MATH1300: CALCULUS I

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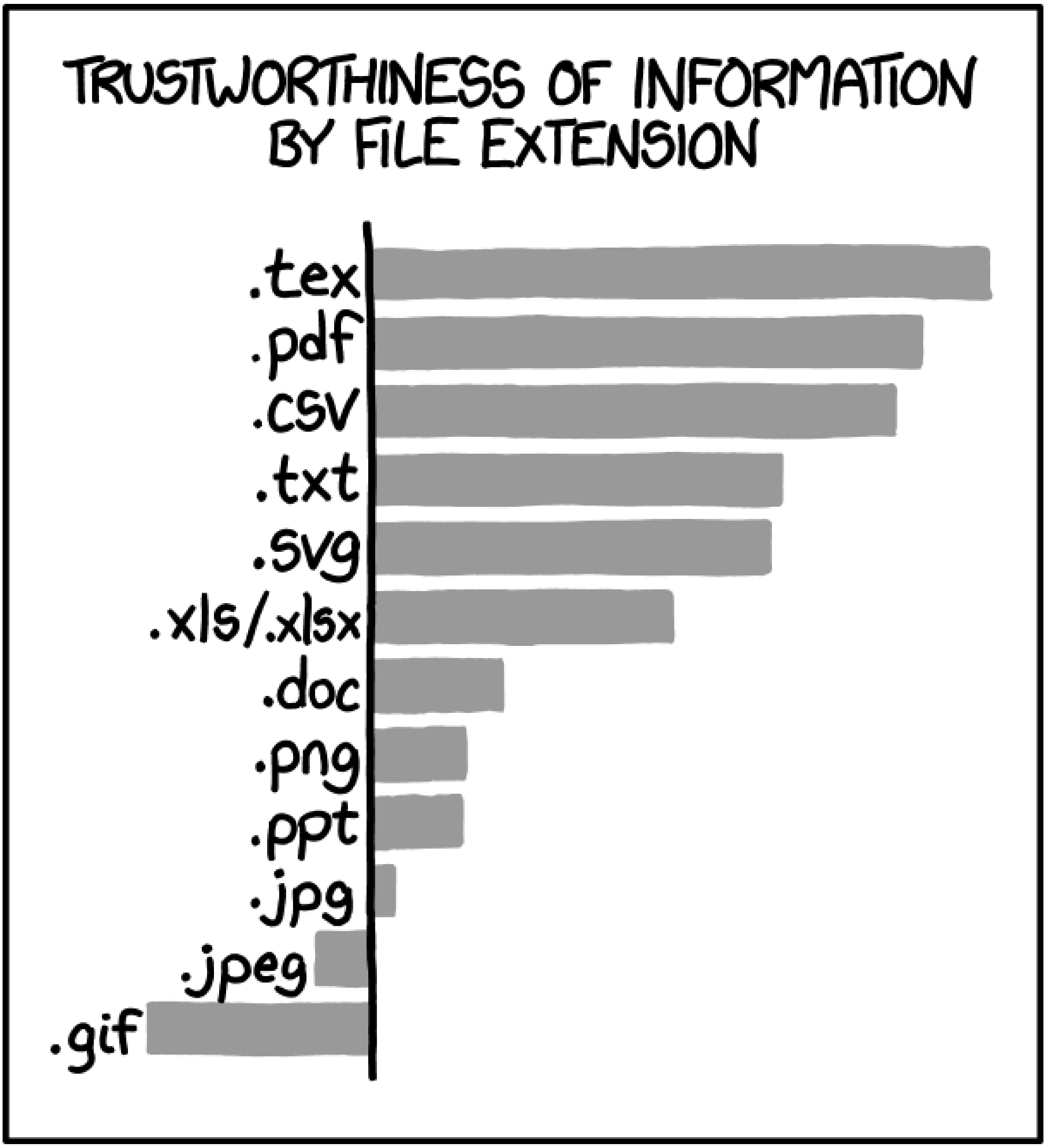
TEXTBOOK: JAMES STEWART

UNIVERSITY OF COLORADO BOULDER

EDITION 1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Preface

To the interested reader,

This document is a compilation of notes taken during the Spring 2023 semester for MATH1300: Calculus I at the University of Colorado Boulder during the author's tenure as a learning assistant for the course. The course used *Calculus – Concepts and Contexts*[1] by James Stewart as its primary text and was coordinated by Harrison Stalvey and Christopher Eblen. Additionally, in creating these notes, the author used *Calculus* by Ron Larson and Bruce Edwards. As such, many theorems, definitions, and examples may be quoted or derived from the aforementioned books. The section names will follow Stewart's book exactly.

The author would like to provide the following resources for students currently taking a Calculus I course:

1. Paul's *Online Math Notes* for Calculus I at Lamar University.
2. Professor Leonard's YouTube Calculus I Lectures.
3. 3Blue1Brown's *Essence of Calculus*.

Theorems, definitions, and examples may be quoted or derived from the aforementioned resources as well.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. If an error needs to be resolved, please contact Adithya Bhaskara at adithya.bhaskara@colorado.edu.

Best Regards,
Adithya Bhaskara

REVISED: January 15, 2023

[1] Stewart, J. (2010). *Calculus – Concepts and Contexts* (4th ed.). Cengage.

1
Functions and Models

1.1 Week 1: January 16 – January 20

1.1.1 Four Ways to Represent a Function

Consider the following definition.

Definition 1.1.1: Functions

A function f is a rule that assigns to each element in a set D exactly one element, called $f(x)$ in a set E.

In the context of Definition 1.1.1, the set D is the domain of f. The range of f is the set of all possible values of $f(x)$ for all x in the domain.

Functions can, for the purposes of this course, be represented in the following ways:

1. Verbally, with a description in words.
2. Numerically, with a table of values.
3. Visually, with a graph.
4. Algebraically, with an explicit formula.

It is often useful to use graphs or arrow diagrams to visualize functions. Consider the following definitions.

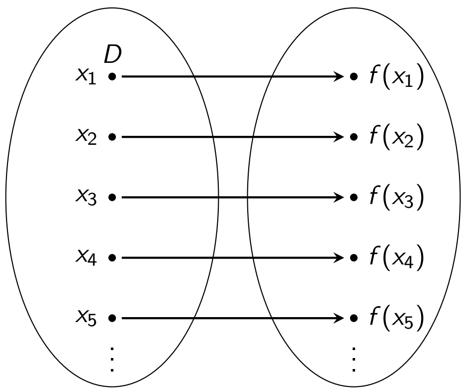
Definition 1.1.2: Functions' Graphs

If f is a function with domain D, the graph of f consists of all points $(x, f(x))$ in the xy plane. Equivalently, the graph of f is the set of ordered pairs

$$\{(x, f(x)) : x \in D\}.$$

Definition 1.1.3: Functions' Arrow Diagrams

If f is a function with domain D, the arrow diagram of f consists of a selection of points $(x, f(x))$ organized in the following manner.



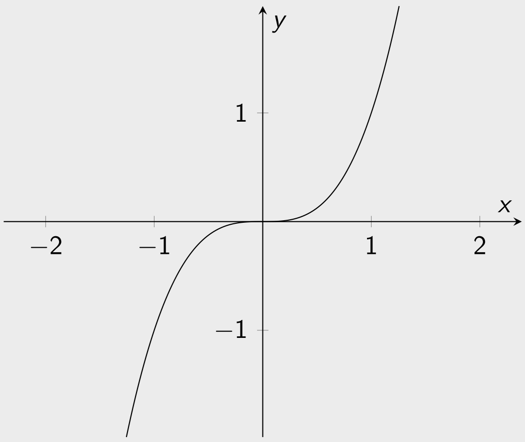
Often, it is not practical, or impossible, to add all elements of D to the left ellipse, so a useful selection of elements of D is used instead.

Consider the following examples.

Example 1.1.1: Creating a Graph 1

Graph the function $f(x) = x^3$ on the interval $[-2, 2]$.

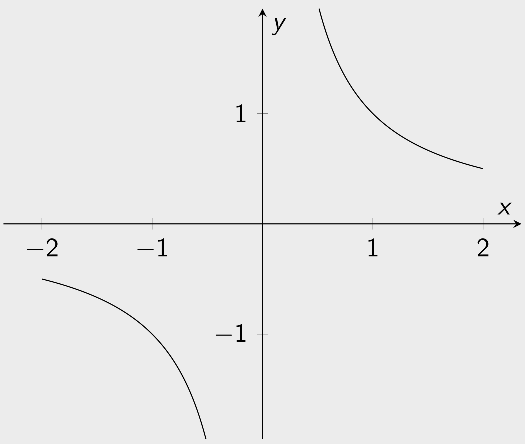
Consider the following graph.



Example 1.1.2: Creating a Graph 2

Graph the function $f(x) = \frac{1}{x}$ on the interval $[-2, 2]$.

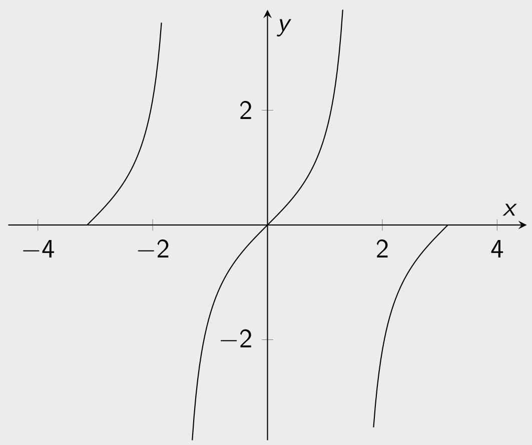
Consider the following graph.



Example 1.1.3: ☢☢ Creating a Graph 3

Graph the function $f(x) = \tan x$ on the interval $[-\pi, \pi]$.

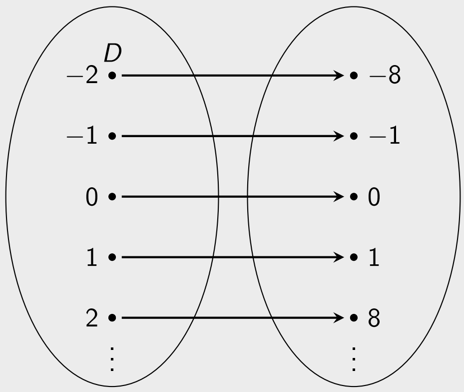
Consider the following graph.



Example 1.1.4: ☢☢ Creating an Arrow Diagram 1

Create an arrow diagram for the function $f(x) = x^3$ with the integers $\{-2, -1, 0, 1, 2\}$.

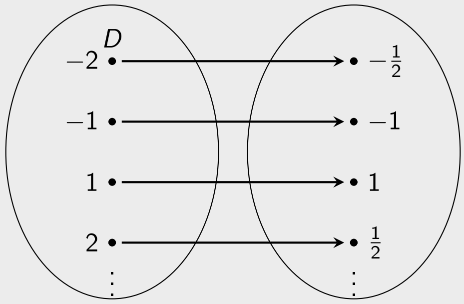
Consider the following diagram.



Example 1.1.5: Creating an Arrow Diagram 2

Create an arrow diagram for the function $f(x) = \frac{1}{x}$ with the integers $\{-2, -1, 1, 2\}$.

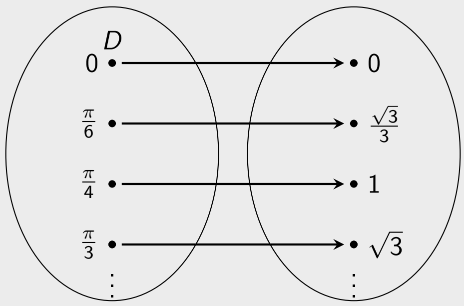
Consider the following diagram.



Example 1.1.6: Creating an Arrow Diagram 3

Create an arrow diagram for the function $f(x) = \frac{1}{x}$ with the values $\left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$.

Consider the following arrow diagram.



Often, given a graph, we must be able to tell whether the curve is a function. Consider the following theorem.

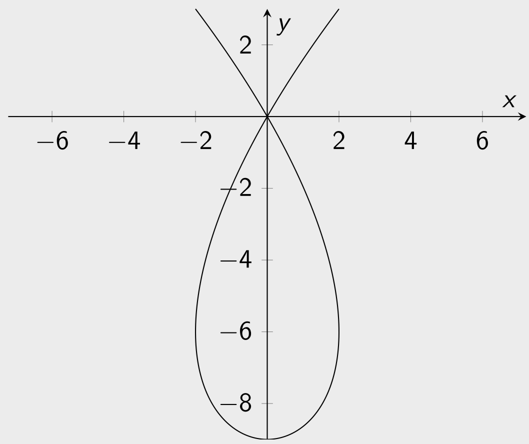
Theorem 1.1.1: The Vertical Line Test

A curve in the xy plane is the graph of a function if and only if no verticl line intersects the curve more than once.

Consider the following examples.

Example 1.1.7: ☢☢ Is the Curve the Graph of a Function 1

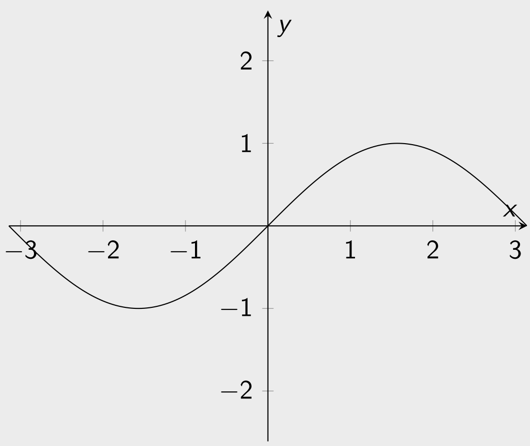
Is the graph below a function or not?



No. The graph does not correspond to a function.

Example 1.1.8: ☢☢ Is the Curve the Graph of a Function 2

Is the graph below a function or not?



Yes. The graph corresponds to a function.

Consider the following definition.

Definition 1.1.4: Piecewise Functions

Piecewise functions are those that have multiple assignment rules for $f(x)$ depending on the interval x is in.

It is often useful to graph piecewise functions. Consider the following examples.

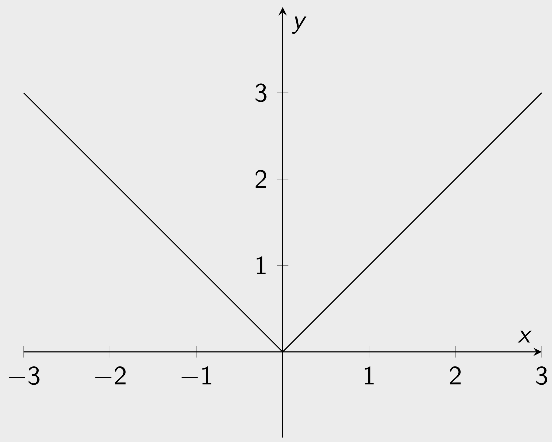
Example 1.1.9: Graphing a Piecewise Function 1

Graph the function

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

on the interval $[-3, 3]$.

We see that $f(x)$ is really the familiar absolute value function, $|x|$. We graph the result of applying the rule for $f(x)$ on its corresponding inequality, which denotes the interval for which the rule is valid. Consider the following graph.



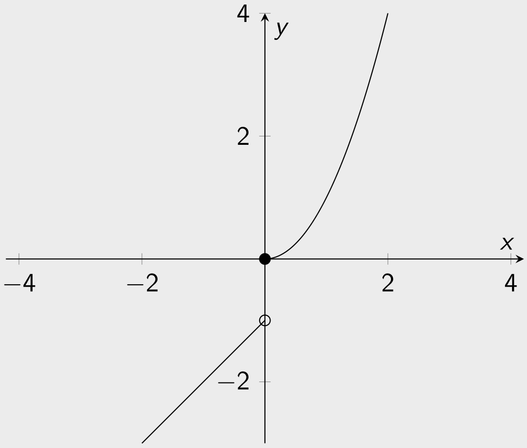
Example 1.1.10: ☢☢ Graphing a Piecewise Function 2

Graph the function

$$f(x) = \begin{cases} x-1 & x<0 \\ x^2 & x \geq 0 \end{cases}$$

on the interval $[-2, 2]$.

We graph the result of applying the rule for $f(x)$ on its corresponding inequality, which denotes the interval for which the rule is valid. Consider the following graph.



We now turn to even and odd functions. Consider the following definition.

Definition 1.1.5: Even and Odd Functions

A function $f(x)$ is even if and only if $f(-x) = f(x)$ and odd if and only if $f(-x) = -f(x)$. Functions that don't have either property are called neither.

Consider the following examples.

Example 1.1.11: Is it Even, Odd, or Neither 1

Is $f(x) = \sin x$ even, odd, or neither?

Consider, for some real numbers a and b such that $a - b = -1$,

$$\begin{aligned} f(-x) = \sin(-x) &= \sin(ax - bx) \\ &= \sin(ax)\cos(bx) - \sin(bx)\cos(ax) \\ &= -(-\sin(ax)\cos(bx) + \sin(bx)\cos(ax)) \\ &= -(\sin(bx)\cos(ax) - \sin(ax)\cos(bx)) \\ &= -\sin(bx - ax) \\ &= -\sin((b-a)x) \\ &= -\sin x = -f(x). \end{aligned}$$

Therefore, $f(x)$ is odd.

Example 1.1.12: Is it Even, Odd, or Neither 2

Is $f(x) = 1 - x^2$ even, odd, or neither?

Consider

$$\begin{aligned} f(-x) &= 1 - (-x)^2 \\ &= 1 - x^2 = f(x). \end{aligned}$$

Therefore, $f(x)$ is even.

Example 1.1.13: Is it Even, Odd, or Neither 3

Is $f(x) = e^{2x}$ even, odd, or neither?

Consider

$$\begin{aligned} f(-x) &= e^{-2x} \\ &= \frac{1}{e^{2x}}. \end{aligned}$$

Therefore, $f(x)$ is neither even nor odd.

Graphically, even functions are symmetric about the y axis, and odd functions are symmetric about the origin.

Consider the following definition.

Definition 1.1.6: Increasing and Decreasing Functions

A function f is increasing on an interval I if and only if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for $x_1, x_2 \in I$. Similarly, f is decreasing on an interval I if and only if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ for $x_1, x_2 \in I$.

1.1.2 Mathematical Models: A Catalog of Essential Functions

Consider the following definitions.

Definition 1.1.7: Mathematical Models

A mathematical model is a mathematical description of a real-world phenomenon that is used for analysis of the phenomenon. Mathematical models are never fully accurate and seek to balance simplification to permit calculation with accuracy to provide valuable information.

We will now define various functions that will allow us to utilize mathematical modelling.

Definition 1.1.8: Stop Linear Functions

A function $f(x)$ is linear if and only if

$$f(x) = mx + b$$

for real numbers m and b. Graphically, m is the slope of $f(x)$ and b is the y intercept.

Definition 1.1.9: Polynomials

A function $f(x)$ is a polynomial if and only if

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

for real numbers $a_1, \ldots, a_n$ and a nonnegative integer n. If the leading coefficient, a_n is nonzero, the degree of the polynomial is n.

If the degree of a polynomial is 2, it is called quadratic and has the form

$$Q(x) = ax^2 + bx + c.$$

If the degree of a polynomial is 3, it is called cubic and has the form

$$C(x) = ax^3 + bx^2 + cx + d.$$

Definition 1.1.10: Power Functions

A function $f(x)$ is a power function if and only if

$$f(x) = x^a$$

for some real number a. Let n be a positive integer. If $a = n$, $f(x)$ is a polynomial. If $a = \frac{1}{n}$, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is a root function. If $a = -1$, $f(x)$ is the reciprocal function $\frac{1}{x}$.

Definition 1.1.11: Rational Functions

A function $f(x)$ is a rational function if and only if

$$f(x) = \frac{P(x)}{Q(x)}$$

for polynomials P and Q. The function $f(x)$ is defined for all x where $Q(x) \neq 0$.

Definition 1.1.12: Algebraic Functions

A function $f(x)$ is a rational function if and only if $f(x)$ can be constructed using only addition, subtraction, multiplication, division, and taking roots to manipulate polynomials.

Definition 1.1.13: Trigonometric Functions

A function $f(x)$ is a trigonometric function if and only if $f(x)$ involves the sine, cosine, tangent, cosecant, secant, or cotangent functions. For this text, radians will always be used in lieu of degrees, unless explicitly stated.

Definition 1.1.14: Exponential Functions

A function $f(x)$ is a rational function if and only if

$$f(x) = a^x$$

for some real number a.

Definition 1.1.15: Logarithmic Functions

A function $f(x)$ is a rational function if and only if

$$f(x) = \log_a x$$

for some real number a.

A familiarity with the above definitions is crucial to understanding this text.

1.1.3 New Functions from Old Functions

Consider the following function transformations.

Theorem 1.1.2: Function Transformations

Let f be a function and $c > 0$. Then,

1. To find the graph of $f(x) + c$, shift the graph of $f(x)$ c units upward.
2. To find the graph of $f(x) - c$, shift the graph of $f(x)$ c units downward.
3. To find the graph of $f(x - c)$, shift the graph of $f(x)$ c units rightward.
4. To find the graph of $f(x + c)$, shift the graph of $f(x)$ c units leftward.

Now, let $c > 1$. Then,

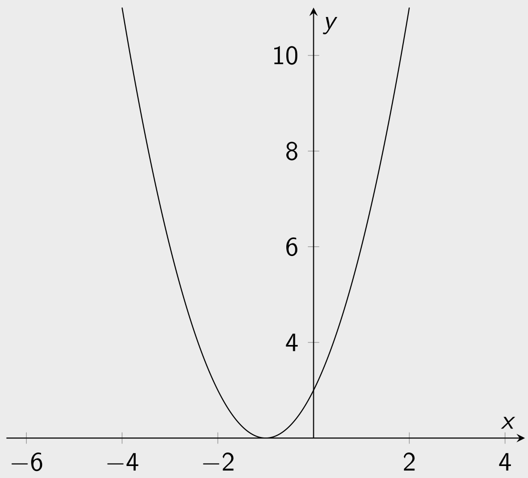
1. To find the graph of $cf(x)$, stretch the graph of $f(x)$ vertically by a factor of c.
2. To find the graph of $\frac{1}{c}f(x)$, compress the graph of $f(x)$ vertically by a factor of c.
3. To find the graph of $f\left(\frac{1}{c}x\right)$, stretch the graph of $f(x)$ horizontally by a factor of c.
4. To find the graph of $f(cx)$, compress the graph of $f(x)$ horizontally by a factor of c.
5. To find the graph of $-f(x)$, reflect the graph of $f(x)$ about the x axis.
6. To find the graph of $f(-x)$, reflect the graph of $f(x)$ about the y axis.

Theorem 1.1.2 is extremely useful for graphing functions when it is easy to rewrite a function in terms of a transformation of a simpler function. Consider the following example.

Example 1.1.14: Using Function Transformations to Graph a Function

Graph the function
$$f(x) = x^2 + 2x + 3$$
on the interval $[-4, 2]$.

Notice that $f(x) = x^2 + 2x + 3 = (x^2 + 2x + 1) + 2 = (x + 1)^2 + 2$. Therefore, the graph of $f(x)$ is the graph of $g(x) = x^2$ shifted one unit leftward and two units upward. Consider the following graph.



We can also define certain combinations of functions.

Definition 1.1.16: Algebraic Combinations of Functions

Let f and g be functions. Then,
$$(f + g)(x) = f(x) + g(x), \quad (f - g)(x) = f(x) - g(x)$$
and
$$(fg)(x) = f(x)g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$
In all cases, the domain of the combination is the set of values that are in both the domains of f and g.

Definition 1.1.17: Compositions of Functions

Let f and g be functions. Then,

$$(f \circ g)(x) = f(g(x)).$$

The domain of the composition is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Consider the following examples.

Example 1.1.15: A Sum of Functions

Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Find $(f + g)(x)$.

We see that

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= \sin^2 x + \cos^2 x\\ &= 1.\end{aligned}$$

Example 1.1.16: A Product of Functions

Let $f(x) = 2\sin x$ and $g(x) = \cos x$. Find $(fg)(x)$.

We see that

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= 2\sin x\cos x\\ &= \sin(2x).\end{aligned}$$

Example 1.1.17: A Composition of Functions 1

Let $f(x) = 2\sin x$ and $g(x) = e^{2x+3}$. Find $(f \circ g)(x)$.

We see that

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= 2\sin(e^{2x+3}).\end{aligned}$$

Example 1.1.18: A Composition of Functions 2

If $h(x) = (x + 3\cos x)^2$. Find functions f and g such that $(f \circ g)(x) = h(x)$.

We see that $f(x) = x^2$ and $g(x) = x + 3\cos x$ satisfy the desired property.

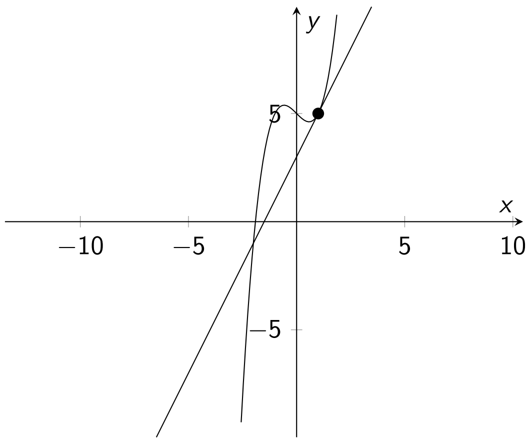
2

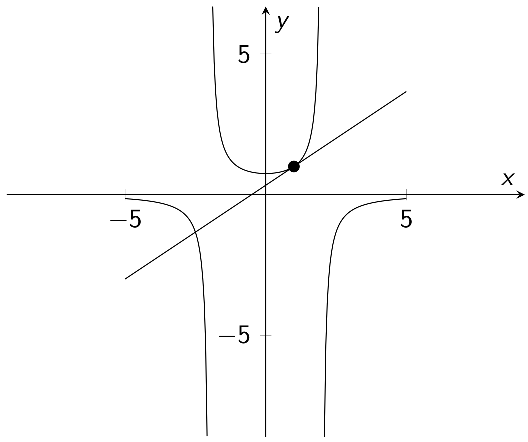
Limits and Derivatives

2.1 Week 1: January 16 – January 20

2.1.1 The Tangent and Velocity Problems

We seek to find the equation of a line $L(x)$ that touches the curve created by $f(x)$ at only one point $(x_0, f(x_0))$ *near* $x = x_0$. To better visualize this, imagine $f(x)$ to be the path a car drives. We seek to find the line that the car would follow if the driver immediately jerked the wheel to the straight position at some point x_0. Consider the following graphs.





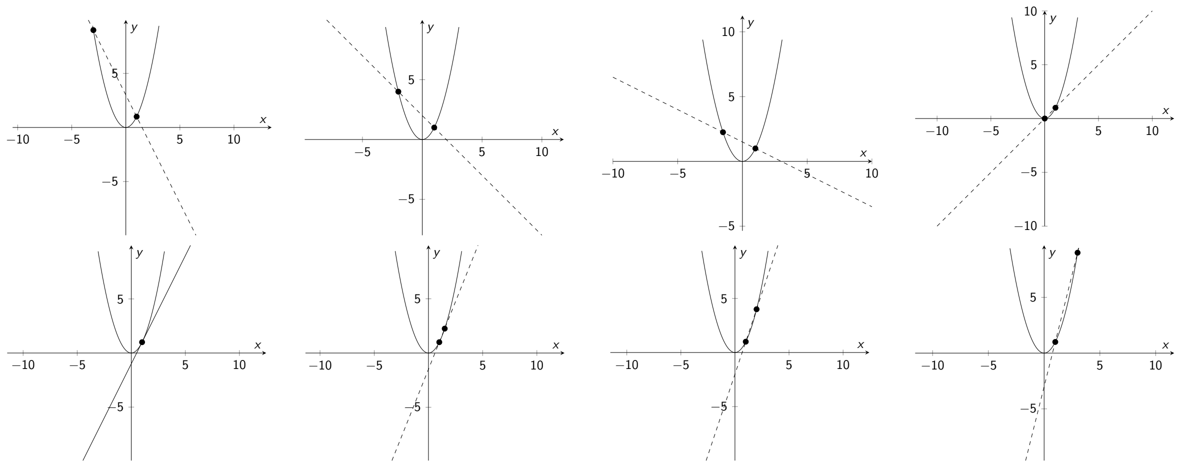
While it can be easy to sketch this line, finding the equation is certainly not trivial. The difficulty lies in finding the slope m of the line. Then, the equation is given by the point-slope formula. That is,

$$L(x) - f(x_0) = m(x - x_0) \implies L(x) = f(x_0) + m(x - x_0).$$

We only have one point, x_0 at our disposal, yet we need two points to calculate the slope. We will choose our second point, x_1, to be somewhere near x_0 to approximate the slope. Recall the slope of the line formed by x_0 and x_1, or equivalently, the average slope of $f(x)$ on the interval $[x_0, x_1]$ is

$$\overline{m} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Our method will be to gradually move the point x_1 closer to x_0 until the two points are indistinguishable from each other. The two points will be *arbitrarily close* to each other. Then, the slope between the points will be the slope of the tangent line. The following graphs will visualize this method and find a tangent line for $f(x) = x^2$ at $(1, 1)$.



We can say that the slope of the tangent line is the limit of the slopes of the secant lines $\overline{m}$ as x_1 approaches x_0. That is,

$$m = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Many texts will instead use

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

to mean the same thing–the slope of the tangent line of $f(x)$ at $(x_0, f(x_0))$, or equivalently, $(a, f(a))$. That means $L(x)$ is given by

$$L(x) = f(x_0) + \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

The specific answer to our question about $f(x) = x^2$ is not important for now, but the process described certainly is. We will defer the computation to a later section.

We will now turn to the velocity problem. Speedometers in cars are useful, as they help the driver keep track of the velocity they are going at very moment t_0 in their journey. How is this instantaneous velocity defined, given that the average velocity of an object is given by its change in position divided by the elapsed time? The notion of "elapsed time" seems to require two distinct timestamps to measure. Again, the difficulty lies in that we are only given one value, t_0, to determine the instantaneous velocity at. Suppose we are given a position function $x(t)$. Then, the time elapsed between t_1 and t_0 is $t_1 - t_0$. The change in position is $x(t_1) - x(t_0)$, meaning that the average velocity on the interval $[t_0, t_1]$ is

$$\overline{v} = \frac{x(t_1) - x(t_0)}{t_1 - t_0}.$$

To get a better approximation of the instantaneous velocity at t_0, we can decrease the elapsed time by choosing t_1 near t_0 and take the limit as t_1 approaches t_0. Then, we have

$$v = \lim_{t_1 \to t_0} \frac{x(t_1) - x(t_0)}{t_1 - t_0}.$$

Similarly, some texts may instead use

$$v = \lim_{t \to a} \frac{x(t) - x(a)}{t - a}.$$

We will end with the notion that your car's speedometer uses the above technique with extremely small time intervals.

2.1.2 The Limit of a Function

To solve the problems described in the previous section, we must learn to evaluate limits. With limits, we care not about the value of a function at some value $x = a$. We instead care about how the function *behaves* close to that value. Consider the following definition.

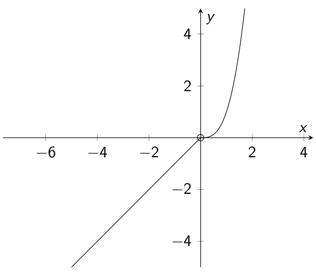
Definition 2.1.1: Limits

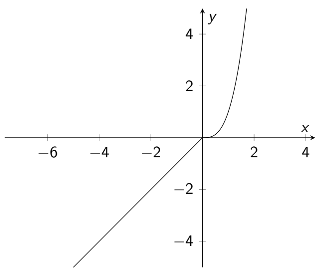
We write

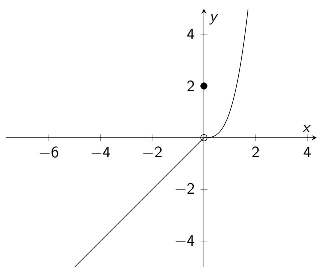
$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a" is L if and only if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a, on either side, while $x \neq a$.

The above definition means that the values of $f(x)$ get closer and closer to L as x gets closer and closer to a, on either side. If this is not true, the limit does not exist. We will often abbreviate "does not exist" to "DNE." The definition makes clear that the value of $f(a)$ is irrelevant. We care only about the value of $f(x)$ for x near a. Consider the following graphs.







At $x = 0$, all of the above graphs have different behavior; however, the limit, as x approaches 0 is the same regardless: 0. To estimate a limit as x approaches a with a graph of $f(x)$, simply use a pen or other writing utensil to trace the graph toward $x = a$, from both sides, and record the value that the function approaches, if there is one.

We can often estimate the limit $\lim_{x\to a} f(x)$ by creating a table of values for x and $f(x)$ for x near a. Consider the following examples.

Example 2.1.1: Estimating a Limit 1

Estimate the limit

$$\lim_{x\to 0} \frac{\sin x}{x}.$$

Consider the following table.

x	$f(x)$
-1	0.84147098
-0.1	0.99833417
-0.01	0.99998333
0.01	0.99998333
0.1	0.99833417
1	0.84147098

Therefore, we estimate that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Example 2.1.2: Estimating a Limit 2

Estimate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{x}.$$

Consider the following table.

x	$f(x)$
-1	-0.45969769
-0.1	-0.049958347
-0.01	-0.0049999583
0.01	0.0049999583
0.1	0.049958347
1	0.45969769

Therefore, we estimate that

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Example 2.1.3: ☢☢ Estimating a Limit 3

Estimate the limit

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

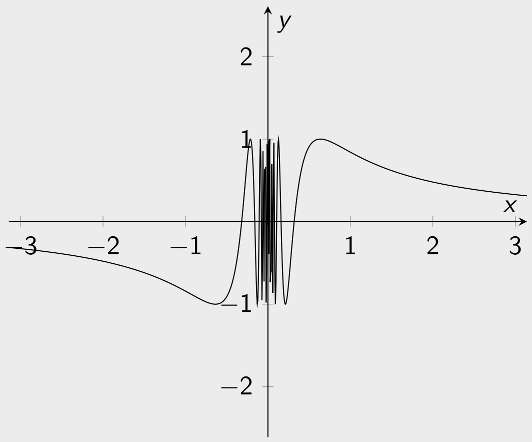
Consider the following table.

x	$f(x)$
-1	0
-0.1	0
-0.01	0
0.01	0
0.1	0
1	0

Therefore, we estimate that

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = 0.$$

However, this is wrong, and a good reminder to be cautious while estimating. Consider the following graph of $\sin\left(\frac{1}{x}\right)$.



We see that as x approaches zero, it is not true that $\sin\left(\frac{1}{x}\right)$ approaches zero. Instead, the function oscillates between -1 and 1 infinitely many times. Therefore

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ DNE.}$$

Often, it is useful to only scrutinize the limiting behavior of one side of some a instead of looking at both sides. Consider the following definition and related theorem.

Definition 2.1.2: One-Sided Limits

We write

$$\lim_{x \to a^-} f(x) = L$$

and say "the left-hand limit of $f(x)$ as x approaches a" is L if and only if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x < a$.

We write

$$\lim_{x \to a^+} f(x) = L$$

and say "the right-hand limit of $f(x)$ as x approaches a" is L if and only if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and $x > a$.

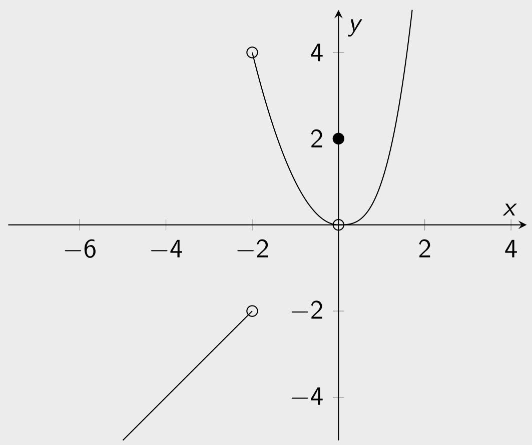
Theorem 2.1.1: The Relationship Between Limits and One-Sided Limits

For some function $f(x)$, $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$.

Consider the following example.

Example 2.1.4: ☢☢ Determining Limits Graphically

Let $f(x)$ be defined by the following graph.



Find the quantities

$$\lim_{x\to -2^-} f(x), \quad \lim_{x\to -2^+} f(x), \quad \lim_{x\to 0^-} f(x), \quad \lim_{x\to 0^+} f(x),$$

and

$$\lim_{x\to -2} f(x), \quad \lim_{x\to 0} f(x).$$

We see that

$$\lim_{x\to -2^-} f(x) = -2, \quad \lim_{x\to -2^+} f(x) = 4, \quad \lim_{x\to 0^-} f(x) = 0, \quad \lim_{x\to 0^+} f(x) = 0,$$

so

$$\lim_{x\to -2} f(x) \text{ DNE}, \quad \lim_{x\to 0} f(x) = 0.$$

2.2 Week 2: January 23 – January 27

2.2.1 Calculating Limits Using the Limit Laws

Consider the following properties of limits.

Theorem 2.2.1: Limit Laws

Let a and c be constants and let the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then,

1. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$.
2. $\lim\limits_{x\to a}[cf(x)] = c \lim\limits_{x\to a} f(x)$.
3. $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$.
4. $\lim\limits_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x) \neq 0$.
5. $\lim\limits_{x\to a} c = c$.
6. $\lim\limits_{x\to a} x = a$.

If we take the laws in Theorem 2.2.1 for granted, we have the following consequences.

Theorem 2.2.2: Derived Limit Laws

Let a and c be constants and let the limit $\lim_{x\to a} f(x)$ exist. Let n be a positive integer. Then,

1. $\lim\limits_{x\to a}[f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$.
2. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$.
3. $\lim\limits_{x\to a} x^n = a^n$.
4. $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$.

Theorems 2.2.1 and 2.2.2 allow us to claim the following.

Theorem 2.2.3: Direct Substitution

If f is a polynomial or rational function and a is in the domain of f,

$$\lim_{x\to a} f(x) = f(a).$$

There exist other functions with the property described in Theorem 2.2.3, but we will postpone their discussion to a later section.

Consider the following examples.

Example 2.2.1: ☢☢ Direct Substitution 1

Evaluate $\lim_{x\to 2}(x^3 + 2x^2 - 11x - 7)$.

We see that

$$\begin{aligned}\lim_{x\to 2}(x^3 + 2x^2 - 11x - 7) &= (2)^3 + 2(2)^2 - 11(2) - 7\\ &= 8 + 8 - 22 - 7\\ &= -13.\end{aligned}$$

Example 2.2.2: ☢☢ Direct Substitution 2

Evaluate $\lim_{x\to 3}\frac{2x}{x-4}$.

We see that

$$\begin{aligned}\lim_{x\to 3}\frac{2x}{x-4} &= \frac{2(3)}{(3)-4}\\ &= -6.\end{aligned}$$

Most of the time, direct substitution will not work, but it is often beneficial to try it before trying anything else. It will certainly not work when the function is not defined at the value the limit is being evaluated at, and it is beneficial to use caution when considering piecewise functions. Consider the following theorem.

Theorem 2.2.4: Limit Equivalence of Two Functions

If $f(x) = r(x)$ whenever $x \neq a$,

$$\lim_{x\to a} f(x) = \lim_{x\to a} r(x).$$

When given a function $f(x)$ where direct substitution fails, we will find a function $r(x)$ such that $f(x) = r(x)$ whenever $x \neq a$ and compute $\lim_{x\to a} r(x)$. Then, we will use Theorem 2.2.4 to conclude that $\lim_{x\to a} r(x) = \lim_{x\to a} f(x)$. This process is called removing a discontinuity.

Consider the following examples.

Example 2.2.3: ☢☢ Removing a Discontinuity 1

Evaluate $\lim_{x\to-3} \frac{x^2+6x+6}{x+3}$.

Let $f(x) = \frac{x^2+6x+6}{x+3}$. We note that direct substitution fails on $f(x)$ since $f(x)$ is not defined at $x = -3$. We cannot use our limit laws since $\lim_{x\to-3}(x+3) = 0$. We must find $r(x)$ such that $r(x) = f(x)$ whenever $x \neq -3$. Consider

$$\begin{aligned} f(x) &= \frac{x^2+5x+6}{x+3} \\ &= \frac{(x+3)(x+2)}{x+3} \\ &= x+2, \quad x \neq -3. \end{aligned}$$

Then $r(x) = x+2 = f(x)$, except at $x = -3$; $r(-3) = -1$ while $f(-3)$ is not defined. Then we can apply direct substitution on $r(x)$ to obtain

$$\begin{aligned} \lim_{x\to-3} \frac{x^2+6x+6}{x+3} &= \lim_{x\to-3}(x+2) \\ &= -3+2 \\ &= 1. \end{aligned}$$

Example 2.2.4: ☢☢ Removing a Discontinuity 2

Evaluate $\lim_{x\to-5} \frac{x^3+5x^2}{x+5}$.

Let $f(x) = \frac{x^3+5x^2}{x+5}$. We note that direct substitution fails on $f(x)$ since $f(x)$ is not defined at $x = -5$. We cannot use our limit laws since $\lim_{x\to-5}(x+5) = 0$. We must find $r(x)$ such that $r(x) = f(x)$ whenever $x \neq -5$. Consider

$$\begin{aligned} f(x) &= \frac{x^3+5x^2}{x+5} \\ &= \frac{x^2(x+5)}{x+5} \\ &= x^2, \quad x \neq 5. \end{aligned}$$

Then $r(x) = x^2 = f(x)$, except at $x = -5$; $r(-5) = 25$ while $f(-5)$ is not defined. Then we can apply direct substitution on $r(x)$ to obtain

$$\begin{aligned} \lim_{x\to-5} \frac{x^3+5x^2}{x+5} &= \lim_{x\to-5} x^2 \\ &= 25. \end{aligned}$$

Example 2.2.5: ☢☢ Removing a Discontinuity 3

Evaluate $\lim_{x\to 7}\frac{\sqrt{x+9}-4}{x-7}$.

Let $f(x)=\frac{\sqrt{x+9}-4}{x-7}$. We note that direct substitution fails on $f(x)$ since $f(x)$ is not defined at $x=7$. We cannot use our limit laws since $\lim_{x\to 7}(x-7)=0$. We must find $r(x)$ such that $r(x)=f(x)$ whenever $x\neq 7$. Consider

$$\begin{aligned}
f(x) &= \frac{\sqrt{x+9}-4}{x-7} \\
&= \frac{\sqrt{x+9}-4}{x-7}\,\frac{\sqrt{x+9}+4}{\sqrt{x+9}+4} \\
&= \frac{x+9-16}{(x-7)(\sqrt{x+9}+4)} \\
&= \frac{1}{\sqrt{x+9}+4}, \quad x\neq 7.
\end{aligned}$$

Then $r(x)=\frac{1}{\sqrt{x+9}+4}=f(x)$, except at $x=7$; $r(7)=\frac{1}{8}$ while $f(7)$ is not defined. Then we can apply direct substitution on $r(x)$ to obtain

$$\begin{aligned}
\lim_{x\to 7}\frac{\sqrt{x+9}-4}{x-7} &= \lim_{x\to 7}\frac{1}{\sqrt{x+9}+4} \\
&= \frac{1}{8}.
\end{aligned}$$

Example 2.2.6: ☢☢ Piecewise Functions and Direct Substitution

Let $f(x)=\begin{cases} x+5 & x\neq 2 \\ e & x=2 \end{cases}$. Evaluate $\lim_{x\to 2}f(x)$.

If we try direct substitution, we may come to the conclusion that the desired limit is simply e; however, this is not the case, as we can construct $r(x)=x+5$ and note that $r(x)=f(x)$ whenever $x\neq 2$. Therefore,

$$\begin{aligned}
\lim_{x\to 2}f(x) &= \lim_{x\to 2}r(x) \\
&= 2+5 \\
&= 7.
\end{aligned}$$

For piecewise functions in particular, it is often beneficial to compute the one-sided limits when trying to find a limit. Consider the following examples.

Example 2.2.7: Piecewise Functions and One-Sided Limits 1

Let $f(x) = \begin{cases} x+5 & x \neq 2 \\ e & x = 2 \end{cases}$. Evaluate $\lim_{x\to 2} f(x)$.

For $x > 2$, $f(x) = x + 5$. Therefore, $\lim_{x\to 2^+} f(x) = 7$. Similarly, for $x < 2$, $f(x) = x + 5$, so $\lim_{x\to 2^-} f(x) = 7$. Therefore, $\lim_{x\to 2} f(x) = 7$.

Example 2.2.8: Piecewise Functions and One-Sided Limits 2

Let $f(x) = \frac{|x|}{x}$. Evaluate $\lim_{x\to 0} f(x)$.

We will first rewrite $f(x)$ as a piecewise function to obtain

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}.$$

Then, for $x > 0$, $f(x) = 1$, so $\lim_{x\to 0^+} f(x) = 1$. For $x < 0$, $f(x) = -1$, so $\lim_{x\to 0^-} f(x) = -1$. Therefore, $\lim_{x\to 0} f(x)$ DNE.

The next two theorems will be very useful in evaluating limits.

Theorem 2.2.5: Limits Preserve Inequalities

If $f(x) \leq g(x)$ when x is near a, except possibly at $x = a$, and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist,

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Theorem 2.2.6: The Squeeze Theorem

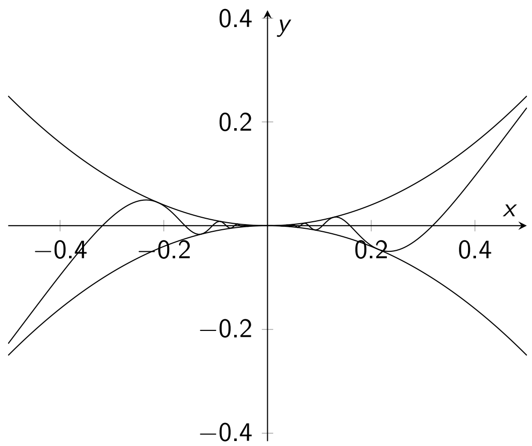
If $f(x) \leq g(x) \leq h(x)$ when x is near a, except possibly at $x = a$ and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then,

$$\lim_{x\to a} g(x) = L.$$

Consider the following diagram to illustrate. Here, $f(x) = -x^2$, $g(x) = x^2 \sin\left(\frac{1}{x}\right)$, and $h(x) = x^2$.



2.3 Week 3: January 30 – February 3

2.4 Week 4: February 6 – February 10

3
Differentiation Rules

3.1 Week 5: February 13 – February 17

3.2 Week 6: February 20 – February 24

4
Applications of Differentiation

4.1 Week 7: February 27 – March 3

4.2 Week 8: March 6 – March 10

4.3 Week 9: March 13 – March 17

4.4 Week 10: March 20 – March 24

4.5 Week 11: March 27 – March 31

5
Integrals

5.1 Week 12: April 3 – April 7

5.2 Week 13: April 10 – April 14

5.3 Week 14: April 17 – April 21

5.4 Week 15: April 24 – April 28

5.5 Week 16: May 1 – May 5