MATH1300: CALCULUS I

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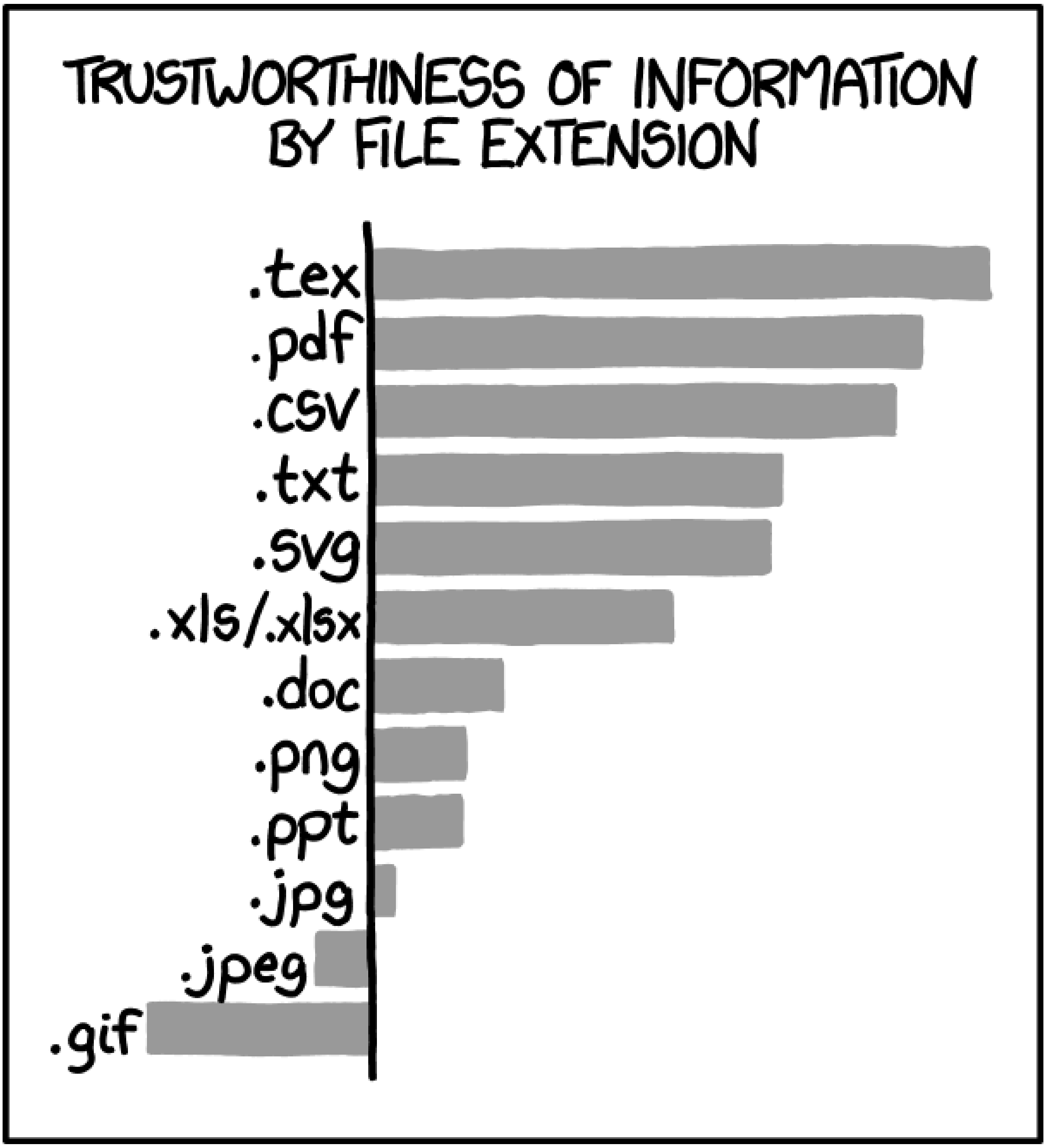
TEXTBOOK: JAMES STEWART

UNIVERSITY OF COLORADO BOULDER

EDITION 1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Preface

To the interested reader,

This document is a compilation of notes taken during the Spring 2023 semester for MATH1300: Calculus I at the University of Colorado Boulder during the author's tenure as a learning assistant for the course. The course used *Calculus – Concepts and Contexts*[1] by James Stewart as its primary text and was coordinated by Harrison Stalvey and Christopher Eblen. Additionally, in creating these notes, the author used *Calculus* by Ron Larson and Bruce Edwards. As such, many theorems, definitions, and examples may be quoted or derived from the aforementioned books.

The author would like to provide the following resources for students currently taking a Calculus I course:

1. Paul's *Online Math Notes* for Calculus I at Lamar University.
2. Professor Leonard's YouTube Calculus I Lectures.
3. 3Blue1Brown's *Essence of Calculus*.

Theorems, definitions, and examples may be quoted or derived from the aforementioned resources as well.

While much effort has been put in to remove typos and mathematical errors, it is very likely that some errors, both small and large, are present. If an error needs to be resolved, please contact Adithya Bhaskara at adithya.bhaskara@colorado.edu.

Best Regards,
Adithya Bhaskara

REVISED: January 15, 2023

[1] Stewart, J. (2010). *Calculus – Concepts and Contexts* (4th ed.). Cengage.

1

Functions and Models

1.1 Week 1: January 16 – January 20

1.1.1 Four Ways to Represent a Function

Consider the following definition.

Definition 1.1.1: Functions

A function f is a rule that assigns to each element in a set D exactly one element, called $f(x)$ in a set E.

In the context of Definition 1.1.1, the set D is the domain of f. The range of f is the set of all possible values of $f(x)$ for all x in the domain.

Functions can, for the purposes of this course, be represented in the following ways:

1. Verbally, with a description in words.
2. Numerically, with a table of values.
3. Visually, with a graph.
4. Algebraically, with an explicit formula.

It is often useful to use graphs or arrow diagrams to visualize functions. Consider the following definitions.

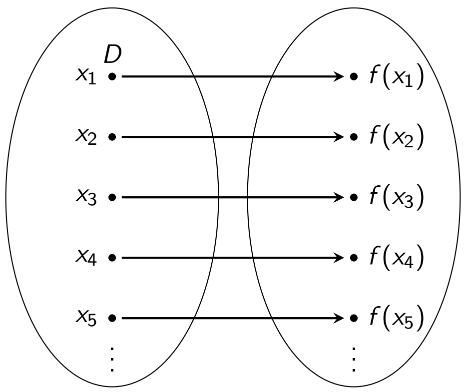
Definition 1.1.2: Functions' Graphs

If f is a function with domain D, the graph of f consists of all points $(x, f(x))$ in the xy plane. Equivalently, the graph of f is the set of ordered pairs

$$\{(x, f(x)) : x \in D\}.$$

Definition 1.1.3: Functions' Arrow Diagrams

If f is a function with domain D, the arrow diagram of f consists of a selection of points $(x, f(x))$ organized in the following manner.



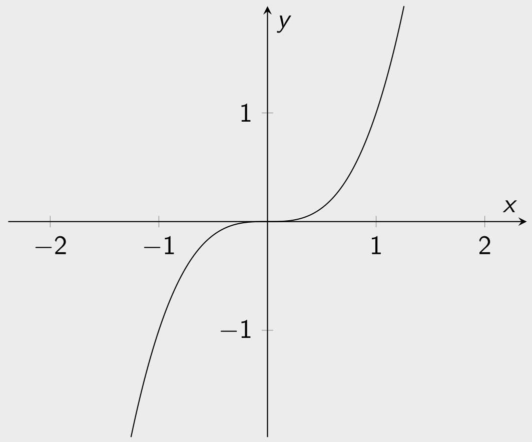
Often, it is not practical, or impossible, to add all elements of D to the left ellipse, so a useful selection of elements of D is used instead.

Consider the following examples.

Example 1.1.1: ☢☢ Creating a Graph 1

Graph the function $f(x) = x^3$ on the interval $[-2, 2]$.

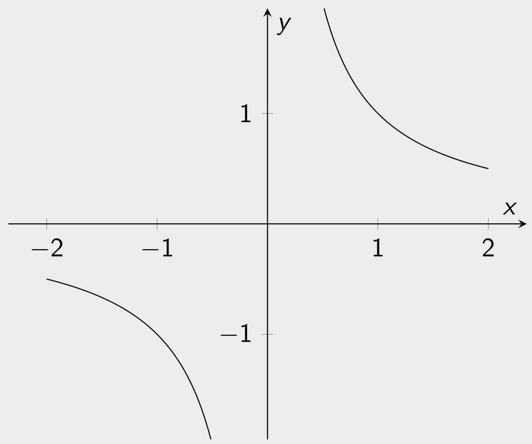
Consider the following graph.



Example 1.1.2: ☢☢ Creating a Graph 2

Graph the function $f(x) = \frac{1}{x}$ on the interval $[-2, 2]$.

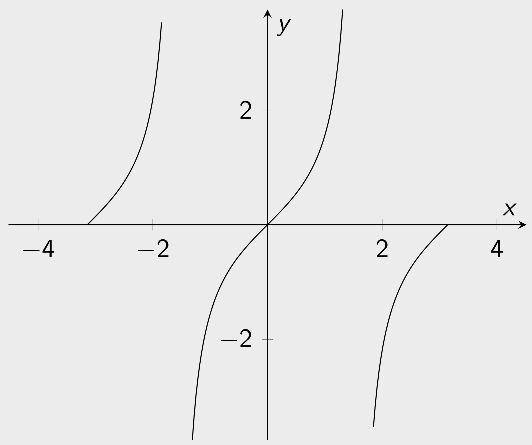
Consider the following graph.



Example 1.1.3: ☢☢ Creating a Graph 3

Graph the function $f(x) = \tan x$ on the interval $[-\pi, \pi]$.

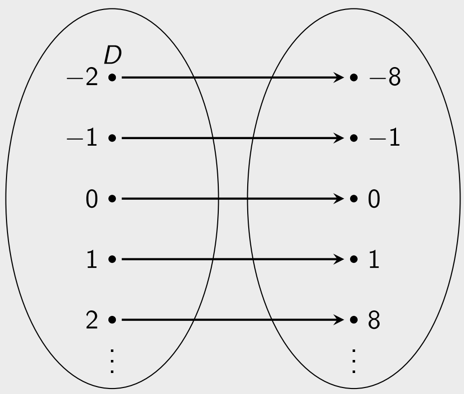
Consider the following graph.



Example 1.1.4: ☢☢ Creating an Arrow Diagram 1

Create an arrow diagram for the function $f(x) = x^3$ with the integers $\{-2, -1, 0, 1, 2\}$.

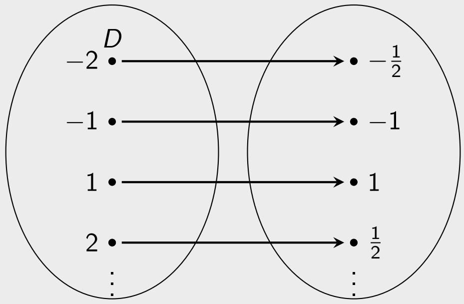
Consider the following diagram.



Example 1.1.5: Creating an Arrow Diagram 2

Create an arrow diagram for the function $f(x) = \frac{1}{x}$ with the integers $\{-2, -1, 1, 2\}$.

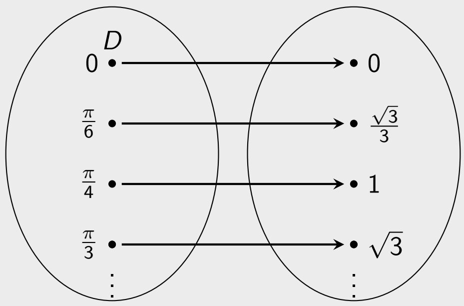
Consider the following diagram.



Example 1.1.6: Creating an Arrow Diagram 3

Create an arrow diagram for the function $f(x) = \frac{1}{x}$ with the values $\left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$.

Consider the following arrow diagram.



Often, given a graph, we must be able to tell whether the curve is a function. Consider the following theorem.

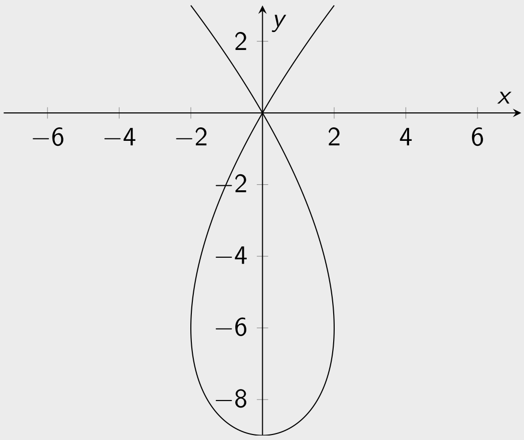
Theorem 1.1.1: The Vertical Line Test

A curve in the xy plane is the graph of a function if and only if no verticl line intersects the curve more than once.

Consider the following examples.

Example 1.1.7: ☢☢ Is the Curve the Graph of a Function 1

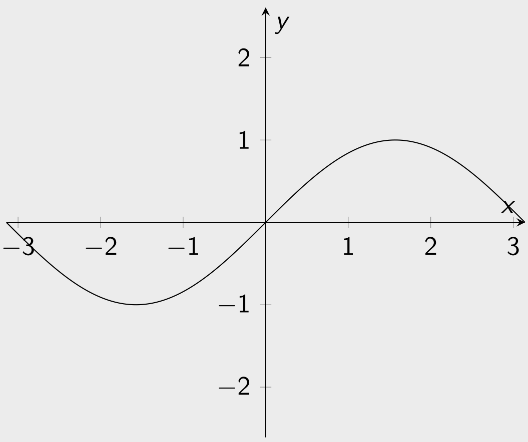
Is the graph below a function or not?



No. The graph does not correspond to a function.

Example 1.1.8: ☢☢ Is the Curve the Graph of a Function 2

Is the graph below a function or not?



Yes. The graph corresponds to a function.

Consider the following definition.

Definition 1.1.4: Piecewise Functions

Piecewise functions are those that have multiple assignment rules for $f(x)$ depending on the interval x is in.

It is often useful to graph piecewise functions. Consider the following examples.

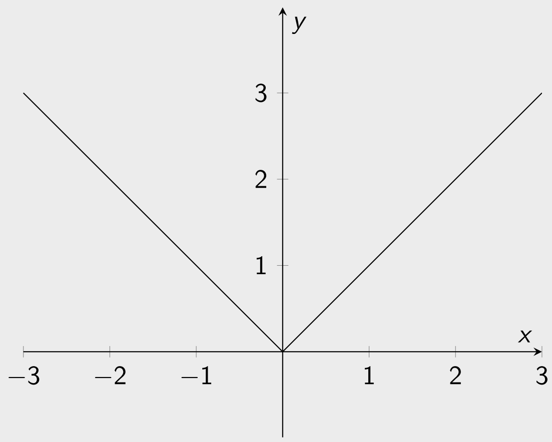
Example 1.1.9: Graphing a Piecewise Function 1

Graph the function

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

on the interval $[-3, 3]$.

We see that $f(x)$ is really the familiar absolute value function, $|x|$. We graph the result of applying the rule for $f(x)$ on its corresponding inequality, which denotes the interval for which the rule is valid. Consider the following graph.



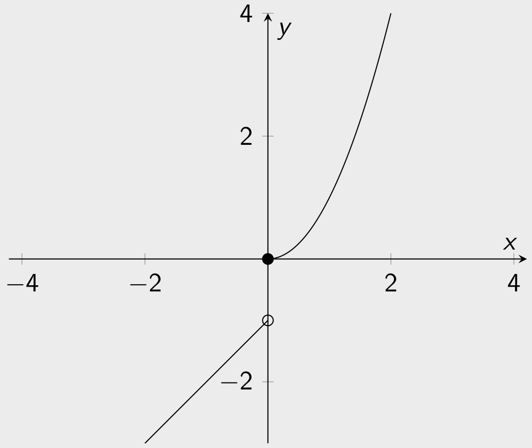
Example 1.1.10: ☢☢ Graphing a Piecewise Function 2

Graph the function

$$f(x) = \begin{cases} x - 1 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

on the interval $[-2, 2]$.

We graph the result of applying the rule for $f(x)$ on its corresponding inequality, which denotes the interval for which the rule is valid. Consider the following graph.



We now turn to even and odd functions. Consider the following definition.

Definition 1.1.5: Even and Odd Functions

A function $f(x)$ is even if and only if $f(-x) = f(x)$ and odd if and only if $f(-x) = -f(x)$. Functions that don't have either property are called neither.

Consider the following examples.

Example 1.1.11: ☢☢ Is it Even, Odd, or Neither 1

Is $f(x) = \sin x$ even, odd, or neither?

Consider, for some real numbers a and b such that $a - b = -1$,

$$\begin{aligned} f(-x) = \sin(-x) &= \sin(ax - bx) \\ &= \sin(ax)\cos(bx) - \sin(bx)\cos(ax) \\ &= -(-\sin(ax)\cos(bx) + \sin(bx)\cos(ax)) \\ &= -(\sin(bx)\cos(ax) - \sin(ax)\cos(bx)) \\ &= -\sin(bx - ax) \\ &= -\sin((b-a)x) \\ &= -\sin x = -f(x). \end{aligned}$$

Therefore, $f(x)$ is odd.

Example 1.1.12: ☢☢ Is it Even, Odd, or Neither 2

Is $f(x) = 1 - x^2$ even, odd, or neither?

Consider

$$\begin{aligned} f(-x) &= 1 - (-x)^2 \\ &= 1 - x^2 = f(x). \end{aligned}$$

Therefore, $f(x)$ is even.

Example 1.1.13: ☢☢ Is it Even, Odd, or Neither 3

Is $f(x) = e^{2x}$ even, odd, or neither?

Consider

$$\begin{aligned} f(-x) &= e^{-2x} \\ &= \frac{1}{e^{2x}}. \end{aligned}$$

Therefore, $f(x)$ is neither even nor odd.

Graphically, even functions are symmetric about the y axis, and odd functions are symmetric about the origin.

Consider the following definition.

Definition 1.1.6: Increasing and Decreasing Functions

A function f is increasing on an interval I if and only if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for $x_1, x_2 \in I$. Similarly, f is decreasing on an interval I if and only if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ for $x_1, x_2 \in I$.

1.1.2 Mathematical Models: A Catalog of Essential Functions

Consider the following definitions.

Definition 1.1.7: Mathematical Models

A mathematical model is a mathematical description of a real-world phenomenon that is used for analysis of the phenomenon. Mathematical models are never fully accurate and seek to balance simplification to permit calculation with accuracy to provide valuable information.

We will now define various functions that will allow us to utilize mathematical modelling.

Definition 1.1.8: Stop Linear Functions

A function $f(x)$ is linear if and only if

$$f(x) = mx + b$$

for real numbers m and b. Graphically, m is the slope of $f(x)$ and b is the y intercept.

Definition 1.1.9: Polynomials

A function $f(x)$ is a polynomial if and only if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

for real numbers $a_1, \ldots, a_n$ and a nonnegative integer n. If the leading coefficient, a_n is nonzero, the degree of the polynomial is n.

If the degree of a polynomial is 2, it is called quadratic and has the form

$$Q(x) = ax^2 + bx + c.$$

If the degree of a polynomial is 3, it is called cubic and has the form

$$C(x) = ax^3 + bx^2 + cx + d.$$

Definition 1.1.10: Power Functions

A function $f(x)$ is a power function if and only if

$$f(x) = x^a$$

for some real number a. Let n be a positive integer. If $a = n$, $f(x)$ is a polynomial. If $a = \frac{1}{n}$, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is a root function. If $a = -1$, $f(x)$ is the reciprocal function $\frac{1}{x}$.

Definition 1.1.11: Rational Functions

A function $f(x)$ is a rational function if and only if

$$f(x) = \frac{P(x)}{Q(x)}$$

for polynomials P and Q. The function $f(x)$ is defined for all x where $Q(x) \neq 0$.

Definition 1.1.12: Algebraic Functions

A function $f(x)$ is a rational function if and only if $f(x)$ can be constructed using only addition, subtraction, multiplication, division, and taking roots to manipulate polynomials.

Definition 1.1.13: Trigonometric Functions

A function $f(x)$ is a trigonometric function if and only if $f(x)$ involves the sine, cosine, tangent, cosecant, secant, or cotangent functions. For this text, radians will always be used in lieu of degrees, unless explicitly stated.

Definition 1.1.14: Exponential Functions

A function $f(x)$ is a rational function if and only if

$$f(x) = a^x$$

for some real number a.

Definition 1.1.15: Logarithmic Functions

A function $f(x)$ is a rational function if and only if

$$f(x) = \log_a x$$

for some real number a.

A familiarity with the above definitions is crucial to understanding this text.

1.1.3 New Functions from Old Functions

Consider the following function transformations.

Theorem 1.1.2: Function Transformations

Let f be a function and $c > 0$. Then,

1. To find the graph of $f(x) + c$, shift the graph of $f(x)$ c units upward.
2. To find the graph of $f(x) - c$, shift the graph of $f(x)$ c units downward.
3. To find the graph of $f(x - c)$, shift the graph of $f(x)$ c units rightward.
4. To find the graph of $f(x + c)$, shift the graph of $f(x)$ c units leftward.

Now, let $c > 1$. Then,

1. To find the graph of $cf(x)$, stretch the graph of $f(x)$ vertically by a factor of c.
2. To find the graph of $\frac{1}{c}f(x)$, compress the graph of $f(x)$ vertically by a factor of c.
3. To find the graph of $f\left(\frac{1}{c}x\right)$, stretch the graph of $f(x)$ horizontally by a factor of c.
4. To find the graph of $f(cx)$, compress the graph of $f(x)$ horizontally by a factor of c.
5. To find the graph of $-f(x)$, reflect the graph of $f(x)$ about the x axis.
6. To find the graph of $f(-x)$, reflect the graph of $f(x)$ about the y axis.

Theorem 1.1.2 is extremely useful for graphing functions when it is easy to rewrite a function in terms of a transformation of a simpler function. Consider the following example.

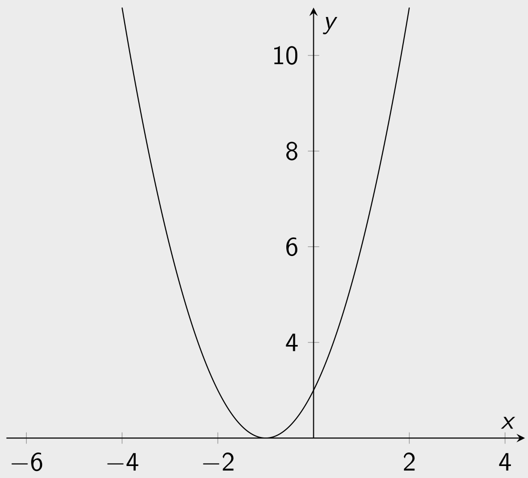
Example 1.1.14: Using Function Transformations to Graph a Function

Graph the function

$$f(x) = x^2 + 2x + 3$$

on the interval $[-4, 2]$.

Notice that $f(x) = x^2 + 2x + 3 = (x^2 + 2x + 1) + 2 = (x + 1)^2 + 2$. Therefore, the graph of $f(x)$ is the graph of $g(x) = x^2$ shifted one unit leftward and two units upward. Consider the following graph.



We can also define certain combinations of functions.

Definition 1.1.16: Algebraic Combinations of Functions

Let f and g be functions. Then,

$$(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x)$$

and

$$(fg)(x) = f(x)g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

In all cases, the domain of the combination is the set of values that are in both the domains of f and g.

Definition 1.1.17: Compositions of Functions

Let f and g be functions. Then,

$$(f \circ g)(x) = f(g(x)).$$

The domain of the composition is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Consider the following examples.

Example 1.1.15: A Product of Functions

Let $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$. Find $(f+g)(x)$.

We see that

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= \sin^2 x + \cos^2 x\\ &= 1.\end{aligned}$$

Example 1.1.16: A Product of Functions

Let $f(x) = 2\sin x$ and $g(x) = \cos x$. Find $(fg)(x)$.

We see that

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= 2\sin x \cos x\\ &= \sin(2x).\end{aligned}$$

Example 1.1.17: A Composition of Functions 1

Let $f(x) = 2\sin x$ and $g(x) = e^{2x+3}$. Find $(f \circ g)(x)$.

We see that

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= 2\sin(e^{2x+3}).\end{aligned}$$

Example 1.1.18: A Composition of Functions 2

If $h(x) = (x + 3\cos x)^2$. Find functions f and g such that $(f \circ g)(x) = h(x)$.

We see that $f(x) = x^2$ and $g(x) = x + 3\cos x$ satisfy the desired property.

2
Limits and Derivatives

3
Differentiation Rules

3.1 Week 5: February 13 – February 17

3.2 Week 6: February 20 – February 24

4

Applications of Differentiation

4.1 Week 7: February 27 – March 3

4.2 Week 8: March 6 – March 10

4.3 Week 9: March 13 – March 17

4.4 Week 10: March 20 – March 24

4.5 Week 11: March 27 – March 31

5
Integrals

5.1 Week 12: April 3 – April 7

5.2 Week 13: April 10 – April 14

5.3 Week 14: April 17 – April 21

5.4 Week 15: April 24 – April 28

5.5 Week 16: May 1 – May 5