MATH3430: DIFFERENTIAL EQUATIONS

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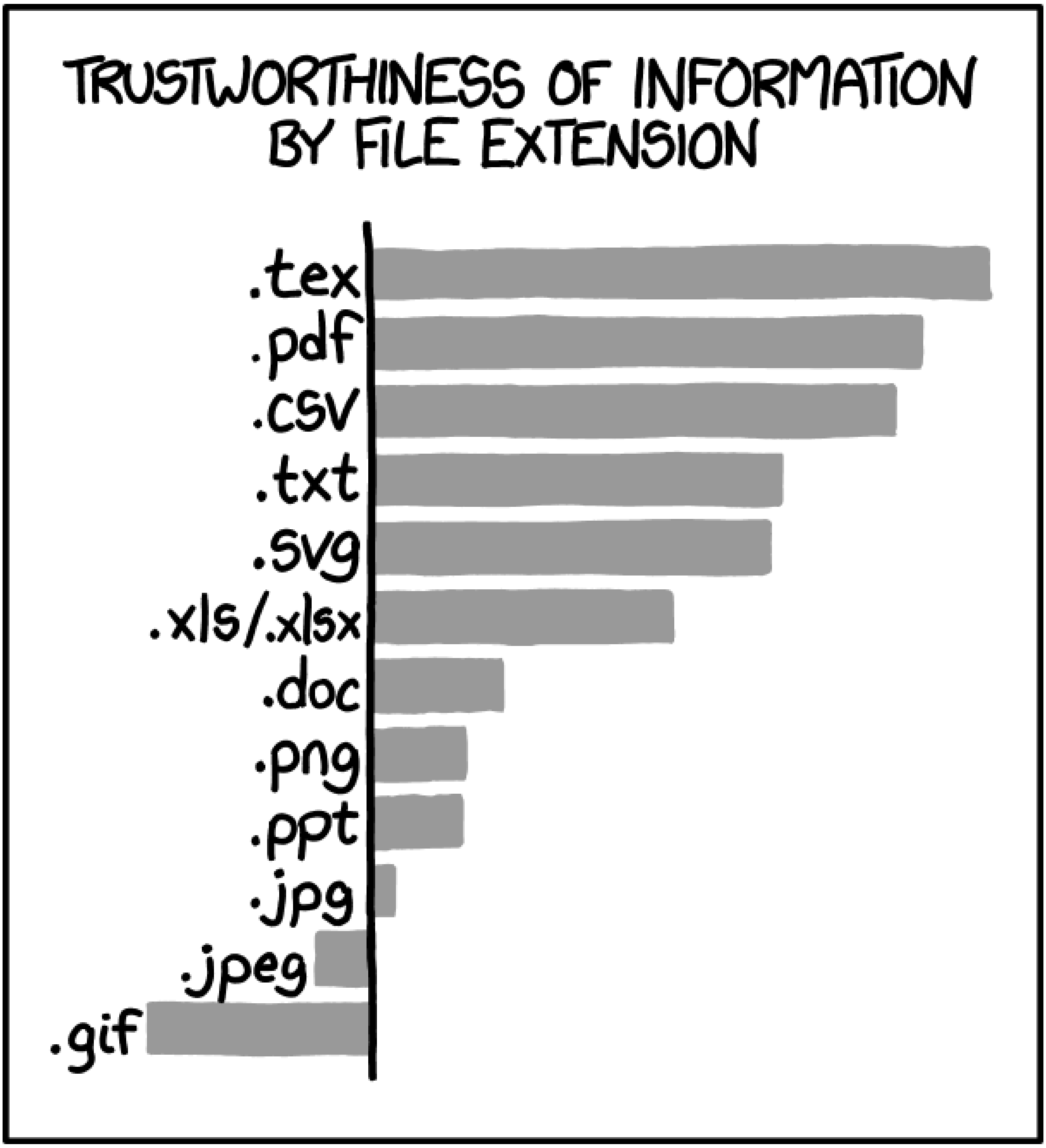
EDITION 1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Preface

To the interested reader,

This document is a compilation of lecture notes taken during the Spring 2023 semester for MATH3430: Ordinary Differential Equations at the University of Colorado Boulder. The course used *Ordinary Differential Equations*[1] by Morris Tenenbaum and Harry Pollard as its primary text. Supplemental texts included Martin Braun's *Differential Equations and Their Applications*[2]. As such, many theorems, definitions, and content may be quoted or derivdd from the aforementioned books. This course was taught by Sheagan John, Ph. D.

The author would like to provide the following resources for students currently taking a differential equations course:

1. Martin Braun's *Differential Equations and Their Applications*.
2. Paul's Online Notes for Differential Equations
3. MIT OpenCourseWare Differential Equations Lectures From Spring 2006.
4. 3Blue1Brown's *Overview of Differential Equations*.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. Please keep in mind that the author wrote this resource during his second semester of his undergraduate studies. If an error needs to be resolved, please contact Adithya Bhaskara at adithya.bhaskara@colorado.edu.

Finally, the author would like to dedicate this set of lecture notes to *Aidan Janney*, *Erika Sjöblom*, *Tate McDonald* and *Benjamin Braun*, four of the author's closest friends who plan to take Differential Equations in the upcoming semester; Fall 2023, at the time of writing.

Best Regards,
Adithya Bhaskara

REVISED: April 26, 2023

[1]Tenenbaum, M., & Pollard, H. (1985). *Ordinary Differential Equations*. Dover Publications.
[2]Braun, M. (1993). Differential Equations and Their Applications an Introduction to Applied Mathematics. Springer.

1
Basic Notions

1.1 Lecture 1: January 20, 2023

1.1.1 Definition of an Ordinary Differential Equation

Consider the following definitions.

Definition 1.1.1: Ordinary Differential Equations

An ordinary differential equation is an equation of the form

$$F(x, y(x), y'(x), \dots, y^{(n)}(x)) = 0$$

where x is an independent variable and y is nth order differentiable.

We remark that every ordinary differential equation is valid as an expression only when we specify the values of x for which it is defined.

Definition 1.1.2: Order of a Differential Equation

The order of an ordinary differential equation is the highest nontrivial derivative present in the equation.

Consider the following ordinary differential equations. From now on, we may omit the term "ordinary," as partial differential equations are not considered in this text.

1. $F(x, y, y') = \cos(xy') + y^2y' + x^2 = 0$ is a first order differential equation.
2. $F(x, y, y', y'') = -\frac{1}{1-x^2} + y'' = 0$ is a second order differential equation.
3. $F(x, y, y', y'', y''', y'''') = e^{xy}y'''' - x^2y'' - \sin x = 0$ is a fourth order differential equation.

1.2 Lecture 2: January 23, 2023

1.2.1 Explicit and Implicit Solutions

We will now consider explicit and implicit solutions to differential equations.

Definition 1.2.1: Explicit Solutions to Ordinary Differential Equations

Let $F(x, y, y', \dots, y^{(n)}) = 0$ be a differential equation defined on the interval I. Then, an explicit solution to F is a function

$$f : \mathbb{R} \to \mathbb{C}$$

for which $f(x)$ is well-defined on some set X such that $I \cap X \neq \emptyset$ and, for all $x \in I$,

$$F(x, f, f', \dots, f^{(n)}) = 0.$$

Definition 1.2.2: Implicit Functions

Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(x, y) = 0$. Then, f defines y as an implicit function of x if and only if

1. There exists an explicit function $g(x)$ such that $y = g(x)$ on some interval $I \subseteq \mathbb{R}$.
2. For all $x \in I$, $f(x, g(x)) = 0$.

Consider the following examples.

Example 1.2.1: Is it an Implicit Function? 1

Does $f(x, y) = x^2 + y^2 - 25 = 0$ define an implicit function of x on $\{x \in \mathbb{R} : -5 \leq x \leq 5\}$?

We consider solving for y. We obtain $y = \pm\sqrt{25 - x^2}$. Therefore we define $y_1(x) = \sqrt{25 - x^2}$ and $y_2(x) = -\sqrt{25 - x^2}$. For $y_1(x)$, we see that

$$f(x, y_1(x)) = x^2 + (\sqrt{25 - x^2})^2 - 25 = x^2 + 25 - x^2 - 25 = 0,$$

as desired. Also, $y_1(x)$ is defined for on $\{x \in \mathbb{R} : -5 \leq x \leq 5\}$. Therefore, $f(x, y)$ does define an implicit function of x. Upon further inspection, $f(x, y)$ defines more than one implicit function of x.

Example 1.2.2: ☢☢ Is it an Implicit Function? 2

Does $f(x,y) = \sqrt{x^2 - y^2} + \arccos\left(\frac{x}{y}\right) = 0, \quad y \neq 0$ define an implicit function?

Here, it is too difficult to solve for y. Instead, we notice that $\arccos(1) = 0$. Therefore, we try $y(x) = x$. By substitution,

$$f(x, y(x)) = \sqrt{x^2 - x^2} + \arccos\left(\frac{x}{x}\right) = 0,$$

as desired. Also, $y(x)$ is defined on $\mathbb{R}$. Therefore, $f(x,y)$ defines an implicit function on $\mathbb{R} - \{0\}$.

Consider the following definition.

Definition 1.2.3: Implicit Solutions to Ordinary Differential Equations

Let $F(x, y, y', \dots, y^{(n)}) = 0$ be a differential equation defined on the interval I. Then, an implicit solution to F is a relation $f(x,y) = 0$ if and only if f defines y as an implicit function of x on I, and if $y(x)$ is an explicit solution to the differential equation.

Consider the following examples.

Example 1.2.3: ☢☢ Verifying Explicit Solutions 1

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y', y'') = x^2 y'' + xy' = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = \log x$, where $\log x = \log_e x$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{1}{x}$. The second derivative of $y(x)$ is $y''(x) = -\frac{1}{x^2}$. Then, we have

$$\begin{aligned} x^2 y''(x) + xy'(x) &= x^2 \cdot -\frac{1}{x^2} + x \cdot \frac{1}{x} \\ &= -1 + 1 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x > 0\}$. The differential equation is defined on $\{x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x \in \mathbb{R}\} \cap \{x \in \mathbb{R} : x > 0\} \cap \{x \in \mathbb{R} : x \neq 0\} = \{x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = \log x$ is an explicit solution for F on $\{x \in \mathbb{R} : x > 0\}$. $\square$

Example 1.2.4: ☢☢ Verifying Explicit Solutions 2

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y') = yy' - 4 = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = 2\sqrt{2x}$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \frac{2}{\sqrt{2x}}$. Then, we have

$$\begin{aligned} y(x)y'(x) - 4 &= 2\sqrt{2x} \cdot \frac{2}{\sqrt{2x}} - 4 \\ &= 4 - 4 = 0, \quad x \neq 0, \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x \in \mathbb{R} : x \geq 0\}$. The differential equation is defined on $\{x \in \mathbb{R}\}$. Additionally, we have the restriction $x \neq 0$. Therefore, the solution set is

$$\{x \in \mathbb{R}\} \cap \{x \in \mathbb{R} : x \geq 0\} \cap \{x \in \mathbb{R} : x \neq 0\} = \{x \in \mathbb{R} : x > 0\}.$$

We can then state that $y(x) = 2\sqrt{2x}$ is an explicit solution for F on $\{x \in \mathbb{R} : x > 0\}$. □

Example 1.2.5: ☢☢ Verifying Explicit Solutions 3

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x, y, y', y'') = y''^3 + y'^2 - y - 3x^2 - 8.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = x^2$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = 2x$. The second derivative of $y(x)$ is $y''(x) = 2$. Then, we have

$$\begin{aligned} y''^3 + y'^2 - y - 3x^2 - 8 &= (2)^3 + (2x)^2 - x^2 - 3x^2 - 8 \\ &= 8 + 4x^2 - x^2 - 3x^2 - 8 = 0 \end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\mathbb{R}$. The differential equation is defined on $\mathbb{R}$. We have no additional restrictions. Therefore, the solution set is $\mathbb{R}$. We can then state that $y(x) = x^2$ is an explicit solution for F on $\mathbb{R}$. □

Example 1.2.6: ☢☢ Verifying Explicit Solutions 4

Show that $y(x)$ is an explicit solution to the differential equation

$$F(x,y,y') = (x+y)^2 - y' = 0.$$

Provide the domain of definition for $y(x)$ along with the solution set. Let $y(x) = \tan x - x$.

Proof. We first take the derivative of $y(x)$ to obtain $y'(x) = \sec^2 x - 1 = \tan^2 x$. Then, we have

$$\begin{aligned}(x + \tan x - x)^2 - \tan^2 x &= (\tan x)^2 - \tan^2 x \\ &= \tan^2 x - \tan^2 x = 0\end{aligned}$$

as desired. The domain of definition for $y(x)$ is $\{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. The differential equation is defined on $\{x \in \mathbb{R}\}$. We have no additional restrictions. Therefore, the solution set is

$$\{x \in \mathbb{R}\} \cap \{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\} = \{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}.$$

We can then state that $y(x) = \tan x - x$ is an explicit solution for F on $\{x : x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. □

Example 1.2.7: ☢☢ Verifying Implicit Solutions 1

Determine whether $f(x,y) = x^2 + y^2 + 4 = 0$ provides an implicit solution to

$$F(x,y,y') = 2x + 2y'' = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x,y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = \sqrt{-x^2-4}$ and $g_2(x) = -\sqrt{-x^2-4}$; these functions are defined nowhere on $\mathbb{R}$. Thus, $f(x,y) = 0$ does not provide an implicit solution to the differential equation.

Example 1.2.8: ☢☢ Verifying Implicit Solutions 2

Determine whether $f(x, y) = xy - y^2 = 0$ provides an implicit solution to

$$F(x, y, y', y'') = \frac{1}{y - x^2} y'' + yy' - y = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y) = xy - y^2 = y(x - y)$ defines y as an implicit function of x. Consider the functions $g_1(x) = 0$ and $g_2(x) = x$. Both g_1 and g_2 are defined on $\mathbb{R}$. Note that F has the restriction $y - x^2 \neq 0$. We see that $f(x, g_1(x)) = 0$ for all $x \in \mathbb{R}$ and $f(x, g_2(x)) = x^2 - x^2 = 0$ for all $x \in \mathbb{R}$. Therefore, f defines y as an implicit function of x. Taking $y = g_1(x)$ gives

$$F(x, g_1(x), g_1'(x), g_1''(x)) = \frac{1}{0 - x^2}(0) + (0)(0) - (0) = -\frac{1}{x^2} = 0, \quad x \neq 0.$$

Then, if we take $y = g_2(x)$, we have

$$F(x, g_2(x), g_2'(x), g_2''(x)) = \frac{1}{x - x^2} y'' + xy' - x = \frac{1}{x(1 - x)}(0) + x(1) - x = 0, \quad x \neq 0, x \neq 1.$$

Therefore, $f(x, y)$ provides an implicit solution to the differential equation. When providing the intervals of solution, we must explicitly pick which solution, g_1 or g_2, to provide the interval with respect to. For $g_1(x)$, this is

$$\{x \in \mathbb{R} : x \neq 0\}.$$

and for $g_2(x)$, it is

$$\{x \in \mathbb{R} : x \neq 0, x \neq 1\}.$$

Note that we did not need to consider $y = g_2(x)$ to show that $f(x, y)$ provides an implicit solution.

Note that it is bad practice to immediately differentiate the relation $f(x, y)$. For example, in Example 1.2.7, if we immediately differentiated f, we would indeed obtain a symbolic equivalent to the differential equation, but we would not account for the domain restrictions.

Example 1.2.9: ☢☢ Verifying Implicit Solutions 3

Determine whether $f(x, y) = e^{2y} + e^{2x} - 1 = 0$ provides an implicit solution to

$$F(x, y, y') = e^{x-y} + y'e^{y-x} = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x, y) = e^{2y} + e^{2x} = 0$ defines y as an implicit function of x. Consider the function $g(x) = \frac{1}{2}\log(1 - e^{2x})$. We see that $g(x)$ is defined on $\{x \in \mathbb{R} : x < 0\}$. We see that $f(x, g(x)) = 0$ for all $\{x \in \mathbb{R} : x < 0\}$. Therefore, f defines g as an implicit function of x. Now, we have

$$\begin{aligned}
F(x, g(x), g'(x)) &= e^{x-\frac{1}{2}\log(1-e^{2x})} - \frac{e^{2x}}{1-e^{2x}} e^{\frac{1}{2}\log(1-e^{2x})-x} \\
&= \frac{e^x}{\sqrt{1-e^{2x}}} - \frac{e^{2x}}{1-e^{2x}} \frac{\sqrt{1-e^{2x}}}{e^x} \\
&= \frac{e^{2x}\sqrt{1-e^{2x}}}{e^x(1-e^{2x})} - \frac{e^{2x}}{1-e^{2x}} \frac{\sqrt{1-e^{2x}}}{e^x} = 0.
\end{aligned}$$

Note that F has no additional restrictions, so the interval of solution for $g(x)$ is simply $\{x \in \mathbb{R} : x < 0\}$.

Example 1.2.10: ☢☢ Verifying Implicit Solutions 4

Determine whether $f(x,y) = -(x+2)^2 + y^2 - 1 = 0$ provides an implicit solution to

$$F(x,y,y') = y^2 - 1 - y'(2y + xy) = 0.$$

Provide the intervals of solution.

First, we determine whether $f(x,y) = -(x+2)^2 + y^2 - 1 = 0 = 0$ defines y as an implicit function of x. Consider the functions $g_1(x) = \sqrt{1+(x+2)^2}$ and $g_2(x) = -\sqrt{1+(x+2)^2}$. We see that $g_1(x)$ and $g_2(x)$ are defined on $\mathbb{R}$. We see that $f(x, g_1(x)) = f(x, g_2(x)) = 0$ on $\mathbb{R}$. Therefore, f defines g_1 and g_2 as implicit functions of x. If we take $y = g_1(x)$, we have

$$\begin{aligned}
F(x, g_1(x), g_1'(x)) &= 1 + (x+2)^2 - 1 - \frac{2(x+2)}{2\sqrt{1+(x+2)^2}}(2\sqrt{1+(x+2)^2} + x\sqrt{1+(x+2)^2}) \\
&= (x+2)^2 - \frac{x+2}{\sqrt{1+(x+2)^2}}(x+2)(\sqrt{1+(x+2)^2}) \\
&= (x+2)^2 - (x+2)^2 = 0.
\end{aligned}$$

Then, if we take $y = g_2(x)$, we have

$$\begin{aligned}
F(x, g_2(x), g_2''(x)) &= 1 + (x+2)^2 - 1 + \frac{2(x+2)}{2\sqrt{1+(x+2)^2}}(-2\sqrt{1+(x+2)^2} - x\sqrt{1+(x+2)^2}) \\
&= (x+2)^2 - \frac{x+2}{\sqrt{1+(x+2)^2}}(x+2)(\sqrt{1+(x+2)^2}) \\
&= (x+2)^2 - (x+2)^2 = 0.
\end{aligned}$$

Note that F has no additional restrictions, so the interval of solution for both $g_1(x)$ and $g_2(x)$ is $\mathbb{R}$. Note that we did not to consider $y = g_2(x)$ to show that $f(x,y)$ provides an implicit solution.

1.3 Lecture 3: January 25, 2023

1.3.1 General and Particular Solutions

Consider the following definitions.

Definition 1.3.1: n-Parameter Families of Solutions

A differential equation $F(x, y, y', \dots, y^{(n)}) = 0$ possesses an n-parameter family of solutions $y(x, c_1, \dots, c_n)$ if and only if y is a solution for any choice of $c_1, \dots, c_n \in \mathbb{F}$.

Definition 1.3.2: Particular Solutions of Differential Equations

Let $y(x, c_1, \dots, c_n)$ be an n-parameter family of solutions to $F(x, y, y', \dots, y^{(n)}) = 0$. Then, for each choice of $c_1, \dots, c_n$, we obtain one particular solution.

Consider the following example.

Example 1.3.1: Finding an n-Parameter Family of Solutions

Consider $F(x, y, y', y'') = y'' = 0$. Note that F has solutions $y(x) = x$ and $y(x) = \pi$ on $\mathbb{R}$. Both these solutions are particular, as they contain no arbitrary constants. If we take the linear combination of the solutions to obtain

$$y(x, c_1, c_2) = c_1 x + c_2,$$

as our 2-parameter family of solutions.

Note that we will often rewrite $y(x, c_1, \dots, c_n)$ as $y(x)$ even though this is an abuse of notation.

Definition 1.3.3: General Solutions of Differential Equations

Let $y(x, c_1, \dots, c_n)$ be an n-parameter family of solutions to $F(x, y, y', \dots, y^{(n)}) = 0$. Then, y is a general solution if and only if every solution to F can be obtained from some choice of $c_1, \dots, c_n$.

In various engineering applications, the terms defined in Definition 1.3.1 and Definition 1.3.3 are equivalent; however, this construction can break. Consider the following examples.

1. The differential equation $F(x, y, y', y'') = y'' = 0$ has the general solution $y(x, c_1, c_2) = c_1 x + c_2$.
2. The differential equation $F(x, y, y') = y'^2 + y^2 = 0$ has only one particular solution $y(x) = 0$.
3. Examples 1.3.2 and 1.3.3 demonstrate where the differential equations has an n-parameter family of solutions but no general solution.

Consider the following examples.

Example 1.3.2: ☢☢ n-Parameter Families and General Solutions 1

Show that $F(x, y, y') = y'^2 - 3y' = 0$ has a 1-parameter family of solutions but no general solution.

Note that $y'^2 - 3y' = y'(y' - 3) = 0$. Therefore, either $y' = 0$ or $y' = 3$. We have that $y' = 0$ implies

$$y(x, c_1) = c_1.$$

For $y' = 3$, we have that $y(x) = 3x$. Both $y(x)$ and $y(x, c_1)$ are valid on $\mathbb{R}$. The particular solution $y(x)$ cannot be obtained from $y(x, c_1)$. But, we can take the linear combination of both solutions $y_?(x, c_1) = 3x + c_1$ because, then, we have

$$y_?'^2 - 3y_?' = (3)^2 - 3(3) = 0$$

on $\mathbb{R}$. Therefore, we redefine $y(x, c_1) = y_?(x, c_1)$. Still, there is no choice of c_1 which produces the particular solution $y(x) = 5$ for $y(x, c_1) = 3x + c_1$. Therefore, we have found that not all solutions to F can be obtained from $y(x, c_1)$, so y is not a general solution.

Example 1.3.3: ☢☢ n-Parameter Families and General Solutions 2

Show that $F(x, y, y') = y'^2 + (y - 2)y' - 2y = 0$ has two distinct 1-parameter families of solutions. Does F have a general solution?

Note that $y'^2 + (y - 2)y' - 2y = (y' + y)(y' - 2) = 0$. Therefore, $y' = 2$ or $y' = -y$. For $y' = 2$, we have the 1-parameter family $y_1(x, c_{1_1}) = 2x + c_{1_1}$. For $y' = -y$, we have $y_2(x, c_{1_2}) = c_{1_2}e^{-x}$. Both 1-parameter families are valid on $\mathbb{R}$. Note that both n-parameter families are distinct; they cannot be obtained from each other. They cannot be combined into a single general solution.

1.4 Lecture 4: January 27, 2023

1.4.1 Initial Conditions

Consider the following definition.

Definition 1.4.1: Initial Conditions

Let $F(x, y, y', \dots, y^{(n)}) = 0$ possess an n-parameter family of solutions. Any system of n equations which determine unique values for the arbitrary constants is called a set of initial conditions.

Consider the following example.

Example 1.4.1: Finding a Particular Solution Given Initial Conditions 1

Recall that $F(x, y, y', y'') = y'' = 0$ has a general solution $y(x, c_1, c_2) = c_1 x + c_2$. Find the particular solution satisfying $y(0) = 5$ and $y'(1) = 3$.

We have that $y(0) = 5$ implies that $c_2 = 5$ and $y'(1) = 3$ implies that $c_1 = 3$. Our particular solution is then

$$y(x) = 3x + 5.$$

Example 1.4.2: Finding a Particular Solution Given Initial Conditions 2

Recall that $F(x, y, y', y'') = y'' = 0$ has a general solution $y(x, c_1, c_2) = c_1 x + c_2$. Find the particular solution satisfying $y(2) = 2$ and $y(1) = 3$.

Now, we have

$$\begin{bmatrix} 2 & 1 & | & 2 \\ 1 & 1 & | & 3 \end{bmatrix} \underbrace{\rightarrow}_{\text{RREF}} \begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 4 \end{bmatrix}.$$

Thus $c_1 = -1$ and $c_2 = 4$. Here, our particular solution is

$$y(x) = -x + 4.$$

1.5 Lecture 5: January 30, 2023

1.5.1 Finding an nth Order Differential Equation Given an n-Parameter Family of Solutions

We now direct our attention to finding an nth order differential equation when given an n-parameter family of solutions. To this, we will look at $y(x, c_1, \dots, c_n)$ and its derivatives and find relations between them. There is no *efficient* technique that will work for all n-parameter families, but usually, we can take n derivatives of our family and find relationships between them. Recall that differential equations must not contain any arbitrary constants.

Consider the following examples.

Example 1.5.1: ☢☢ Find a Differential Equation With an n-Parameter Family 1

For the 2-parameter family $y = c_1e^{3x} + c_2e^{-3x}$, determine the 2nd-order differential equation which has the family as a solution.

If $y = c_1e^{3x}+c_2e^{-3x}$, $y' = 3c_1e^{3x}-3c_2e^{-3x} = 3(c_1e^{3x}-c_2e^{-3x})$. Then, $y'' = 3(3c_1e^{3x}+3c_2e^{-3x}) = 9(c_1e^{3x} + c_2e^{3x}) = 9y$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = y'' - 9y = 0,$$

as desired.

Example 1.5.2: ☢☢ Find a Differential Equation With an n-Parameter Family 2

For the 2-parameter family $y = c_1 + c_2e^{2x}$, determine the 2nd-order differential equation which has the family as a solution.

If $y = c_1 + c_2e^{2x}$, $y' = 2c_2e^{2x}$ and $y'' = 4c_2e^{2x}$. Then, $y'' = 2y'$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = y'' - 2y' = 0,$$

as desired.

Example 1.5.3: ☢☢ Find a Differential Equation With an n-Parameter Family 3

For the 2-parameter family $y = \frac{c_1}{x} + c_2$, determine the 2nd-order differential equation which has the family as a solution.

If $y = \frac{c_1}{x} + c_2$, $y' = -\frac{c_1}{x^2}$ and $y'' = \frac{2c_1}{x^3}$. Then, $c_1 = -x^2y = \frac{1}{2}x^3y''$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = x^3y'' + 2x^2y = 0,$$

as desired.

Example 1.5.4: ☢☢ Find a Differential Equation With an n-Parameter Family 4

For the 2-parameter family $y = c_1 \cos(2x) + c_2 \sin(2x)$, determine the 2nd-order differential equation which has the family as a solution.

If $y = c_1 \cos(2x) + c_2 \sin(2x)$, $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$ and $y'' = -4c_1 \cos(2x) - 4c_2 \sin(2x)$. Then, $y'' = -4y$. Therefore, our 2nd-order differential equation is given by

$$F(x, y, y', y'') = y'' + 4y = 0,$$

as desired.

2
First Order Differential Equations

2.1 Lecture 6: February 1, 2023

2.1.1 An Introduction to Separable Differential Equations

We wish to solve differential equations of the form $F(x, y, y') = f(y)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x) = 0$ where f and g are continuous on a common interval I. But, what is $\mathrm{d}x$, and what is $\mathrm{d}y$? Consider the following definition.

Definition 2.1.1: Differentials

Let $y(x)$ be a differentiable function. Then, if Δx represents any small change in x, we define

$$\mathrm{d}y(x, \Delta x) = y'(x)\Delta x$$

We wish to apply Definiton 2.1.1 to the function $y(x) = x$. Consider the following theorem.

Theorem 2.1.1: A Useful Lemma for a Property of Differentials

If $y(x) = x$, $\mathrm{d}y(x, \Delta x) = \mathrm{d}x(x, \Delta x) = \Delta x$.

Proof. If $y(x) = x$, $y'(x) = 1$, so by Definition 2.1.1, $\mathrm{d}y(x, \Delta x) = \Delta x$. $\square$

Theorem 2.1.2: A Useful Property of Differentials

Let $y(x)$ be a differentiable function. Then, $\mathrm{d}y(x, \Delta x) = y'(x)\,\mathrm{d}x(x, \Delta x)$.

Proof. By Definition 2.1.1, $\mathrm{d}y(x, \Delta x) = y'(x)\Delta x$. But, by Theorem 2.1.1, $\mathrm{d}x(x, \Delta x)$, so

$$\mathrm{d}y(x, \Delta x) = y'(x)\Delta x = y'(x)\,\mathrm{d}x(x, \Delta x),$$

as desired. □

More familiarly, $\mathrm{d}y = y'(x)\,\mathrm{d}x$.

We may use Theorem 2.1.2 to prove a version of the familar chain rule.

Theorem 2.1.3: The Chain Rule for Differentials

Let $y = f(x)$ be a differentiable function, and $x(t) = g(t)$. Therefore, $y(x) = f(g(t))$. Then,

$$\mathrm{d}y(t, \Delta t) = f'(x(t))\,\mathrm{d}x(t, \Delta t).$$

Proof. Since $x = g(t)$, we have $\mathrm{d}x(t, \Delta t) = g'(t)\,\mathrm{d}t(t, \Delta t)$. Then, using the chain rule for derivatives, we obtain the chain rule for differentials below

$$\mathrm{d}y(t, \Delta t) = f'(x(t))g'(t)(t, \Delta t),$$

as desired. □

Note that if $z(x, y)$ is a function differentiable with respect to both x and y, we obtain

$$\mathrm{d}z(x, y, \Delta x, \Delta y) = \frac{\partial z}{\partial y}\,\mathrm{d}y + \frac{\partial z}{\partial x}\,\mathrm{d}x.$$

Suppose we have a first order differential equation in the form $F(x, y, y') = f(y)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x) = 0$. We use that fact that $\mathrm{d}y$ and $\mathrm{d}x$ are differential functions to write $F(x, y, y') = f(y)\,\mathrm{d}y + g(x)\,\mathrm{d}x = 0$. Then, we obtain the 1-parameter family of solutions

$$\int f(y)\,\mathrm{d}y + \int g(x)\,\mathrm{d}x = c.$$

Consider the following examples.

Example 2.1.1: ☢☢ Separation of Variables 1

Consider $F(x, y, y') = \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$, therefore, $\mathrm{d}y = -y\,\mathrm{d}x$. Assuming that $y(x) \neq 0$, we can divide by y to give

$$\frac{1}{y}\,\mathrm{d}y = -\,\mathrm{d}x.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{1}{y}\,\mathrm{d}y = \int -\,\mathrm{d}x$$

meaning that

$$\log|y| = -x + c_1.$$

This provides an implicit solution. Note that it is a 1-parameter family of solutions, but is not a general function since $y(x) = 0$ is a particular solution to the differential equation, not obtainable from the family. We seek to find an explicit solution. Consider

$$e^{\log|y|} = e^{-x+c_1} = e^{-x}e^{c_1}$$

Then, let $c_2 = e^{c_1}$. Note that $c_2 > 0$. Then, we have $|y| = c_2e^{-x}$. This 1-parameter family can be made a general solution by allowing $c_2 = 0$. Our general solution is then

$$|y| = c_2e^{-x}, \quad c_2 \geq 0, \quad x \in \mathbb{R}.$$

If we allow $c_2 \in \mathbb{R}$, we can also write that a general solution is

$$y = c_2e^{-x}, \quad c_2 \in \mathbb{R}, \quad x \in \mathbb{R}.$$

This method does not always work.

Example 2.1.2: ☢☢ Separation of Variables 2

Consider $F(x, y, y') = x^2(y-2)\frac{\mathrm{d}y}{\mathrm{d}x} - y^3 = 0$. Assuming that $x \neq 0$ and $y(x) \neq 0$, we can write

$$\frac{1}{x^2}\,\mathrm{d}x = \frac{y-2}{y^3}\,\mathrm{d}y.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{1}{x^2}\,\mathrm{d}x = \int \frac{y-2}{y^3}\,\mathrm{d}y$$

meaning that

$$\begin{aligned} -\frac{1}{x} &= \int \left(\frac{1}{y^2} - \frac{2}{y^3}\right)\mathrm{d}y \\ &= -\frac{1}{y} + \frac{1}{y^2} + c_1. \end{aligned}$$

We may multiply by y^2 to obtain

$$-\frac{y^2}{x} = -y + 1 + c_1 y^2,$$

and by x to obtain

$$-y^2 = -xy + x + c_1 x y^2,$$

so

$$y^2(c_1 x + 1) = x(y-1), \quad y(x) \neq 0, \quad x \neq 0$$

is a 1-parameter family of solutions, but it is not general because $y(x) = 0$ is a valid particular solution of F on $\mathbb{R}$.

Example 2.1.3: Separation of Variables 3

Consider $F(x, y, y') = x\sqrt{1-y} - y'\sqrt{1-x^2} = 0$. Assuming that $x^2 \neq 1$ and $y \neq 1$, we can write

$$\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{1-y}}\,\mathrm{d}y.$$

We have a separable differential equation, and by integration, we have

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-y}}\,\mathrm{d}y.$$

meaning that

$$-\sqrt{1-x^2} = -2\sqrt{1-y} + c_1$$

is our 1-parameter family of solutions. We must restrict our solution for $|x| < 1$ and $y < 1$ due to the square roots. Our 1-parameter family of solutions is not general because $y(x) = 1$ is a valid particular solution of F on the interval.

Example 2.1.4: Separation of Variables 4

Consider $F(x, y, y') = x^2y' + 1 = 0$. In differential form, we obtain $x^2\,\mathrm{d}y + \mathrm{d}x = 0$. With the restriction $x \neq 0$, we have $\mathrm{d}y = -\frac{1}{x^2}\,\mathrm{d}x$. By integration, we have the 1-parameter family of solutions $y(x) = \frac{1}{x} + c$ on the interval $\mathbb{R} - \{0\}$.

Example 2.1.5: Separation of Variables 5

Consider $F(x, y, y') = x^2y' + y - 1 = 0$. In differential form, we obtain $x^2\,\mathrm{d}y + (y-1)\,\mathrm{d}x = 0$. With the restrictions $x \neq 0$ and $y(x) \neq 1$, we have $\frac{1}{y-1}\,\mathrm{d}y = -\frac{1}{x^2}\,\mathrm{d}x$. By integration, we have the 1-parameter family of solutions $\log|y-1| = \frac{1}{x} + c$ on the interval $\mathbb{R} - \{0\}$.

2.2 Lecture 7: February 3, 2023

2.2.1 Differential Equations with Homogeneous Coefficients

Consider the following definitions.

Definition 2.2.1: Homogeneous Functions

The function $f(x, y)$ is homogeneous, of order n, on some region $B \subseteq \mathbb{R}^2$ if and only if for all $x, y \in B$, either of the below hold.

1. The function $f(tx, ty) = t^n f(x, y)$ for some $n \in \mathbb{N}$.
2. The function $f(x, y) = x^n g(u)$ for some $u = \frac{y}{x}$ and $n \in \mathbb{N}$.
3. The function $f(x, y) = y^n h(u)$ for some $u = \frac{x}{y}$ and $n \in \mathbb{N}$.

Definition 2.2.2: Differential Equations With Homogeneous Coefficients

A first order differential equation $F(x, y, y') = Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x, y) = 0$ has homogeneous coefficients if and only if both $P(x, y)$ and $Q(x, y)$ are both homogeneous functions of equal order.

Consider the following examples.

Example 2.2.1: Is it Homogeneous? 1

Determine whether $f(x, y) = 3x^2y - y^3$ is homogeneous on its domain.

Consider

$$\begin{aligned} f(tx, ty) &= 3(tx)^2(ty) - (ty)^3 \\ &= 3t^3x^2y - t^3y^3 \\ &= t^3(3x^2y - y^3) \\ &= t^3 f(x, y). \end{aligned}$$

Therefore, $f(x, y)$ is homogeneous of order 3.

Example 2.2.2: Is it Homogeneous? 2

Determine whether $f(x, y) = xy\sin(xy)$ is homogeneous on its domain.

Consider $f(tx, ty) = t^2xy\sin(t^2xy)$. There is no way in which this can be reduced to satisfy Definition 2.2.1.

Example 2.2.3: ☢☢ Is it Homogeneous? 3

Determine whether $f(x, y) = xy \sin\left(\frac{x}{y}\right) - x^2$ is homogeneous on its domain.

Consider

$$\begin{aligned} f(tx, ty) &= (tx)(ty)\sin\left(\frac{tx}{ty}\right) - (tx)^2 \\ &= t^2 xy \sin\left(\frac{x}{y}\right) - t^2x^2 \\ &= t^2\left(xy\sin\left(\frac{x}{y}\right) - x^2\right) \\ &= t^2 f(x, y). \end{aligned}$$

Therefore, $f(x, y)$ is homogeneous of order 2.

While the first condition is, usually, easiest to use to show that a function is homogeneous, the other conditions are very useful in edge cases and for proofs. Consider the following theorem.

Theorem 2.2.1: Differential Equations With Homogeneous Coefficients are Separable

If $F(x, y, y') = Q(x, y)y' + P(x, y) = 0$ has homogeneous coefficients, F can be solved using separation of variables.

Proof. If $F(x, y, y')$ has homogeneous coefficients, $Q(x, y) = x^n g_1(u)$ and $P(x, y) = x^n g_2(u)$. We may make the substitution

$$F(x, y, y') = x^n g_1(u)\frac{\mathrm{d}y}{\mathrm{d}x} + x^n g_2(u) = 0.$$

Then, since $u = \frac{y}{x}$, $y = ux$, so $y' = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$. Therefore,

$$\begin{aligned} F(x, y, y') = x^n g_1(u)\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) + x^n g_2(u) = 0 \\ = ux^n g_1(u) + x^{n+1}g_1(u)\frac{\mathrm{d}u}{\mathrm{d}x} + x^n g_2(u) = 0 \\ = ux^n g_1(u)\,\mathrm{d}x + x^{n+1}g_1(u)\,\mathrm{d}u + x^n g_2(u)\,\mathrm{d}x = 0 \\ = \frac{g_1(u)}{x}\,\mathrm{d}x + \frac{g_1(u)}{u}\,\mathrm{d}u + \frac{g_2(u)}{ux}\,\mathrm{d}x = 0, \quad ux^{n+1} \neq 0 \\ = \frac{\mathrm{d}x}{x}\left(g_1(u) + \frac{g_2(u)}{u}\right) + \frac{g_1(u)}{u}\,\mathrm{d}u = 0 \\ = \frac{1}{x}\,\mathrm{d}x + \frac{g_1(u)}{ug_1(u) + g_2(u)}\,\mathrm{d}u = 0, \quad g_1(u) + \frac{g_2(u)}{u} \neq 0, \quad u \neq 0 \end{aligned}$$

so $F(x, y, y')$ is separable. □

2.3 Lecture 8: February 6, 2023

2.3.1 Using the Homogeneous Substitution to Solve Differential Equations

Consider the following example.

Example 2.3.1: ☢☢ A Homogeneous Substitution 1

Find a 1-parameter family of solutions for

$$F(x, y, y') = 2xy\frac{\mathrm{d}y}{\mathrm{d}x} - (x^2 + y^2) = 0.$$

We may rewrite the above as

$$2xy\,\mathrm{d}y - (x^2 + y^2)\,\mathrm{d}x = 0.$$

We see that $2(tx)(ty) = t^2(2xy)$ and $-((tx)^2 + (ty)^2) = -(t^2x^2 + t^2y^2) = -t^2(x^2 + y^2)$ so our differential equation has homogeneous coefficients of order 2. Let $y = ux$, meaning $\frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$. Then, our differential equation is

$$\begin{aligned}
0 &= 2ux^2\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) - (x^2 + u^2x^2) \\
&= 2u^2x^2 + 2ux^3\frac{\mathrm{d}u}{\mathrm{d}x} - x^2 - u^2x^2 \\
&= x^2(u^2 - 1)\,\mathrm{d}x + 2ux^3\,\mathrm{d}u \\
&= \frac{1}{x}\,\mathrm{d}x + \frac{2u}{u^2 - 1}, \quad x \neq 0, \quad u^2 - 1 \neq 0.
\end{aligned}$$

Then, we may integrate to obtain

$$c_1 = \log|x| + \log|u^2 - 1|.$$

If we let $c_2 = e^{c_1}$, we further obtain

$$c_2 = |x||u^2 - 1|$$

By our earlier substitution, we have

$$c_2 = |x|\left|\left(\frac{y}{x}\right)^2 - 1\right|.$$

Our restrictions are $x \neq 0$, $y(x) \neq x$; note that $y(x) = x$ is a particular solution, so our 1-parameter solution is not general.

Example 2.3.2: ☢☢ A Homogeneous Substitution 2

Find a 1-parameter family of solutions for

$$F(x,y,y') = \left(x\log\frac{y}{x} - x\right)y' + y = 0.$$

We may rewrite the above as

$$\left(x\log\frac{y}{x} - x\right)\,\mathrm{d}y + y\,\mathrm{d}x = 0.$$

We see that $tx\log\frac{ty}{tx} - tx = t\left(x\log\frac{y}{x} - x\right)$ and $ty = ty$ so our differential equation has homogeneous coefficients of order 2. Let $y = ux$, meaning $\mathrm{d}y = u\,\mathrm{d}x + x\,\mathrm{d}u$. Then, our differential equation is

$$\begin{aligned}
0 &= \left(x\log\left(\frac{ux}{x}\right) - x\right)(u\,\mathrm{d}x + x\,\mathrm{d}u) + ux\,\mathrm{d}x \\
&= (x\log(u) - x)(u\,\mathrm{d}x + x\,\mathrm{d}u) + ux\,\mathrm{d}x \\
&= ux\log(u)\,\mathrm{d}x + x^2\log(u)\,\mathrm{d}u - ux\,\mathrm{d}x - x^2\,\mathrm{d}u + ux\,\mathrm{d}x \\
&= ux\log(u)\,\mathrm{d}x + x^2(\log(u) - 1)\,\mathrm{d}u.
\end{aligned}$$

With the restrictions $x \neq 0$ and $u\log(u) \neq 0$, we have

$$\frac{1}{x}\,\mathrm{d}x = -\frac{\log(u)-1}{u\log(u)}\,\mathrm{d}u = \left(-\frac{1}{u} + \frac{1}{u\log(u)}\right)\mathrm{d}u.$$

By integration, we have

$$\log|x| = -\log|u| + \log|\log|u|| + c,$$

which by our previous substitution, gives

$$\log|x| = -\log\left|\frac{y}{x}\right| + \log\left|\log\left|\frac{y}{x}\right|\right| + c,$$

on the interval $\mathbb{R} - \{0\}$.

Note that in Example 2.3.1, we used $\frac{\mathrm{d}y}{\mathrm{d}x} = u + x\frac{\mathrm{d}u}{\mathrm{d}x}$, but in Example 2.3.2, we used $\mathrm{d}y = u\,\mathrm{d}x + x\,\mathrm{d}u$. The latter is much cleaner and will therefore be used onwards.

2.3.2 Exactness

Consider the following definitions.

Definition 2.3.1: Simply Connected Regions

A set $B \subseteq \mathbb{R}^2$ is simply connected if and only if every non-intersecting closed curve lying in B contains only points of B.

Definition 2.3.2: Exact Differential Equations

The differential equation $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0$ is exact if and only if $P(x,y)$, $\frac{\partial P}{\partial y}$, $Q(x,y)$, and $\frac{\partial Q}{\partial x}$ are all continuous on a common simply connected region B and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 2.3.1: The Existence of a Potential

If the differential equation $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0$ is exact, there exists $f(x,y)$ such that $\frac{\partial f}{\partial x} = P(x,y)$ and $\frac{\partial f}{\partial y} = Q(x,y)$. The function $f(x,y)$ can be considered as a potential function for the vector field $\vec{F} = [P(x,y), Q(x,y)]$.

As motivation, let $z = f(x,y)$. Recall that

$$\mathrm{d}z = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y.$$

If we take $\frac{\partial f}{\partial x} = P(x,y)$ and $\frac{\partial f}{\partial y} = Q(x,y)$, as stipulated in Definition 2.3.2, the right hand side of the above becomes

$$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y.$$

2.4 Lecture 9: February 8, 2023

2.4.1 Solving Exact Differential Equations

Recall the Fundamental Theorem of Calculus. That is, for some function $H(x)$, we have

$$H(x) = \int_{t_0}^{t} H'(t)\,\mathrm{d}t.$$

Then, if $H(x) = x$, we have $x = \int_{t_0}^{t} \mathrm{d}x$. Recall that for $z = f(x, y)$,

$$\mathrm{d}z = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y,$$

and if we take $\frac{\partial f}{\partial x} = P(x, y)$ and $\frac{\partial f}{\partial y} = Q(x, y)$,

$$\mathrm{d}z = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y.$$

Notice that $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0$ is exact. Let the associated simply connected region be B. From the total differential of z, we have

$$f(x, y) = \int_B \mathrm{d}z = \int_B P(x, y)\,\mathrm{d}x + \int_B Q(x, y)\,\mathrm{d}y.$$

Note that all the above integrals exist, as by Definition 2.3.2, $P(x, y)$ and $Q(x, y)$ are continuous. Consider the following examples.

Example 2.4.1: ☢☢ Exact Differential Equation 1

Solve $F(x, y, y') = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - \cos x\frac{\mathrm{d}y}{\mathrm{d}x} + y\sin x = 0$.

We must first write the differential equation in differential form to produce

$$(2y - \cos x)\,\mathrm{d}y + y\sin x\,\mathrm{d}x = 0.$$

Let $P(x, y) = y\sin x$ and $Q(x, y) = 2y - \cos x$. Then, note that $\frac{\partial P}{\partial y} = \sin x$ and $\frac{\partial Q}{\partial x} = \sin x$. Since the partial derivatives are equal, and P, Q, $\frac{\partial Q}{\partial x}$, and $\frac{\partial P}{\partial y}$ are all continuous on the simply connected region $\mathbb{R}^2$, the differential equation is exact. We must now find some $z = f(x, y)$ such that $\frac{\partial f}{\partial x} = y\sin x$ and $\frac{\partial f}{\partial y} = 2y - \cos x$. Then,

$$\begin{aligned} f(x, y) = \int P(x, y)\,\mathrm{d}x &= \int y\sin x\,\mathrm{d}x \\ &= -y\cos x + c(y). \end{aligned}$$

Note that, now, we have $\frac{\partial f}{\partial y} = -\cos x + c'(y) = Q(x, y) = 2y - \cos x$. Therefore, $c'(y) = 2y$, and

$$\begin{aligned} f(x, y) &= \int c'(y)\,\mathrm{d}y - y\cos x \\ &= y^2 - y\cos x + c_1. \end{aligned}$$

Then, the 1-parameter family of solutions $f(x, y, c_1) = y^2 - y\cos x + c_1 = 0$ defines an implicit solution to $F(x, y, y')$ on $\mathbb{R}$.

2.5 Lecture 10: Februrary 10, 2023

2.5.1 Integrating Factors

Consider the differential equation

$$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0,$$

and let $P(x,y)$, $\frac{\partial P}{\partial x}$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$ all be continuous on some common simply connected region B; however, $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ and there does not exist some $z = f(x,y)$ such that $\mathrm{d}z = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$. This motivates the following definition. We will be able to find some exact differential equation possessing the same solutions as the original differential equation.

Definition 2.5.1: Integrating Factors

Let $P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}x = 0$, and let $P(x,y)$, $\frac{\partial P}{\partial x}$, $Q(x,y)$, $\frac{\partial Q}{\partial x}$ all be continuous on some common simply connected region B. The function $\mu(x,y)$ is an integrating factor if and only if the differential equation

$$\mu(x,y)P(x,y)\,\mathrm{d}x + \mu(x,y)Q(x,y)\,\mathrm{d}y = 0.$$

is exact.

Theorem 2.5.1: Sameness of Solutions

If the integrating factor $\mu(x,y) \neq 0$ for all (x,y) and has continuous first order partial derivatives on the simply connected region B. Then,

$$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = 0$$

has the same solution as

$$\mu(x,y)P(x,y)\,\mathrm{d}x + \mu(x,y)Q(x,y)\,\mathrm{d}y = 0$$

Proof. Let $y(x)$ be a solution to the original differential equation. Then, consider

$$\begin{aligned}\mu(x,y(x))P(x,y(x))\,\mathrm{d}x + \mu(x,y(x))Q(x,y(x))\,\mathrm{d}y &= \mu(x,y(x))(P(x,y(x))\,\mathrm{d}x + \mu(x,y)Q(x,y(x))\,\mathrm{d}y)\\ &= \mu(x,y(x))(0)\\ &= 0,\end{aligned}$$

so $y(x)$ is also a solution to

$$\mu(x,y)P(x,y)\,\mathrm{d}x + \mu(x,y)Q(x,y)\,\mathrm{d}y = 0$$

on the same intervals of solution. On the other hand, if $y(\tilde{x})$ is a solution to the exact differential equation, we have that

$$\mu(x,\tilde{y}(x))(P(x,\tilde{y}(x))\,\mathrm{d}x + \mu(x,\tilde{y})Q(x,\tilde{y}(x))\,\mathrm{d}y) = 0,$$

so either $\mu(x,\tilde{y}(x)) = 0$ or $P(x,\tilde{y}(x))\,\mathrm{d}x + \mu(x,\tilde{y})Q(x,\tilde{y}(x))\,\mathrm{d}y = 0$. By supposition, $\mu(x,y) \neq 0$ for all (x,y), so it must be the case that

$$P(x,\tilde{y}(x))\,\mathrm{d}x + \mu(x,\tilde{y})Q(x,\tilde{y}(x))\,\mathrm{d}y = 0,$$

meaning that $\tilde{y}(x)$ is a solution to the original differential equation. $\square$

We seek to find the integrating factor $\mu(x, y)$ when it is purely a function of x. That is, $\mu(x, y) = \mu(x)$. If $\mu(x)$ is an integrating factor, we have that

$$\mu(x)P(x, y)\,\mathrm{d}x + \mu(x)Q(x, y)\,\mathrm{d}y = 0,$$

where

$$\frac{\partial}{\partial y}(\mu(x)P(x, y)) = \frac{\partial}{\partial x}(\mu(x)Q(x, y)).$$

By differentiation, we have

$$\mu(x)\frac{\partial P}{\partial y} = \mu(x)\frac{\partial Q}{\partial x} + Q\frac{\partial \mu}{\partial x}.$$

By rearrangement, we obtain

$$\mu(x)\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\mathrm{d}\mu}{\mathrm{d}x}.$$

By the suppositions $h(x) \neq 0$ and $Q(x, y) \neq 0$, we may divide and see

$$\frac{1}{\mu(x)}\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{1}{Q(x, y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

By recognizing the left hand side as the natural logarithm's derivative, we multiply by the differential function dx and antidifferentiate to obtain

$$\int \frac{1}{\mu(x)}\frac{\mathrm{d}\mu}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{Q(x, y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}x,$$

which implies

$$\log|\mu(x)| = \int \frac{1}{Q(x, y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}x.$$

Let $H(x)$ be the integrand of the right hand side. We may exponentiate to get

$$|\mu(x)| = e^{\int H(x)\,\mathrm{d}x}.$$

We need only consider $\mu(x) = e^{\int H(x)\,\mathrm{d}x}$, as any constants multiplied by $\mu(x)$ will not change the above derivation significantly. For other cases, where $\mu(x, y)$ is not purely a function of x, the derivation is much more complicated. Note that in the case that $\mu(x, y) = \mu(y)$, let

$$\tilde{H}(x) = \frac{1}{P(x, y)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right),$$

and $\mu(x, y) = \mu(y) = e^{\int \tilde{H}(y)\,\mathrm{d}y}$.

As a review of the previous solution techniques, consider the following example.

Example 2.5.1: ☢☢ 3 Different Solution Techniques

Solve $F(x, y, y') = 2yy' + 6x = 0$ using separation of variables and the homogeneous substitution.

In differential form, we have

$$2y\,\mathrm{d}y = -6x\,\mathrm{d}x,$$

which integrates to give us $y^2 = -3x^2 + c_1$ as our implicit solution on $\mathbb{R}$. Before performing the homogeneous substitution, note that the equation $2y\,\mathrm{d}y + 6x\,\mathrm{d}x = 0$ has homogeneous coefficients of order 1 trivially. We make the substitution $y = ux$ to obtain

$$\begin{aligned} 0 &= 2ux(u\,\mathrm{d}x + x\,\mathrm{d}u) + 6x\,\mathrm{d}x \\ &= 2u^2x\,\mathrm{d}x + 2ux^2\,\mathrm{d}u + 6x\,\mathrm{d}x \\ &= x(2u^2 + 6)\,\mathrm{d}x + 2ux^2\,\mathrm{d}u. \end{aligned}$$

With the restriction $x \neq 0$, we may divide by x^2 and $2u^2 + 6$ to obtain

$$\frac{1}{x}\,\mathrm{d}x + \frac{u}{u^2+3}\,\mathrm{d}u,$$

and integration gives $\log|x| = \log\left|\sqrt{u^2+3}\right| + c_2$ meaning, if $\ell = e^{c_2}$, $|x| = \ell\sqrt{\frac{y^2}{x^2} + 3} = \sqrt{\frac{y^2+3x^2}{x^2}}$. Then, if $k = \ell^2$,

$$x^2 = k\left(\frac{y^2+3x^2}{x^2}\right)$$

on $\{x \in \mathbb{R} : x \neq 0\}$. Note that the differential equation is also trivially exact on $\mathbb{R}^2$.

Example 2.5.2: An Differential Equation With an Integrating Factor Not a Function of x

Show that $2xy^3\,\mathrm{d}x + (2y + x^2y^2)\,\mathrm{d}y = 0$ is not exact and $\mu(x, y) \neq \mu(x)$.

Let $P(x, y) = 2xy^3$, so $\frac{\partial P}{\partial x} = 6xy^2$. Let $Q(x, y) = 2y + x^2y^2$, so $\frac{\partial Q}{\partial x} = 2xy^2$. Therefore, the differential equation is not exact. To show that the differential equation's integrating factor is not purely a function of x, suppose, for the sake of contradiction, that $\mu(x, y) = \mu(x)$. Then,

$$\mu(x) = e^{\int H(x)\,\mathrm{d}x}, \quad H(x) = \frac{1}{Q(x,y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{4xy^2}{2y + x^2y^2}.$$

Since $H(x)$ cannot be written as a pure function of x, we have a contradiction, and $\mu(x, y) \neq \mu(x)$. No such issue occurs if we assume

$$\mu(x, y) = \mu(y) = e^{\int \tilde{H}(y)\,\mathrm{d}y}, \quad \tilde{H}(y) = \frac{1}{P(x,y)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = -\frac{2}{y}.$$

Note that another integrating factor is $\hat{\mu}(x, y) = e^{x^2y}$.

Example 2.5.3: ☢☢ Solving a Differential Equation Using an Integrating Factor

Solve $F(x,y,y') = (ye^y - xe^x)y' + e^x(x+1) = 0$ by integrating factors and exactness.

We first can rewrite the equation in differential form to produce

$$(ye^y - xe^x)\,\mathrm{d}y + e^x(x+1)\,\mathrm{d}x = 0.$$

If we define $P(x,y) = e^x(x+1)$ and $Q(x,y) = ye^y - xe^x$. We see that $\frac{\partial P}{\partial y} = 0$ and $\frac{\partial Q}{\partial x} = -e^x - xe^x$. We see that F is not exact. We seek to find an integrating factor $\mu(x)$. Then, we have

$$\begin{aligned}\mu(x) &= \exp\left(\int \frac{1}{Q(x,y)}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathrm{d}x\right)\\ &= \exp\left(\int \frac{e^x + xe^x}{ye^y - xe^x}\,\mathrm{d}x\right).\end{aligned}$$

We see that this is a lost cause, and instead try to find $\mu(y)$, and see that

$$\begin{aligned}\mu(y) &= \exp\left(\int \frac{1}{P(x,y)}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}y\right)\\ &\exp\left(\int \frac{-e^x - xe^x}{e^x(x+1)}\,\mathrm{d}y\right)\\ &= \exp\left(\int \frac{-e^x(x+1)}{e^x(x+1)}\,\mathrm{d}y\right)\\ &= e^{-y}.\end{aligned}$$

Now, note $\mu(y)F(x,y,y') = (y - xe^{x-y})\,\mathrm{d}y + e^{x-y}(x+1)\,\mathrm{d}x = 0$. Let $\tilde{P}(x,y) = e^{x-y}(x+1)$ and $\tilde{Q}(x,y) = y - xe^{x-y}$. Then, $\frac{\partial \tilde{P}}{\partial y} = -e^{x-y}(x+1)$ and $\frac{\partial \tilde{Q}}{\partial x} = -xe^{x-y} - e^{x-y} = -e^{x-y}(x+1) = \frac{\partial P}{\partial y}$. We must now find some $z = f(x,y)$ such that $\frac{\partial f}{\partial x} = e^{x-y}(x+1)$ and $\frac{\partial f}{\partial y} = y - xe^{x-y}$. We proceed by integration to obtain

$$\begin{aligned}f(x,y) = \int \tilde{Q}(x,y)\,\mathrm{d}y &= \int (y - xe^{x-y})\,\mathrm{d}y\\ &= \frac{y^2}{2} + xe^{x-y} + c(x).\end{aligned}$$

Now, noting that $\frac{\partial f}{\partial x} = \tilde{P}(x,y)$, we have that

$$\begin{aligned}\frac{\partial f}{\partial x} &= xe^{x-y} + e^{x-y} + c'(x)\\ &= e^{x-y}(x+1) + c'(x) = \tilde{P}(x,y).\end{aligned}$$

Thus, $c'(x) = 0$, so $c(x) = 0$. Thus, our implicit solution is

$$\frac{y^2}{2} + xe^{x-y} = c$$

on $\mathbb{R} - \{-1\}$.

3
Higher Order Differential Equations

3.1 Lecture 11: February 13, 2023

3.1.1 Linear Ordinary Differential Equations

Consider the following definition.

Definition 3.1.1: Linear Ordinary Differential Equations

An nth order linear ordinary differential equation, with respect to I, is

$$F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x) = 0$$

where all f_i, $0 \leq i \leq n$, and Q are continuous on some common interval I and $f_n(x) \neq 0$ for all $x \in I$.

Note that every first-order linear differential equation can be written as

$$F(x, y, y') = \tilde{P}(x)y + \tilde{Q}(x) = 0.$$

and has the integrating factor $\mu(x, y) = \mu(x) = e^{\int \tilde{P}(x)\,\mathrm{d}x}$. After multiplying through, we use the product rule for differentials $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$.

Consider the following non-example of a linear ordinary differential equation.

Example 3.1.1: A Non-Example of a Linear Ordinary Differential Equation

Consider $F(x, y, y', y'', y'') = y''' + y'' \log(x-1) + y \arccos(x) - \log(x-2) = 0$.

The differential equation $F(x, y, y', y'', y''')$ is not a linear ordinary differential equation since $\log(x-1)$, $\arccos(x)$, and $\log(x-2)$ are not continuous on any common interval.

We now provide an example of solving a first-order linear differential equation using integrating factors.

Example 3.1.2: ☢☢ Solving a First-Order Linear Equation Using Integrating Factors

Solve $F(x, y, y') = y' - 4y\tan x + \sin x = 0$.

We have that the integrating factor is given by

$$\begin{aligned}\mu(x) &= \exp\left(\int -4\tan x \, \mathrm{d}x\right) \\ &= \exp(-4\log(\sec x)) \\ &= \sec^{-4} x.\end{aligned}$$

Then,

$$\mu(x)F(x, y, y') = y' \sec^{-4} x - 4y\tan x \sec^{-4} x + \sin x \sec^{-4} x = 0.$$

In differential form, we have

$$\sec^{-4} x \, \mathrm{d}y - 4y\tan x \sec^{-4} x \, \mathrm{d}x + \sin x \sec^{-4} x \, \mathrm{d}x = 0.$$

Then, we can obtain, by the product rule for differentials,

$$\mathrm{d}(y\sec^{-4} x) + \sin x \sec^{-4} x \, \mathrm{d}x = 0.$$

By integration, we obtain

$$\begin{aligned}c &= y\sec^{-4} x + \int \sin x \cos^4 x \, \mathrm{d}x \\ &= y\sec^{-4} x - \frac{\cos^5 x}{5}\end{aligned}$$

Then, we have

$$\begin{aligned}y(x) &= \frac{1}{\sec^{-4} x}\left(c + \frac{\cos^5 x}{5}\right) \\ &= c\sec^4 x + \frac{\cos x}{5},\end{aligned}$$

on $\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$, as desired.

3.2 Lecture 12: February 20, 2023

3.2.1 Existence and Uniqueness of Solutions for Linear Differential Equations

Consider the following definition.

Theorem 3.2.1: Linear Independence

Let $S = \{f_i : X \to \mathbb{C} : 1 \leq i \leq n\}$ be a set of continuous complex-valued functions defined on $X \subseteq \mathbb{R}$. Then, S is linearly independent if and only if, for all $x \in X$,

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0$$

with $c_1, \ldots, c_n \in \mathbb{C}$, is satisfied only if $c_1 = \cdots = c_n = 0$. Note that S is linearly dependent if and only if S is not linearly independent.

Consider the following theorems.

Theorem 3.2.2: Existence of Solutions to nth Order Linear Differential Equations (I)

Let $F(x, y, \ldots, y^{(n)})$ be an nth order linear differential equation with respect to I. Then, there exists $y(x)$ such that

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

are initial conditions on F satisfied for any $x_0 \in I$.

Theorem 3.2.3: Existence of Solutions to nth Order Linear Differential Equations (II)

Let $F(x, y, \ldots, y^{(n)})$ be an nth order linear differential equation with respect to I. Then, there exist n 1-parameter families of solutions $c_i y_i(x)$ for $c_i \in \mathbb{C}$ and some particular solution $y_i(x)$ valid on I. Moreover, the set $S = \{y_1(x), \ldots, y_n(x)\}$ is linearly independent and $c_1 y_1(x) + \cdots + c_n y_n(x)$ is an n-parameter family of solutions valid on I.

Theorem 3.2.4: Uniqueness of Solutions to nth Order Linear Differential Equations

Let $F(x, y, \ldots, y^{(n)})$ be an nth order linear differential equation with respect to I. Then, F has a unique solution valid on I satisfying n arbitrary initial conditions.

Therefore, by Theorem 3.2.4, the n-parameter family of solutions specified in Theorem 3.2.3 is the $y(x)$ specified in Theorem 3.2.2; note that $y(x)$ is a general solution.

Definition 3.2.1: Wronskians

Let $S = \{f_i : X \to \mathbb{C} : 1 \leq i \leq n\}$ be a set of continuous complex-valued functions defined on $X \subseteq \mathbb{R}$. Suppose that each $f_i \in S$, $1 \leq i \leq n$ us $(n-1)$-times continuously differentiable. Then, the Wronskian of S is given by

$$W(S)(x) = \det \begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x). \end{bmatrix}$$

Theorem 3.2.5: Wronskians Determine Linear Independence*

If S is linearly dependent on X, $W(S)(x) = 0$ for all $x \in X$. If $W(S)(x) \neq 0$ for some $x \in X$, S is linearly independent on X.

*Wronksians are not necessary and sufficient to determine linear independence.

Consider the following example.

Example 3.2.1: Showing Linear Independence With the Wronskian

Show that $\{x, e^{2x}, e^{3x}\}$ are linearly independent on $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$.

Note that all elements of S are $(n-1)$-times continuously differentiable. Then,

$$\begin{aligned} W(S)(x) &= \det \begin{bmatrix} x & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 0 & 4e^{2x} & 9e^{3x} \end{bmatrix} \\ &= -6xe^{5x} + 5e^{5x}. \end{aligned}$$

Note that $W(S)(0) = 5 \neq 0$, so S is linearly independent.

Theorem 3.2.6: Wronksians are Necessary and Sufficient in Some Cases

If $S = \{y_i : X \to \mathbb{C}\}$ and all y_i, $1 \leq i \leq n$, are solutions to the same nth order linear differential equation, $W(S)(x) = 0$ on X if and only if S is linearly dependent.

3.3 Lecture 13: Feburary 22, 2023

3.3.1 Linear Differential Equations With Constant Coefficients

We will now study linear differential equations with constant coefficients. Consider the following definitions.

Definition 3.3.1: Linear Differential Equations With Constant Coefficients

An nth order linear differential equation, with respect to I, is

$$F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x) = 0,$$

where all $f_i = a_i$, $0 \leq i \leq n$, where $a_i \in \mathbb{C}$ and Q is continuous on some common interval I and $f_n(x) \neq 0$ for all $x \in I$.

Definition 3.3.2: Homogeneous Linear Differential Equations

An nth order linear homogeneous differential equation, with respect to I, is

$$F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x) = 0,$$

where all f_i, $0 \leq i \leq n$, and Q are continuous on some common interval I and $f_n(x) \neq 0$ and $Q(x) = 0$ for all $x \in I$.

Consider the following theorem.

Theorem 3.3.1: Decomposing a Solution of a Linear Differential Equation

Let $F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y = 0$ have the solution

$$y_c(x, c_1, \dots, c_n) = c_1 y_1(x) + \cdots + c_n y_n(x).$$

Then, the solution to

$$\tilde{F}(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x) - Q(x) = 0$$

is given by

$$y(x, c_1, \dots, c_n) = y_c(x, c_1, \dots, c_n) + y_p(x)$$

where $y_p(x)$ is any particular solution to $\tilde{F}$.

3.4 Lecture 14: February 24, 2023

3.4.1 Solving Linear Homogeneous Equations With Constant Coefficients: Part I

Consider the second order homogeneous linear differential equation with constant coefficients

$$F(x, y, y', y'') = ay'' + by' + cy = 0,$$

for $a, b, c \in \mathbb{C}$. We may, without loss of generality, rewrite this as

$$\tilde{F}(x, y, y', y'') = y'' + Ay' + By = 0,$$

since $a \neq 0$. Choose some $\alpha, \beta \in \mathbb{C}$ such that $-(\alpha+\beta) = A$ and $\alpha\beta = B$. Now, consider $z = y'(x) - \alpha y(x)$. Then, note that $\frac{\mathrm{d}z}{\mathrm{d}x} - \beta z = 0$. That is,

$$\begin{aligned} 0 &= y'' - \alpha y'(x) - \beta y'(x) + \alpha\beta y(x) \\ &= y'' - (\alpha+\beta)y' + 2\beta y \\ &= y'' + Ay' + By. \end{aligned}$$

Now, since $\frac{\mathrm{d}z}{\mathrm{d}x} - \beta z = 0$, we have, by separation of variables, $z = c_1 e^{\beta x}$ where $c_1 \in \mathbb{C}$ and $x \in \mathbb{R}$. Now, we have

$$z = y'(x) - \alpha y(x) = c_1 e^{\beta x}.$$

We may use the integrating factor $\mu(x) = e^{\int -\alpha \, \mathrm{d}x} = e^{-\alpha x}$. Consider

$$\begin{aligned} y'(x)e^{-\alpha x} - \alpha y(x)e^{-\alpha x} &= c_1 e^{\beta x} e^{-\alpha x} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(y(x)e^{-\alpha x}). \end{aligned}$$

By integration,

$$\begin{aligned} y(x) &= e^{\alpha x} \int c_1 e^{(\beta-\alpha)x} \, \mathrm{d}x \\ &= \begin{cases} e^{\alpha x}(c_1 x + c_2) = c_1 x e^{\alpha x} + c_2 e^{\alpha x} & \alpha = \beta \\ e^{\alpha x}\left(\frac{c_1 e^{(\beta-\alpha)x}}{\beta-\alpha} + c_2\right) = \frac{c_1}{\beta-\alpha} e^{\beta x} + c_2 e^{\alpha x} = \tilde{c}_1 e^{\beta x} + c_2 e^{\alpha x} & \alpha \neq \beta \end{cases}. \end{aligned}$$

In the first case $\alpha = \beta$, we have that $\alpha = \beta$ is a double root of the polynomial $m^2 - 2\alpha m + \alpha^2 = 0$. In the second case, α and β are distinct solutions to the polynomial $m^2 - (\alpha+\beta)m + \alpha\beta = 0$. Both solutions are truly general solutions, because there are $n = 2$ linearly independent terms in the linear combination.

3.5 Lecture 15: February 28, 2023

3.5.1 Solving Linear Homogeneous Equations With Constant Coefficients: Part II

With the discoveries in the previous section, we provide the following definition and theorem.

Definition 3.5.1: Characteristic Polynomials

Let $F(x, y, \dots, y^{(n)})_{\text{hom}} = a_n y^{(n)} + \cdots + a_0 y = 0$ be an ordinary differential equation. The characteristic polynomial of F is given by

$$p_F(m) = a_n m^n + \cdots + a_0 m^0.$$

Theorem 3.5.1: Finding Solutions With Characteristic Polynomials

If $F(x, y, \dots, y^{(n)})_{\text{hom}} = a_n y^{(n)} + \cdots + a_0 y = 0$ has a characteristic polynomial with factors into distinct factors,

$$y(x, c_1, \dots c_n) = c_1 e^{m_1 x} + \cdots + c_n e^{m_n x},$$

If $F(x, y, \dots, y^{(n)})_{\text{hom}} = a_n y^{(n)} + \cdots + a_0 y = 0$ has a characteristic polynomial that factors into k distinct roots, $k < n$, that is,

$$p_F(m) = (m - m_1)^{r_1} + \cdots + (m - m_k)^{r_k},$$

where $r_1 + \cdots + r_k = n$. We have

$$y(x, c_1, \dots, c_n) = (c_1 x^0 + \cdots + c_{r_1} x^{r_1 - 1}) e^{m_1 x} + \cdots + (c_{n - r_k} x^0 + \cdots + c_n x^{n - r_k}) e^{m_n x}.$$

We will now provide an important remark about Theorem 3.5.1 for the ease of understanding. Let $p_F(m)$ be a characteristic polynomial with root $m_k \in \mathbb{C}$. We may use Euler's Formula to remove imaginary exponents from the nth parameter family given by Theorem 3.5.1. For convenience, a statement of Euler's Formula is given in Theorem 3.5.2.

Theorem 3.5.2: Euler's Formula

For $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof. Recall that $i^0 = i^4 = i^8 = \cdots = 1$, $i^1 = i^5 = i^9 = \cdots = i$, $i^2 = i^6 = i^10 = \cdots = -1$, and $i^3 = i^7 = i^{11} = \cdots = -i$. If we define, for $z \in \mathbb{C}$,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

we have that

$$\begin{aligned}
e^{i\theta} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\
&= \cos\theta + i\sin\theta,
\end{aligned}$$

as desired. $\square$

To use Euler's Formula, suppose that $p_F(m)$ is our characteristic polynomial with root $m_k = a + bi \in \mathbb{C}$. Then, $\overline{m_\ell} = a - bi \in \mathbb{C}$ is also a root of $p_F(m)$. Without loss of generality, suppose both m_k and m_ℓ have multiplicity 1. Let our solution components be

$$y_k(x) = e^{(a+bi)x} = e^{ax}(\cos(bx) + i\sin(bx)), \quad y_\ell(x) = e^{(a-bi)x} = e^{ax}(\cos(bx) - i\sin(bx)).$$

Since the linear combination of y_k and y_ℓ is a solution, $\frac{1}{2}y_k(x) + \frac{1}{2}y_\ell(x) = e^{ax}\cos(bx)$ is a solution, and $\frac{1}{2i}y_k(x) - \frac{1}{2i}y_\ell(x) = e^{ax}\sin(bx)$ is also a solution. Thus,

$$y_{k,\ell} = c_k e^{ax}\cos(bx) + c_\ell e^{ax}\sin(bx)$$

is a solution.

Consider the following examples. Note that only Examples 3.5.8, 3.5.9, and 3.5.10 were worked in Spring 2023, so these three examples are the best examplars. Nonetheless, the other examples are correct, but may be missing small details, such as omitting the interval of solution.

Example 3.5.1: Real Roots 1

Find a general solution to

$$F(x, y, y', y'') = 2y'' + y' - 6y = 0.$$

The characteristic polynomial is

$$2m^2 + m - 6 = 0,$$

which can be rewritten as

$$(2m - 3)(m + 2) = 0.$$

We can see that $m = -2$ and $m = \frac{3}{2}$ are solutions to the characteristic polynomial. Therefore,

$$y(x) = c_1 e^{-2x} + c_2 e^{\frac{3}{2}x}.$$

Example 3.5.2: Real Roots 2

Find a general solution to

$$F(x, y, \dots, y''') = y''' - 6y'' + 11y' - 6y = 0.$$

The characteristic polynomial is

$$m^3 - 6m^2 + 11m - 6 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$\begin{array}{r l}
 & x^2 \quad - 5x + 6. \\
x - 1 \big) & \overline{x^3 - 6x^2 + 11x - 6} \\
 & \underline{- x^3 \quad + x^2} \\
 & \quad - 5x^2 + 11x \\
 & \quad \underline{5x^2 \quad - 5x} \\
 & \qquad\qquad 6x - 6 \\
 & \qquad\qquad \underline{- 6x + 6} \\
 & \qquad\qquad\qquad 0
\end{array}$$

We can see that $m = 1$, $m = 2$, and $m = 3$ are solutions to the characteristic polynomial. Therefore,

$$v = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

Example 3.5.3: ☢☢ Real Roots 3

Find a general solution to

$$F(x, y, \ldots, y''') = 2y''' - 7y'' + 4y' + 4y = 0.$$

The characteristic polynomial is

$$2m^3 - 7m^2 + 4m + 4 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$\begin{array}{rl}
 & 2x^2 - 3x - 2. \\
x - 2 \big) & 2x^3 - 7x^2 + 4x + 4 \\
 & -2x^3 + 4x^2 \\
 \hline
 & -3x^2 + 4x \\
 & 3x^2 - 6x \\
 \hline
 & -2x + 4 \\
 & 2x - 4 \\
 \hline
 & 0
\end{array}$$

Therefore, the characteristic polynomial may be rewritten as

$$(m-2)(2m^2 - 3m - 2) = 0.$$

The above quadratic can be factored as

$$(2m+1)(m-2).$$

Therefore, the characteristic polynomial is

$$(m-2)^2(2m+1) = 0.$$

Therefore,

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-\frac{1}{2}x}.$$

Example 3.5.4: ☢☢ Real Roots 4

Find a general solution to

$$F(x, y, \ldots, y'''') = y'''' - y''' - 13y'' + y' + 12y = 0.$$

The characteristic polynomial is

$$m^4 - m^3 - 13m^2 + m + 12 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$\begin{array}{r l}
 & x^3 \phantom{{}-x^3} \phantom{{}-13x^2} - 13x - 12. \\
\cline{2-2}
x - 1 \big) & x^4 - x^3 - 13x^2 + x + 12 \\
 & -x^4 + x^3 \\
\cline{2-2}
 & - 13x^2 + x \\
 & \phantom{-x^4 + x^3 -{}} 13x^2 - 13x \\
\cline{2-2}
 & - 12x + 12 \\
 & \phantom{-x^4 + x^3 - 13x^2 -{}} 12x - 12 \\
\cline{2-2}
 & 0
\end{array}$$

Therefore, the characteristic polynomial may be rewritten as

$$(m - 1)(m^3 - 13m - 12) = 0.$$

We may also perform polynomial long division with the above cubic, with the same substitution as before, producing

$$\begin{array}{r l}
 & \;x^2 - x - 12. \\
\cline{2-2}
x + 1 \big) & x^3 \phantom{{}-x^2} - 13x - 12 \\
 & -x^3 - x^2 \\
\cline{2-2}
 & - x^2 - 13x \\
 & \phantom{-x^3 -{}} x^2 + x \\
\cline{2-2}
 & - 12x - 12 \\
 & \phantom{-x^3 - x^2 -{}} 12x + 12 \\
\cline{2-2}
 & 0
\end{array}$$

Therefore, we have

$$(m - 1)(m + 1)(m - 4)(m + 3) = 0.$$

Therefore,

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{4x} + c_4 e^{-3x}.$$

Example 3.5.5: ☢☢ Real Roots 5

Find a general solution to

$$F(x, y, \dots, y''''') = y''''' - 4y'''' - 8y''' + 14y'' + 7y' - 10y = 0.$$

The characteristic polynomial is

$$m^5 - 4m^4 - 8m^3 + 14m^2 + 7m - 10 = 0.$$

We may carry out polynomial long division, with the substitution $x = m$, as follows

$$\begin{array}{r|rrrrrr}
 & x^4 & -3x^3 & -11x^2 & +3x & +10. & \\
\hline
x-1 & x^5 - 4x^4 & -8x^3 & +14x^2 & +7x & -10 & \\
 & -x^5 + x^4 & & & & & \\
\cline{2-2}
 & -3x^4 & -8x^3 & & & & \\
 & 3x^4 & -3x^3 & & & & \\
\cline{2-3}
 & & -11x^3 & +14x^2 & & & \\
 & & 11x^3 & -11x^2 & & & \\
\cline{3-4}
 & & & 3x^2 & +7x & & \\
 & & & -3x^2 & +3x & & \\
\cline{4-5}
 & & & & 10x & -10 & \\
 & & & & -10x & +10 & \\
\cline{5-6}
 & & & & & 0 &
\end{array}$$

Therefore, the characteristic polynomial may be rewritten as

$$(m-1)(m^4 - 3m^3 - 11m^2 + 3m + 10) = 0.$$

We may also perform polynomial long division with the above quartic, with the same substitution as before, producing

$$\begin{array}{r|rrrrr}
 & x^3 & -5x^2 & -x & +5. & \\
\hline
x+2 & x^4 - 3x^3 & -11x^2 & +3x & +10 & \\
 & -x^4 - 2x^3 & & & & \\
\cline{2-2}
 & -5x^3 & -11x^2 & & & \\
 & 5x^3 & +10x^2 & & & \\
\cline{2-3}
 & & -x^2 & +3x & & \\
 & & x^2 & +2x & & \\
\cline{3-4}
 & & & 5x & +10 & \\
 & & & -5x & -10 & \\
\cline{4-5}
 & & & & 0 &
\end{array}$$

We then obtain

$$(m-1)(m+2)(m^3 - 5m^2 - m + 5) = 0.$$

We may proceed similarly to produce

$$(m-1)^2(m+2)(m-5)(m+1) = 0.$$

Therefore,

$$y(x) = c_1 e^x + c_2 x e^x + c_3 e^{-2x} + c_4 e^{-x} + c_5 e^{5x}.$$

Example 3.5.6: ☢☢ Complex Roots 1

Find a general solution to

$$F(x, y, y', y'') = 9y'' + 6y' + 4y = 0.$$

The characteristic polynomial is

$$9m^2 + 6m + 4.$$

By the quadratic formula,

$$m = -\frac{1}{3} \pm \frac{\sqrt{3}}{3}.$$

Therefore,

$$y(x) = e^{-\frac{1}{3}} \left(c_1 \cos\left(x\frac{\sqrt{3}}{3} \right) + c_2 \sin\left(x\frac{\sqrt{3}}{3} \right) \right).$$

Example 3.5.7: ☢☢ Complex Roots 2

Find a general solution to

$$F(x, y, \dots, y''') = y''' + 27y' = 0.$$

The characteristic polynomial is

$$m^3 + 27m = 0,$$

or

$$m(m^2 + 27) = 0,$$

which can further be factored as

$$m(m + 3i\sqrt{3})(m - 3i\sqrt{3}) = 0.$$

Solutions are then $m = 0$ and $m = \pm 3i\sqrt{3}$. Therefore,

$$y(x) = c_1 + c_2 \cos(3x\sqrt{3}) + c_3 \sin(3x\sqrt{3}).$$

Consider the following mixed practice exercises.

Example 3.5.8: ☢☢ Mixed Practice With Characteristic Polynomials 1

Find a general solution to

$$F(x, y, \dots, y''') = 2y''' + 3\sqrt{2}y'' - 4y' = 0.$$

The characteristic polynomial is

$$\begin{aligned} p_F(m) &= 2m^3 + 3\sqrt{2}m^2 - 4m \\ &= m(2m^2 - 2\sqrt{2}m - 4) \\ &= m(2m - \sqrt{2})(m + \sqrt{8}) = 0. \end{aligned}$$

Therefore,

$$y(x) = c_1 + c_2 e^{\frac{x\sqrt{2}}{2}} + c_3 e^{-x\sqrt{8}}$$

on $\mathbb{R}$.

Example 3.5.9: ☢☢ Mixed Practice With Characteristic Polynomials 2

Find a general solution to

$$F(x, y, \dots, y''''') = y''''' - 2y'''' + y''' = 0.$$

The characteristic polynomial is

$$\begin{aligned} p_F(m) &= m^5 - 2m^4 + m^3 \\ &= m^3(m^2 - 2m + 1) \\ &= m^3(m - 1)^2 = 0. \end{aligned}$$

Therefore,

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 x e^x$$

on $\mathbb{R}$.

Example 3.5.10: ☢☢ Mixed Practice With Characteristic Polynomials 3

Find a general solution to

$$F(x, y, \ldots, y''') = y''' + 3y'' + 3y' + 2y = 0.$$

The characteristic polynomial is

$$\begin{aligned} p_F(m) &= m^3 + 3m^2 + 3m + 2 \\ &= (m+2)(m^2 + m + 1) \\ &= (m+2)\left(m - \left(\frac{-1 + i\sqrt{3}}{2}\right)\right)\left(m - \left(\frac{-1 - i\sqrt{3}}{2}\right)\right) \end{aligned}$$

Therefore,

$$y(x) = c_1 e^{-2x} + e^{-\frac{x}{2}}\left(c_2 \cos\left(\frac{x\sqrt{3}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{3}}{2}\right)\right)$$

on $\mathbb{R}$.

3.6 Lectures 16, February 27, 2023 & Lecture 17, March 1, 2023

3.6.1 The Method of Undetermined Coefficients: Part I

Consider $F(x, y \ldots, y^{(n)}) = a_n y^{(n)} + \cdots + a_1 y' + a_0 y - Q(x) = 0$. Recall that the general solution

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ is the general solution to $F(x, y, \ldots, y^{(n)})_{\text{hom}} = 0$. We wish to find $y_p(x)$. If $Q(x)$ is an elementary function that can be expressed as a sum of terms, each of which has finitely many linearly independent derivatives, we may use the Method of Undetermined Coefficients.

Note that the Method of Undetermined Coefficients will only work if $Q(x)$ only contains terms of the form $a, x^k, e^{ax}, \sin ax, \cos ax$, and combinations of the previous terms with $a \in \mathbb{R}$ and $k \in \mathbb{Z}^+$.

Consider the following cases that often arise with the Method of Undetermined Coefficients. In the Method of Undetermined Coefficients, we compare the terms of $y_c(x)$ to those of $Q(x)$ to find $y_p(x)$. Consider the following theorem.

Theorem 3.6.1: Cases of the Method of Undetermined Coefficients

The following cases describe the Method of Undetermined Coefficients.

1. If no term of the forcing term $Q(x)$ is the same as a term of $y_c(x)$, $y_p(x)$ will be a linear combination of the terms of $Q(x)$ and all its linearly independent derivatives.

2. If the forcing term $Q(x)$ contains a term which, ignoring constant coefficients, is $x^k u(x)$, where $u(x)$ is a term of $y_c(x)$ and $k \in \mathbb{N}$, $y_p(x)$ will be a linear combination of $x^{k+1}u(x)$ and all of its linearly independent derivatives, again ignoring constant coefficients. Note that if $Q(x)$ contains terms which belong to Case 1, include them appropriately in $y_p(x)$.

For the Method of Undetermined Coefficients, to form the necessary linear combinations, consider the following table.

Forcing Term $Q(x)$	**General Form of** $y_p(x)$
$ae^{\beta x}$	$Ae^{\beta x}$
$a\cos(\beta x) + b\sin(\beta x)$	$A\cos(\beta x) + B\sin(\beta x)$
$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 + A_0$
$e^{\beta x}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$	$e^{\beta x}(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 + A_0)$
$\cos(\beta x)(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$	$\cos(\beta x)(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 + A_0)$
$\sin(\beta x)(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$	$\sin(\beta x)(A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 + A_0)$

Table 3.1: Table of the General Forms of $y_p(x)$ for the Method of Undetermined Coefficients

3.7 Lecture 18, March 3, 2023

3.7.1 The Method of Undetermined Coefficients: Part II

Theorem 3.6.1 is quite technical, so we provide the following examples.

Example 3.7.1: ☢☢ Case 1 of the Method of Undetermined Coefficients 1

FInd a general solution to $F(x, y, y', y'') = y'' + 4y' + 4y = 4x^2 + 6e^x$ given that

$$y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}.$$

By Theorem 3.6.1, $y_p(x)$ is a linear combination of the terms of $Q(x) = 4x^2 + 6e^x$ and its derivatives. Suppose, then, that $y_p(x) = Ax^2 + Bx + Ce^x + D$. Then, $y_p'(x) = 2Ax + Ce^x + B$ and $y_p''(x) = Ce^x + 2A$. Then,

$$\begin{aligned} F(x, y_p, y_p', y_p'') &= Ce^x + 2A + 4(2Ax + Ce^x + B) + 4(Ax^2 + Bx + Ce^x + D) &&= 4x^2 + 6e^x \\ &= Ce^x + 2A + 8Ax + 4Ce^x + 4B + 4Ax^2 + 4Bx + 4Ce^x + 4D &&= 4x^2 + 6e^x \\ &= 4Ax^2 + (8A + 4B)x + 9Ce^x + 2A + 4B + 4D &&= 4x^2 + 6e^x. \end{aligned}$$

By matching coefficients, $4Ax^2 = 4x^2$, so $A = 1$. Then, $8A + 4B = 0$, so $B = -2$. Then, $9Ce^x = 6$, so $C = \frac{2}{3}$. Finally, since $2A + 4B + 4D = 0$, so $D = \frac{3}{2}$. Therefore, our general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 2x + \frac{2}{3}e^x + \frac{3}{2}$$

on $\mathbb{R}$.

Example 3.7.2: ☢☢ Case 1 of the Method of Undetermined Coefficients 2

Find a general solution to $F(x, y, y', y'') = y'' - 3y' + 2y = 2xe^{3x} + 3\sin x$ given that

$$y_c(x) = c_1 e^{3x} + c_2 e^{2x}$$

By Theorem 3.6.1, $y_p(x)$ is a linear combination of the terms of $Q(x) = 2xe^{3x} + 3\sin x$ and its derivatives. Suppose, then, that $y_p(x) = A\sin x + B\cos x + Ce^{3x} + Dxe^{3x}$. Then, $y_p'(x) = A\cos x - B\sin x + 3Ce^{3x} + 3Dxe^{3x} + De^{3x}$ and $y_p''(x) = -A\sin x - B\cos x + 9Ce^{3x} + 9Dxe^{3x} + 6De^{3x}$. Then,

$$F(x, y_p, y_p', y_p'') = 2Dxe^{3x} + (2C + 3D)e^{3x} + (A + 3B)\sin x + (B - 3A)\cos x \quad = 2xe^{3x} + 3\sin x.$$

By matching coefficients, $2Dx = 2x$, so $D = 1$. Then, $2C + 3D = 0$, so $C = -\frac{3}{2}$. We also have $A + 3B = 3$ and $B - 3A = 0$, so $A = \frac{3}{10}$ and $B = \frac{9}{10}$. Therefore, our general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{2x} + \frac{3}{10}\sin x + \frac{9}{10}\cos x - \frac{3}{2}e^{3x} + xe^{3x}$$

on $\mathbb{R}$.

Example 3.7.3: ☢☢ Case 1 of the Method of Undetermined Coefficients 3

Find a general solution to $F(x, y, y', y'') = y'' - y = 5x$.

We see that $p_F(m) = m^2 - 1$, so $y_c(x) = c_1 e^x + c_2 e^{-x}$. Then, by Theorem 3.6.1, $y_p(x)$ is a linear combination of $Q(x) = 5x$ and its derivatives. Suppose, then, that $y_p(x) = Ax + B$. Then, $y_p'(x) = A$ and $y_p''(x) = 0$. Then,

$$F(x, y_p, y_p', y_p'') = 0 - Ax - B \qquad = 5x.$$

Thus, $A = -5$ and $B = 0$. Therefore, our general solution is

$$y(x) = c_1 e^x + c_2 e^{-x} - 5x$$

on $\mathbb{R}$.

Example 3.7.4: ☢☢ Case 2 of the Method of Undetermined Coefficients 1

Find a general solution to $F(x, y, y', y'') = y'' + y = \sin x$ given that

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

By Theorem 3.6.1, since $\sin x = x^0 \sin x$, $y_p(x)$ will be a linear combination of $x \sin x$ and its linearly independent derivatives. We have that $y_p(x) = Ax \sin x + Bx \cos x$. Then, $y_p'(x) = A \sin x + Ax \cos x - Bx \sin x + B \cos x$ and $y_p''(x) = A \cos x + A \cos x - Ax \sin x - Bx \cos x - B \sin x - B \sin x$. Then,

$$F(x, y_p, y_p', y_p'') = 2A \cos x - 2B \sin x \qquad = \sin x.$$

Thus, $A = 0$ and $B = -\frac{1}{2}$. Therefore, our general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$$

on $\mathbb{R}$.

Example 3.7.5: ☢☢ Case 2 of the Method of Undetermined Coefficients 2

Find a general solution to $y''' - 4y'' + 4y' = 3x^3$ given that

$$y_c(x) = c_1 + c_2 e^{2x} + c_3 x e^{2x}.$$

By Theorem 3.6.1, since $3x^3 = c_1 x^3(1)$ ignoring constant coefficients, $y_p(x)$ wil be a linear combination of x^4 and all its linearly independent derivatives. We have that $y_p(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$. Then $y_p'(x) = 4Ax^3 + 3Bx^2 + 2Cx + D$, $y_p''(x) = 12Ax^2 + 6Bx + 2C$, and $y_p'''(x) = 24Ax + 6B$. Then,

$$\begin{aligned} F(x, y_p, y_p', y_p'', y_p''') &= 24Ax + 6B - 48Ax^2 - 24Bx - 8C + 16Ax^3 + 12Bx^2 + 8Cx + 4D &= 3x^3 \\ &= (6B - 8C + 4D) + (24A - 24B + 8C)x + (-48A + 12B)x^2 + (16A)x^3 &= 3x^3. \end{aligned}$$

Thus, $A = \frac{3}{16}$, $B = \frac{3}{4}$, $C = \frac{27}{16}$, and $D = \frac{9}{4}$. Therefore, our general solution is

$$y(x) = c_1 + c_2 e^{2x} + c_3 x e^{2x} + \frac{3}{16}x^4 + \frac{3}{4}x^3 + \frac{27}{16}x^2 + \frac{9}{4}x$$

on $\mathbb{R}$.

3.8 Lecture 19, March 6, 2023

3.8.1 Variation of Parameters: Part I

Recall that the Method of Undetermined Coefficients only works if the forcing term $Q(x)$ in

$$F(x, y, \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x) = 0$$

can be expressed as a sum of terms, each of which has finitely many linearly independent derivatives. This severely restricts the solution technique, as $x^{-n}, \log(ax), \tan(x), \dots$ have infinitely many derivatives. Consider the following outline.

1. Suppose $F(x, y, \dots, y^{(n)}) = 0$ has associated $F(x, y, \dots, y^{(n)})_{\text{hom}} = 0$ with general solution $y_c(x) = c_1y_1(x) + \cdots + c_ny_n(x)$.
2. Suppose that $y_p(x)$ has the form $y_p(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ where $u_1(x), \dots, u_n(x)$ are continuously differentiable. If we take n derivatives and substitute into our original differential equation, we can form the system of n equations

$$\begin{cases} u_1'(x)y_1(x) + \cdots + u_n'(x)y_n(x) & = 0 \\ u_1'(x)y_1'(x) + \cdots + u_n'(x)y_n'(x) & = 0 \\ & \vdots \\ u_1'(x)y_1^{(n-2)}(x) + \cdots + u_n'(x)y_n^{(n-2)}(x) & = 0 \\ u_1'(x)y_1^{(n-1)}(x) + \cdots + u_n'(x)y_n^{(n-1)}(x) & = \frac{Q(x)}{f_n(x)} \end{cases}.$$

We wish to find $u_i'(x)$ such that it is possible to solve for $u_i(x) = \int u_i'(x)\,\mathrm{d}x$. Consider

$$\begin{bmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} u_1'(x) \\ u_2'(x) \\ \vdots \\ u_{n-1}'(x) \\ u_n'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{Q(x)}{f_n(x)} \end{bmatrix}$$

We can solve this system using Cramer's Rule or matrix inverses. This is possible since $S = \{y_1 \dots, y_n\}$ is linearly independent, so $W(S)(x) \neq 0$ on I. Thus, our coefficient matrix is invertible and $\det A \neq 0$. For convenience, Cramer's Rule is provided below.

Theorem 3.8.1: Cramer's Rule

Suppose we have a system $A\vec{X} = \vec{B}$, and $\det A \neq 0$. Let $\vec{X} = [x_1, \dots, x_n]^T$ be the solution vector. Then,

$$x_i = \frac{\det A_i}{\det A}$$

where A_i is obtained by replacing the ith column of A with $\vec{b}$.

Consider the following example.

Example 3.8.1: ☢☢ Variation of Parameters 1

Find a solution component of $y^{(5)} - 2y^{(4)} + y''' + x - e^{-x} = 0$.

Note that here, $Q(x) = e^{-x} - x$ and $f_5(x) = 1$. Note

$$y_c(x) = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + c_4y_4(x) + c_5y_5(x) = c_1 + c_2x + c_3x^2 + c_4e^x + c_5xe^x.$$

Thus,

$$A = \begin{bmatrix} 1 & x & x^2 & e^x & xe^x \\ 0 & 1 & 2x & e^x & e^x + xe^x \\ 0 & 0 & 2 & e^x & 2e^x + xe^x \\ 0 & 0 & 0 & e^x & 3e^x + xe^x \\ 0 & 0 & 0 & e^x & 4e^x + xe^x \end{bmatrix} \underbrace{\rightarrow}_{\text{REF}} \begin{bmatrix} 1 & x & x^2 & e^x & xe^x \\ 0 & 1 & 2x & e^x & e^x + xe^x \\ 0 & 0 & 2 & e^x & 2e^x + xe^x \\ 0 & 0 & 0 & e^x & 3e^x + xe^x \\ 0 & 0 & 0 & 0 & -3e^x - xe^x + 4e^x + xe^x \end{bmatrix}$$

Note that

$$\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ e^{-x} - x \end{bmatrix}.$$

Since ref $A \in \mathcal{U}$, and our row operation didn't change the value of the determinant of A, we have that $\det A = 2e^{2x}$. By Cramer's Rule,

$$\begin{aligned} u_5'(x) = \frac{\det A_5}{\det A} &= \frac{1}{2e^{2x}} \det \begin{bmatrix} 1 & x & x^2 & e^x & 0 \\ 0 & 1 & 2x & e^x & 0 \\ 0 & 0 & 2 & e^x & 0 \\ 0 & 0 & 0 & e^x & 0 \\ 0 & 0 & 0 & e^x & e^{-x} - x \end{bmatrix} \\ &= \frac{1}{2e^{2x}} \det \begin{bmatrix} 1 & x & x^2 & e^x & 0 \\ 0 & 1 & 2x & e^x & 0 \\ 0 & 0 & 2 & e^x & 0 \\ 0 & 0 & 0 & e^x & 0 \\ 0 & 0 & 0 & 0 & e^{-x} - x \end{bmatrix} \\ &= \frac{2e^x(e^{-x} - x)}{2e^{2x}} = \frac{1 - xe^x}{e^{2x}} = e^{-2x} - xe^{-x}. \end{aligned}$$

Then, we have that

$$u_5(x) = \int (e^{-2x} - xe^{-x})\, dx.$$

3.9 Lecture 20, March 8, 2023

3.9.1 Variation of Parameters: Part II

Consider the following examples.

Example 3.9.1: ☢☢ Variation of Parameters 2

Find a general solution to $F(x, y, y', y'') = y'' - 2y' - \log x = 0$.

Here $Q(x) = \log x$. Note that we cannot use the method of Undetermined Coefficients, as $\log x$ has infinitely many derivatives. Since $F(x, y, y', y'')_{\text{hom}} = y'' - 2y' = 0$, $p_F(m) = m^2 - 2m = 0$, so

$$y_c(x) = c_1 + c_2 e^{2x}.$$

Thus,

$$A = \begin{bmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{bmatrix}.$$

Note that

$$\vec{b} = \begin{bmatrix} 0 \\ \log x \end{bmatrix}.$$

We have that $\det A = 2e^{2x}$, so by Cramer's Rule,

$$\begin{aligned} u_1'(x) = \frac{\det A_1}{\det A} &= \frac{1}{2e^{2x}} \det \begin{bmatrix} 0 & e^{2x} \\ \log x & 2e^{2x} \end{bmatrix} \\ &= \frac{-e^{2x} \log x}{2e^{2x}} = -\frac{\log x}{2} \end{aligned}$$

and

$$\begin{aligned} u_2'(x) = \frac{\det A_2}{\det A} &= \frac{1}{2e^{2x}} \det \begin{bmatrix} 1 & 0 \\ 0 & \log x \end{bmatrix} \\ &= \frac{\log x}{2e^{2x}}. \end{aligned}$$

Then,

$$\begin{aligned} y_p(x) &= -\frac{1}{2} \int \log x \, \mathrm{d}x + e^{2x} \int \frac{\log x}{e^{2x}} \, \mathrm{d}x \\ &= -\frac{1}{2} (x \log x - x) + \frac{e^{2x}}{2} \int \frac{\log x}{e^{2x}} \, \mathrm{d}x. \end{aligned}$$

Therefore,

$$y(x) = c_1 + c_2 e^{2x} = -\frac{1}{2} (x \log x - x) + \frac{e^{2x}}{2} \int \frac{\log x}{e^{2x}} \, \mathrm{d}x,$$

on $\{x \in \mathbb{R} : x > 0\}$.

Example 3.9.2: ☢☢ Variation of Parameters 3

Find a general solution to $y'' + y = \log(5x+2)$.

Here $Q(x) = \log(5x+2)$. Note that we cannot use the Method of Undetermined Coefficients, for a similar reason as Example 3.9.1. We note that $y_c(x) = c_1 \cos x + c_2 \sin x$. Then,

$$A = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}.$$

Note that

$$\vec{b} = \begin{bmatrix} 0 \\ \log(5x+2) \end{bmatrix}.$$

We have that $\det A = 2e^{2x}$, so by Cramer's Rule,

$$u_1'(x) = \frac{1}{\det A} \det \begin{bmatrix} 0 & \sin(x) \\ \log(5x+2) & \cos(x) \end{bmatrix} = -\sin(x)\log(5x+2)$$

and

$$u_2'(x) = \frac{1}{\det A} \begin{bmatrix} \cos(x) & 0 \\ -\sin(x) & \log(5x+2) \end{bmatrix} = \cos(x)\log(5x+2).$$

Then,

$$y_p(x) = -\cos(x) \int \sin(x)\log(5x+2)\,\mathrm{d}x + \sin(x) \int \cos(x)\log(5x+2)\,\mathrm{d}x.$$

Therefore,

$$y(x) = c_1\cos(x) + c_2\sin(x) - \cos(x)\int \sin(x)\log(5x+2)\,\mathrm{d}x + \sin(x)\int \cos(x)\log(5x+2)\,\mathrm{d}x$$

valid on $\left\{x \in \mathbb{R} : x > -\frac{2}{5}\right\}$.

3.9.2 Reduction of Order

Consider the following theorems.

Theorem 3.9.1: A Useful Lemma for Reduction of Order

Given $n-1$ linearly independent solutions of an nth order linear differential equation, the nth 1-parameter family of solutions can be obtained.

Theorem 3.9.2: Reduction of Order, in the $n=2$ Case

Let $F(x,y,y,',y'') = f_2(x)y'' + f_1(x)y' + f_0(x)y - Q(x) = 0$ has the known 1-parameter family $c_1y_1(x)$, a linearly independent second solution is given by

$$y_2(x) = y_1(x)\int \frac{e^{-\int \frac{f_1(x)}{f_2(x)}\,\mathrm{d}x}}{(y_1(x))^2}\,\mathrm{d}x.$$

Example 3.9.3: Reduction of Order 1

Use Reduction of Order to find the general solution to $x^2y'' - 2xy' + 2y = 0$ given that $y_1(x) = x$.

By Theorem 3.9.2,

$$\begin{aligned} y_2(x) &= x\int \frac{e^{-\frac{-2x}{x^2}}}{x^2}\,\mathrm{d}x \\ &= x\int \frac{e^{\frac{2}{x}}}{x^2}\,\mathrm{d}x \\ &= x\int \mathrm{d}x \\ &= x^2, \end{aligned}$$

so $y_c(x) = c_1x + c_2x^2$ on $\{x \in \mathbb{R} : x \neq 0\}$.

We may now use Variation of Parameters on Example 3.9.3 with any forcing term $Q(x)$. Consider the following example.

Example 3.9.4: ☢☢ Variation of Parameters With Reduction of Order

Find a general solution to $x^2y'' - 2xy' + 2y = x\log x$. Here, $Q(x) = x\log x$, and recall $y_c(x) = c_1x + c_2x^2$. Thus,

$$A = \begin{bmatrix} x & x^2 \\ 1 & 2x \end{bmatrix}.$$

Note that

$$\vec{b} = \begin{bmatrix} 0 \\ \frac{x\log x}{x^2} \end{bmatrix}.$$

We have that $\det A = x^2$, so by Cramer's Rule,

$$\begin{aligned} u_1'(x) &= \frac{1}{x^2}\det\begin{bmatrix} 0 & x^2 \\ \frac{x\log x}{x^2} & 2x \end{bmatrix} \\ &= -\frac{1}{x}\log x \end{aligned}$$

and

$$\begin{aligned} u_2'(x) &= \frac{1}{x^2}\det\begin{bmatrix} x & 0 \\ 1 & \frac{x\log x}{x^2} \end{bmatrix} \\ &= \frac{1}{x^2}\log x. \end{aligned}$$

Then,

$$\begin{aligned} y_p(x) &= x\int -\frac{1}{x}\log x\,\mathrm{d}x + x^2\int \frac{1}{x^2}\log x\,\mathrm{d}x \\ &= -\frac{x(\log x)^2}{2} + x(-\log x + 1) \end{aligned}$$

Thus, the general solution is

$$y(x) = c_1 + c_2x - \frac{x(\log x)^2}{2} + x(-\log x + 1).$$

3.10 Lecture 21: March 10, 2023

3.10.1 The Laplace Transform

Recall the definition of an improper integral.

Definition 3.10.1: Improper Integrals

Let $f \in C(\{x \in \mathbb{R} : x \geq \alpha, \alpha \in \mathbb{R}\})$. Then,

$$\int_{\alpha}^{\infty} f(x)\,\mathrm{d}x = \lim_{\beta \to \infty} \int_{\alpha}^{\beta} f(x)\,\mathrm{d}x$$

exists if and only if the limit exists, and is finite.

Consider the following definition of the Laplace Transform and its related theorems.

Definition 3.10.2: The Laplace Transform

Let $s_0 > 0 \in \mathbb{R}$ and $f \in C(\{t \in \mathbb{R} : x \geq s_0\})$. Then, the Laplace Transform of f is defined as

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{t=0}^{t=\infty} e^{-st} f(t)\,\mathrm{d}t.$$

The improper integral may diverge. To alleviate this, we restrict the Laplace Transform to functions $f(t)$ of exponential growth order or less.

Theorem 3.10.1: The Linearity of the Laplace Transform

Let $f(t)$ and $g(t)$ be functions such that $\mathcal{L}\left[f(t)\right] = F(s)$ exists for $s > s_1$ and $\mathcal{L}\left[g(t)\right] = G(s)$ exists for $s > s_2$ and $\alpha, \beta \in \mathbb{C}$. Then, choosing $s > \max(\{s_1, s_2\})$,

$$\mathcal{L}\left[\alpha f(t) + \beta g(t)\right] = \alpha\mathcal{L}\left[f(t)\right] + \beta\mathcal{L}\left[g(t)\right].$$

Proof. The result follows from the linearity of the Riemann integral. $\square$

To compute Laplace Transforms, we can use Definition 3.10.2, but in practice, it is much easier to use a table. A bare-bones table is provided in Table A.1. Consider the following examples.

Example 3.10.1: ☢☢ Computing a Laplace Transform 1

Compute $\mathcal{L}\left[6e^{-5t} + e^{3t} + 5t^3 - 9\right]$.

We have that

$$\begin{aligned}\mathcal{L}\left[6e^{-5t} + e^{3t} + 5t^3 - 9\right] &= 6\mathcal{L}\left[e^{-5t}\right] + \mathcal{L}\left[e^{3t}\right] + 5\mathcal{L}\left[t^3\right] - \mathcal{L}\left[9\right] \\ &= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}.\end{aligned}$$

Example 3.10.2: ☢☢ Computing a Laplace Transform 2

Compute $\mathcal{L}\left[t^2 \sin 2t\right]$.

We have that

$$\begin{aligned}\mathcal{L}\left[t^2 \sin 2t\right] &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\frac{2}{s^2+4} \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{4s}{(s^2+4)^2} \\ &= -\left(\frac{4}{(s^2+4)^2} - \frac{16s^2}{(s^2+4)^3}\right).\end{aligned}$$

Example 3.10.3: ☢☢ Computing a Laplace Transform 3

Compute $\mathcal{L}\left[4\cos 4t - 9\sin 4t + 2\cos 10t\right]$.

We have that

$$\begin{aligned}\mathcal{L}\left[4\cos 4t - 9\sin 4t + 2\cos 10t\right] &= 4\mathcal{L}\left[\cos 4t\right] - 9\mathcal{L}\left[\sin 4t\right] + 2\mathcal{L}\left[\cos 10t\right] \\ &= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}.\end{aligned}$$

Example 3.10.4: ☢☢ Computing a Laplace Transform 4

Compute $\mathcal{L}\left[e^{3t} + \cos 6t - e^{3t}\cos 6t\right]$.

We have that

$$\begin{aligned}\mathcal{L}\left[e^{3t} + \cos 6t - e^{3t}\cos 6t\right] &= \mathcal{L}\left[e^{3t}\right] + \mathcal{L}\left[\cos 6t\right] - \mathcal{L}\left[e^{3t}\cos 6t\right] \\ &= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}\end{aligned}$$

3.11 Lecture 22: March 13, 2023

3.11.1 The Inverse Laplace Transform

THe following theorems are both essential in using the Laplace Transform in order to solve differential equations.

Theorem 3.11.1: The Laplace of a Derivative

The Laplace Transform turns differentiation into multiplication. That is,

$$\begin{aligned}\mathcal{L}\left[f'(t)\right] &= \int_{t=0}^{t=\infty} e^{-st} f'(t)\,\mathrm{d}t \\ &= \left(\frac{f(t)}{e^{st}}\right)_{t=0}^{t=\infty} + s\int_{t=0}^{t=\infty} e^{-st} f(t)\,\mathrm{d}t \\ &= \lim_{b\to\infty}\left(\frac{f(t)}{e^{sb}}\right) - f(0) + s\int_{t=0}^{t=\infty} e^{-st} f(t)\,\mathrm{d}t \\ &= s\mathcal{L}(f(t)) - f(0).\end{aligned}$$

We must note that this is only possible for functions $f(t)$ of exponential growth order or less.

Theorem 3.11.2: The Injectivity of the Laplace Transform (Lerch's Theorem)

Let $f(t), g(t) \in C(\{x \in \mathbb{R} : x \geq 0\})$ such that $\mathcal{L}\left[f(t)\right] = \mathcal{L}\left[g(t)\right]$. Then, $f(t) = g(t)$ almost everywhere[a] on $\{x \in \mathbb{R} : t \geq 0\}$.

[a] Measure Theory.

Theorem 3.11.2 tells us that if we have $F(s) = G(s)$, where F and G are Laplace Transforms, the functions that were transformed to get F and G are indeed equal. This means we can construct the following definition.

Definition 3.11.1: The Inverse Laplace Transform

If $\mathcal{L}\left[f(t)\right] = F(s)$, we define

$$\mathcal{L}^{-1}\left[F(s)\right] = f(t).$$

When computing the Inverse Laplace Transform, it is again useful to use a table, such as one provided in Table A.1. It is also critical to use partial fraction decomposition techniques commonly taught in either a precalculus or second-semester calculus course. For convenience, we provide the below table.

Factor	**Term**
$(ax+b)^n$	$\dfrac{A_1}{(ax+b)} + \cdots + \dfrac{A_n}{(ax+b)^n}$
$(ax^2+bx+c)^n$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \cdots + \dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

Table 3.2: Table of Partial Fraction Decompositions

Consider the following examples.

Example 3.11.1: ☢☢ Computing an Inverse Laplace Transform 1

Compute $\mathcal{L}^{-1}\left[\frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}\right]$.

We have that

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}\right] &= \mathcal{L}^{-1}\left[\frac{6}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-8}\right] + \mathcal{L}^{-1}\left[\frac{4}{s-3}\right] \\ &= 6 - e^{8t} + 4e^{3t}.\end{aligned}$$

Example 3.11.2: ☢☢ Computing an Inverse Laplace Transform 2

Compute $\mathcal{L}^{-1}\left[\frac{6s}{s^2+25} + \frac{3}{s^2+25}\right]$.

We have that

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{6s}{s^2+25} + \frac{3}{s^2+25}\right] &= \mathcal{L}^{-1}\left[\frac{6s}{s^2+25}\right] + \mathcal{L}^{-1}\left[\frac{3}{s^2+25}\right] \\ &= 6\cos 5t + \frac{3}{5}\sin 5t.\end{aligned}$$

Example 3.11.3: ☢☢ Computing an Inverse Laplace Transform 3

Compute $\mathcal{L}^{-1}\left[\frac{1}{s^3+4s^2+3s}\right]$.

We have that

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^3+4s^2+3s}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s(s^2+4s+3)}\right]\\ &= \mathcal{L}^{-1}\left[\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}\right].\end{aligned}$$

Then, $1 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1)$, so $A = \frac{1}{3}$, $B = -\frac{1}{2}$, and $C = \frac{1}{6}$. Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^3+4s^2+3s}\right] &= \mathcal{L}^{-1}\left[\frac{1}{3}\frac{1}{s}-\frac{1}{2}\frac{1}{s+1}+\frac{1}{6}\frac{1}{s+3}\right]\\ &= \frac{1}{3}-\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t}.\end{aligned}$$

Example 3.11.4: ☢☢ Computing an Inverse Laplace Transform 4

Compute $\mathcal{L}^{-1}\left[\frac{-s^5-2s^4+12}{2s^4(s+1)^2}\right]$.

We have that

$$\mathcal{L}^{-1}\left[\frac{-s^5-2s^4+12}{2s^4(s+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s^4}+\frac{E}{s+1}+\frac{F}{(s+1)^2}\right].$$

Then,

$$-s^5-2s^4+12 = As^3(s+1)^2+Bs^2(s+1)^2+Cs(s+1)^2+D(s+1)^2+Es^4(s+1)+Fs^4.$$

By expanding the right hand side, and matching coefficients, we obtain $A = -24$, $B = 18$, $C = -12$, $D = 6$, $E = \frac{47}{2}$, and $F = \frac{11}{2}$. Thus,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{-s^5-2s^4+12}{2s^4(s+1)^2}\right] &= \mathcal{L}^{-1}\left[-\frac{24}{s}+\frac{18}{s^2}-\frac{12}{s^3}+\frac{6}{s^4}+\frac{47}{2}\frac{1}{s+1}+\frac{11}{2}\frac{1}{(s+1)^2}\right]\\ &= -24+18t-12t^2+6t^3+\frac{47}{2}e^{-t}+\frac{11}{2}te^{-t}.\end{aligned}$$

3.12 Lecture 23: March 15, 2023

3.12.1 Solving Differential Equations With the Laplace Transform

We are now ready to solve linear differential equations using the Laplace Transform. Let

$$F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x) = 0.$$

Consider the following steps.

1. Form $\mathcal{L}\left[F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x)\right] = \mathcal{L}\,[0]$.
2. Solve the resulting algebraic equation for $\mathcal{L}\,[y(x)]$.
3. Find $\mathcal{L}^{-1}\,[\mathcal{L}\,[y(x)]]$ in order to find $y(x)$.

Consider the following examples.

Example 3.12.1: Using the Laplace Transform as a Solution Method 1

3.13 Lecture 24: March 17, 2023

3.13.1 Generalized Differential Operators: Part I

Consider the following definition.

Definition 3.13.1: Differential Operators

Let X and Y be spaces and $D : X \to Y$. Then, D is a differential operator if and only if it is linear and satisfies the Leibniz Rule. That is,

$$D(x_1, x_2) = x_1 D(x_2) + x_2 D(x_1).$$

For example, for $C^k(\mathbb{R})$, the space of k-continuously differentiable functions on $\mathbb{R}$, the regular differential operator $D = \frac{\mathrm{d}}{\mathrm{d}x}$ satisfies the Leibniz Rule. Note $\frac{\mathrm{d}}{\mathrm{d}x} : C^k(\mathbb{R}) \to C^{k-1}(\mathbb{R})$.

Example 3.13.1: A Differential Operator

Given any $\varphi \in C(\mathbb{R})$, consider $D = \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}$. Show that D is a differential operator.

For $f, g \in C^k(\mathbb{R})$,

$$\begin{aligned} D(c_1 f(x) + c_2 g(x)) &= \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}(c_1 f(x) + c_2 g(x)) \\ &= c_1\varphi(x) f(x) + c_2\varphi(x) g(x), \end{aligned}$$

so D is linear. Then,

$$\begin{aligned} D(f(x), g(x)) &= \varphi(x)\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) \\ &= \varphi(x)f'(x)g(x) + \varphi(x)f(x)g'(x) \\ &= \varphi(x)f(x)g'(x) + \varphi(x)f'(x)g(x) \\ &= f(x)D(g(x)) + g(x)D(f(x)), \end{aligned}$$

so D satisfies the Leibniz Rule and satisfies the Leibniz Rule.

We will now introduce 2 common generalized differential operators:

$$\tilde{D} = \varphi_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + \cdots + \varphi_n\frac{\mathrm{d}}{\mathrm{d}x}, \quad \hat{D} = \varphi_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + \cdots + \varphi_n\frac{\mathrm{d}^n}{\mathrm{d}x^n}.$$

Consider the following example of computing a differential operator.

Example 3.13.2: ☢☢ Computing a Differential Operator 1

Consider $D^n = \frac{\mathrm{d}^n}{\mathrm{d}x^n}$. Then, find $((x^2+5)D^2 - e^x D^3)(y(x) + \sin x)$.

We have that

$$\begin{aligned}((x^2+5)D^2 - e^x D^3)(y(x)+\sin x) &= ((x^2+5)D^2 - e^x D^3)y(x) + ((x^2+5)D^2 - e^x D^3)\sin x \\ &= (x^2+5)D^2 y(x) - e^x D^3 y(x) + (x^2+5)D^2 \sin x - e^x D^3 \sin x \\ &= (x^2+5)y''(x) - e^x y'''(x) - (x^2+5)\sin x + e^x \cos x.\end{aligned}$$

Example 3.13.3: ☢☢ Computing a Differential Operator 2

Consider $D^n = \frac{\mathrm{d}^n}{\mathrm{d}x^n}$. Then, find $x^2 D(\log(x)D(y(x)))$ and $\log(x)D(x^2 D(y(x)))$.

We have that

$$\begin{aligned}x^2 D(\log(x)D(y(x))) &= x^2\left(\log(x)D^2(y(x)) + \frac{1}{x}D(y(x))\right) \\ &= x^2\left(y''(x)\log(x) + y'(x)\frac{1}{x}\right)\end{aligned}$$

and

$$\begin{aligned}\log(x)D(x^2 D(y(x))) &= \log(x)\left(x^2 D^2(y(x)) + 2xD(y(x))\right) \\ &= \log(x)\left(x^2 y''(x) + 2xy'(x)\right).\end{aligned}$$

Note that $x^2 D(\log(x)D(y(x))) \neq \log(x)D(x^2 D(y(x)))$.

Example 3.13.4: ☢☢ Computing a Differential Operator 3

Consider $D^n = \frac{\mathrm{d}^n}{\mathrm{d}x^n}$. Then, find $(f_n(x)D^n + \cdots + f_0(x)D^0)y(x) - Q(x)$.

We have that

$$(f_n(x)D^n + \cdots + f_0(x)D^0)y(x) - Q(x) = f_n(x)y^{(n)}(x) + \cdots + f_0(x)y(x) - Q(x).$$

Note that then,

$$(f_n(x)D^n + \cdots + f_0(x)D^0)y(x) - Q(x) = 0$$

is an nth order linear differential equation.

3.14 Lecture 25: March 20, 2023

3.14.1 Generalized Differential Operators: Part II

Consider the following properties of generalized differential operators.

Theorem 3.14.1: Properties of Generalized Differential Operators

The following properties hold true for all generalized differential operators.

1. Commutativity of Addition: $F(D) + G(D) = G(D) + F(D)$.
2. Associativity of Composition: $F(D)[G(D)H(D)] = (F(D)G(D))[H(D)]$.
3. Distributivity I: $F(D)[G(D) + H(D)] = F(D)[G(D)] + F(D)[H(D)]$.
4. Distributivity II: $(F(D) + G(D))[H(D)] = F(D)[H(D)] + G(D)[H(D)]$.
5. Additive Identity $0 \cdot \frac{\mathrm{d}}{\mathrm{d}x} = 0$.
6. Compositional Identity: $1 \cdot D^0 = 1$.

Consider the following definitions and theorems.

Definition 3.14.1: Polynomial Differential Operators

We define $P_n(D) = a_n D^n + \cdots + a_0$, where $a_0, \dots, a_n \in \mathbb{C}$, as a polynomial differential operator of order n.

Theorem 3.14.2: Polynomial Operators Satisfy Compositional Commutativity

For polynomial differential operators $P_m(D)$ and $P_n(D)$,

$$P_m(D)[P_n(D)] = P_n(D)[P_m(D)] = P_{m+n}(D).$$

Moreover, if $P_m(D) = a_n D^n + \cdots + a_0$ and $P_n(D) = b_n D^n + \cdots + b_0$, $P_{m+n}(D)$ has coefficients obtained through ordinary polynomial multiplication.

Theorem 3.14.3: Factorization of Polynomial Operators

If $a_n m^n + \cdots + a_0 = (m - m_1) \cdots (m - m_n)$ for $m_1, \dots, m_n \in \mathbb{C}$,

$$P_n(D) = (D - m_1) \cdots (D - m_n).$$

Consider the following example.

Example 3.14.1: An Example Involving Theorem 3.14.3

Show that

$$P_3(D) = D^3 - 3D^2 + 2D = D[(D-2)[D-1]].$$

By Theorem 3.14.3, this is true since $m^3 - 3m^2 + 2m = m(m-2)(m-1)$. But, to explicitly show this, consider

$$(D^3 - 3D^2 + 2D)[y(x)] = y'''(x) - 3y''(x) + 2y'(x),$$

and then

$$\begin{aligned} D[(D-2)[D-1]][y(x)] &= D[(D-2)[y'(x) - y(x)]] \\ &= D[y''(x) - 3y'(x) + 2y(x)] \\ &= y'''(x) - 3y''(x) + 2y'(x), \end{aligned}$$

as desired.

Since $P_n(D)[y] = 0$ defines a linear ordinary differential equation with constant coefficients, the characteristic polynomial is precisely obtained by replacing D with m. Note that factorization is still sometimes possible for some general differential operator $F(D)$. Take $x^2y'' + 2xy' = 0$ as an example. Then, $x^2D^2 + 2xD = D[x^2D]$.

3.15 Lecture 26, April 3, 2023

3.15.1 The Method of Operators

Suppose $F(x, y, \dots, y^{(n)}) = 0$ is linear and can be "factored" in terms of generalized differential operators such that we have
$$F_1(D) \cdots F_n(D) + Q(x) = 0.$$
Then, $F(x, y, \dots, y^{(n)}) = 0$ can be solved recursively by solving n first order equations in sequence. Consider the following examples.

Example 3.15.1: ☢☢ The Method of Operators 1

Solve $F(x, y, y', y'') = y'' - y' - 2e^x = 0$ using the Method of Operators.

We have that $y_c(x) = c_1 + c_2 e^x$. Note that $y'' - y' = (D^2 - D)y(x) = D(D-1)(y(x))$. Therefore,
$$F(x, y, y', y'') = D(D-1)(y_p(x)) - 2e^x = 0.$$
Let $v(x) = (D-1)(y_p(x))$. Then, we have the two first order equations
$$Dv(x) - 2e^x = 0, \quad (D-1)y_p(x) - v(x) = 0.$$
The first equation gives us $v'(x) = 2e^x$, so $v(x) = 2e^x + c$. Since we are only looking for a particular solution, let $c = 0$ so $v(x) = 2e^x$. The second equation gives us $y_p'(x) - y_p(x) - 2e^x = 0$. Thus, $y_p(x) = \frac{2x}{e^{-x}} = 2xe^x$. Thus, we have that
$$y(x) = c_1 + c_2 e^x + 2xe^x$$
on $\mathbb{R}$.

Example 3.15.2: ☢☢ The Method of Operators 2

Solve $F(x, y, y', y'') = x^2 y'' - 2xy' + 5 = 0$ using the Method of Operators.

We have that $(x^2D^2 - 2xD)(y(x)) + 5 = (x^2D - 2x)(D(y(x))) + 5$. Then,
$$F(x, y, y', y'') = (x^2D - 2x)(D(y(x))) + 5 = 0.$$
Let $v(x) = D(y(x))$. Then we have the two first order equations
$$(x^2D - 2x)v(x) + 5 = 0, \quad Dy(x) - v(x) = 0.$$
The first equation gives us $x^2v'(x) - 2xv(x) + 5 = 0$. With $x \neq 0$, we have $v'(x) - \frac{2}{x}v(x) + \frac{5}{x^2} = 0$. Thus, $v(x) = x^2\left(\int -\frac{5}{x^4}\,\mathrm{d}x\right) + c_1 = \frac{5}{3x} + c_1x^2$. The second equation gives us $y'(x) - \frac{5}{3x} - c_1x^2 = 0$. Then,
$$y(x) = \int \frac{5}{3x}\,\mathrm{d}x + \int c_1x^2\,\mathrm{d}x = \frac{5}{3}\log|x| + c_2 + \tilde{c}_1x^3$$
on $\{x \in \mathbb{R} : x \neq 0\}$.

4

Systems of Differential Equations

4.1 Lecture 27: April 5, 2023

4.1.1 An Introduction to Linear Systems of Differential Equations

We now introduce systems of differential equations. Consider the following definition.

Definition 4.1.1: Systems of First Order Differential Equations

A system of m first order differential equations in k dependent variables is of the form

$$F_1(x, y_1, \dots, y_k, y_1', \dots, y_k') = 0, \quad \dots, \quad F_m(x, y_1, \dots, y_k, y_1', \dots, y_k').$$

Note that the analog for nth order equations simply includes all derivatives up to $y_i^{(n)}$, $1 \leq i \leq n$. Consider the following definitions and theorems; in the below cases, linear algebraic techniques can be employed.

Definition 4.1.2: Linear Systems of First Order Differential Equations

A system of differential equations is linear if and only if each equation contains only terms $f_{1,i}(x)y_i'(x)$, $f_{0,i}(x)y_i(x)$, and $Q(x)$.

Definition 4.1.3: Reduced Linear Systems of First Order Differential Equations

A linear system of first order differential equations is reduced if and only if no nonzero equation is a linear combination of other equations.

Theorem 4.1.1: Number of Solutions to Reduced Systems

If a reduced linear system is square, the system has a unique solution for each $y_i(x)$, or no solution exists. If a reduced linear system has more variables than equations, either no solution exists or some $y_i(x)$ has no unique p-parameter family.

Consider the following examples.

Example 4.1.1: Substitution System 1

Find a solution to the system

$$\begin{cases} y_2' - xy_2 & = 0 \\ x^2 y_1 - e^x \sin(2x) & = 0 \\ y_1' - 807y_2 & = 0 \end{cases}.$$

From the second equation, we have $y_1 = \frac{e^x \sin(2x)}{x^2}$ with $x \neq 0$. If we substitute this into the third equation, we have

$$\begin{aligned} y_2 = \frac{y_1'}{807} &= \frac{1}{807} \frac{\mathrm{d}}{\mathrm{d}x} \frac{e^x \sin(2x)}{x^2} \\ &= \frac{-2xe^x \sin(2x) + x^2(e^x \sin(2x) + 2e^x \cos(2x))}{807x^4}. \end{aligned}$$

The first equation gives us

$$\begin{aligned} y_2 &= e^{-\int -x \, \mathrm{d}x} \left(\int e^{\int -x \, \mathrm{d}x} \cdot 0 \, \mathrm{d}x + c_1 \right) \\ &= c_1 e^{\frac{1}{2}x^2}. \end{aligned}$$

Thus, we don't have a solution. Note that the second equation is not first order.

Example 4.1.2: ☢☢ Substitution System 2

Find a solution to the system

$$\begin{cases} y_2' - xy_2 & = 0 \\ x^2 y_1' - e^x & = 0 \\ y_1' - 807y_2 & = 0 \end{cases}.$$

From the second equation, we have $y_1' = \frac{e^x}{x^2}$, so $y_1 = \int \frac{e^x}{x^2}\, \mathrm{d}x + c_1$. Then, substituting into the third equation, we have $\frac{e^x}{x^2} - 807y_2 = 0$, so $y_2 = \frac{e^x}{807x^2}$. But, by the first equation, we have $y_2 = e^{\int x \, \mathrm{d}x}\left(\int e^{\int -x\, \mathrm{d}x} \cdot 0 \, \mathrm{d}x + c_2\right) = c_2 e^{\frac{1}{2}x^2}$. This doesn't match; thus, we don't have a solution.

The following theorem, which is not at all new, but is useful to know, describes the solution to a first order linear differential equation.

Theorem 4.1.2: Solution of a First Order Linear Differential Equation

If we have $F(x, y, y') = y' + Py(x) = Q(x)$,

$$y(x) = e^{-\int P(x)\, \mathrm{d}x}\left(\int e^{\int P(x)\, \mathrm{d}x} Q(x) + c\right).$$

4.2 Lecture 28: April 7, 2023

4.2.1 Gauss-Jordan Elimination for Systems of Differential Equations: Part I

If we use operator notation, an analog of Gauss-Jordan Elimination can be employed to put linear systems in triangular form. Recall that the three row operations are swapping rows, adding two rows, and applying $f(x)D$. We have covered substitution in the previous lecture. We will move to elimination. But, before that, consider the following example using substitution.

Example 4.2.1: ☢☢ Substitution System 3

Find a solution to the system

$$\begin{cases} y_2' - xy_2 &= 0 \\ y_1' - e^x &= 0 \\ y_1' - 807y_2 &= 0 \end{cases}.$$

From the second equation, we have $y_1' = e^x$, so $y_1 = e^x + c_1$. Then, by the third equation, we have $e^x - 807y_2 = 0$, so $y_2 = \frac{e^x}{807}$. But, by the first equation, we have $y_2 = e^{\int x \, dx}\left(\int e^{\int -x \, dx} \cdot 0 \, dx + c_2\right) = c_2 e^{\frac{1}{2}x^2}$. This doesn't match; thus, we don't have a solution.

For Gauss-Jordan Elimination for Systems of Differential Equations, we wish to write our system in terms of differential operators, and then row reduce to triangular form. The three allowable row operations are:

1. Swap row i and row j; denote by $\langle i \rangle \leftrightarrow \langle j \rangle$.
2. Add row i and row j; denote by $\langle i \rangle + \langle j \rangle \to \langle i \rangle$.
3. Apply a differential operator to row i; $F(D)\langle i \rangle \to \langle i \rangle$.

This is best seen through examples. Consider the following.

Example 4.2.2: ☢☢ Elimination System 1

4.3 Lecture 29: April 10, 2023

4.3.1 Gauss-Jordan Elimination for Systems of Differential Equations: Part II

If we have the triangular form

$$\begin{aligned} Dy_1 + D(D-x)y_2 &= 0 \\ (D(D-x)+D)y_2 &= -e^x, \end{aligned}$$

Note that $(D(D-x)+D) = D((D-x)+1)$. We may now use the Method of Operators to solve the second equation. We have $v(x) = (D-(x+1))y_2(x)$ and $Dv(x) = -e^x$. Since $v'(x) = -e^x$, $v(x) = -e^x + c_1$ by separation of variables. Then, we have

$$-e^x + c_1 = y_2'(x) + y_2(x)(1-x),$$

so

$$\begin{aligned} y_2(x) &= e^{\int -(1-x)\,\mathrm{d}x}\left(\int e^{\int (1-x)\,\mathrm{d}x}(-e^x + c_1)\,\mathrm{d}x\right) \\ &= e^{\frac{x^2}{2}-x}\left(\int e^{x-\frac{x^2}{2}}(-e^x + c_1)\,\mathrm{d}x + c_2\right) \\ &= e^{\frac{x^2}{2}-x}\left(\int e^{x-\frac{x^2}{2}} + c_1 e^{x-\frac{x^2}{2}}\,\mathrm{d}x + c_2\right). \end{aligned}$$

Now, all we need to do is substitute into the first equation and solve algorithmically. This is possible, but tedious. Note that $(D(D-x)+D)y_2 = -e^x$ can be written as

$$\begin{aligned} -e^x &= D^2y_2 - D(xy_2) + Dy_2 \\ &= y_2''(x) - xy_2'(x) - y_2(x) + y_2'(x) \\ &= y_2''(x) + (1-x)y_2'(x) - y_2(x). \end{aligned}$$

We can solve the above by "guessing properly" for a solution for the homogeneous solution and applying Reduction of Order.

4.4 Lecture 30: April 12, 2023

4.4.1 From an nth Order Equation to an n Dimensional System of First Order Equations

As motivation, consider a second order linear equation. That is,

$$F(x, y, y', y'') = f_2(x)y'' + f_1(x)y' + f_0(x)y = Q(x).$$

Denote $y_1(x) = y(x)$. Then denote $y_2(x) = y_1'(x)$. We then have $y_2'(x) = y_1''(x)$. From $F(x, y, y', y'')$, we have $f_2(x)y_2'(x) + f_1(x)y_2(x) + f_0(x)y_1(x) = Q(x)$. Our system is then

$$\begin{cases} y_1'(x) - y_2(x) &= 0 \\ f_2(x)y_2'(x) + f_1(x)y_2(x) + f_0(x)y_1(x) &= Q(x) \end{cases}.$$

For $n = 3$,

$$\begin{cases} y_1'(x) - y_2(x) &= 0 \\ y_2'(x) - y_3(x) &= 0 \\ f_3(x)y_3'(x) + f_2(x)y_3(x) + f_1(x)y_2(x) + f_0(x)y_1(x) &= Q(x) \end{cases}.$$

Consider the following theorem, illustrating the general case.

Theorem 4.4.1: Reducing an nth Order Linear Equation Into an n Dimensional System

Let $F(x, y, y', \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x) = 0$ be an nth order linear differential equation. Denote $y_1(x) = y(x)$ and

$$y_1'(x) = y_2(x), \quad , \dots , \quad y_1^{(n-1)}(x) = \cdots = y_{n-1}'(x) = y_n(x).$$

Note that $y_n'(x) = y^{(n)}(x)$, so substituting this into F ensures that y_n' is the only derivative term. We have

$$f_n(x)y_n'(x) + f_{n-1}(x)y_n(x) + f_{n-2}(x)y_{n-1}(x) + \cdots + f_2(x)y_3(x) + f_1(x)y_2(x) + f_0(x)y_1(x) = 0.$$

Our system is then

$$\begin{cases} y_1'(x) - y_2(x) &= 0 \\ \vdots & \vdots \\ y_{n-1}'(x) - y_n(x) &= 0 \\ f_n(x)y_n'(x) + f_{n-1}(x)y_n(x) + f_{n-2}(x)y_{n-1}(x) + \cdots + f_2(x)y_3(x) + f_1(x)y_2(x) + f_0(x)y_1(x) &= 0 \end{cases}.$$

Consider the following example.

Example 4.4.1: ☢☢ Reduction Into Triangular Form 1

Express $F(x, y, y', y'', y'') = xy''' - y'' = 0$ as a first order system. Then, place the system into triangular form.

We have the system

$$\begin{cases} y_1'(x) - y_2(x) & = 0 \\ y_2'(x) - y_3(x) & = 0 \\ xy_3'(x) - y_3(x) & = 0 \end{cases}.$$

Then, using operator notation, we have

$$\begin{cases} Dy_1 - y_2 + 0y_3 & = 0 \\ 0y_1 + Dy_2 - y_3 & = 0 \\ 0y_1 + 0y_2 + (xD - 1)y_3 & = 0 \end{cases}.$$

We realize that we are already in triangular form.

Example 4.4.2: ☢☢ Reduction Into Triangular Form 2

Express $F(x, y, y', y'', y'') = xy''' - y' = 0$ as a first order system. Then, place the system into triangular form.

We have the system

$$\begin{cases} y_1'(x) - y_2(x) &= 0 \\ y_2'(x) - y_3(x) &= 0 \\ xy_3'(x) - y_2(x) &= 0 \end{cases}.$$

Then, using operator notation, we have

$$\begin{cases} Dy_1 - y_2 + 0y_3 &= 0 \\ 0y_1 + Dy_2 - y_3 &= 0 \\ 0y_1 - y_2 + (xD)y_3 &= 0 \end{cases}.$$

To place the above into triangular form, consider the augmented matrix

$$\begin{bmatrix} D & -1 & 0 & | & 0 \\ 0 & D & -1 & | & 0 \\ 0 & -1 & xD & | & 0 \end{bmatrix}$$

We perform $R_3 = DR_3 + R_2$ to obtain

$$\begin{bmatrix} D & -1 & 0 & | & 0 \\ 0 & D & -1 & | & 0 \\ 0 & 0 & D(xD) - 1 & | & 0 \end{bmatrix}.$$

Thus, the triangular system is

$$\begin{cases} Dy_1 - y_2 + 0y_3 &= 0 \\ 0y_1 + Dy_2 - y_3 &= 0 \\ 0y_1 + 0y_2 + (D(xD) - 1)y_3 &= 0 \end{cases},$$

and simplifying gives

$$\begin{cases} y_1' - y_2 &= 0 \\ y_2' - y_3 &= 0 \\ xy_3'' + y_3' - y_3 &= 0 \end{cases}.$$

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5
Miscellaneous Techniques

5.1 Lecture 31: April 14, 2023

5.1.1 A Review of Power Series

We seek to solve differential equations using power series; this is due to the fact that, excluding constant coefficient equations, we have seen enormous difficulty finding closed form solutions to "nice"-looking equations. For example, consider the following.

1. $x^2y'' - 2xy_2y = 0$.
2. $y'' - xy = 0$.
3. $xy''' - y'' = 0$.
4. $xy''' - y' = 0$.
5. $xy'' + y' - y = 0$.

Note that some of the above equations are not too difficult, but distinguishing these from the equations that are is very difficult; sometimes, we may not be able to compute antiderivatives at all. We now will consider power series. Consider the following definition.

Definition 5.1.1: Power Series

Let $a_i \in \mathbb{C}$. The power series centered at $x_0 \in \mathbb{R}$ is given by

$$P(x - x_0) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where $P(x - x_0)$ converges at x if and only if

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_k (x - x_0)^k$$

exists and is finite.

Definition 5.1.2: Radius of Convergence

The power series $P(x - x_0)$ has a radius of convergence $R \in \mathbb{R}^+$ if it converges for all $x \in \{x \in \mathbb{R} : x_0 - R < x < x_0 + R\}$. Note that $P(x - x_0)$ may or may not converge at $x = x_0 \pm R$.

Definition 5.1.3: Valid Power Series Representation

We define $f(x)$ has a valid power series representation $P(x - x_0)$ on some interval I if for all $x \in I$, $P(x - x_0)$ converges to $f(x)$. That is,

$$\lim_{n \to \infty} \left| f(x) - \sum_{k=0}^{n} a_k (x - x_0)^k \right| = 0.$$

If the above equation is true, we write $f(x) \simeq P(x - x_0)$.

Definition 5.1.4: Absolute Convergence

If $\sum_{k=0}^{\infty} |a_k|(x - x_0)^k$ is convergent, $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ converges absolutely.

Theorem 5.1.1: A Useful Lemma for Absolute Convergence

If $P(x - x_0)$ is absolutely convergent, it is convergent. Moreover, $P(x - x_0)$ is convergent regardless of the ordering of a_k.

5.2 Lecture 32: April 17, 2023

5.2.1 Existence and Uniqueness for Power Series Solutions

Consider the following theorems.

Theorem 5.2.1: Uniqueness of Valid Power Series Expansions

If $f(x)$ possesses a valid expansion $P(x-x_0)$ for some $x \in \{x \in \mathbb{R} : x_0 - R < x < x_0 + R\}$, this expansion is unique. Moreover, the coefficient terms are

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

This is just the usual Taylor expansion.

Theorem 5.2.2: Taylor Series Remainder

Recall that by Definition 5.1.3, $f(x)$ has a valid power series representation $P(x-x_0)$ if it converges to $f(x)$; that is, the remainder must approach zero as n approaches infinity. We can give the remainder as

$$R_n(x) = \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!}$$

where ξ is in the interval between x and x_0.

Consider the following example of finding a Taylor series.

Example 5.2.1: Finding a Taylor Series

Find the formal Taylor series expansion for $f(x) = \frac{1}{1-x}$ centered at $x_0 = 0$.

Note $f(x_0) = \frac{1}{1-x}, f'(x_0) = \frac{1}{(1-x)^2}, f''(x_0) = \frac{2}{(1-x)^3}, \ldots, f^{(k)}(x_0) = \frac{k!}{(1-x)^{k+1}}$. At $x_0 = 0$, $f^{(k)}(0) = k!$. Then, the coefficients are given by $a_k = \frac{k!}{k!} = 1$, and so we have the expected expansion

$$f(x) \simeq \sum_{k=0}^{\infty} x^k.$$

The following notion of analytic functions is important.

Definition 5.2.1: Analytic Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is analytic at x if and only if there exists a valid power series expansion on some neighborhood $(x_0 - R, x_0 + R)$. Given some interval I, if f is analytic at every point $x \in I$, it is analytic on I.

Consider the following theorem.

Theorem 5.2.3: Analytic Implies Existence of Power Series General Solution

Let $F(x, y, \dots, y^{(n)}) = f_n(x)y^{(n)} + \cdots + f_1(x)y' + f_0(x)y - Q(x) = 0$ be an nth order linear equation with the additional restrictions that f_i, $1 \leq i \leq n$ and $Q(x)$ are analytic on some common interval I. Under these stipulations, the general solution $y(x)$ possesses a power series solution valid on I.

This leads us to a question. What is the relationship between $y'(x)$ and $y(x) \simeq P(x - x_0)$?

If $P(x - x_0) = \sum_{k=0}^{\infty} a_k(x - x_0)^k = \sum_{k=0}^{\infty} p_k(x)$ and $P'(x - x_0) = \sum_{k=0}^{\infty} p_k'(x)$, then, if $P'(x - x_0)$ converges uniformly to some function,

$$y'(x) \simeq P'(x - x_0).$$

Note p_i is a polynomial. Now, consider the following definitions.

Definition 5.2.2: Singularities

Let $F(x, y, \dots, y^{(n)}) = 0$ be an nth order homogeneous linear equation. A point x_0 is ordinary if and only if all f_i, $1 \leq i \leq n$, are analytic at x_0. Then, x_0 is singular if and only if at least one f_i is not analytic at x_0.

A singularity is regular if and only if, after placing the equation into standard form, all g_i are analytic at x_0. Note

$$g_i(x) = f_i(x)(x - x_0)^{n-i}.$$

A singularity is irregular if and only if it is not regular.

5.3 Lecture 33: April 19, 2023

5.4 Lecture 34: April 21, 2023

5.4.1 Frobenius Equations and Series Solutions: Part I

Recall that $g_0(x) = x - 1$ and $g_1(x) = x$ are polynomials, so both functions are analytic. Thus, $x_0 = 1$ is a regular singularity.

If $x_0 = a$ is an ordinary point, $y(x)$ will jave a power series solution $y(x) \simeq P(x - x_0)$ valid on the common interval of analyticity. In the case x_0 is a regular sngularity, $y(x)$ need not have a power series expansion. We now define Frobenius equations and state an accompanying theorem.

Definition 5.4.1: Frobenius Equations

If $F(x, y \dots, y^{(n)}) = 0$ is in standard form and x_0 is a regular singularity, the associated Frobenius differential equation is

$$(x - x_0)^n F(x, y, \dots, y^{(n)}) = 0.$$

Theorem 5.4.1: Solution to the Frobenius Equation

The Frobenius equation has the same solution as F, given by

$$y(x) \simeq \sum_{k=0}^{\infty} a_{k,m}(x - x_0)^{k+m}$$

with $a_{0,m} \neq 0$ and $m \in \mathbb{C}$. This Frobenius series $P(x - x_0, m)$ is valid on the common interval of analyticity of the coefficient functions of the Frobenius equation.

Consider the following example.

Example 5.4.1: ☢☢ Finding Power Series Coefficients 1

Consider $F(x, y, y', y'') = (1-x)y'' + xy' - y = 0$.

We have proven $x_0 = 0$ is an ordinary point. The interval of analyticity, we have shown to be $I = \{x \in \mathbb{R} : -1 < x < 1\}$. We are expecting a power series solution $y(x) \simeq P(x - x_0)$ valid on I. Since $x_0 = 0$ is ordinary, we will use the easier non-standard form. We now assume

$$y(x) = \sum_{k=0}^{\infty} a_k (x-0)^k = \sum_{k=0}^{\infty} a_k x^k,$$

so

$$y'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}, \quad y''(x) = \sum_{k=0}^{\infty} k(k-1) a_k x^{k-2}.$$

We may now substitute this into F to obtain

$$\begin{aligned}
(1-x)\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} + x\sum_{k=0}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k &= 0 \\
\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1)a_k x^{k-1} + \sum_{k=0}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k &= \\
\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)a_k x^{k-1} + \sum_{k=0}^{\infty} (k-1) a_k x^k &= \\
\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^{k} - \sum_{k=0}^{\infty} k(k+1)a_{k+1} x^{k} + \sum_{k=0}^{\infty} (k-1) a_k x^k &=
\end{aligned}$$

Then,

$$\sum_{k=0}^{\infty} ((k-1)a_k - k(k+1)a_{k+1} + (k+2)(k+1)a_{k+2})x^k = 0 \simeq \sum_{k=0}^{\infty} 0x^k.$$

We now have the equation $(k-1)a_k - k(k+1)a_{k+1} + (k+2)(k+1)a_{k+2} = 0$, a recurrence relation. We test small values of k to see if we can find a pattern. We wish to identify all coefficients in terms of a_0.

- $k = 0$: $a_0 - 0 + 2 \cdot 1 \cdot a_2 = 0$ so $2a_2 = a_0$.
- $k = 1$: $-2 \cdot 1 \cdot a_2 + 6a_3 = 0$ so $a_3 = \frac{1}{3}a_2 = \frac{1}{6}a_0$.
- $k = 2$: $a_2 - 6a_3 + 4 \cdot 3a_4 = 0$ so $a_4 = \frac{1}{2}a_3 - \frac{1}{12}a_2 = \frac{1}{24}a_0$.
- $\cdots$: $\cdots$ so $\cdots$.
- $k = n$: $\cdots$ so $a_{n+2} = \frac{1}{(n+2)!}a_0$.

Note that a_1 is arbitrary and is not dependent on any other a_k. Thus, on $(-1, 1)$,

$$y(x) \simeq \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + \sum_{k=2}^{\infty} \frac{a_k}{k!}x^k = a_0 + a_0 x + (a_1 - a_0)x + \sum_{k=2}^{\infty} \frac{a_0}{k!}x^k = (a_1 - a_0)x + \sum_{k=0}^{\infty} \frac{a_0}{k!}a^k = \tilde{a}_1 x + a_0 e^x.$$

5.5 Lecture 35: April 24, 2023

5.5.1 Frobenius Equations and Series Solutions: Part II

We now proceed by working out more examples.

Example 5.5.1: ☢☢ Finding Power Series Coefficients 2

Consider $F(x, y, y', y'') = x^2y'' - 3xy' + 2y = 0 : x_0 = 0$.

We first determine the nature of $x_0 = 0$ by first writing F in standard form. We obtain

$$y'' - \frac{3}{x}y' + \frac{2}{x^2}y = 0.$$

Note that $f_0(x) = \frac{2}{x^2}$. Since $f_0(x)$ is not defined at $x_0 = 0$, it cannot ba analytic at x_0. Thus, x_0 is a singular point by Definition 5.2.2. Then, we construct $g_0(x) = f_0(x)(x - x_0)^2 = \frac{2}{x^2}x^2 = 2$, $g_1(x) = f_1(x)(x-x_0)^2 = -\frac{3}{x}x = -3$. Since both g_0 and g_1 are constant, they are analytic everythere. Thus, F has a Frobenius solution valid on $\mathbb{R}$, except perhaps at $x = 0$. We have a regular singularity at $x_0 = 0$. Now, we consider the Frobenius equation

$$\begin{aligned} F(x, y, y', y'')_{\text{frob}} &= (x - x_0)^2F(x, y, y', y'') \\ &= x^2\left(y'' - \frac{3}{x}y' + \frac{2}{x^2}\right) = 0 \\ &= x^2y'' - 3xy' + 2y = 0. \end{aligned}$$

Note that in this case, F_{frob} is the same as F, but this is not true in general. Now, we suppose

$$y(x) = \sum_{k=0}^{\infty} a_{k,m}(x - x_0)^{k+m} = \sum_{k=0}^{\infty} a_{k,m}x^{k+m},$$

where $a_{0,m} \neq 0$ and $m \in \mathbb{C}$. Then,

$$y'(x) = \sum_{k=0}^{\infty} a_{k,m}(k + m)a_{k,m}x^{k+m-1}$$

and

$$y''(x) = \sum_{k=0}^{\infty} a_{k,m}(k + m)(k + m - 1)x^{k+m-2}.$$

Substituting into F_{frob}, we have

$$x^2\sum_{k=0}^{\infty} a_{k,m}(k + m)(k + m - 1)x^{k+m-2} - 3x\sum_{k=0}^{\infty} a_{k,m}(k + m)a_{k,m}x^{k+m-1} + 2\sum_{k=0}^{\infty} a_{k,m}(x - x_0)^{k+m} = 0.$$

We can simplify the above to produce

$$\sum_{k=0}^{\infty} a_{k,m}(k + m)(k + m - 1)x^{k+m} - 3\sum_{k=0}^{\infty} a_{k,m}(k + m)a_{k,m}x^{k+m} + 2\sum_{k=0}^{\infty} a_{k,m}(x - x_0)^{k+m} = 0.$$

Now, we can combine terms to obtain

$$\sum_{k=0}^{\infty}((k+m)(k+m-1)-3(k+m)+2)a_{k,m}x^{k+m}=0\simeq\sum_{k=0}^{\infty}0x^{k+m}.$$

Thus, $((k+m)(k+m-1)-3(k+m)+2)a_{k,m}=0$. Since $a_{0,m}\neq 0$, evaluating at $k=0$, we get

$$(m(m-1)-3m+2)a_{0,m}=0,$$

meaning $m^2-4m+2=0$, which is true for $m_1=2+\sqrt{2}$ and $m_2=2-\sqrt{2}$. For m_1, and $k\geq 1$, we have

$$((k+2+\sqrt{2})(k+1+\sqrt{2})-3(k+2+\sqrt{2})+2)a_{k,m_1}=0$$

which has no integer roots for k. Thus, $a_{k,m_1}=0$ for all $k\geq 1$. For m_2, and $k\geq 1$, we have

$$((k+2-\sqrt{2})(k+1-\sqrt{2})-3(k+2-\sqrt{2})+2)a_{k,m_2}=0$$

which has no integer roots for k. Thus, $a_{k,m_2}=0$ for all $k\geq 1$. Thus, m_1, we obtain the series

$$y_1(x)\simeq a_{0,m_1}x^{0+m_1}+\sum_{k=1}^{\infty}0x^{k+m},$$

so

$$y_1(x)=a_{0,m_1}x^{2+\sqrt{2}}.$$

Similarly, for m_2, we have

$$y_2(x)=a_{0,m_2}x^{2-\sqrt{2}}.$$

Since both y_1 and y_2 are both linearly independent 1-parameter families of solutions, the existence and uniqueness for linear equations gives

$$y(x)=a_{0,m_1}x^{2+\sqrt{2}}+a_{0,m_2}x^{2-\sqrt{2}}$$

where $a_{0,m_i}\in\mathbb{C}$. We must check this solution at $x=0$. This is valid since $y(x)$ is well-defined at $x=0$ and so was the original equation F.

5.6 Lecture 35: April 24, 2023

5.6.1 Frobenius Equations and Series Solutions: Part III

In the best case, where the roots of the indical polynomial differ by more than an integer, each root m_i gives rise to at least one 1-parameter family of solutions. Sometimes, with repeated roots, we can get a k-parameter family from a single root for $k \geq 2$.

However, when $m_i - m_j \in \mathbb{Z} - \{0\}$, it is possible that a "logarithmic" family exists not obtainable in the standard way. To illustrate, consider the following example.

Example 5.6.1: ☢☢ Finding Power Series Coefficients 3

Consider $F(x, y, y', y'', y''') = x^2y''' - 2xy'' - 4y' = 0 : x_0 = 0$.

We first deterine the nature of $x_0 = 0$ by first writing F in standard form. We obtain

$$y''' - \frac{2}{x}y'' - \frac{4}{x^2}y' = 0.$$

Note that $f_1(x) = -\frac{4}{x^2}$. Since $f_1(x)$ is not defined at $x_0 = 0$, it cannot be analytic at x_0. Thus x_0 is a singular point by Definition 5.2.2. Then, we construct $g_0(x) = 0$, $g_1(x) = -\frac{4}{x^2}x^2 = -4$, $g_2 = (x) = -\frac{2}{x}x = -2$. Since all g_i are constant, they are analytic everywhere. Thus, F has a Fronbenius solution valid on $\mathbb{R}$, except perhaps at $x = 0$. We have a regular singularity at $x_0 = 0$. Now, we consider the Frobenius equation

$$\begin{aligned} F(x, y, y', y'', y''')_{\text{frob}} &= (x - x_0)^3 F(x, y, y', y'', y''') \\ &= x^3y''' - 2x^2y'' - 4xy' = 0. \end{aligned}$$

Now, we suppose

$$y(x) = \sum_{k=0}^{\infty} a_{k,m}x^{k+m}$$

where $a_{0,m} \neq 0$ and $m \in \mathbb{C}$. Then,

$$y'(x) = \sum_{k=0}^{\infty} a_{k,m}(k + m)x^{k+m-1}, \quad y''(x) = \sum_{k=0}^{\infty} a_{k,m}(k + m)(k + m - 1)x^{k+m-2},$$

and

$$y'''(x) = \sum_{k=0}^{\infty} a_{k,m}(k + m)(k + m - 1)(k + m - 2)x^{k+m-3}.$$

Substituting into F_{frob}, we have

$$\begin{aligned} x^3\sum_{k=0}^{\infty} a_{k,m}(k + m)(k + m - 1)(k + m - 2)x^{k+m-3} - 2x^2\sum_{k=0}^{\infty} a_{k,m}(k + m)(k + m - 1)x^{k+m-2} \\ - 4x\sum_{k=0}^{\infty} a_{k,m}(k + m)x^{k+m-1} = 0 \end{aligned}$$

or, equivalently,

$$\sum_{k=0}^{\infty} a_{k,m}(k+m)(k+m-1)(k+m-2)x^{k+m} - 2\sum_{k=0}^{\infty} a_{k,m}(k+m)(k+m-1)x^{k+m} - 4\sum_{k=0}^{\infty} a_{k,m}(k+m)x^{k+m} = 0.$$

Now we can combine terms to obtain

$$\sum_{k=0}^{\infty}((k+m)(k+m-1)(k+m-2) - 2(k+m)(k+m-1) - 4(k+m))a_{k,m}x^{k+m} = 0 \simeq \sum_{k=0}^{\infty} 0x^{k+m}.$$

Thus, $((k+m)(k+m-1)(k+m-2) - 2(k+m)(k+m-1) - 4(k+m))a_{k,m} = 0$. Since $a_{0,m} \neq 0$, evaluating at $k = 0$, we get

$$(m(m-1)(m-2) - 2m(m-1) - 4m)a_{0,m} = 0,$$

meaning $m^2(m-5) = 0$, which is true for $m_1 = 0$ and $m_2 = 5$. For m_1, and $k \geq 1$, we have

$$(k(k-1)(k-2) - 2k(k-1) - 4k)a_{k,m_1} = k^2(k-5)a_{k,m_1} = 0.$$

For $k = 5$, $a_{5,m_1} \in \mathbb{C}$, but for $k \neq 5$, $a_{k,m_1} = 0$. Thus, $y_1(x) \simeq a_{0,m_1}x^{0+m_1} + \sum_{k=1}^{\infty} a_{k,m_1}x^{k+m_1} = a_{0,m_1}x^0 + a_{5,m_1}x^5 = a_{0,m_1} + a_{5,m_1}x^5$. For m_2, and $k \geq 1$, we have

$$((k+5)(k+4)(k+3) - 2(k+5)(k+4) - 4(k+5))a_{k,m_2} = k(k+5)^2.$$

Thus, $y_2(x) = a_{0,m_2}x^{0+m_2} + \sum_{k=1}^{\infty} a_{k,m_2}x^{k+m_2} = a_{0,m_2}x^5$. This is information we had previously. We must find some third linearly independent 1-parameter family solutions, since F is of order 3. A fact is $y_3(x) = \log x$.

Appendices

A
Table of Laplace Transforms

Function	**Transform** $(F(s) = \mathcal{L}[f(t)])$	**Function**	**Transform** $(F(s) = \mathcal{L}[f(t)])$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	$t^a,\ a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\ d\sigma$	$\int_0^t f(\tau)\ d\tau$	$\frac{F(s)}{s}$
e^{at}	$\frac{1}{s-a}$	$e^{at}f(t)$	$F(s-a)$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$t^nf(t)$	$(-1)^nF^{(n)}(s)$

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