MATH3430: DIFFERENTIAL EQUATIONS

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TEXTBOOK: MORRIS TENENBAUM & HARRY POLLARD

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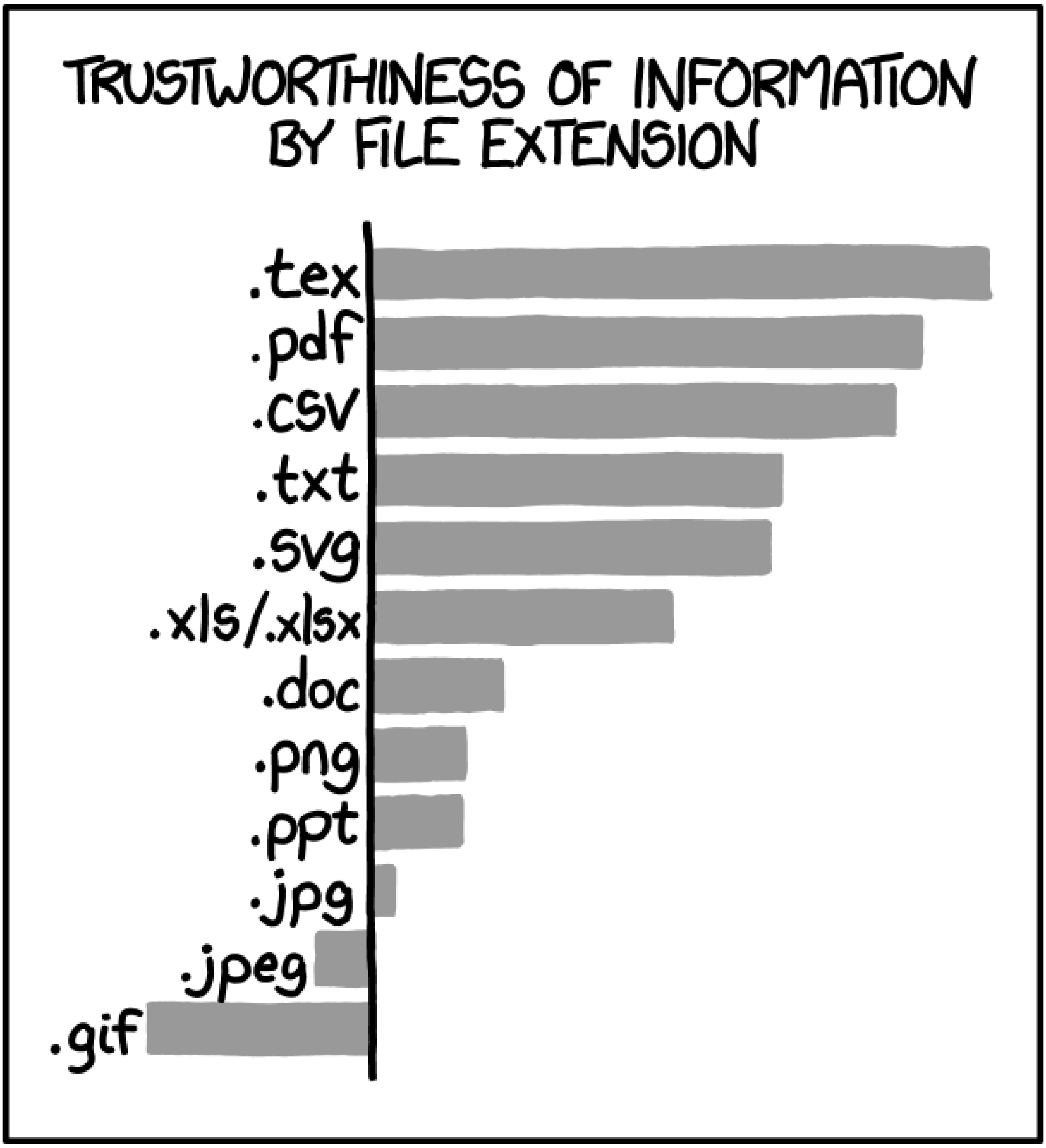
EDITION 1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Preface

To the interested reader,

This document is a compilation of lecture notes taken during the Spring 2023 semester for MATH3430: Ordinary Differential Equations at the University of Colorado Boulder. The course used *Ordinary Differential Equations*[1] by Morris Tenenbaum and Harry Pollard as its primary text. As such, many theorems, definitions, and content may be quoted from the book. This course was taught by Sheagan John, Ph. D.

The author would like to provide the following resources for students currently taking a differential equations course:

1. MIT OpenCourseWare Differential Equations Lectures From Spring 2006.
2. 3Blue1Brown's *Overview of Differential Equations*.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. Please keep in mind that the author wrote this resource during his second semester of his undergraduate studies. If an error needs to be resolved, please contact Adithya Bhaskara at adithya.bhaskara@colorado.edu.

Finally, the author would like to dedicate this set of lecture notes to *Aidan Janney*, *Erika Sjöblom*, and *Tate McDonald*, three of the author's closest friends who plan to take Differential Equations in the upcoming semester; Fall 2023, at the time of writing.

Best Regards,
Adithya Bhaskara

REVISED: January 24, 2023

[1] Tenenbaum, M., & Pollard, H. (1985). *Ordinary Differential Equations*. Dover Publications.

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Basic Notions

1.1 Lecture 1: January 20, 2023

1.1.1 Definition of an Ordinary Differential Equation

Consider the following definitions.

Definition 1.1.1: Ordinary Differential Equations

An ordinary differential equation is an equation of the form

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

where x is an independent variable and y is nth order differentiable.

We remark that every ordinary differential equation is valid as an expression only when we specify the values of x for which it is defined.

Definition 1.1.2: Order of a Differential Equation

The order of an ordinary differential equation is the highest nontrivial derivative present in the equation.

Consider the following ordinary differential equations. From now on, we may omit the term "ordinary," as partial differential equations are not considered in this text.

1. $F(x, y, y') = \cos(xy') + y^2y' + x^2 = 0$ is a first order differential equation.
2. $F(x, y, y', y'') = -\frac{1}{1-x^2} + y'' = 0$ is a second order differential equation.
3. $F(x, y, y', y'', y''', y'''') = e^{xy}y'''' - x^2y'' - \sin x = 0$ is a fourth order differential equation.

1.2 Lecture 2: January 23, 2023

1.2.1 Explicit and Implicit Solutions

We will now consider explicit and implicit solutions to differential equations.

Definition 1.2.1: Explicit Solutions to Ordinary Differential Equations

Let $F(\cdots) = 0$ be a differential equation defined on the interval I. Then, an explicit solution to F is a function

$$y : \mathbb{R} \to \mathbb{C}$$

for which $y(x)$ is well-defined on some set X such that $I \cap X \neq \emptyset$ and y satisfies the differential equation for all $x \in I$.

Definition 1.2.2: Implicit Functions

Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(x, y) = 0$. Then, f defines y as an implicit function of x if and only if

1. There exists an explicit function $g(x)$ such that $y = g(x)$ on some interval $I \subseteq \mathbb{R}$.
2. For all $x \in I$, $f(x, g(x)) = 0$.

Definition 1.2.3: Implicit Solutions to Ordinary Differential Equations

Let $F(\cdots) = 0$ be a differential equation defined on the interval I. Then, an implicit solution to F is a relation $f(x, y) = 0$ if and only if f defines y as an implicit function of x on I, and if $y(x)$ is an explicit solution to the differential equation.

Consider the following examples.

Example 1.2.1: ☢☢ Verifying Explicit Solutions 1

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y', y'') = x^2 y'' + xy' = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = \log x$, where $\log x = \log_e x$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{1}{x}$. The second derivative of $y(x)$ is $y''(x) = -\frac{1}{x^2}$. Then, we have

$$\begin{aligned} x^2 y''(x) + xy'(x) &= x^2 \cdot -\frac{1}{x^2} + x \cdot \frac{1}{x} \\ &= -1 + 1 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x > 0\}$. The differential equation is defined on $\{x : x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x : x \in \mathbb{R}\} \cap \{x : x \in \mathbb{R} : x > 0\} \cap \{x : x \in \mathbb{R} : x \neq 0\} = \{x : x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = \log x$ is an explicit solution for F on $\{x : x \in \mathbb{R} : x > 0\}$. □

Example 1.2.2: ☢☢ Verifying Explicit Solutions 2

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y') = yy' - 4 = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = 2\sqrt{2x}$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{2}{\sqrt{2x}}$. Then, we have

$$\begin{aligned} y(x)y'(x) - 4 &= 2\sqrt{2x} \cdot \frac{2}{\sqrt{2x}} - 4 \\ &= 4 - 4 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x \geq 0\}$. The differential equation is defined on $\{x : x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x : x \in \mathbb{R}\} \cap \{x : x \in \mathbb{R} : x \geq 0\} \cap \{x : x \in \mathbb{R} : x \neq 0\} = \{x : x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = 2\sqrt{2x}$ is an explicit solution for F on $\{x : x \in \mathbb{R} : x > 0\}$. □

Example 1.2.3: ☢☢ Verifying Implicit Solutions 1

Determine whether $f(x, y) = x^2 + y^2 + 4 = 0$ provides an implicit solution to

$$F(x, y, y') = 2x + 2y'' = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = \sqrt{-x^2 - 4}$ and $g_2(x) = -\sqrt{-x^2 - 4}$; these functions are defined nowhere on $\mathbb{R}$. Thus, $f(x, y) = 0$ does not provide an implicit solution to the differential equation.

Example 1.2.4: ☢☢ Verifying Implicit Solutions 2

Determine whether $f(x, y) = xy - y^2 = 0$ provides an implicit solution to

$$F(x, y, y', y'') = \frac{1}{y - x^2} y'' + yy' - y = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y) = xy - y^2 = y(x - y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = 0$ and $g_2(x) = x$. Both g_1 and g_2 are defined on $\mathbb{R}$. We see that $f(x, g_1(x)) = 0$ for all $x \in \mathbb{R}$ and $f(x, g_2(x)) = x^2 - x^2 = 0$ for all $x \in \mathbb{R}$. Therefore, f defines y as an implicit function of x.

Note that it is bad practice to immediately differentiate the relation $f(x, y)$. For example, in Example 1.2.3, if we immediately differentiated f, we would indeed obtain a symbolic equivalent to the differential equation, but we would not account for the domain restrictions.

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First Order Differential Equations

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Higher Order Differential Equations

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Systems of Differential Equations