MATH3430: DIFFERENTIAL EQUATIONS

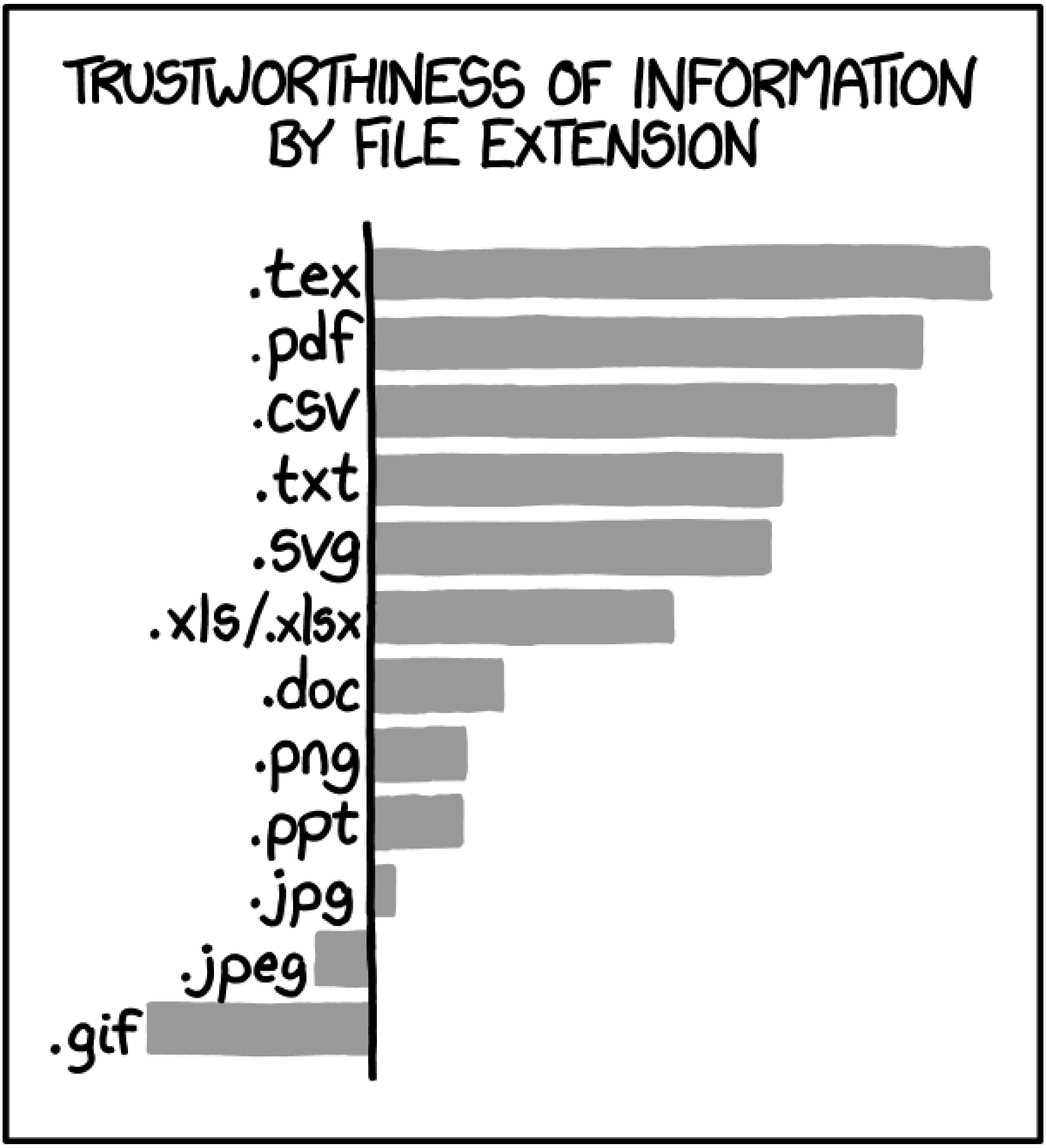
ADITHYA BHASKARA
PROFESSOR: SHEAGAN JOHN

TEXTBOOK: MORRIS TENENBAUM & HARRY POLLARD

UNIVERSITY OF COLORADO BOULDER

Draft: February 3, 2023

EDITION 1


TRUSTWORTHINESS OF INFORMATION BY FILE EXTENSION
.tex
.pdf
.csv
.txt
.svg
.xls/.xlsx
.doc
.png
.ppt
.jpg
.jpeg
.gif

Contents

Draft: February 3, 2023

Preface

To the interested reader,

This document is a compilation of lecture notes taken during the Spring 2023 semester for MATH3430: Ordinary Differential Equations at the University of Colorado Boulder. The course used *Ordinary Differential Equations*[1] by Morris Tenenbaum and Harry Pollard as its primary text. As such, many theorems, definitions, and content may be quoted from the book. This course was taught by Sheagan John, Ph. D.

The author would like to provide the following resources for students currently taking a differential equations course:

1. MIT OpenCourseWare Differential Equations Lectures From Spring 2006.
2. 3Blue1Brown's *Overview of Differential Equations*.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. Please keep in mind that the author wrote this resource during his second semester of his undergraduate studies. If an error needs to be resolved, please contact Adithya Bhaskara at adithya.bhaskara@colorado.edu.

Finally, the author would like to dedicate this set of lecture notes to *Aidan Janney*, *Erika Sjöblom*, and *Tate McDonald*, three of the author's closest friends who plan to take Differential Equations in the upcoming semester; Fall 2023, at the time of writing.

Best Regards,
Adithya Bhaskara

REVISED: February 3, 2023

[1] Tenenbaum, M., & Pollard, H. (1985). *Ordinary Differential Equations.* Dover Publications.

1
Basic Notions

1.1 Lecture 1: January 20, 2023

1.1.1 Definition of an Ordinary Differential Equation

Consider the following definitions.

Definition 1.1.1: Ordinary Differential Equations

An ordinary differential equation is an equation of the form

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

where x is an independent variable and y is nth order differentiable.

We remark that every ordinary differential equation is valid as an expression only when we specify the values of x for which it is defined.

Definition 1.1.2: Order of a Differential Equation

The order of an ordinary differential equation is the highest nontrivial derivative present in the equation.

Consider the following ordinary differential equations. From now on, we may omit the term "ordinary," as partial differential equations are not considered in this text.

1. $F(x, y, y') = \cos(xy') + y^2y' + x^2 = 0$ is a first order differential equation.
2. $F(x, y, y', y'') = -\frac{1}{1-x^2} + y'' = 0$ is a second order differential equation.
3. $F(x, y, y', y'', y''', y'''') = e^{xy}y'''' - x^2y'' - \sin x = 0$ is a fourth order differential equation.

1.2 Lecture 2: January 23, 2023

1.2.1 Explicit and Implicit Solutions

We will now consider explicit and implicit solutions to differential equations.

Definition 1.2.1: Explicit Solutions to Ordinary Differential Equations

Let $F(x, y, y', \dots, y^{(n)}) = 0$ be a differential equation defined on the interval I. Then, an explicit solution to F is a function

$$f : \mathbb{R} \to \mathbb{C}$$

for which $f(x)$ is well-defined on some set X such that $I \cap X \neq \emptyset$ and f satisfies the differential equation for all $x \in I$. That is,

$$F(x, f, f', \dots, f^{(n)}) = 0.$$

Definition 1.2.2: Implicit Functions

Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(x, y) = 0$. Then, f defines y as an implicit function of x if and only if

1. There exists an explicit function $g(x)$ such that $y = g(x)$ on some interval $I \subseteq \mathbb{R}$.
2. For all $x \in I$, $f(x, g(x)) = 0$.

Consider the following examples.

Example 1.2.1: Is it an Implicit Function? 1

Does $f(x, y) = x^2 + y^2 - 25 = 0$ define an implicit function of x on $\{x : x \in \mathbb{R} : -5 \leq x \leq 5\}$?

We consider solving for y. We obtain $y = \pm\sqrt{25 - x^2}$. Therefore we define $y_1(x) = \sqrt{25 - x^2}$ and $y_2(x) = -\sqrt{25 - x^2}$. For $y_1(x)$, we see that

$$f(x, y_1(x)) = x^2 + (\sqrt{25 - x^2})^2 - 25 = x^2 + 25 - x^2 - 25 = 0,$$

as desired. Also, $y_1(x)$ is defined for on $\{x : x \in \mathbb{R} : -5 \leq x \leq 5\}$. Therefore, $f(x, y)$ does define an implicit function of x. Upon further inspection, $f(x, y)$ defines more than one implicit function of x.

Example 1.2.2: ☢☢ Is it an Implicit Function? 2

Does $f(x, y) = \sqrt{x^2 - y^2} + \arccos\left(\frac{x}{y}\right) = 0, \quad y \neq 0$ define an implicit function?

Here, it is too difficult to solve for y. Instead, we notice that $\arccos(1) = 0$. Therefore, we try $y(x) = x$. By substitution,

$$f(x, y(x)) = \sqrt{x^2 - x^2} + \arccos\left(\frac{x}{x}\right) = 0,$$

as desired. Also, $y(x)$ is defined on $\mathbb{R}$. Therefore, $f(x, y)$ defines an implicit function on $\mathbb{R} - \{0\}$.

Consider the following definition.

Definition 1.2.3: Implicit Solutions to Ordinary Differential Equations

Let $F(x, y, y', \dots, y^{(n)}) = 0$ be a differential equation defined on the interval I. Then, an implicit solution to F is a relation $f(x, y) = 0$ if and only if f defines y as an implicit function of x on I, and if $y(x)$ is an explicit solution to the differential equation.

Consider the following examples.

Example 1.2.3: ☢☢ Verifying Explicit Solutions 1

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y', y'') = x^2 y'' + xy' = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = \log x$, where $\log x = \log_e x$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{1}{x}$. The second derivative of $y(x)$ is $y''(x) = -\frac{1}{x^2}$. Then, we have

$$\begin{aligned} x^2 y''(x) + xy'(x) &= x^2 \cdot -\frac{1}{x^2} + x \cdot \frac{1}{x} \\ &= -1 + 1 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x > 0\}$. The differential equation is defined on $\{x : x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x : x \in \mathbb{R}\} \cap \{x : x \in \mathbb{R} : x > 0\} \cap \{x : x \in \mathbb{R} : x \neq 0\} = \{x : x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = \log x$ is an explicit solution for F on $\{x : x \in \mathbb{R} : x > 0\}$. □

Example 1.2.4: ☢☢ Verifying Explicit Solutions 2

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y') = yy' - 4 = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = 2\sqrt{2x}$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{2}{\sqrt{2x}}$. Then, we have

$$\begin{aligned} y(x)y'(x) - 4 &= 2\sqrt{2x} \cdot \frac{2}{\sqrt{2x}} - 4 \\ &= 4 - 4 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x \geq 0\}$. The differential equation is defined on $\{x : x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x : x \in \mathbb{R}\} \cap \{x : x \in \mathbb{R} : x \geq 0\} \cap \{x : x \in \mathbb{R} : x \neq 0\} = \{x : x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = 2\sqrt{2x}$ is an explicit solution for F on $\{x : x \in \mathbb{R} : x > 0\}$. □

Example 1.2.5: ☢☢ Verifying Explicit Solutions 3

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y', y'') = y''^3 + y'^2 - y - 3x^2 - 8.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = x^2$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = 2x$. The second derivative of $y(x)$ is $y''(x) = 2$. Then, we have

$$\begin{aligned} y''^3 + y'^2 - y - 3x^2 - 8 &= (2)^3 + (2x)^2 - x^2 - 3x^2 - 8 \\ &= 8 + 4x^2 - x^2 - 3x^2 - 8 = 0 \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\mathbb{R}$. The differential equation is defined on $\mathbb{R}$. We have no additional restrictions. Therefore, the solution set is $\mathbb{R}$. We can then state that $y(x) = x^2$ is an explicit solution for F on $\mathbb{R}$. □

Example 1.2.6: Verifying Explicit Solutions 4

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y') = (x + y)^2 - y' = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = \tan x - x$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \sec^2 x - 1 = \tan^2 x$. Then, we have

$$\begin{aligned}(x + \tan x - x)^2 - \tan^2 x &= (\tan x)^2 - \tan^2 x \\ &= \tan^2 x - \tan^2 x = 0\end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. The differential equation is defined on $\{x : x \in \mathbb{R}\}$. We have no additional restrictions. Therefore, the solution set is

$$\{x : x \in \mathbb{R}\} \cap \{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\} = \{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}.$$

We can then state that $y(x) = \tan x - x$ is an explicit solution for F on $\{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. $\square$

Example 1.2.7: Verifying Implicit Solutions 1

Determine whether $f(x, y) = x^2 + y^2 + 4 = 0$ provides an implicit solution to

$$F(x, y, y') = 2x + 2y'' = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = \sqrt{-x^2 - 4}$ and $g_2(x) = -\sqrt{-x^2 - 4}$; these functions are defined nowhere on $\mathbb{R}$. Thus, $f(x, y) = 0$ does not provide an implicit solution to the differential equation.

Example 1.2.8: ☢☢ Verifying Implicit Solutions 2

Determine whether $f(x, y) = xy - y^2 = 0$ provides an implicit solution to

$$F(x, y, y', y'') = \frac{1}{y - x^2} y'' + yy' - y = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y) = xy - y^2 = y(x - y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = 0$ and $g_2(x) = x$. Both g_1 and g_2 are defined on $\mathbb{R}$. Note that F has the restriction $y - x^2 \neq 0$. We see that $f(x, g_1(x)) = 0$ for all $x \in \mathbb{R}$ and $f(x, g_2(x)) = x^2 - x^2 = 0$ for all $x \in \mathbb{R}$. Therefore, f defines y as an implicit function of x. Taking $y = g_1(x)$ gives

$$F(x, g_1(x), g_1'(x), g_1''(x)) = \frac{1}{0 - x^2}(0) + (0)(0) - (0) = -\frac{1}{x^2} = 0, \quad x \neq 0.$$

Then, if we take $y = g_2(x)$, we have

$$F(x, g_2(x), g_2'(x), g_2''(x)) = \frac{1}{x - x^2} y'' + xy' - x = \frac{1}{x(1 - x)}(0) + x(1) - x = 0, \quad x \neq 0, x \neq 1.$$

Therefore, $f(x, y)$ provides an implicit solution to the differential equation. When providing the intervals of solution, we must explicitly pick which solution, g_1 or g_2, to provide the interval with respect to. For $g_1(x)$, this is

$$\{x : x \in \mathbb{R} : x \neq 0\}.$$

and for $g_2(x)$, it is

$$\{x : x \in \mathbb{R} : x \neq 0, x \neq 1\}.$$

Note that we did not need to consider $y = g_2(x)$ to show that $f(x, y)$ provides an implicit solution.

Note that it is bad practice to immediately differentiate the relation $f(x, y)$. For example, in Example 1.2.7, if we immediately differentiated f, we would indeed obtain a symbolic equivalent to the differential equation, but we would not account for the domain restrictions.

Example 1.2.9: ☢☢ Verifying Implicit Solutions 3

Determine whether $f(x,y) = e^{2y} + e^{2x} - 1 = 0$ provides an implicit solution to

$$F(x,y,y') = e^{x-y} + y'e^{y-x} = 0$$

Provide the intervals of solution.

First, we determine whether $f(x,y) = e^{2y} + e^{2x} = 0$ defines y as an implicit function of x. Consider the function $g(x) = \frac{1}{2}\ln(1-e^{2x})$. We see that $g(x)$ is defined on $\{x : x \in \mathbb{R} : x < 0\}$. We see that $f(x, g(x)) = 0$ for all $\{x : x \in \mathbb{R} : x < 0\}$. Therefore, f defines g as an implicit function of x. Now, we have

$$\begin{aligned} F(x, g(x), g'(x)) &= e^{x-\frac{1}{2}\ln(1-e^{2x})} - \frac{e^{2x}}{1-e^{2x}} e^{\frac{1}{2}\ln(1-e^{2x})-x} \\ &= \frac{e^x}{\sqrt{1-e^{2x}}} - \frac{e^{2x}}{1-e^{2x}}\frac{\sqrt{1-e^{2x}}}{e^x} \\ &= \frac{e^{2x}\sqrt{1-e^{2x}}}{e^x(1-e^{2x})} - \frac{e^{2x}}{1-e^{2x}}\frac{\sqrt{1-e^{2x}}}{e^x} = 0. \end{aligned}$$

Note that F has no additional restrictions, so the interval of solution for $g(x)$ is simply $\{x : x \in \mathbb{R} : x < 0\}$.

1.3 Lecture 3: January 25, 2023

1.3.1 General and Particular Solutions

Consider the following definitions.

Definition 1.3.1: n-Parameter Families of Solutions

A differential equation $F(x, y, y', \dots, y^{(n)}) = 0$ possesses an n-parameter family of solutions $y(x, c_1, \dots, c_n)$ if and only if y is a solution for any choice of $c_1, \dots, c_n \in \mathbb{F}$.

Definition 1.3.2: Particular Solutions of Differential Equations

Let $y(x, c_1, \dots, c_n)$ be an n-parameter family of solutions to $F(x, y, y', \dots, y^{(n)}) = 0$. Then, for each choice of $c_1, \dots, c_n$, we obtain one particular solution.

Consider the following example.

Example 1.3.1: Finding an n-Parameter Family of Solutions

Consider $F(x, y, y', y'') = y'' = 0$. Note that F has solutions $y(x) = x$ and $y(x) = \pi$ on $\mathbb{R}$. Both these solutions are particular, as they contain no arbitrary constants. If we take the linear combination of the solutions to obtain

$$y(x, c_1, c_2) = c_1 x + c_2,$$

as our 2-parameter family of solutions.

Note that we will often rewrite $y(x, c_1, \dots, c_n)$ as $y(x)$ even though this is an abuse of notation.

Definition 1.3.3: General Solutions of Differential Equations

Let $y(x, c_1, \dots, c_n)$ be an n-parameter family of solutions to $F(x, y, y', \dots, y^{(n)}) = 0$. Then, y is a general solution if and only if every solution to F can be obtained from some choice of $c_1, \dots, c_n$.

In various engineering applications, the terms defined in Definition 1.3.1 and Definition 1.3.3 are equivalent; however, this construction can break. Consider the following examples.

1. The differential equation $F(x, y, y', y'') = y'' = 0$ has the general solution $y(x, c_1, c_2) = c_1 x + c_2$.
2. The differential equation $F(x, y, y') = y'^2 + y^2 = 0$ has only one particular solution $y(x) = 0$.
3. Examples 1.3.2 and 1.3.3 demonstrate where the differential equations has an n-parameter family of solutions but no general solution.

Consider the following examples.

Example 1.3.2: *n*-Parameter Families and General Solutions 1

Show that $F(x, y, y') = y'^2 - 3y' = 0$ has a 1-parameter family of solutions but no general solution.

Note that $y'^2 - 3y' = y'(y' - 3) = 0$. Therefore, either $y' = 0$ or $y' = 3$. We have that $y' = 0$ implies

$$y(x, c_1) = c_1.$$

For $y' = 3$, we have that $y(x) = 3x$. Both $y(x)$ and $y(x, c_1)$ are valid on $\mathbb{R}$. The particular solution $y(x)$ cannot be obtained from $y(x, c_1)$. But, we can take the linear combination of both solutions $y_?(x, c_1) = 3x + c_1$ because, then, we have

$$y_?'^2 - 3y_?' = (3)^2 - 3(3) = 0$$

on $\mathbb{R}$. Therefore, we redefine $y(x, c_1) = y_?(x, c_1)$. Still, there is no choice of c_1 which produces the particular solution $y(x) = 5$ for $y(x, c_1) = 3x + c_1$. Therefore, we have found that not all solutions to F can be obtained from $y(x, c_1)$, so y is not a general solution.

Example 1.3.3: *n*-Parameter Families and General Solutions 2

Show that $F(x, y, y') = y'^2 + (y - 2)y' - 2y = 0$ has two distinct 1-parameter families of solutions. Does F have a general solution?

Note that $y'^2 + (y - 2)y' - 2y = (y' + y)(y' - 2) = 0$. Therefore, $y' = 2$ or $y' = -y$. For $y' = 2$, we have the 1-parameter family $y_1(x, c_{1_1}) = 2x + c_{1_1}$. For $y' = -y$, we have $y_2(x, c_{1_2}) = c_{1_2} e^{-x}$. Both 1-parameter families are valid on $\mathbb{R}$. Note that both *n*-parameter families are distinct; they cannot be obtained from each other. They cannot be combined into a single general solution.

1.4 Lecture 4: January 27, 2023

1.4.1 Initial Conditions

Consider the following definition.

Definition 1.4.1: Initial Conditions

Let $F(x, y, y', \dots, y^{(n)}) = 0$ possess an n-parameter family of solutions. Any system of n equations which determine unique values for the arbitrary constants is called a set of initial conditions.

Consider the following example.

Example 1.4.1: Finding a Particular Solution Given Initial Conditions 1

Recall that $F(x, y, y', y'') = y'' = 0$ has a general solution $y(x, c_1, c_2) = c_1 x + c_2$. Find the particular solution satisfying $y(0) = 5$ and $y'(1) = 3$.

We have that $y(0) = 5$ implies that $c_2 = 5$ and $y'(1) = 3$ implies that $c_1 = 3$. Our particular solution is then

$$y(x) = 3x + 5.$$

Example 1.4.2: Finding a Particular Solution Given Initial Conditions 2

Recall that $F(x, y, y', y'') = y'' = 0$ has a general solution $y(x, c_1, c_2) = c_1 x + c_2$. Find the particular solution satisfying $y(2) = 2$ and $y(1) = 3$.

Now, we have

$$\begin{bmatrix} 2 & 1 & | & 2 \\ 1 & 1 & | & 3 \end{bmatrix} \underbrace{\rightarrow}_{\text{RREF}} \begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 4 \end{bmatrix}.$$

Thus $c_1 = -1$ and $c_2 = 4$. Here, our particular solution is

$$y(x) = -x + 4.$$

1.5 Lecture 5: January 30, 2023

1.5.1 Finding an nth Order Differential Equation Given an n-Parameter Family of Solutions

We now direct our attention to finding an nth order differential equation when given an n-parameter family of solutions. To this, we will look at $y(x, c_1, \dots, c_n)$ and its derivatives and find relations between them. Consider the following examples.

Example 1.5.1: Find a Differential Equation With an n-Parameter Family of Solutions 1

Find an nth order differential equation given the n-parameter family $y(x, c_1) = c_1 x$.

Note that $y(x)$ is a 1-parameter family of solutions. If $y(x) = c_1 x$, $y'(x) = c_1$. Note that $y(x) = xy'(x)$, which gives the differential equation $F(x, y, y') = xy' - y = 0$.

Example 1.5.2: Find a Differential Equation With an n-Parameter Family of Solutions 2

Find an nth order differential equation given the n-parameter family $y(x, c_1, c_2) = c_1 x + c_2 e^x$.

Note that $y(x)$ is a 2-parameter family of solutions. If $y(x) = c_1 x + c_2 e^x$, $y'(x) = c_1 + c_2 e^x$, and $y''(x) = c_2 e^x$. Consider the system

$$\begin{bmatrix} x & e^x & | & y(x) \\ 1 & e^x & | & y'(x) \\ 0 & e^x & | & y''(x) \end{bmatrix} \underbrace{\rightarrow}_{\text{RREF}} \begin{bmatrix} 1 & 0 & | & -y''(x) + y'(x) \\ 0 & 1 & | & \frac{-xy'(x)+y(x)}{-xe^x+e^x} \\ 0 & 0 & | & \frac{-xy'(x)+y(x)}{x-1} + y''(x). \end{bmatrix}$$

Thus, we have the differential equation $F(x, y, y'') = \frac{-xy'(x)+y(x)}{x-1} + y''(x) = 0$.

Example 1.5.3: Find a Differential Equation With an n-Parameter Family of Solutions 3

Find an nth order differential equation given the n-parameter family $y(x, c_1, c_2) = c_1 e^{3x} + c_2 e^{-2x} + x$.

Note that $y(x)$ is a 2-parameter family of solutions. If $y(x) = c_1 e^{3x} + c_2 e^{-2x} + x$, $y'(x) = 3c_1 e^{3x} - 2c_2 e^{-2x} + 1$, and $y''(x) = 9c_1 e^{3x} + 4c_2 e^{-2x}$. Consider the system

$$\begin{bmatrix} e^{3x} & e^{-2x} & | & -x + y(x) \\ 3e^{3x} & -2e^{-2x} & | & -1 + y'(x) \\ 9e^{3x} & 4e^{-2x} & | & y''(x) \end{bmatrix} \underbrace{\rightarrow}_{\text{RREF}} \begin{bmatrix} 1 & \frac{e^{-2x}}{e^{3x}} & | & \frac{-x+y(x)}{e^{3x}} \\ 0 & 1 & | & \frac{-3(-x+y(x))-1+y'(x)}{-5e^{-2x}} \\ 0 & 0 & | & -\frac{-3(-x+y(x))-1+y'(x)}{-5e^{-2x}} + \frac{-9(-x+y(x))+y''(x)}{e^{-2x}(4-9e^{3x})} \end{bmatrix}$$

2
First Order Differential Equations

2.1 Lecture 6: February 1, 2023

2.1.1 An Introduction to Separable Differential Equations

We wish to solve differential equations of the form $F(x, y, y') = f(y)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x) = 0$ where f and g are continuous on a common interval I. But, what is $\mathrm{d}x$, and what is $\mathrm{d}y$? Consider the following definition.

Definition 2.1.1: Differentials

Let $y(x)$ be a differentiable function. Then, if Δx represents any small change in x, we define

$$\mathrm{d}y(x, \Delta x) = y'(x)\Delta x$$

We wish to apply Definiton 2.1.1 to the function $y(x) = x$. Consider the following theorem.

Theorem 2.1.1: A Useful Lemma for a Property of Differentials

If $y(x) = x$, $\mathrm{d}y(x, \Delta x) = \mathrm{d}x(x, \Delta x) = \Delta x$.

Proof. If $y(x) = x$, $y'(x) = 1$, so by Definition 2.1.1, $\mathrm{d}y(x, \Delta x) = \Delta x$. $\square$

Theorem 2.1.2: A Useful Property of Differentials

Let $y(x)$ be a differential function. Then, $\mathrm{d}y(x, \Delta x) = y'(x)\,\mathrm{d}x(x, \Delta x)$.

Proof. By Definition 2.1.1, $\mathrm{d}y(x, \Delta x) = y'(x)\Delta x$. But, by Theorem 2.1.1, $\mathrm{d}x(x, \Delta x)$, so

$$\mathrm{d}y(x, \Delta x) = y'(x)\Delta x = y'(x)\,\mathrm{d}x(x, \Delta x),$$

as desired. □

More familiarly, $\mathrm{d}y = y'(x)\,\mathrm{d}x$.

We may use Theorem 2.1.2 to prove a version of the familar chain rule.

Theorem 2.1.3: The Chain Rule for Differentials

Let $y = f(x)$ be a differentiable function, and $x(t) = g(t)$. Therefore, $y(x) = f(g(t))$. Then,

$$\mathrm{d}y(t, \Delta t) = f'(x(t))\,\mathrm{d}x(t, \Delta t).$$

Proof. Since $x = g(t)$, we have $\mathrm{d}x(t, \Delta t) = g'(t)\,\mathrm{d}t(t, \Delta t)$. Then, using the chain rule for derivatives, we obtain the chain rule for differentials below

$$\mathrm{d}y(t, \Delta t) = f'(x(t))g'(t)(t, \Delta t),$$

as desired. □

Note that if $z(x, y)$ is a function differentiable with respect to both x and y, we obtain

$$\mathrm{d}z(x, y, \Delta x, \Delta y) = \frac{\partial z}{\partial y}\,\mathrm{d}y + \frac{\partial z}{\partial x}\,\mathrm{d}x.$$

Suppose we have a first order differential equation in the form $F(x, y, y') = f(y)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x) = 0$. We use that fact that $\mathrm{d}y$ and $\mathrm{d}x$ are differential functions to write $F(x, y, y') = f(y)\,\mathrm{d}y + g(x)\,\mathrm{d}x = 0$. Then, we have that $f(y)\,\mathrm{d}x = -f(x)\,\mathrm{d}x$, on some common interval I. Let H be a function such that $H'(y) = f(y)$. By the differential chain rule,

$$\mathrm{d}H(x, \Delta x) = H'(y)\,\mathrm{d}y(x, \Delta x) = -g(x)\,\mathrm{d}x(x, \Delta x).$$

Integration gives

$$\int \mathrm{d}H(x, \Delta x) = \int H'(y)\,\mathrm{d}y(x, \Delta x) = \int -g(x)\,\mathrm{d}x(x, \Delta x),$$

so $H(y) + c = \int f(y)\,\mathrm{d}y = \int -f_0(x)\,\mathrm{d}x$.

Consider the following examples.

Example 2.1.1: ☢☢ Separation of Variables 1

Consider $F(x, y, y') = \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$, therefore, $\mathrm{d}y = -y\,\mathrm{d}x$. Assuming that $y(x) \neq 0$, we can divide by y to give

$$\frac{1}{y}\,\mathrm{d}y = -\,\mathrm{d}x.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{1}{y}\,\mathrm{d}y = \int -\,\mathrm{d}x$$

meaning that

$$\log|y| = -x + c_1.$$

This provides an implicit solution. Note that it is a 1-parameter family of solutions, but is not a general function since $y(x) = 0$ is a particular solution to the differential equation, not obtainable from the family. We seek to find an explicit solution. Consider

$$e^{\log|y|} = e^{-x+c_1} = e^{-x}e^{c_1}$$

Then, let $c_2 = e^{c_1}$. Note that $c_2 > 0$. Then, we have $|y| = c_2e^{-x}$. This 1-parameter family can be made a general solution by allowing $c_2 = 0$. Our general solution is then

$$|y| = c_2e^{-x}, \quad c_2 \geq 0, \quad x \in \mathbb{R}.$$

If we allow $c_2 \in \mathbb{R}$, we can also write that a general solution is

$$y = c_2e^{-x}, \quad c_2 \in \mathbb{R}, \quad x \in \mathbb{R}.$$

This method does not always work.

Example 2.1.2: ☢☢ Separation of Variables 2

Consider $F(x, y, y') = x^2(y-2)\frac{dy}{dx} - y^3 = 0$. Assuming that $x \neq 0$ and $y(x) \neq 0$, we can write

$$\frac{1}{x^2}\,dx = \frac{y-2}{y^3}\,dy.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{1}{x^2}\,dx = \int \frac{y-2}{y^3}\,dy$$

meaning that

$$\begin{aligned}-\frac{1}{x} &= \int \left(\frac{1}{y^2} - \frac{2}{y^3}\right) dy \\ &= -\frac{1}{y} + \frac{1}{y^2} + c_1.\end{aligned}$$

We may multiply by y^2 to obtain

$$-\frac{y^2}{x} = -y + 1 + c_1 y^2,$$

and by x to obtain

$$-y^2 = -xy + x + c_1 xy^2,$$

so

$$y^2(c_1 x + 1) = x(y-1), \quad y(x) \neq 0, x \neq 0$$

is a 1-parameter family of solutions, but it is not general because $y(x) = 0$ is a valid particular solution of F on $\mathbb{R}$.

2.2 Lecture 7: February 3, 2023

2.2.1 Differential Equations with Homogeneous Coefficients

Consider the following definitions.

Definition 2.2.1: Homogeneous Functions

The function $f(x, y)$ is homogeneous, of order n, on some region $B \subseteq \mathbb{R}^2$ if and only if for all $x, y \in B$, either of the below hold.

1. The function $f(tx, ty) = t^n f(x, y)$ for some $n \in \mathbb{N}$.
2. The function $f(x, y) = x^n g(u)$ for some $u = \frac{y}{x}$ and $n \in \mathbb{N}$.
3. The function $f(x, y) = y^n h(u)$ for some $u = \frac{x}{y}$ and $n \in \mathbb{N}$.

Definition 2.2.2: Differential Equations With Homogeneous Coefficients

A first order differential equation $F(x, y, y') = Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x, y) = 0$ has homogeneous coefficients if and only if both $P(x, y)$ and $Q(x, y)$ are both homogeneous functions of equal order.

Consider the following examples.

Example 2.2.1: Is it Homogeneous? 1

Determine whether $f(x, y) = 3x^2y - y^3$ is homogeneous on its domain.

Consider

$$\begin{aligned} f(tx, ty) &= 3(tx)^2(ty) - (ty)^3 \\ &= 3t^3x^2y - t^3y^3 \\ &= t^3(3x^2y - y^3) \\ &= t^3 f(x, y). \end{aligned}$$

Therefore, $f(x, y)$ is homogeneous of order 3.

Example 2.2.2: Is it Homogeneous? 2

Determine whether $f(x, y) = xy\sin(xy)$ is homogeneous on its domain.

Consider $f(tx, ty) = t^2xy\sin(t^2xy)$. There is no way in which this can be reduced to satisfy Definition 2.2.1.

Example 2.2.3: ☢☢ Is it Homogeneous? 3

Determine whether $f(x,y) = xy\sin\left(\frac{x}{y}\right) - x^2$ is homogeneous on its domain.

Consider

$$\begin{aligned} f(tx,ty) &= (tx)(ty)\sin\left(\frac{tx}{ty}\right) - (tx)^2 \\ &= t^2 xy\sin\left(\frac{x}{y}\right) - t^2x^2 \\ &= t^2(xy\sin\left(\frac{x}{y}\right) - x^2) \\ &= t^2 f(x,y). \end{aligned}$$

Therefore, $f(x,y)$ is homogeneous of order 2.

While the first condition is, usually, easiest to use to show that a function is homogeneous, the other conditions are very useful in edge cases and for proofs. Consider the following theorem.

Theorem 2.2.1: 🛑 Differential Equations With Homogeneous Coefficients are Separable

If $F(x,y,y') = Q(x,y)y' + P(x,y) = 0$ has homogeneous coefficients, F can be solved using separation of variables.

Proof. If $F(x,y,y')$ has homogeneous coefficients, $Q(x,y) = x^n g_1(u)$ and $P(x,y) = x^n g_2(u)$. We may make the substitution

$$F(x,y,y') = x^n g_1(u)\frac{\mathrm{d}y}{\mathrm{d}x} + x^n g_2(u) = 0.$$

Then, since $u = \frac{y}{x}$, $y = ux$, so $y' = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$. Therefore,

$$\begin{aligned} F(x,y,y') &= x^n g_1(u)\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) + x^n g_2(u) = 0 \\ &= ux^n g_1(u) + x^{n+1} g_1(u)\frac{\mathrm{d}u}{\mathrm{d}x} + x^n g_2(u) = 0 \\ &= ux^n g_1(u)\,\mathrm{d}x + x^{n+1} g_1(u)\,\mathrm{d}u + x^n g_2(u)\,\mathrm{d}x = 0 \\ &= \frac{g_1(u)}{x}\,\mathrm{d}x + \frac{g_1(u)}{u}\,\mathrm{d}u + \frac{g_2(u)}{ux}\,\mathrm{d}x = 0, \quad ux^{n+1} \neq 0 \end{aligned}$$

🛑🛑🛑🛑🛑🛑🛑🛑🛑🛑🛑🛑🛑🛑🛑

so $F(x,y,y')$ is separable. □

3

Higher Order Differential Equations

4
Systems of Differential Equations

Draft: February 3, 2023