F}$ be a field. The $a \times b \times c$ matrix multiplication tensor $\langle a, b, c \rangle$ is given by

$$\sum_{i=1}^{a}\sum_{k=1}^{b}\sum_{j=1}^{c} x_{i,k}y_{k,j}z_{i,j} = \sum_{i=1}^{a}\sum_{j=1}^{c} z_{i,j}\left(\sum_{k=1}^{b} x_{i,k}y_{k,j}\right),$$

corresponding to a bilinear map $\mathbb{F}^{ac} \times \mathbb{F}^{cb} \to \mathbb{F}^{ab}$.

Remark. *If we substitute in the entries of x and y from the matrices, the coefficient of $z_{i,j}$ corresponds to the i, j entry in the matrix product.*

Definition 2.2.3: Tensor Rank

Let T be a tensor over finite sets of variables X, Y, and Z. A tensor over $\mathbb{F}$ has rank 1 if there exist coefficients $\alpha_x, \beta_y, \gamma_z \in \mathbb{F}$ with

$$T = \left(\sum_{x \in X} \alpha_x X\right)\left(\sum_{y \in Y} \beta_y y\right)\left(\sum_{z \in Z} \gamma_z z\right).$$

Then, the rank, $\text{rank}(T)$, of a general tensor T is the minimum of rank 1 tensors that sum up to T.

We now detail an intimate connection between tensor rank and fast matrix multiplication.

Theorem 2.2.1: Upper Bounding Matrix Multiplication Tensor Rank → Fast Algorithms

If $\text{rank}(\langle q, q, q \rangle) \leq r$, then there exists a $O\left(n^{\log_q r}\right)$-time matrix multiplication algorithm.

Proof. By the definition of tensor rank, we have that

$$\begin{aligned}\langle q, q, q \rangle &= \sum_{i=1}^{q}\sum_{j=1}^{q}\sum_{k=1}^{q} x_{i,k}y_{k,j}z_{i,j} \\ &= \sum_{\ell=1}^{r}\left(\sum_{i=1}^{q}\sum_{k=1}^{q}\alpha_{i,k}^{(\ell)}x_{i,k}\right)\left(\sum_{k=1}^{q}\sum_{j=1}^{q}\beta_{k,j}^{(\ell)}y_{k,j}\right)\left(\sum_{i=1}^{q}\sum_{j=1}^{q}\gamma_{i,j}^{(\ell)}y_{i,j}\right)\end{aligned}$$

for coefficients $\alpha_{i,k}^{(\ell)}, \beta_{k,j}^{(\ell)}, \gamma_{i,j}^{(\ell)} \in \mathbb{F}$. Assume $q|n$ without loss of generality, and form $q \times q$ blocks of $X, Y \in \mathbb{F}^{n \times n}$. Then, each block $X_{i,k}, Y_{k,j} \in \mathbb{F}^{\frac{n}{q} \times \frac{n}{q}}$. Then, substitute in the blocks into the right-hand side of the above equation, and compute the coefficient of each $z_{i,j}$ to compute $Z_{i,j}$. Recurse to compute products of the $\frac{n}{q} \times \frac{n}{q}$ matrices. Each term corresponding to a rank 1 tensor indexed by ℓ takes $3q^2$ additions and one multiplication of $\frac{n}{q} \times \frac{n}{q}$ matrices. This gives us the recurrence

$$T(n) = rT\left(\frac{n}{q}\right) + O(n^2) = O\left(n^{\log_q r}\right),$$

as desired. ∎

Applying Theorem 2.2.1 from above, we have the following results.

Corollary 2.2.1: [Str69]'s Bound

Note $\text{rank}(\langle 2,2,2\rangle) \leq 7$, so we have an $O\left(n^{\log_2 7}\right)$-time matrix multiplication algorithm.

Corollary 2.2.2: [Pan78]'s Bound

Note $\text{rank}(\langle 70,70,70\rangle) \leq 143640$, so we have an $O\left(n^{\log_{70} 143640}\right)$-time matrix multiplication algorithm.

Definition 2.2.4: Rectangular Matrix Multiplication Exponent

The RECTANGULAR MATRIX MULTIPLICATION exponent ω is

$$\omega(a,b,c) = \inf\left\{\omega' > 0 : \forall \epsilon > 0, \exists \text{ an algorithm for multiplying } A \in \mathbb{R}^{n^a \times n^b} \text{ and } B \in \mathbb{R}^{n^b \times n^c} \text{ in } O\left(n^{\omega'+\epsilon} \text{ time}\right)\right\}.$$

Definition 2.2.5: Dual Matrix Multiplication Exponent

The dual MATRIX MULTIPLICATION exponent α is

$$\alpha = \sup\{\alpha' > 0 : \omega(1,1,\alpha') = 2\}.$$

Remark. *By [WXXZ23], $\alpha \geq 0.321334$. Also, $\alpha = 1$ if and only if $\omega = 2$.*

2.3 Lecture 8: February 6, 2025

2.3.1 Matrix Multiplication Verification

Consider the following problem.

Computational Problem 7 (Matrix Multiplication Verification)**.**

- *Given matrices* $A, B, C \in \mathbb{R}^{n \times n}$,
- *Decide:* $AB = C$*?*

Naively, we immediately have an $O(n^\omega)$ algorithm by an easy reduction to Matrix Multiplication.

Question. *Can we do better?*

Answer. *Yes, we can!*

We present a natural randomized algorithm, due to [Fre79].

Algorithm 2.3.2 Freivalds' Matrix Multiplication Verification

```
1: procedure FREIVALDS_MMV(A, B, C)
2:     Sample x ~ {0,1}^n uniformly at random.
3:     return ABx = Cx                          ▷ Compute ABx as A(Bx)
```

For correctness, let $D = AB - C$ and observe that $Dx = 0$ if and only if $ABx = Cx$. If $D = 0$, then $Dx = 0$ with probability 1. If $D \neq 0$, D has a nonzero row d. Then,

$$\Pr_x[Dx = 0] \leq \Pr_x[\langle d, x \rangle = 0] \leq \frac{1}{2}.$$

Algorithm 2.3.2 runs in $O(n^2)$ time. Note that Freivalds' procedure uses n random bits. Naturally, we ask how to partially derandomize while still retaining $o(n^\omega)$ time.

2.4 Lecture 9: February 11, 2025

2.4.1 Sparse Matrix Multiplication

Another natural question is how we can do better in the case of sparse matrices. Here, we follow [YZ05].

Computational Problem 8 (Sparse Matrix Multiplication)**.**

- *Given matrices $A, B \in \mathbb{R}^{n \times n}$, each with at most m nonzero entries,*
- *Find $C = AB$.*

We have two critical observations that we rely on. First, let $A, B \in \mathbb{R}^{n \times n}$ where A has columns a_i and B has rows b_i. Then, $AB = \sum_{i=1}^{n} a_i b_i$; this is an outer product characterization of matrix multiplication. Also, for any permutation π, $\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_{\pi(i)} b_{\pi(i)}$. We may compute this product using $O\left(+n^2\right)$ [AB: todo].

We present the algorithm by [YZ05]. [AB: TODO!!!]

3
NP-Hard Graph Problems

3.1 Lecture 10: February 13, 2025

3.1.1 Graph Coloring

We introduce the notion of coloring a graph and the associated graph coloring problem.

Definition 3.1.1: k-Coloring of a Graph

Let $G = (V, E)$ be an unweighted and undirected graph. A k-coloring of G is a function $c : V \to \{1, \dots, k\}$. We say c is admissible if $c(u) \neq c(v)$ whenever $(u, v) \in E$.

Definition 3.1.2: Chromatic Number of a Graph

Let $G = (V, E)$ be an unweighted and undirected graph. The chromatic number $\chi(G)$ of G is the minimum k such that G has an admissible k-coloring.

Computational Problem 9 (Graph k-Coloring)**.**

- *Given a unweighted and undirected graph $G = (V, E)$ and $k \in \mathbb{Z}^+$,*
- *Decide: Does there exist an admissible k-coloring of G?*

Remark. *It is easy to see that any admissible k-coloring of G immediately partitions G into k independent sets, one for each color class.*

Note that $k = 2$ corresponds to determining whether the given graph is bipartite, easily done in polynomial time. For $k \geq 3$, Graph k-Coloring is **NP**-complete. We follow the exposition of [Fei11] to describe fast k-coloring algorithms. Often, we'll take $k = 3$ and refer to "colors" $c(V)$ as $\{\text{Red}, \text{Green}, \text{Blue}\}$.

There are k^n k-colorings of G. The naive algorithm is to check all of them in $O^*(k^n)$ time for admissiblity.

Question. *Can we do better?*

Answer. *Yes, we can!*

For our first improvement, we'll observe that a tree with n vertices has $k(k-1)^{n-1}$ admissible k-colorings. Hence, we may, in polynomial time, find a spanning tree of our graph G, and then check $k(k-1)^{n-1}$ admissible k-colorings of the spanning tree for admissiblity in G. This gives rise to an $O^*((k-1)^n)$-time algorithm.

Our first nontrivial algorithm for 3-coloring is given in Algorithm 3.1.1. We reduce 3-coloring to 2-coloring.

Algorithm 3.1.1 3-Coloring via a Reduction to 2-Coloring

Require: $G = (V, E)$.
1: **procedure** 3-Color-Reduction-2-Color(G)
2: **for** each subset $R \subseteq V$ with $|R| \leq \frac{n}{3}$ **do**
3: Attempt to 2-color $(V \backslash R, E)$ with Green and Blue
4: **return** the coloring if successful.

We can think of $R \subseteq V$ as a possible Red color class. The smallest color class must have size at most $\frac{n}{3}$. Then, we can attempt to color the remaining edges with Green and Blue. We must check $\sum_{i=1}^{\binom{n}{3}} \binom{n}{i}$ sets. Therefore, the running time of Algorithm 3.1.1 is

$$\begin{aligned}\left(\sum_{i=1}^{\binom{n}{3}} \binom{n}{i}\right) \text{poly}(n) &\leq \binom{n}{\frac{n}{3}} \text{poly}(n) \\ &= \frac{n!}{(\frac{n}{3})!\,(\frac{2n}{3})!} \text{poly}(n) \\ &\approx 3^{\frac{n}{3}} \left(\frac{3}{2}\right)^{\frac{2n}{3}} \text{poly}(n) = O^*\left(\left(\frac{27}{4}\right)^{\frac{n}{3}}\right) = O(1.89^n).\end{aligned}$$

Another approach is to reduce 3-coloring to 2-SAT, illustrated in Algorithm 3.1.2.

Algorithm 3.1.2 3-Coloring via a Reduction to 2-SAT

Require: $G = (V, E)$.
1: **procedure** 3-Color-Reduction-2-SAT(G)
2: **for** $v \in V$ **do**
3: Repeatedly randomly choose an assignment $\ell(v)$ of two colors to v
4: Find a 3-coloring respecting $\ell(v_1), \ldots, \ell(v_n)$ by a reduction to 2-SAT.

The idea behind Algorithm 3.1.2 is to fix a legal 3-coloring $c^\star$ of G, and for each vertex, we can randomly select two colors from $\{\text{Red}, \text{Green}, \text{Blue}\}$. We want one of them to correspond with $c^\star$. Then,

$$\Pr[\text{all } n \text{ guesses agree with } c^\star] = \left(\frac{2}{3}\right)^n$$

Lemma. *Suppose that we have guesses $\ell(v) \in \{\text{RED}, \text{GREEN}, \text{BLUE}\}^2$ that agree with $c^\star$ for all n vertices. Then, we can find a 3-coloring of G in $\text{poly}(n)$ time.*

Proof. Construct a 2-SAT instance with one variable x_i for each vertex $v_i \in G$. The values that can be assigned to x_i are in one-to-one correspondence with the colors in $\ell(v_i)$. Then, for each edge (v_i, v_j), add either one or two 2-clauses, depending on $|\ell(v_i) \cap \ell(v_j)|$. ■

By our probabilistic analysis, and using a polynomial time 2-SAT algorithm [AB: cite: e.g., papadimitriou], we see that Algorithm 3.1.2 runs in $O^*\left(\left(\frac{3}{2}\right)^n\right)$ time.

We now provide an algorithm for k-coloring in general. But first, consider the following theorems.

Theorem 3.1.1: Fast Multiplication of Multivariate Polynomials

There exists an algorithm that multiplies polynomials $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$ with degree at most d in each variable using $O^*(2^n d)$ arithmetic operations.

Proof. The proof is by reduction to univariate polynomial multiplication and applying the fast Fourier transform. ■

Definition 3.1.3: Independent Set Polynomial of a Graph

Define the independent set polynomial of $G = ([n], E)$ as

$$p_G(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} (\mathbf{1}_{S \text{ is an independent set}}) \prod_{i \in S} x_i.$$

Theorem 3.1.2: A Necessary and Sufficient Condition for k-Colorability

The graph $G = ([n], E)$ is k-colorable if and only if $p_G(x_1, \ldots, x_n)^k$ contains the monomial $z(x_1 \cdots x_n)$ for some $z \in \mathbb{Z}^+$.

Proof. A k-coloring of G is a partition of the vertices of G into k independent sets $S_1, \ldots, S_k$. If $S_1, \ldots, S_k$ corresponds to a legal k-coloring, then $p_G(x_1, \ldots, x_k)$ contains each of the k monomials $\prod_{i \in S_j} x_i$. So, $p_G(x_1 \ldots, x_n)^k$ contains $x_1 \cdots x_n$. On the other hand, if $p_G(x_1, \ldots, x_n)^k$ contains $x_1 \cdots x_n$, then p_G contains k disjoint monomials corresponding to a partition of the input graph into k independent sets. ■

The running time is $O^*(2^n)$ to compute p_G, and $O^*(2^n)$ to compute $p_G(x_1, \ldots, x_k)^n$.

3.2 Lecture 11: February 19, 2025

3.2.1 Approximate Graph Coloring

Consider the following problem.

Computational Problem 10 (Graph k-Coloring, Relax)**.**

- *Given a unweighted and undirected graph $G = (V, E)$ and $k \in \mathbb{Z}^+$ such that G is k-colorable,*
- *Find: the smallest $k'(n)$ such that finding an admissible $k'(n)$-coloring takes* poly(n) *time.*

Again, we focus on $k = 3$. Note trivially, that we can pick $k'(n) = n$. We give a better algorithm, due to [Joh74].

Algorithm 3.2.3 Coloring in Polynomial Time with $O\left(\frac{n}{\log n}\right)$ Colors, [Joh74]

Require: $G = (V, E)$ is 3-Colorable
1: **procedure** 3-Color-Relax(G)
2: Choose $S \subseteq V$ with $|S| = \log n$.
3: Check all $3^{|S|}$ possible colorings of $G_{|S}$ for admissibility.
4: Color S according to an admissible coloring with 3 new colors, and recuse on $G_{|V \setminus S}$.

We recurse $\frac{n}{\log n}$ times, and use 3 new colors each time. Now, we present Wigderson's algorithm, [Wig83].

Algorithm 3.2.4 Coloring in Polynomial Time with $O(\sqrt{n})$ Colors, [Wig83]

Require: $G = (V, E)$ is 3-Colorable.
1: **procedure** 3-Color-Relax-Wigderson(G)
2: **while** $\exists v \in V, \deg(v) \geq \sqrt{n}$ where v is not colored **do**
3: Color v with a new color. ▷ Observe that the neighborhood $N(v)$ of v is 2-colorable.
4: Color $N(v)$ with two new colors.
5: Color the remaining vertices using $\sqrt{n}$ new colors.

A key observation in Algorithm 3.2.4 is that we can color a graph G with maximum degree d with at most $d + 1$ colors in polynomial time. Each vertex v has at most d neighbors, so even if they are all colored with different colors, we can pick a $d + 1$th color for v. Therefore, Algorithm 3.2.4.

In each iteration of the while loop, we use 3 colors and color $\sqrt{n}$ vertices. For the low-degree vertices, we use $\sqrt{n}$ colors. So, in sum, we use $3\sqrt{n} + \sqrt{n} = O(\sqrt{n})$ colors.

We'll return to the relaxation of graph coloring presented, but first, we'll start an interlude on constraint satisfaction problems.

3.2.2 Interlude on Constraint Satisfaction Problems

Consider the following definition.

Definition 3.2.1: Constraint Satisfaction Problems (CSPs)

A k-ary constraint satisfaction problem (k-CSP) consists of

1. n variables $x_1, \dots, x_n$,
2. a finite universe U of values that the variables can take,
3. a family C of constraints which are functions $f : U^k \to \{0, 1\}$.

An instance of a k-CSP C consists of m constraints

$$f_1(x_{i,1}, \dots, x_{i,k}), \dots, f_m(x_{m,1}, \dots, x_{m,k}).$$

We hope to find an assignment $x_i = a_i$ for $a_1, \dots, a_n \in V$ satisfying all m constraints.

We provide some examples.

Example 5 (3-SAT as a CSP)**.** *We have that* 3-SAT *is a* 3*-CSP with* $U = \{0, 1\}$ *and constraints are disjunctions of three literals.*

Example 6 (3-Coloring as a CSP)**.** *We have that* 3-Coloring *is a* 2*-CSP with* $U = \{1, 2, 3\}$ *and variables corresponding to colors assigned to vertices. The constraints correspond to edges and are of the form* $f(x_i, x_j) = 1$ *if* $x_i \neq x_j$ *and* $f(x_i, x_j) = 0$ *if* $x_i = x_j$*.*

Computational Problem 11 (MaxCut)**.**

- *Given a unweighted and undirected graph* $G = (V, E)$*,*
- *Find:* V^- *and* V^+ *with* $V^- \sqcup V^+ = V$ *such that* $|\{(u, v) : u \in V^-, v \in V^+\}|$ *is maximized.*

Example 7 (MaxCut as a CSP)**.** *We have that* MaxCut *is a* 2*-CSP with* $U = \{0, 1\}$ *and variables corresponding to an assignment of vertices to* V^- *or* V^+*. The constraints correspond to edges and are of the form* $f(x_i, x_j) = 1$ *if* $x_i \neq x_j$ *and* $f(x_i, x_j) = 0$ *if* $x_i = x_j$*.*

Remark. *Note that if we wish to satisfy* ℓ *constraints, taking* $\ell = m$ *gives* **NP***-completeness for the CSPs corresponding to* 3-SAT *and* 3-Coloring*, but* $\ell = m$ *for the* MaxCut *CSP is simply* 2-Coloring*, which is in* **P***.*

Remark. *The optimization form of a CSP hopes to satisfy as many constraints as is possible. Suppose* $\mathsf{OPT} \leq m$ *is the maximal number of satisfied constraints, then, we call an* α*-approximate solution satisfying at least* $\alpha\mathsf{OPT}$ *clauses.*

Theorem 3.2.1: Randomized Approximation Algorithms for CSPs

A random assignment of variables satisfies $p_0 m$ constraints in a CSP C in expectation where

$$p_0 = \min_{f \in C} \Pr_{(a_1,\dots,a_k)\sim U^k}[f(a_1, \dots, a_k) = 1].$$

Proof. Define an indicator random variable $\mathbf{1}_i$ to be 1 if f_i is satisfied and 0 otherwise. Then,

$$\begin{aligned}
\mathbb{E}[|f : f(x) = 1|] &= \mathbb{E}\sum_{i=1}^{m}[\mathbf{1}_i] \\
&= \sum_{i=1}^{m}\mathbb{E}[\mathbf{1}_i] \\
&\geq \sum_{i=1}^{m} p_0 \\
&= p_0 m,
\end{aligned}$$

as desired. ■

Remark. *For* 3-SAT, *random assignment gives a* $\frac{7}{8}$*-approximation algorithm, and it is known that* $\left(\frac{7}{8} + \epsilon\right)$ *is* **NP***-complete. For* MaxCut, *random assignment gives a* $\frac{1}{2}$ *approximation algorithm,*

Question. *Can we do better?*

Answer. *In some cases, yes, we can!*

3.3 Lecture 12: February 25, 2025

3.3.1 Semidefinite Programming and Approximation

In this section, we present the results of [GW95] for MaxCut and discuss relations to graph coloring. We follow the exposition of [WS11]. We use the technique of semidefinite programming. Recall the following definition from linear algebra.

Definition 3.3.1: Positive Semidefinite Matrices

Let $X \in \mathbb{R}^{n\times n}$. We say X is positive semidefinite if all eigenvalues of X are nonnegative. If X is positive semidefinite, we write $X \succeq \mathbf{0}$. The set of symmetric matrices is denoted $\mathbf{S}^n \subseteq \mathbb{R}^{n\times n}$.

Remark. *Equivalently, $X \in \mathbb{R}^{n\times n}$ is positive semidefinite if and only if*

1. *for all $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{y}^\top X\mathbf{y} \geq 0$, or*
2. *$X = V^\top V$ for some $V \in \mathbb{R}^{n\times n}$.*

Definition 3.3.2: Semidefinite Programs

A semidefinite program (SDP) is an optimization problem that can be written as

$$\begin{aligned} &\max_{X\in\mathbf{S}^n} \langle C, X\rangle \\ &\text{s.t. } AX = b, \\ &\quad X \succeq \mathbf{0}. \end{aligned}$$

where $C \in \mathbf{S}^n$ and the inner product is $\langle C, X\rangle = \operatorname{trace}(C^\top X)$.

Remark. *Semidefinite programs are just like linear programs, though in addition to having a linear objective and affine constraints, we also require that the matrix X of variables $x_{i,j}$ is positive semidefinite.*

Theorem 3.3.1: Solving SDPs

Given some technical conditions, we can solve SDPs to within additive error $\epsilon > 0$ in poly $\left(\log\left(\frac{1}{\epsilon}\right), n\right)$ time.

Remark. *We'll just say that SDPs can be solved exactly in polynomial time, since this is the "moral" claim of Theorem 3.3.1.*

Throughout, we will work with vector programs, those of the form

$$\begin{aligned} &\max \sum_{i,j} c_{i,j} \langle \mathbf{v}_i, \mathbf{v}_j\rangle \\ &\text{s.t. } \sum_{i,j} a_{i,j,k} \langle \mathbf{v}_i, \mathbf{v}_j\rangle = b_k, \quad \forall k \\ &\quad \mathbf{v}_i \in \mathbb{R}^n. \end{aligned}$$

The above vector program is equivalent to the SDP

$$\begin{aligned} \max & \sum_{i,j} c_{i,j} x_{i,j} \\ \text{s.t.} & \sum_{i,j} a_{i,j,k} x_{i,j} = b_k, \quad \forall k \\ & x_{i,j} = x_{j,i}, \quad \forall i, j \\ & X = (x_{i,j}) \succeq \mathbf{0}. \end{aligned}$$

This can be seen by establishing a correspondence between $x_{i,j}$ in the SDP to the inner product $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ in the vector program.

- Given a solution X to the SDP, we can turn it into a solution V to the vector program by computing $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ such that $X = V^\top V$.
- Given a solution V to the vector program, we can turn it into a solution $X = V^\top V$ to the SDP.

Before we move to MaxCut, we define quadratic programs.

Definition 3.3.3: Quadratic Programs

A quadratic program (QP) is an optimization problem that can be written as

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^n} & \frac{1}{2} \mathbf{x}^\top Q \mathbf{x} + \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \ A\mathbf{x} \leq \mathbf{b}. \end{aligned}$$

We can formulate MaxCut as a quadratic program. Given a graph $G = (V, E)$, we have the problem

$$\begin{aligned} \max & \frac{1}{2} \sum_{(i,j) \in E} (1 - y_i y_j) \\ \text{s.t.} & \ y_i \in \{-1, 1\}, \quad i \in \{1, \dots, n\}. \end{aligned}$$

Consider the following vector program relaxation to the MaxCut QP. We have

$$\begin{aligned} \max & \frac{1}{2} \sum_{(i,j) \in E} (1 - \langle \mathbf{v}_i, \mathbf{v}_j \rangle) \\ \text{s.t.} & \ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 1, \quad i \in \{1, \dots, n\}, \\ & \mathbf{v}_i \in \mathbb{R}^n, \quad i \in \{1, \dots, n\}. \end{aligned}$$

Any feasible solution $\mathbf{y}$ to the QP can be turned into a feasible solution $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ to the vector program with the same objective value; consider $y_i \mapsto \mathbf{v}_i = [y_i, 0, \dots, 0]^\top$. If OPT is optimal for the QP, and hence MaxCut, and Z_{VP} is optimal for the vector problem, then $\text{OPT} \leq Z_{\text{VP}}$.

Remark. *We can think of the vector program as the n-dimensional relaxation of the* 1*-dimensional QP.*

We wish to obtain a $\{-1, 1\}$ assignment to vertices of G. We use a "randomized rounding" technique. We give the Geomans-Williamson algorithm in Algorithm 3.3.5.

Algorithm 3.3.5 Goemans-Williamson Approximation Algorithm for MaxCut

```
1: procedure GoemansWilliamson(G)
2:     V⁻ ← ∅
3:     V⁺ ← ∅
4:     Solve the vector program relaxation of the MaxCut QP for V = [v_1, ..., v_n].
5:     Sample r ~ {x ∈ ℝ^n : ||x||_2 = 1} uniformly at random.
6:     for i ∈ {1, ..., n} do
7:         if ⟨v_i, r⟩ ≤ 0 then
8:             V⁻ ← V⁻ ∪ {i}
9:         else
10:            V⁺ ← V⁺ ∪ {i}
11:    return (V⁻, V⁺)
```

The inner product $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ for $i \neq j$ corresponds to the angle formed by $\mathbf{v}_i$ and $\mathbf{v}_j$, and therefore to the chance that the corresponding edge (i, j) is cut. Then,

$$\Pr[(i,j) \text{ is a cut edge}] = \Pr[\text{proj}_{\text{span}\{\mathbf{v}_i, \mathbf{v}_j\}} \mathbf{r}]$$

Appendices

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