MATH3430: DIFFERENTIAL EQUATIONS

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TEXTBOOK: MORRIS TENENBAUM & HARRY POLLARD

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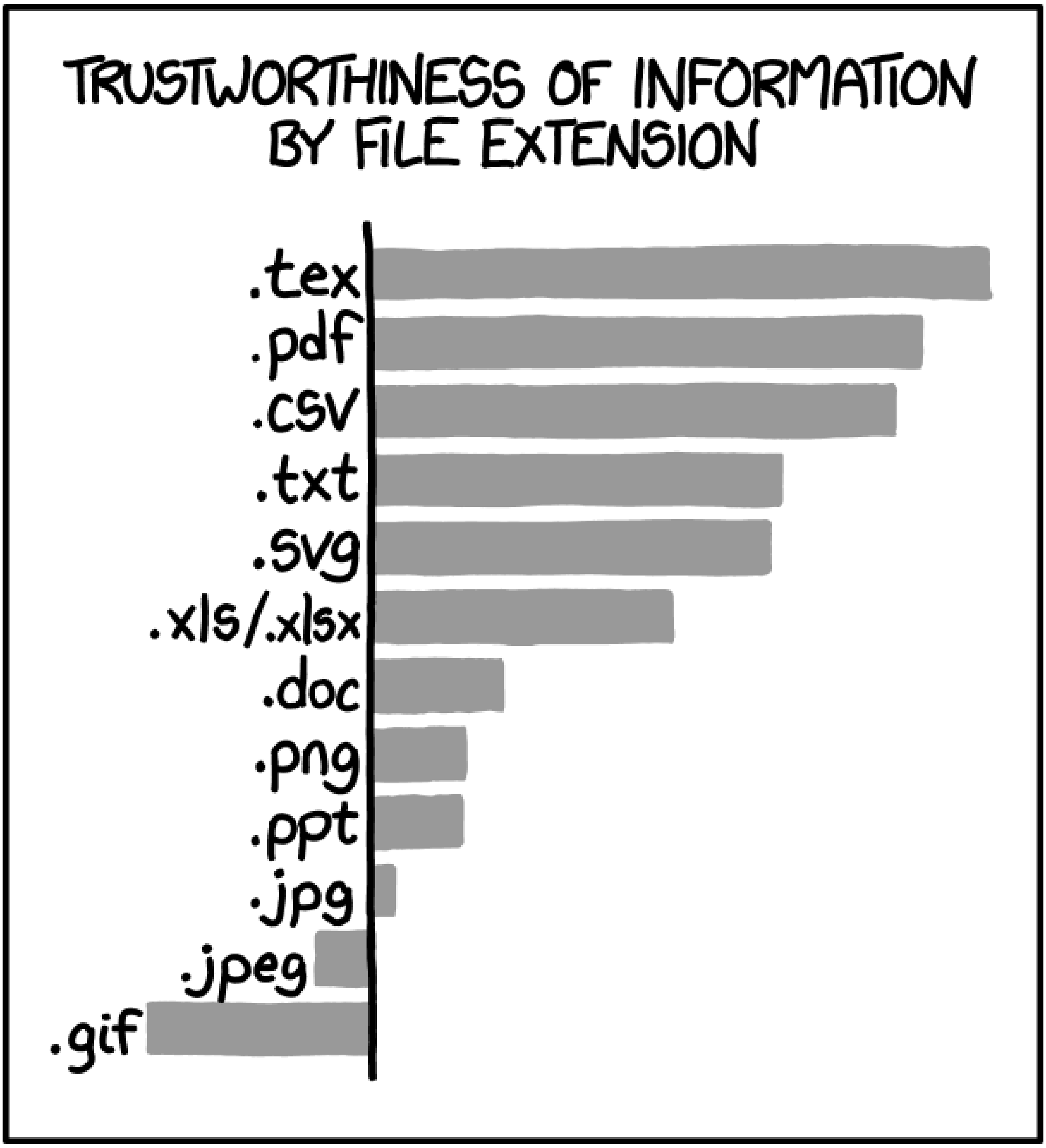
EDITION 1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Preface

To the interested reader,

This document is a compilation of lecture notes taken during the Spring 2023 semester for MATH3430: Ordinary Differential Equations at the University of Colorado Boulder. The course used *Ordinary Differential Equations*[1] by Morris Tenenbaum and Harry Pollard as its primary text. As such, many theorems, definitions, and content may be quoted from the book. This course was taught by Sheagan John, Ph. D.

The author would like to provide the following resources for students currently taking a differential equations course:

1. MIT OpenCourseWare Differential Equations Lectures From Spring 2006.
2. 3Blue1Brown's *Overview of Differential Equations*.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. Please keep in mind that the author wrote this resource during his second semester of his undergraduate studies. If an error needs to be resolved, please contact Adithya Bhaskara at adithya.bhaskara@colorado.edu.

Finally, the author would like to dedicate this set of lecture notes to *Aidan Janney*, *Erika Sjöblom*, *Tate McDonald* and *Benjamin Braun*, three of the author's closest friends who plan to take Differential Equations in the upcoming semester; Fall 2023, at the time of writing.

Best Regards,
Adithya Bhaskara

REVISED: February 26, 2023

[1] Tenenbaum, M., & Pollard, H. (1985). *Ordinary Differential Equations.* Dover Publications.

1

Basic Notions

1.1 Lecture 1: January 20, 2023

1.1.1 Definition of an Ordinary Differential Equation

Consider the following definitions.

Definition 1.1.1: Ordinary Differential Equations

An ordinary differential equation is an equation of the form

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

where x is an independent variable and y is nth order differentiable.

We remark that every ordinary differential equation is valid as an expression only when we specify the values of x for which it is defined.

Definition 1.1.2: Order of a Differential Equation

The order of an ordinary differential equation is the highest nontrivial derivative present in the equation.

Consider the following ordinary differential equations. From now on, we may omit the term "ordinary," as partial differential equations are not considered in this text.

1. $F(x, y, y') = \cos(xy') + y^2y' + x^2 = 0$ is a first order differential equation.
2. $F(x, y, y', y'') = -\frac{1}{1-x^2} + y'' = 0$ is a second order differential equation.
3. $F(x, y, y', y'', y''', y'''') = e^{xy}y'''' - x^2y'' - \sin x = 0$ is a fourth order differential equation.

1.2 Lecture 2: January 23, 2023

1.2.1 Explicit and Implicit Solutions

We will now consider explicit and implicit solutions to differential equations.

Definition 1.2.1: Explicit Solutions to Ordinary Differential Equations

Let $F(x, y, y', \dots, y^{(n)}) = 0$ be a differential equation defined on the interval I. Then, an explicit solution to F is a function

$$f : \mathbb{R} \to \mathbb{C}$$

for which $f(x)$ is well-defined on some set X such that $I \cap X \neq \emptyset$ and, for all $x \in I$,

$$F(x, f, f', \dots, f^{(n)}) = 0.$$

Definition 1.2.2: Implicit Functions

Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(x, y) = 0$. Then, f defines y as an implicit function of x if and only if

1. There exists an explicit function $g(x)$ such that $y = g(x)$ on some interval $I \subseteq \mathbb{R}$.
2. For all $x \in I$, $f(x, g(x)) = 0$.

Consider the following examples.

Example 1.2.1: Is it an Implicit Function? 1

Does $f(x, y) = x^2 + y^2 - 25 = 0$ define an implicit function of x on $\{x \in \mathbb{R} : -5 \leq x \leq 5\}$?

We consider solving for y. We obtain $y = \pm\sqrt{25 - x^2}$. Therefore we define $y_1(x) = \sqrt{25 - x^2}$ and $y_2(x) = -\sqrt{25 - x^2}$. For $y_1(x)$, we see that

$$f(x, y_1(x)) = x^2 + (\sqrt{25 - x^2})^2 - 25 = x^2 + 25 - x^2 - 25 = 0,$$

as desired. Also, $y_1(x)$ is defined for on $\{x \in \mathbb{R} : -5 \leq x \leq 5\}$. Therefore, $f(x, y)$ does define an implicit function of x. Upon further inspection, $f(x, y)$ defines more than one implicit function of x.

Example 1.2.2: Is it an Implicit Function? 2

Does $f(x,y) = \sqrt{x^2 - y^2} + \arccos\left(\frac{x}{y}\right) = 0, \quad y \neq 0$ define an implicit function?

Here, it is too difficult to solve for y. Instead, we notice that $\arccos(1) = 0$. Therefore, we try $y(x) = x$. By substitution,

$$f(x, y(x)) = \sqrt{x^2 - x^2} + \arccos\left(\frac{x}{x}\right) = 0,$$

as desired. Also, $y(x)$ is defined on $\mathbb{R}$. Therefore, $f(x,y)$ defines an implicit function on $\mathbb{R} - \{0\}$.

Consider the following definition.

Definition 1.2.3: Implicit Solutions to Ordinary Differential Equations

Let $F(x, y, y', \ldots, y^{(n)}) = 0$ be a differential equation defined on the interval I. Then, an implicit solution to F is a relation $f(x,y) = 0$ if and only if f defines y as an implicit function of x on I, and if $y(x)$ is an explicit solution to the differential equation.

Consider the following examples.

Example 1.2.3: Verifying Explicit Solutions 1

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y', y'') = x^2 y'' + xy' = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = \log x$, where $\log x = \log_e x$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{1}{x}$. The second derivative of $y(x)$ is $y''(x) = -\frac{1}{x^2}$. Then, we have

$$\begin{aligned} x^2 y''(x) + xy'(x) &= x^2 \cdot -\frac{1}{x^2} + x \cdot \frac{1}{x} \\ &= -1 + 1 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x > 0\}$. The differential equation is defined on $\{x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x \in \mathbb{R}\} \cap \{x \in \mathbb{R} : x > 0\} \cap \{x \in \mathbb{R} : x \neq 0\} = \{x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = \log x$ is an explicit solution for F on $\{x \in \mathbb{R} : x > 0\}$. $\square$

Example 1.2.4: ☢☢ Verifying Explicit Solutions 2

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y') = yy' - 4 = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = 2\sqrt{2x}$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{2}{\sqrt{2x}}$. Then, we have

$$\begin{aligned} y(x)y'(x) - 4 &= 2\sqrt{2x} \cdot \frac{2}{\sqrt{2x}} - 4 \\ &= 4 - 4 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x \in \mathbb{R} : x \geq 0\}$. The differential equation is defined on $\{x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x \in \mathbb{R}\} \cap \{x \in \mathbb{R} : x \geq 0\} \cap \{x \in \mathbb{R} : x \neq 0\} = \{x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = 2\sqrt{2x}$ is an explicit solution for F on $\{x \in \mathbb{R} : x > 0\}$. □

Example 1.2.5: ☢☢ Verifying Explicit Solutions 3

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y', y'') = y''^3 + y'^2 - y - 3x^2 - 8.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = x^2$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = 2x$. The second derivative of $y(x)$ is $y''(x) = 2$. Then, we have

$$\begin{aligned} y''^3 + y'^2 - y - 3x^2 - 8 &= (2)^3 + (2x)^2 - x^2 - 3x^2 - 8 \\ &= 8 + 4x^2 - x^2 - 3x^2 - 8 = 0 \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\mathbb{R}$. The differential equation is defined on $\mathbb{R}$. We have no additional restrictions. Therefore, the solution set is $\mathbb{R}$. We can then state that $y(x) = x^2$ is an explicit solution for F on $\mathbb{R}$. □

Example 1.2.6: ☢☢ Verifying Explicit Solutions 4

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y') = (x + y)^2 - y' = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = \tan x - x$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \sec^2 x - 1 = \tan^2 x$. Then, we have

$$\begin{aligned}(x + \tan x - x)^2 - \tan^2 x &= (\tan x)^2 - \tan^2 x \\ &= \tan^2 x - \tan^2 x = 0\end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. The differential equation is defined on $\{x \in \mathbb{R}\}$. We have no additional restrictions. Therefore, the solution set is

$$\{x \in \mathbb{R}\} \cap \{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\} = \{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}.$$

We can then state that $y(x) = \tan x - x$ is an explicit solution for F on $\{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. □

Example 1.2.7: ☢☢ Verifying Implicit Solutions 1

Determine whether $f(x, y) = x^2 + y^2 + 4 = 0$ provides an implicit solution to

$$F(x, y, y') = 2x + 2y'' = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = \sqrt{-x^2 - 4}$ and $g_2(x) = -\sqrt{-x^2 - 4}$; these functions are defined nowhere on $\mathbb{R}$. Thus, $f(x, y) = 0$ does not provide an implicit solution to the differential equation.

Example 1.2.8: ☢☢ Verifying Implicit Solutions 2

Determine whether $f(x, y) = xy - y^2 = 0$ provides an implicit solution to

$$F(x, y, y', y'') = \frac{1}{y - x^2}y'' + yy' - y = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y) = xy - y^2 = y(x - y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = 0$ and $g_2(x) = x$. Both g_1 and g_2 are defined on $\mathbb{R}$. Note that F has the restriction $y - x^2 \neq 0$. We see that $f(x, g_1(x)) = 0$ for all $x \in \mathbb{R}$ and $f(x, g_2(x)) = x^2 - x^2 = 0$ for all $x \in \mathbb{R}$. Therefore, f defines y as an implicit function of x. Taking $y = g_1(x)$ gives

$$F(x, g_1(x), g_1'(x), g_1''(x)) = \frac{1}{0 - x^2}(0) + (0)(0) - (0) = -\frac{1}{x^2} = 0, \quad x \neq 0.$$

Then, if we take $y = g_2(x)$, we have

$$F(x, g_2(x), g_2'(x), g_2''(x)) = \frac{1}{x - x^2}y'' + xy' - x = \frac{1}{x(1 - x)}(0) + x(1) - x = 0, \quad x \neq 0, x \neq 1.$$

Therefore, $f(x, y)$ provides an implicit solution to the differential equation. When providing the intervals of solution, we must explicitly pick which solution, g_1 or g_2, to provide the interval with respect to. For $g_1(x)$, this is

$$\{x \in \mathbb{R} : x \neq 0\}.$$

and for $g_2(x)$, it is

$$\{x \in \mathbb{R} : x \neq 0, x \neq 1\}.$$

Note that we did not need to consider $y = g_2(x)$ to show that $f(x, y)$ provides an implicit solution.

Note that it is bad practice to immediately differentiate the relation $f(x, y)$. For example, in Example 1.2.7, if we immediately differentiated f, we would indeed obtain a symbolic equivalent to the differential equation, but we would not account for the domain restrictions.

Example 1.2.9: ☢☢ Verifying Implicit Solutions 3

Determine whether $f(x,y) = e^{2y} + e^{2x} - 1 = 0$ provides an implicit solution to

$$F(x,y,y') = e^{x-y} + y'e^{y-x} = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x,y) = e^{2y} + e^{2x} = 0$ defines y as an implicit function of x. Consider the function $g(x) = \frac{1}{2}\log(1-e^{2x})$. We see that $g(x)$ is defined on $\{x \in \mathbb{R} : x < 0\}$. We see that $f(x,g(x)) = 0$ for all $\{x \in \mathbb{R} : x < 0\}$. Therefore, f defines g as an implicit function of x. Now, we have

$$\begin{aligned} F(x,g(x),g'(x)) &= e^{x-\frac{1}{2}\log(1-e^{2x})} - \frac{e^{2x}}{1-e^{2x}} e^{\frac{1}{2}\log(1-e^{2x})-x} \\ &= \frac{e^x}{\sqrt{1-e^{2x}}} - \frac{e^{2x}}{1-e^{2x}}\frac{\sqrt{1-e^{2x}}}{e^x} \\ &= \frac{e^{2x}\sqrt{1-e^{2x}}}{e^x(1-e^{2x})} - \frac{e^{2x}}{1-e^{2x}}\frac{\sqrt{1-e^{2x}}}{e^x} = 0. \end{aligned}$$

Note that F has no additional restrictions, so the interval of solution for $g(x)$ is simply $\{x \in \mathbb{R} : x < 0\}$.

Example 1.2.10: ☢☢ Verifying Implicit Solutions 4

Determine whether $f(x, y) = -(x+2)^2 + y^2 - 1 = 0$ provides an implicit solution to

$$F(x, y, y') = y^2 - 1 - y'(2y + xy) = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y) = -(x+2)^2 + y^2 - 1 = 0 = 0$ defines y as an implicit function of x. Consider the functions $g_1(x) = \sqrt{1+(x+2)^2}$ and $g_2(x) = -\sqrt{1+(x+2)^2}$. We see that $g_1(x)$ and $g_2(x)$ are defined on $\mathbb{R}$. We see that $f(x, g_1(x)) = f(x, g_2(x)) = 0$ on $\mathbb{R}$. Therefore, f defines g_1 and g_2 as implicit functions of x. If we take $y = g_1(x)$, we have

$$\begin{aligned} F(x, g_1(x), g_1'(x)) &= 1 + (x+2)^2 - 1 - \frac{2(x+2)}{2\sqrt{1+(x+2)^2}}(2\sqrt{1+(x+2)^2} + x\sqrt{1+(x+2)^2}) \\ &= (x+2)^2 - \frac{x+2}{\sqrt{1+(x+2)^2}}(x+2)(\sqrt{1+(x+2)^2}) \\ &= (x+2)^2 - (x+2)^2 = 0. \end{aligned}$$

Then, if we take $y = g_2(x)$, we have

$$\begin{aligned} F(x, g_2(x), g_2''(x)) &= 1 + (x+2)^2 - 1 + \frac{2(x+2)}{2\sqrt{1+(x+2)^2}}(-2\sqrt{1+(x+2)^2} - x\sqrt{1+(x+2)^2}) \\ &= (x+2)^2 - \frac{x+2}{\sqrt{1+(x+2)^2}}(x+2)(\sqrt{1+(x+2)^2}) \\ &= (x+2)^2 - (x+2)^2 = 0. \end{aligned}$$

Note that F has no additional restrictions, so the interval of solution for both $g_1(x)$ and $g_2(x)$ is $\mathbb{R}$. Note that we did not to consider $y = g_2(x)$ to show that $f(x, y)$ provides an implicit solution.

1.3 Lecture 3: January 25, 2023

1.3.1 General and Particular Solutions

Consider the following definitions.

Definition 1.3.1: n-Parameter Families of Solutions

A differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ possesses an n-parameter family of solutions $y(x, c_1, \ldots, c_n)$ if and only if y is a solution for any choice of $c_1, \ldots, c_n \in \mathbb{F}$.

Definition 1.3.2: Particular Solutions of Differential Equations

Let $y(x, c_1, \ldots, c_n)$ be an n-parameter family of solutions to $F(x, y, y', \ldots, y^{(n)}) = 0$. Then, for each choice of $c_1, \ldots, c_n$, we obtain one particular solution.

Consider the following example.

Example 1.3.1: Finding an n-Parameter Family of Solutions

Consider $F(x, y, y', y'') = y'' = 0$. Note that F has solutions $y(x) = x$ and $y(x) = \pi$ on $\mathbb{R}$. Both these solutions are particular, as they contain no arbitrary constants. If we take the linear combination of the solutions to obtain

$$y(x, c_1, c_2) = c_1 x + c_2,$$

as our 2-parameter family of solutions.

Note that we will often rewrite $y(x, c_1, \ldots, c_n)$ as $y(x)$ even though this is an abuse of notation.

Definition 1.3.3: General Solutions of Differential Equations

Let $y(x, c_1, \ldots, c_n)$ be an n-parameter family of solutions to $F(x, y, y', \ldots, y^{(n)}) = 0$. Then, y is a general solution if and only if every solution to F can be obtained from some choice of $c_1, \ldots, c_n$.

In various engineering applications, the terms defined in Definition 1.3.1 and Definition 1.3.3 are equivalent; however, this construction can break. Consider the following examples.

1. The differential equation $F(x, y, y', y'') = y'' = 0$ has the general solution $y(x, c_1, c_2) = c_1 x + c_2$.
2. The differential equation $F(x, y, y') = y'^2 + y^2 = 0$ has only one particular solution $y(x) = 0$.
3. Examples 1.3.2 and 1.3.3 demonstrate where the differential equations has an n-parameter family of solutions but no general solution.

Consider the following examples.

Example 1.3.2: ☢☢ *n*-Parameter Families and General Solutions 1

Show that $F(x, y, y') = y'^2 - 3y' = 0$ has a 1-parameter family of solutions but no general solution.

Note that $y'^2 - 3y' = y'(y' - 3) = 0$. Therefore, either $y' = 0$ or $y' = 3$. We have that $y' = 0$ implies

$$y(x, c_1) = c_1.$$

For $y' = 3$, we have that $y(x) = 3x$. Both $y(x)$ and $y(x, c_1)$ are valid on $\mathbb{R}$. The particular solution $y(x)$ cannot be obtained from $y(x, c_1)$. But, we can take the linear combination of both solutions $y_?(x, c_1) = 3x + c_1$ because, then, we have

$$y_?'^2 - 3y_?' = (3)^2 - 3(3) = 0$$

on $\mathbb{R}$. Therefore, we redefine $y(x, c_1) = y_?(x, c_1)$. Still, there is no choice of c_1 which produces the particular solution $y(x) = 5$ for $y(x, c_1) = 3x + c_1$. Therefore, we have found that not all solutions to F can be obtained from $y(x, c_1)$, so y is not a general solution.

Example 1.3.3: ☢☢ *n*-Parameter Families and General Solutions 2

Show that $F(x, y, y') = y'^2 + (y - 2)y' - 2y = 0$ has two distinct 1-parameter families of solutions. Does F have a general solution?

Note that $y'^2 + (y - 2)y' - 2y = (y' + y)(y' - 2) = 0$. Therefore, $y' = 2$ or $y' = -y$. For $y' = 2$, we have the 1-parameter family $y_1(x, c_{1_1}) = 2x + c_{1_1}$. For $y' = -y$, we have $y_2(x, c_{1_2}) = c_{1_2}e^{-x}$. Both 1-parameter families are valid on $\mathbb{R}$. Note that both *n*-parameter families are distinct; they cannot be obtained from each other. They cannot be combined into a single general solution.

1.4 Lecture 4: January 27, 2023

1.4.1 Initial Conditions

Consider the following definition.

Definition 1.4.1: Initial Conditions

Let $F(x, y, y', \ldots, y^{(n)}) = 0$ possess an n-parameter family of solutions. Any system of n equations which determine unique values for the arbitrary constants is called a set of initial conditions.

Consider the following example.

Example 1.4.1: Finding a Particular Solution Given Initial Conditions 1

Recall that $F(x, y, y', y'') = y'' = 0$ has a general solution $y(x, c_1, c_2) = c_1 x + c_2$. Find the particular solution satisfying $y(0) = 5$ and $y'(1) = 3$.

We have that $y(0) = 5$ implies that $c_2 = 5$ and $y'(1) = 3$ implies that $c_1 = 3$. Our particular solution is then

$$y(x) = 3x + 5.$$

Example 1.4.2: Finding a Particular Solution Given Initial Conditions 2

Recall that $F(x, y, y', y'') = y'' = 0$ has a general solution $y(x, c_1, c_2) = c_1 x + c_2$. Find the particular solution satisfying $y(2) = 2$ and $y(1) = 3$.

Now, we have

$$\left[\begin{array}{cc|c} 2 & 1 & 2 \\ 1 & 1 & 3 \end{array}\right] \underbrace{\rightarrow}_{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right].$$

Thus $c_1 = -1$ and $c_2 = 4$. Here, our particular solution is

$$y(x) = -x + 4.$$

1.5 Lecture 5: January 30, 2023

1.5.1 Finding an nth Order Differential Equation Given an n-Parameter Family of Solutions

We now direct our attention to finding an nth order differential equation when given an n-parameter family of solutions. To this, we will look at $y(x, c_1, \dots, c_n)$ and its derivatives and find relations between them. There is no *efficient* technique that will work for all n-parameter families, but usually, we can take n derivatives of our family and find relationships between them. Recall that differential equations must not contain any arbitrary constants.

Consider the following examples.

Example 1.5.1: Find a Differential Equation With an n-Parameter Family 1

For the 2-parameter family $y = c_1e^{3x} + c_2e^{-3x}$, determine the 2nd-order differential equation which has the family as a solution.

If $y = c_1e^{3x} + c_2e^{-3x}$, $y' = 3c_1e^{3x} - 3c_2e^{-3x} = 3(c_1e^{3x} - c_2e^{-3x})$. Then, $y'' = 3(3c_1e^{3x} + 3c_2e^{-3x}) = 9(c_1e^{3x} + c_2e^{3x}) = 9y$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = y'' - 9y = 0,$$

as desired.

Example 1.5.2: Find a Differential Equation With an n-Parameter Family 2

For the 2-parameter family $y = c_1 + c_2e^{2x}$, determine the 2nd-order differential equation which has the family as a solution.

If $y = c_1 + c_2e^{2x}$, $y' = 2c_2e^{2x}$ and $y'' = 4c_2e^{2x}$. Then, $y'' = 2y'$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = y'' - 2y' = 0,$$

as desired.

Example 1.5.3: Find a Differential Equation With an n-Parameter Family 3

For the 2-parameter family $y = \frac{c_1}{x} + c_2$, determine the 2nd-order differential equation which has the family as a solution.

If $y = \frac{c_1}{x} + c_2$, $y' = -\frac{c_1}{x^2}$ and $y'' = \frac{2c_1}{x^3}$. Then, $c_1 = -x^2y = \frac{1}{2}x^3y''$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = x^3y'' + 2x^2y = 0,$$

as desired.

Example 1.5.4: ☢☢ Find a Differential Equation With an n-Parameter Family 4

For the 2-parameter family $y = c_1\cos(2x) + c_2\sin(2x)$, determine the 2nd-order differential equation which has the family as a solution.

If $y = c_1\cos(2x) + c_2\sin(2x)$, $y' = -2c_1\sin(2x) + 2c_2\cos(2x)$ and $y'' = -4c_1\cos(2x) - 4c_2\sin(2x)$. Then, $y'' = -4y$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = y'' + 4y = 0,$$

as desired.

2

First Order Differential Equations

2.1 Lecture 6: February 1, 2023

2.1.1 An Introduction to Separable Differential Equations

We wish to solve differential equations of the form $F(x, y, y') = f(y)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x) = 0$ where f and g are continuous on a common interval I. But, what is $\mathrm{d}x$, and what is $\mathrm{d}y$? Consider the following definition.

Definition 2.1.1: Differentials

Let $y(x)$ be a differentiable function. Then, if Δx represents any small change in x, we define

$$\mathrm{d}y(x, \Delta x) = y'(x)\Delta x$$

We wish to apply Definiton 2.1.1 to the function $y(x) = x$. Consider the following theorem.

Theorem 2.1.1: A Useful Lemma for a Property of Differentials

If $y(x) = x$, $\mathrm{d}y(x, \Delta x) = \mathrm{d}x(x, \Delta x) = \Delta x$.

Proof. If $y(x) = x$, $y'(x) = 1$, so by Definition 2.1.1, $\mathrm{d}y(x, \Delta x) = \Delta x$. □

Theorem 2.1.2: A Useful Property of Differentials

Let $y(x)$ be a differentiable function. Then, $\mathrm{d}y(x, \Delta x) = y'(x)\,\mathrm{d}x(x, \Delta x)$.

Proof. By Definition 2.1.1, $\mathrm{d}y(x, \Delta x) = y'(x)\Delta x$. But, by Theorem 2.1.1, $\mathrm{d}x(x, \Delta x)$, so

$$\mathrm{d}y(x, \Delta x) = y'(x)\Delta x = y'(x)\,\mathrm{d}x(x, \Delta x),$$

as desired. □

More familiarly, $\mathrm{d}y = y'(x)\,\mathrm{d}x$.

We may use Theorem 2.1.2 to prove a version of the familar chain rule.

Theorem 2.1.3: The Chain Rule for Differentials

Let $y = f(x)$ be a differentiable function, and $x(t) = g(t)$. Therefore, $y(x) = f(g(t))$. Then,

$$\mathrm{d}y(t, \Delta t) = f'(x(t))\,\mathrm{d}x(t, \Delta t).$$

Proof. Since $x = g(t)$, we have $\mathrm{d}x(t, \Delta t) = g'(t)\,\mathrm{d}t(t, \Delta t)$. Then, using the chain rule for derivatives, we obtain the chain rule for differentials below

$$\mathrm{d}y(t, \Delta t) = f'(x(t))g'(t)(t, \Delta t),$$

as desired. □

Note that if $z(x, y)$ is a function differentiable with respect to both x and y, we obtain

$$\mathrm{d}z(x, y, \Delta x, \Delta y) = \frac{\partial z}{\partial y}\,\mathrm{d}y + \frac{\partial z}{\partial x}\,\mathrm{d}x.$$

Suppose we have a first order differential equation in the form $F(x, y, y') = f(y)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x) = 0$. We use that fact that $\mathrm{d}y$ and $\mathrm{d}x$ are differential functions to write $F(x, y, y') = f(y)\,\mathrm{d}y + g(x)\,\mathrm{d}x = 0$. Then, we obtain the 1-parameter family of solutions

$$\int f(y)\,\mathrm{d}y + \int g(x)\,\mathrm{d}x = c.$$

Consider the following examples.

Example 2.1.1: ☢☢ Separation of Variables 1

Consider $F(x, y, y') = \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$, therefore, $\mathrm{d}y = -y\,\mathrm{d}x$. Assuming that $y(x) \neq 0$, we can divide by y to give

$$\frac{1}{y}\,\mathrm{d}y = -\,\mathrm{d}x.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{1}{y}\,\mathrm{d}y = \int -\,\mathrm{d}x$$

meaning that

$$\log|y| = -x + c_1.$$

This provides an implicit solution. Note that it is a 1-parameter family of solutions, but is not a general function since $y(x) = 0$ is a particular solution to the differential equation, not obtainable from the family. We seek to find an explicit solution. Consider

$$e^{\log|y|} = e^{-x+c_1} = e^{-x}e^{c_1}$$

Then, let $c_2 = e^{c_1}$. Note that $c_2 > 0$. Then, we have $|y| = c_2e^{-x}$. This 1-parameter family can be made a general solution by allowing $c_2 = 0$. Our general solution is then

$$|y| = c_2e^{-x}, \quad c_2 \geq 0, \quad x \in \mathbb{R}.$$

If we allow $c_2 \in \mathbb{R}$, we can also write that a general solution is

$$y = c_2e^{-x}, \quad c_2 \in \mathbb{R}, \quad x \in \mathbb{R}.$$

This method does not always work.

Example 2.1.2: ☢☢ Separation of Variables 2

Consider $F(x, y, y') = x^2(y-2)\frac{dy}{dx} - y^3 = 0$. Assuming that $x \neq 0$ and $y(x) \neq 0$, we can write

$$\frac{1}{x^2}\,dx = \frac{y-2}{y^3}\,dy.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{1}{x^2}\,dx = \int \frac{y-2}{y^3}\,dy$$

meaning that

$$\begin{aligned} -\frac{1}{x} &= \int \left(\frac{1}{y^2} - \frac{2}{y^3}\right) dy \\ &= -\frac{1}{y} + \frac{1}{y^2} + c_1. \end{aligned}$$

We may multiply by y^2 to obtain

$$-\frac{y^2}{x} = -y + 1 + c_1 y^2,$$

and by x to obtain

$$-y^2 = -xy + x + c_1 x y^2,$$

so

$$y^2(c_1 x + 1) = x(y-1), \quad y(x) \neq 0, \quad x \neq 0$$

is a 1-parameter family of solutions, but it is not general because $y(x) = 0$ is a valid particular solution of F on $\mathbb{R}$.

Example 2.1.3: Separation of Variables 3

Consider $F(x, y, y') = x\sqrt{1-y} - y'\sqrt{1-x^2} = 0$. Assuming that $x^2 \neq 1$ and $y \neq 1$, we can write

$$\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{1-y}}\,\mathrm{d}y.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-y}}\,\mathrm{d}y.$$

meaning that

$$-\sqrt{1-x^2} = -2\sqrt{1-y} + c_1$$

is our 1-parameter family of solutions. We must restrict our solution for $|x| < 1$ and $y < 1$ due to the square roots. Our 1-parameter family of solutions is not general because $y(x) = 1$ is a valid particular solution of F on the interval.

Example 2.1.4: Separation of Variables 4

Consider $F(x, y, y') = x^2y' + 1 = 0$. In differential form, we obtain $x^2\,\mathrm{d}y + \mathrm{d}x = 0$. With the restriction $x \neq 0$, we have $\mathrm{d}y = -\frac{1}{x^2}\,\mathrm{d}x$. By integration, we have the 1-parameter family of solutions $y(x) = \frac{1}{x} + c$ on the interval $\mathbb{R} - \{0\}$.

Example 2.1.5: Separation of Variables 5

Consider $F(x, y, y') = x^2y' + y - 1 = 0$. In differential form, we obtain $x^2\,\mathrm{d}y + (y-1)\,\mathrm{d}x = 0$. With the restrictions $x \neq 0$ and $y(x) \neq 1$, we have $\frac{1}{y-1}\,\mathrm{d}y = -\frac{1}{x^2}\,\mathrm{d}x$. By integration, we have the 1-parameter family of solutions $\log|y-1| = \frac{1}{x} + c$ on the interval $\mathbb{R} - \{0\}$.

2.2 Lecture 7: February 3, 2023

2.2.1 Differential Equations with Homogeneous Coefficients

Consider the following definitions.

Definition 2.2.1: Homogeneous Functions

The function $f(x, y)$ is homogeneous, of order n, on some region $B \subseteq \mathbb{R}^2$ if and only if for all $x, y \in B$, either of the below hold.

1. The function $f(tx, ty) = t^n f(x, y)$ for some $n \in \mathbb{N}$.
2. The function $f(x, y) = x^n g(u)$ for some $u = \frac{y}{x}$ and $n \in \mathbb{N}$.
3. The function $f(x, y) = y^n h(u)$ for some $u = \frac{x}{y}$ and $n \in \mathbb{N}$.

Definition 2.2.2: Differential Equations With Homogeneous Coefficients

A first order differential equation $F(x, y, y') = Q(x, y)\frac{dy}{dx} + P(x, y) = 0$ has homogeneous coefficients if and only if both $P(x, y)$ and $Q(x, y)$ are both homogeneous functions of equal order.

Consider the following examples.

Example 2.2.1: Is it Homogeneous? 1

Determine whether $f(x, y) = 3x^2y - y^3$ is homogeneous on its domain.

Consider

$$\begin{aligned} f(tx, ty) &= 3(tx)^2(ty) - (ty)^3 \\ &= 3t^3x^2y - t^3y^3 \\ &= t^3(3x^2y - y^3) \\ &= t^3 f(x, y). \end{aligned}$$

Therefore, $f(x, y)$ is homogeneous of order 3.

Example 2.2.2: Is it Homogeneous? 2

Determine whether $f(x, y) = xy\sin(xy)$ is homogeneous on its domain.

Consider $f(tx, ty) = t^2xy\sin(t^2xy)$. There is no way in which this can be reduced to satisfy Definition 2.2.1.

Example 2.2.3: Is it Homogeneous? 3

Determine whether $f(x,y) = xy\sin\left(\frac{x}{y}\right) - x^2$ is homogeneous on its domain.

Consider

$$\begin{aligned} f(tx,ty) &= (tx)(ty)\sin\left(\frac{tx}{ty}\right) - (tx)^2 \\ &= t^2xy\sin\left(\frac{x}{y}\right) - t^2x^2 \\ &= t^2\left(xy\sin\left(\frac{x}{y}\right) - x^2\right) \\ &= t^2 f(x,y). \end{aligned}$$

Therefore, $f(x,y)$ is homogeneous of order 2.

While the first condition is, usually, easiest to use to show that a function is homogeneous, the other conditions are very useful in edge cases and for proofs. Consider the following theorem.

Theorem 2.2.1: Differential Equations With Homogeneous Coefficients are Separable

If $F(x,y,y') = Q(x,y)y' + P(x,y) = 0$ has homogeneous coefficients, F can be solved using separation of variables.

Proof. If $F(x,y,y')$ has homogeneous coefficients, $Q(x,y) = x^n g_1(u)$ and $P(x,y) = x^n g_2(u)$. We may make the substitution

$$F(x,y,y') = x^n g_1(u)\frac{\mathrm{d}y}{\mathrm{d}x} + x^n g_2(u) = 0.$$

Then, since $u = \frac{y}{x}$, $y = ux$, so $y' = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$. Therefore,

$$\begin{aligned} F(x,y,y') = x^n g_1(u)\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) + x^n g_2(u) = 0 \\ = ux^n g_1(u) + x^{n+1}g_1(u)\frac{\mathrm{d}u}{\mathrm{d}x} + x^n g_2(u) = 0 \\ = ux^n g_1(u)\,\mathrm{d}x + x^{n+1}g_1(u)\,\mathrm{d}u + x^n g_2(u)\,\mathrm{d}x = 0 \\ = \frac{g_1(u)}{x}\,\mathrm{d}x + \frac{g_1(u)}{u}\,\mathrm{d}u + \frac{g_2(u)}{ux}\,\mathrm{d}x = 0, \quad ux^{n+1} \neq 0 \\ = \frac{\mathrm{d}x}{x}\left(g_1(u) + \frac{g_2(u)}{u}\right) + \frac{g_1(u)}{u}\,\mathrm{d}u = 0 \\ = \frac{1}{x}\,\mathrm{d}x + \frac{g_1(u)}{ug_1(u) + g_2(u)}\,\mathrm{d}u = 0, \quad g_1(u) + \frac{g_2(u)}{u} \neq 0, \quad u \neq 0 \end{aligned}$$

so $F(x,y,y')$ is separable. □

2.3 Lecture 8: February 6, 2023

2.3.1 Using the Homogeneous Substitution to Solve Differential Equations

Consider the following example.

Example 2.3.1: ☢☢ A Homogeneous Substitution 1

Find a 1-parameter family of solutions for

$$F(x, y, y') = 2xy\frac{\mathrm{d}y}{\mathrm{d}x} - (x^2 + y^2) = 0.$$

We may rewrite the above as

$$2xy\,\mathrm{d}y - (x^2 + y^2)\,\mathrm{d}x = 0.$$

We see that $2(tx)(ty) = t^2(2xy)$ and $-((tx)^2 + (ty)^2) = -(t^2x^2 + t^2y^2) = -t^2(x^2 + y^2)$ so our differential equation has homogeneous coefficients of order 2. Let $y = ux$, meaning $\frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$. Then, our differential equation is

$$\begin{aligned} 0 &= 2ux^2\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) - (x^2 + u^2x^2) \\ &= 2u^2x^2 + 2ux^3\frac{\mathrm{d}u}{\mathrm{d}x} - x^2 - u^2x^2 \\ &= x^2(u^2 - 1)\,\mathrm{d}x + 2ux^3\,\mathrm{d}u \\ &= \frac{1}{x}\,\mathrm{d}x + \frac{2u}{u^2 - 1}, \quad x \neq 0, \quad u^2 - 1 \neq 0. \end{aligned}$$

Then, we may integrate to obtain

$$c_1 = \log|x| + \log|u^2 - 1|.$$

If we let $c_2 = e^{c_1}$, we further obtain

$$c_2 = |x||u^2 - 1|$$

By our earlier substitution, we have

$$c_2 = |x|\left|\left(\frac{y}{x}\right)^2 - 1\right|.$$

Our restrictions are $x \neq 0$, $y(x) \neq x$; note that $y(x) = x$ is a particular solution, so our 1-parameter solution is not general.

Example 2.3.2: ☢☢ A Homogeneous Substitution 2

Find a 1-parameter family of solutions for

$$F(x, y, y') = \left(x \log \frac{y}{x} - x\right) y' + y = 0.$$

We may rewrite the above as

$$\left(x \log \frac{y}{x} - x\right) \mathrm{d}y + y\,\mathrm{d}x = 0.$$

We see that $tx \log \frac{ty}{tx} - tx = t\left(x \log \frac{y}{x} - x\right)$ and $ty = ty$ so our differential equation has homogeneous coefficients of order 2. Let $y = ux$, meaning $\mathrm{d}y = u\,\mathrm{d}x + x\,\mathrm{d}u$. Then, our differential equation is

$$\begin{aligned}
0 &= \left(x \log\left(\frac{ux}{x}\right) - x\right)(u\,\mathrm{d}x + x\,\mathrm{d}u) + ux\,\mathrm{d}x \\
&= (x \log(u) - x)(u\,\mathrm{d}x + x\,\mathrm{d}u) + ux\,\mathrm{d}x \\
&= ux \log(u)\,\mathrm{d}x + x^2 \log(u)\,\mathrm{d}u - ux\,\mathrm{d}x - x^2\,\mathrm{d}u + ux\,\mathrm{d}x \\
&= ux \log(u)\,\mathrm{d}x + x^2(\log(u) - 1)\,\mathrm{d}u.
\end{aligned}$$

With the restrictions $x \neq 0$ and $u \log(u) \neq 0$, we have

$$\frac{1}{x}\,\mathrm{d}x = -\frac{\log(u) - 1}{u \log(u)}\,\mathrm{d}u = \left(-\frac{1}{u} + \frac{1}{u \log(u)}\right)\mathrm{d}u.$$

By integration, we have

$$\log|x| = -\log|u| + \log|\log|u|| + c,$$

which by our previous substitution, gives

$$\log|x| = -\log\left|\frac{y}{x}\right| + \log\left|\log\left|\frac{y}{x}\right|\right| + c,$$

on the interval $\mathbb{R} - \{0\}$.

Note that in Example 2.3.1, we used $\frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$, but in Example 2.3.2, we used $\mathrm{d}y = u\,\mathrm{d}x + x\,\mathrm{d}u$. The latter is much cleaner and will therefore be used onwards.

2.3.2 Exactness

Consider the following definitions.

Definition 2.3.1: Simply Connected Regions

A set $B \subseteq \mathbb{R}^2$ is simply connected if and only if every non-intersecting closed curve lying in B contains only points of B.

Definition 2.3.2: Exact Differential Equations

The differential equation $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$ is exact if and only if $P(x, y)$, $\frac{\partial P}{\partial y}$, $Q(x, y)$, and $\frac{\partial Q}{\partial x}$ are all continuous on a common simply connected region B and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 2.3.1: The Existence of a Potential

If the differential equation $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$ is exact, there exists $f(x, y)$ such that $\frac{\partial f}{\partial x} = P(x, y)$ and $\frac{\partial f}{\partial y} = Q(x, y)$. The function $f(x, y)$ can be considered as a potential function for the vector field $\vec{F} = [P(x, y), Q(x, y)]$.

As motivation, let $z = f(x, y)$. Recall that

$$\mathrm{d}z = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y.$$

If we take $\frac{\partial f}{\partial x} = P(x, y)$ and $\frac{\partial f}{\partial y} = Q(x, y)$, as stipulated in Definition 2.3.2, the right hand side of the above becomes

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y.$$

2.4 Lecture 9: February 8, 2023

2.4.1 Solving Exact Differential Equations

Recall the Fundamental Theorem of Calculus. That is, for some function $H(x)$, we have

$$H(x) = \int_{t_0}^{t} H'(t)\,\mathrm{d}t.$$

Then, if $H(x) = x$, we have $x = \int_{t_0}^{t} \mathrm{d}x$. Recall that for $z = f(x, y)$,

$$\mathrm{d}z = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y,$$

and if we take $\frac{\partial f}{\partial x} = P(x, y)$ and $\frac{\partial f}{\partial y} = Q(x, y)$,

$$\mathrm{d}z = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y.$$

Notice that $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$ is exact. Let the associated simply connected region be B. From the total differential of z, we have

$$f(x, y) = \int_B \mathrm{d}z = \int_B P(x, y)\,\mathrm{d}x + \int_B Q(x, y)\,\mathrm{d}y.$$

Note that all the above integrals exist, as by Definition 2.3.2, $P(x, y)$ and $Q(x, y)$ are continuous. Consider the following examples.

Example 2.4.1: ☢☢ Exact Differential Equation 1

Solve $F(x, y, y') = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - \cos x\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0$.

We must first write the differential equation in differential form to produce

$$(2y - \cos x)\,\mathrm{d}y + y\sin x\,\mathrm{d}x = 0.$$

Then, note that $\frac{\partial}{\partial y} y\sin x = \sin x$ and $\frac{\partial}{\partial x} = \sin x$. Since the partial derivatives are equal, and P, Q, $\frac{\partial Q}{\partial x}$, and $\frac{\partial P}{\partial y}$ are all continuous on the simply connected region $\mathbb{R}^2$, the differential equation is exact. We must now find some $z = f(x, y)$ such that $\frac{\partial f}{\partial x} = y\sin x$ and $\frac{\partial f}{\partial y} = 2y - \cos x$. Then,

$$\begin{aligned} f(x, y) &= \int y\sin x\,\mathrm{d}x \\ &= -y\cos x + c(y). \end{aligned}$$

Note that, now, we have $\frac{\partial f}{\partial y} = -\cos x + c'(y) = 2y - \cos x$. Therefore, $c'(y) = 2y$, and

$$\begin{aligned} f(x, y) &= \int c'(y)\,\mathrm{d}y - y\cos x \\ &= y^2 - y\cos x + c_1. \end{aligned}$$

Then, the 1-parameter family of solutions $f(x, y, c_1) = y^2 - y\cos x + c_1 = 0$ defines an implicit solution to $F(x, y, y')$ on $\mathbb{R}$.

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2.5 Lecture 10: Februrary 10, 2023

2.5.1 Integrating Factors

Consider the differential equation

$$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0,$$

and let $P(x,y)$, $\frac{\partial P}{\partial x}$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$ all be continuous on some common simply connected region B; however, $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ and there does not exist some $z = f(x,y)$ such that $\mathrm{d}z = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$. This motivates the following definition. We will be able to find some exact differential equation possessing the same solutions as the original differential equation.

Definition 2.5.1: Integrating Factors

Let $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}x = 0$, and let $P(x,y)$, $\frac{\partial P}{\partial x}$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$ all be continuous on some common simply connected region B. The function $\mu(x,y)$ is an integrating factor if and only if the differential equation

$$\mu(x,y)P(x,y)\,\mathrm{d}x + \mu(x,y)Q(x,y)\,\mathrm{d}y = 0.$$

is exact.

Theorem 2.5.1: Sameness of Solutions

If the integrating factor $\mu(x,y) \neq 0$ for all (x,y) and has continuous first order partial derivatives on the simply connected region B. Then,

$$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0$$

has the same solution as

$$\mu(x,y)P(x,y)\,\mathrm{d}x + \mu(x,y)Q(x,y)\,\mathrm{d}y = 0$$

Proof. Let $y(x)$ be a solution to the original differential equation. Then, consider

$$\begin{aligned}\mu(x,y(x))P(x,y(x))\,\mathrm{d}x + \mu(x,y(x))Q(x,y(x))\,\mathrm{d}y &= \mu(x,y(x))(P(x,y(x))\,\mathrm{d}x + \mu(x,y)Q(x,y(x))\,\mathrm{d}y)\\ &= \mu(x,y(x))(0)\\ &= 0,\end{aligned}$$

so $y(x)$ is also a solution to

$$\mu(x,y)P(x,y)\,\mathrm{d}x + \mu(x,y)Q(x,y)\,\mathrm{d}y = 0$$

on the same intervals of solution. On the other hand, if $\tilde{y(x)}$ is a solution to the exact differential equation, we have that

$$\mu(x,\tilde{y}(x))(P(x,\tilde{y}(x))\,\mathrm{d}x + \mu(x,\tilde{y})Q(x,\tilde{y}(x))\,\mathrm{d}y) = 0,$$

so either $\mu(x,\tilde{y}(x)) = 0$ or $P(x,\tilde{y}(x))\,\mathrm{d}x + \mu(x,\tilde{y})Q(x,\tilde{y}(x))\,\mathrm{d}y = 0$. By supposition, $\mu(x,y) \neq 0$ for all (x,y), so it must be the case that

$$P(x,\tilde{y}(x))\,\mathrm{d}x + \mu(x,\tilde{y})Q(x,\tilde{y}(x))\,\mathrm{d}y = 0,$$

meaning that $\tilde{y}(x)$ is a solution to the original differential equation. □

We seek to find the integrating factor $\mu(x, y)$ when it is purely a function of x. That is, $\mu(x, y) = \mu(x)$. If $\mu(x)$ is an integrating factor, we have that

$$\mu(x)P(x, y)\,\mathrm{d}x + \mu(x)Q(x, y)\,\mathrm{d}y = 0,$$

where

$$\frac{\partial}{\partial y}(\mu(x)P(x, y)) = \frac{\partial}{\partial x}(\mu(x)Q(x, y)).$$

By differentiation, we have

$$\mu(x)\frac{\partial P}{\partial y} = \mu(x)\frac{\partial Q}{\partial x} + Q\frac{\partial \mu}{\partial x}.$$

By rearrangement, we obtain

$$\mu(x)\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\mathrm{d}\mu}{\mathrm{d}x}.$$

By the suppositions $h(x) \neq 0$ and $Q(x, y) \neq 0$, we may divide and see

$$\frac{1}{\mu(x)}\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{1}{Q(x, y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

By recognizing the left hand side as the natural logarithm's derivative, we multiply by the differential function $\mathrm{d}x$ and antidifferentiate to obtain

$$\int \frac{1}{\mu(x)}\frac{\mathrm{d}\mu}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{Q(x, y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}x,$$

which implies

$$\log|\mu(x)| = \int \frac{1}{Q(x, y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}x.$$

Let $H(x)$ be the integrand of the right hand side. We may exponentiate to get

$$|\mu(x)| = e^{\int H(x)\,\mathrm{d}x}.$$

We need only consider $\mu(x) = e^{\int H(x)\,\mathrm{d}x}$, as any constants multiplied by $\mu(x)$ will not change the above derivation significantly. For other cases, where $\mu(x, y)$ is not purely a function of x, the derivation is much more complicated. Note that in the case that $\mu(x, y) = \mu(y)$, let

$$\tilde{H}(x) = \frac{1}{P(x, y)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$

and $\mu(x, y) = \mu(y) = e^{\int \tilde{H}(y)\,\mathrm{d}y}$.

As a review of the previous solution techniques, consider the following example.

Example 2.5.1: ☢☢ 3 Different Solution Techniques

Solve $F(x, y, y') = 2yy' + 6x = 0$ using separation of variables, homogeneous substitution, and integrating factors and exactness.

In differential form, we have

$$2y\,\mathrm{d}y = -6x\,\mathrm{d}x,$$

which integrates to give us $y^2 = -3x^2 + c_1$ as our implicit solution on $\mathbb{R}$. Before performing the homogeneous substitution, note that the equation $2y\,\mathrm{d}y + 6x\,\mathrm{d}x = 0$ has homogeneous coefficients of order 1 trivially. We make the substitution $y = ux$ to obtain

$$\begin{aligned} 0 &= 2ux(u\,\mathrm{d}x + x\,\mathrm{d}u) + 6x\,\mathrm{d}x \\ &= 2u^2x\,\mathrm{d}x + 2ux^2\,\mathrm{d}u + 6x\,\mathrm{d}x \\ &= x(2u^2 + 6)\,\mathrm{d}x + 2ux^2\,\mathrm{d}u. \end{aligned}$$

With the restriction $x \neq 0$, we may divide by x^2 and $2u^2 + 6$ to obtain

$$\frac{1}{x}\,\mathrm{d}x + \frac{u}{u^2+3}\,\mathrm{d}u,$$

and integration gives $\log|x| = \log|\sqrt{u^2+3}| + c_2$ meaning, if $\ell = e^{c_2}$, $|x| = \ell\sqrt{\frac{y^2}{x^2} + 3} = \sqrt{\frac{y^2+3x^2}{x^2}}$. Then, if $k = \ell^2$,

$$x^2 = k\left(\frac{y^2+3x^2}{x^2}\right)$$

The differential equation is also trivially exact on $\mathbb{R}^2$.

Example 2.5.2: An Differential Equation With an Integrating Factor Not a Function of x

Show that $2xy^3\,\mathrm{d}x + (2y + x^2y^2)\,\mathrm{d}y = 0$ is not exact and $\mu(x, y) \neq \mu(x)$.

Let $P(x, y) = 2xy^3$, so $\frac{\partial P}{\partial x} = 6xy^2$. Let $Q(x, y) = 2y + x^2y^2$, so $\frac{\partial Q}{\partial x} = 2xy^2$. Therefore, the differential equation is not exact. To show that the differential equation's integrating factor is not purely a function of x, suppose, for the sake of contradiction, that $\mu(x, y) = \mu(x)$. Then,

$$\mu(x) = e^{\int H(x)\,\mathrm{d}x}, \quad H(x) = \frac{1}{Q(x,y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{4xy^2}{2y + x^2y^2}.$$

Since $H(x)$ cannot be written as a pure function of x, we have a contradiction, and $\mu(x, y) \neq \mu(x)$. No such issue occurs if we assume

$$\mu(x, y) = \mu(y) = e^{\int \tilde{H}(y)\,\mathrm{d}y}, \quad \tilde{H}(y) = \frac{1}{P(x,y)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -\frac{2}{y}.$$

Note that another integrating factor is $\hat{\mu}(x, y) = e^{x^2y}$.

3

Higher Order Differential Equations

3.1 Lecture 11: February 13, 2023

3.1.1 Linear Ordinary Differential Equations

Consider the following definition.

Definition 3.1.1: Linear Ordinary Differential Equations

An nth order linear ordinary differential equation, with respect to I, is

$$F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y + Q(x) = 0$$

where all f_i, $0 \le i \le n$, and Q are continuous on some common interval I and $f_n(x) \neq 0$ for all $x \in I$.

Note that every first order linear differential equation can be written as

$$F(x, y, y') = f_1(x)y' + f_0(x)y + Q(x) = y' + \tilde{P}(x)y + \tilde{Q}(x) = 0.$$

and has the integrating factor $\mu(x, y) = \mu(x) = e^{\int \tilde{P}(x)\,\mathrm{d}x}$.

Consider the following non-example of a linear ordinary differential equation.

Example 3.1.1: A Non-Example of a Linear Ordinary Differential Equation

Consider $F(x, y, y', y'', y'') = y''' + y'' \log(x-1) + y \arccos(x) - \log(x-2) = 0$.

The differential equation $F(x, y, y', y'', y''')$ is not a linear ordinary differential equation since $\log(x-1)$, $\arccos(x)$, and $\log(x-2)$ are not continuous on any common interval.

3.2 Lecture 12: February 20, 2023

3.2.1 Existence and Uniqueness of Solutions for Linear Differential Equations

Consider the following definition.

Theorem 3.2.1: Linear Independence

Let $S = \{f_i : X \to \mathbb{C} : 1 \leq i \leq n\}$ be a set of continuous complex-valued functions defined on $X \subseteq \mathbb{R}$. Then, S is linearly independent if and only if, for all $x \in X$,

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0$$

with $c_1, \ldots, c_n \in \mathbb{C}$, is satisfied only if $c_1 = \cdots = c_n = 0$. Note that S is linearly dependent if and only if S is not linearly independent.

Consider the following theorems.

Theorem 3.2.2: Existence of Solutions to nth Order Linear Differential Equations (I)

Let $F(x, y, \ldots, y^{(n)})$ be an nth order linear differential equation with respect to I. Then, there exists $y(x)$ such that

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

are initial conditions on F satisfied for any $x_0 \in I$.

Theorem 3.2.3: Existence of Solutions to nth Order Linear Differential Equations (II)

Let $F(x, y, \ldots, y^{(n)})$ be an nth order linear differential equation with respect to I. Then, there exist n 1-parameter families of solutions $c_i y_i(x)$ for $c_i \in \mathbb{C}$ and some particular solution $y_i(x)$ valid on I. Moreover, the set $S = \{y_1(x), \ldots, y_n(x)\}$ is linearly independent and $c_1 y_1(x) + \cdots + c_n y_n(x)$ is an n-parameter family of solutions valid on I.

Theorem 3.2.4: Uniqueness of Solutions to nth Order Linear Differential Equations

Let $F(x, y, \ldots, y^{(n)})$ be an nth order linear differential equation with respect to I. Then, F has a unique solution valid on I satisfying n arbitrary initial conditions.

Therefore, by Theorem 3.2.4, the n-parameter family of solutions specified in Theorem 3.2.3 is the $y(x)$ specified in Theorem 3.2.2; note that $y(x)$ is a general solution.

Definition 3.2.1: Wronskians

Let $S = \{f_i : X \to \mathbb{C} : 1 \leq i \leq n\}$ be a set of continuous complex-valued functions defined on $X \subseteq \mathbb{R}$. Suppose that each $f_i \in S$, $1 \leq i \leq n$ us $(n-1)$-times continuously differentiable. Then, the Wronskian of S is given by

$$W(S)(x) = \det \begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x). \end{bmatrix}$$

Theorem 3.2.5: Wronskians Determine Linear Independence

If S is linearly dependent, $W(S)(x) = 0$ for all $x \in X$. If $W(S)(x) \neq 0$ for some $x \in X$, S is linearly independent on X.

Consider the following example.

Example 3.2.1: Showing Linear Independence With the Wronskian

Show that $\{x, e^{2x}, e^{3x}\}$ are linearly independent on $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$.

Note that all elements of S are $(n-1)$-times continuously differentiable. Then,

$$\begin{aligned} W(S)(x) &= \det \begin{bmatrix} x & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 0 & 4e^{2x} & 9e^{3x} \end{bmatrix} \\ &= -6xe^{5x} + 5e^{5x}. \end{aligned}$$

Note that $W(S)(0) = 5 \neq 0$, so S is linearly independent.

Theorem 3.2.6: Wronksians are Necessary and Sufficient in Some Cases

If $S = \{y_i : X \to \mathbb{C}\}$ and all y_i, $1 \leq i \leq n$, are solutions to the same nth order linear differential equation, $W(S)(x) = 0$ on X if and only if S is linearly dependent.

3.3 Lecture 13: Feburary 22, 2023

3.3.1 Linear Differential Equations With Constant Coefficients

We will now study linear differential equations with constant coefficients. Consider the following definitions.

Definition 3.3.1: Linear Differential Equations With Constant Coefficients

An nth order linear homogeneous differential equation, with respect to I, is

$$F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x) + Q(x) = 0,$$

where all $f_i = a_i$, $0 \leq i \leq n$, where $a_i \in \mathbb{C}$ and Q are continuous on some common interval I.

Definition 3.3.2: Homogeneous Linear Differential Equations

An nth order linear homogeneous differential equation, with respect to I, is

$$F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x) + Q(x) = 0,$$

where all f_i, $0 \leq i \leq n$, and Q are continuous on some common interval I and $f_n(x) \neq 0$ and $Q(x) = 0$ for all $x \in I$.

Consider the following theorem.

Theorem 3.3.1: Decomposing a Solution of a Linear Differential Equation

Let $F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x) = 0$ have the solution

$$y_c(x, c_1, \dots, c_n) = c_1 y_1(x) + \cdots + c_n y_n(x).$$

Then, the solution to

$$\tilde{F}(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x) + Q(x) = 0$$

is given by

$$y(x, c_1, \dots, c_n) = y_c(x, c_1, \dots, c_n) + y_p(x)$$

where $y_p(x)$ is any particular solution to $\tilde{F}$.

3.4 Lecture 14: February 24, 2023

3.4.1 Solving Linear Homogeneous Equations With Constant Coefficients: Part I

Consider the second order homogeneous linear differential equation with constant coefficients

$$F(x, y, y', y'') = ay'' + by' + cy = 0,$$

for $a, b, c \in \mathbb{C}$. We may, without loss of generality, rewrite this as

$$\tilde{F}(x, y, y', y'') = y'' + Ay' + By = 0,$$

since $a \neq 0$. Choose some $\alpha, \beta \in \mathbb{C}$ such that $-(\alpha+\beta) = A$ and $\alpha\beta = B$. Now, consider $z = y'(x) - \alpha y(x)$. Then, note that $\frac{\mathrm{d}z}{\mathrm{d}x} - \beta z = 0$. That is,

$$\begin{aligned} 0 &= y'' - \alpha y'(x) - \beta y'(x) + \alpha\beta y(x) \\ &= y'' - (\alpha+\beta)y' + 2\beta y \\ &= y'' + Ay' + By. \end{aligned}$$

Now, since $\frac{\mathrm{d}z}{\mathrm{d}x} - \beta z = 0$, we have, by separation of variables, $z = c_1 e^{\beta x}$ where $c_1 \in \mathbb{C}$ and $x \in \mathbb{R}$. Now, we have

$$z = y'(x) - \alpha y(x) = c_1 e^{\beta x}.$$

We may use the integrating factor $\mu(x) = e^{\int -\alpha \, \mathrm{d}x} = e^{-\alpha x}$. Consider

$$\begin{aligned} y'(x)e^{-\alpha x} - \alpha y(x)e^{-\alpha x} &= c_1 e^{\beta x} e^{-\alpha x} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(y(x)e^{-\alpha x}). \end{aligned}$$

By integration,

$$\begin{aligned} y(x) &= e^{\alpha x} \int c_1 e^{(\beta-\alpha)x} \, \mathrm{d}x \\ &= \begin{cases} e^{\alpha x}(c_1 x + c_2) = c_1 x e^{\alpha x} + c_2 e^{\alpha x} & \alpha = \beta \\ e^{\alpha x}\left(\frac{c_1 e^{(\beta-\alpha)x}}{\beta-\alpha} + c_2\right) = \frac{c_1}{\beta-\alpha} e^{\beta x} + c_2 e^{\alpha x} = \tilde{c}_1 e^{\beta x} + c_2 e^{\alpha x} & \alpha \neq \beta \end{cases}. \end{aligned}$$

In the first case $\alpha = \beta$, we have that $\alpha = \beta$ is a double root of the polynomial $m^2 - 2\alpha m + \alpha^2 = 0$. In the second case, α and β are distinct solutions to the polynomial $m^2 - (\alpha+\beta)m + \alpha\beta = 0$. Both solutions are truly general solutions, because there are $n = 2$ linearly independent terms in the linear combination.

3.5 Lecture 15: February 28, 2023

3.5.1 Solving Linear Homogeneous Equations With Constant Coefficients: Part II

With the discoveries in the previous section, we provide the following definition and theorem.

Definition 3.5.1: Characteristic Polynomials

Let $F(x, y, \dots, y^{(n)})_{\text{hom}} = a_n y^{(n)} + \cdots + a_0 y = 0$ be an ordinary differential equation. The characteristic polynomial of F is given by

$$p_F(m) = a_n m^n + \cdots + a_0 m^0.$$

Theorem 3.5.1: Finding Solutions With Characteristic Polynomials

If $F(x, y, \dots, y^{(n)})_{\text{hom}} = a_n y^{(n)} + \cdots + a_0 y = 0$ has a characteristic polynomial with factors into distinct factors,

$$y(x, c_1, \dots c_n) = c_1 e^{m_1 x} + \cdots + c_n e^{m_n x},$$

If $F(x, y, \dots, y^{(n)})_{\text{hom}} = a_n y^{(n)} + \cdots + a_0 y = 0$ has a characteristic polynomial that factors into k distinct roots, $k < n$, that is,

$$p_F(m) = (m - m_1)^{r_1} + \cdots + (m - m_k)^{r_k},$$

where $r_1 + \cdots + r_k = n$. We have

$$y(x, c_1, \dots, c_n) = (c_1 x^0 + \cdots + c_{r_1} x^{r_1 - 1}) e^{m_1 x} + \cdots + (c_{n - r_k} x^0 + \cdots + c_n x^{n - r_k}) e^{m_n x}.$$

We will now provide an important remark about Theorem 3.5.1 for the ease of understanding. Let $p_F(m)$ be a characteristic polynomial with root $m_k \in \mathbb{C}$. We may use Euler's Formula to remove imaginary exponents from the nth parameter family given by Theorem 3.5.1. For convenience, a statement of Euler's Formula is given in Theorem 3.5.2.

Theorem 3.5.2: Euler's Formula

For $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Recall that $i^0 = i^4 = i^8 = \cdots = 1$, $i^1 = i^5 = i^9 = \cdots = i$, $i^2 = i^6 = i^10 = \cdots = -1$, and $i^3 = i^7 = i^{11} = \cdots = -i$. If we define, for $z \in \mathbb{C}$,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

we have that

$$\begin{aligned} e^{i\theta} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= \cos\theta + i\sin\theta, \end{aligned}$$

as desired. □

To use Euler's Formula, suppose that $p_L(m)$ is our characteristic polynomial with root $m_k = a + bi \in \mathbb{C}$. Then, $\overline{m_\ell} = a - bi \in \mathbb{C}$ is also a root of $p_L(m)$. Without loss of generality, suppose both m_k and m_ℓ have multiplicity 1. Let our solution components be

$$y_k(x) = e^{(a+bi)x} = e^{ax}(\cos(bx) + i\sin(bx)), \quad y_\ell(x) = e^{(a-bi)x} = e^{ax}(\cos(bx) - i\sin(bx)).$$

Since the linear combination of y_k and y_ℓ is a solution, $\frac{1}{2}y_k(x) + \frac{1}{2}y_\ell(x) = e^{ax}\cos(bx)$ is a solution, and $\frac{1}{2i}y_k(x) - \frac{1}{2i}y_\ell(x) = e^{ax}\sin(bx)$ is also a solution. Thus,

$$y_{k,\ell} = c_k e^{ax}\cos(bx) + c_\ell e^{ax}\sin(bx)$$

is a solution.

Consider the following examples.

Example 3.5.1: ☢☢ Real Roots 1

Find a general solution to

$$2y'' + y' - 6y = 0.$$

The characteristic polynomial is

$$2m^2 + m - 6 = 0,$$

which can be rewritten as

$$(2m-3)(m+2) = 0.$$

We can see that $m = -2$ and $m = \frac{3}{2}$ are solutions to the characteristic polynomial. Therefore,

$$y = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}.$$

Example 3.5.2: ☢☢ Real Roots 2

Find a general solution to

$$y''' - 6y'' + 11y' - 6y = 0.$$

The characteristic polynomial is

$$m^3 - 6m^2 + 11m - 6 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$\begin{array}{r|rrrr}
 & & x^2 & -5x & +6. \\ \hline
x-1 & x^3 & -6x^2 & +11x & -6 \\
 & -x^3 & +x^2 & & \\ \hline
 & & -5x^2 & +11x & \\
 & & 5x^2 & -5x & \\ \hline
 & & & 6x & -6 \\
 & & & -6x & +6 \\ \hline
 & & & & 0
\end{array}$$

We can see that $m = 1$, $m = 2$, and $m = 3$ are solutions to the characteristic polynomial. Therefore,

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

Example 3.5.3: ☢☢ Real Roots 3

Find a general solution to

$$2y''' - 7y'' + 4y' + 4y = 0.$$

The characteristic polynomial is

$$2m^3 - 7m^2 + 4m + 4 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$\begin{array}{r|l}
 & 2x^2 - 3x - 2. \\
\hline
x - 2 & 2x^3 - 7x^2 + 4x + 4 \\
 & -2x^3 + 4x^2 \\
\hline
 & -3x^2 + 4x \\
 & 3x^2 - 6x \\
\hline
 & -2x + 4 \\
 & 2x - 4 \\
\hline
 & 0
\end{array}$$

Therefore, the characteristic polynomial may be rewritten as

$$(m - 2)(2m^2 - 3m - 2) = 0.$$

The above quadratic can be factored as

$$(2m + 1)(m - 2).$$

Therefore, the characteristic polynomial is

$$(m - 2)^2(2m + 1) = 0.$$

Therefore,

$$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-\frac{1}{2}x}.$$

Example 3.5.4: ☢☢ Real Roots 4

Find a general solution to

$$y'''' - y''' - 13y'' + y' + 12y = 0.$$

The characteristic polynomial is

$$m^4 - m^3 - 13m^2 + m + 12 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$\begin{array}{r|rrrrr}
 & x^3 & & -13x & -12. & \\
\cline{2-6}
x-1 & x^4 & -x^3 & -13x^2 & +x & +12 \\
 & -x^4 & +x^3 & & & \\
\cline{2-3}
 & & & -13x^2 & +x & \\
 & & & 13x^2 & -13x & \\
\cline{4-5}
 & & & & -12x & +12 \\
 & & & & 12x & -12 \\
\cline{5-6}
 & & & & & 0
\end{array}$$

Therefore, the characteristic polynomial may be rewritten as

$$(m-1)(m^3 - 13m - 12) = 0.$$

We may also perform polynomial long division with the above cubic, with the same substitution as before, producing

$$\begin{array}{r|rrrr}
 & x^2 & -x & -12. & \\
\cline{2-5}
x+1 & x^3 & & -13x & -12 \\
 & -x^3 & -x^2 & & \\
\cline{2-3}
 & & -x^2 & -13x & \\
 & & x^2 & +x & \\
\cline{3-4}
 & & & -12x & -12 \\
 & & & 12x & +12 \\
\cline{4-5}
 & & & & 0
\end{array}$$

Therefore, we have

$$(m-1)(m+1)(m-4)(m+3) = 0.$$

Therefore,

$$y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{4x} + c_4 e^{-3x}.$$

Example 3.5.5: ☢☢ Real Roots 5

Find a general solution to

$$y''''' - 4y'''' - 8y''' + 14y'' + 7y' - 10y = 0.$$

The characteristic polynomial is

$$m^5 - 4m^4 - 8m^3 + 14m^2 + 7m - 10 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$
\begin{array}{r|l}
 & x^4 - 3x^3 - 11x^2 + 3x + 10. \\ \hline
x - 1 & x^5 - 4x^4 - 8x^3 + 14x^2 + 7x - 10 \\
 & \underline{-x^5 + x^4} \\
 & \quad -3x^4 - 8x^3 \\
 & \quad \underline{3x^4 - 3x^3} \\
 & \qquad -11x^3 + 14x^2 \\
 & \qquad \underline{11x^3 - 11x^2} \\
 & \qquad\qquad 3x^2 + 7x \\
 & \qquad\qquad \underline{-3x^2 + 3x} \\
 & \qquad\qquad\quad 10x - 10 \\
 & \qquad\qquad\quad \underline{-10x + 10} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Therefore, the characteristic polynomial may be rewritten as

$$(m - 1)(m^4 - 3m^3 - 11m^2 + 3m + 10) = 0.$$

We may also perform polynomial long division with the above quartic, with the same substitution as before, producing

$$
\begin{array}{r|l}
 & x^3 - 5x^2 - x + 5. \\ \hline
x + 2 & x^4 - 3x^3 - 11x^2 + 3x + 10 \\
 & \underline{-x^4 - 2x^3} \\
 & \quad -5x^3 - 11x^2 \\
 & \quad \underline{5x^3 + 10x^2} \\
 & \qquad -x^2 + 3x \\
 & \qquad \underline{x^2 + 2x} \\
 & \qquad\quad 5x + 10 \\
 & \qquad\quad \underline{-5x - 10} \\
 & \qquad\qquad 0
\end{array}
$$

We then obtain

$$(m - 1)(m + 2)(m^3 - 5m^2 - m + 5) = 0.$$

We may proceed similarly to produce

$$(m - 1)^2(m + 2)(m - 5)(m + 1) = 0.$$

Therefore,

$$y = c_1e^x + c_2xe^x + c_3e^{-2x} + c_4e^{-x} + c_5e^{5x}.$$

Exercise 3.5.1: ☢☢ Complex Roots 1

Find a general solution to

$$9y'' + 6y' + 4y = 0.$$

The characteristic polynomial is

$$9m^2 + 6m + 4.$$

By the quadratic formula,

$$m = -\frac{1}{3} \pm \frac{\sqrt{3}}{3}.$$

Therefore,

$$y = e^{-\frac{1}{3}} \left(c_1 \cos\left(x\frac{\sqrt{3}}{3} \right) + c_2 \sin\left(x\frac{\sqrt{3}}{3} \right) \right).$$

Exercise 3.5.2: ☢☢ Complex Roots 2

Find a general solution to

$$y''' + 27y' = 0.$$

The characteristic polynomial is

$$m^3 + 27m = 0,$$

or

$$m(m^2 + 27) = 0,$$

which can further be factored as

$$m(m + 3i\sqrt{3})(m - 3i\sqrt{3}) = 0.$$

Solutions are then $m = 0$ and $m = \pm 3i\sqrt{3}$. Therefore,

$$y = c_1 + c_2 \cos(3x\sqrt{3}) + c_3 \sin(3x\sqrt{3}).$$

4
Systems of Differential Equations

Function	Transform	Function	Transform
1	$\frac{1}{s}$	$tf(t)$	$-F'(s)$
t	$\frac{1}{s^2}$	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$t^a,\ a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\ d\sigma$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
e^{at}	$\frac{1}{s-a}$	$e^{at} f(t)$	$F(s-a)$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$f'(t)$	$sF(s)-f(0)$	$\int_0^t f(\tau)\ d\tau$	$\frac{F(s)}{s}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$u_c(t)$	$\frac{e^{-cs}}{s}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$(f*g)(t)$	$F(s)G(s)$	$\delta(t-c)$	e^{-cs}