MATH2135: LINEAR ALGEBRA

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I go by the name of Vector. It's a mathematical term, represented by an arrow with both direction and magnitude. Vector! That's me, because I commit crimes with both direction and magnitude. Oh yeah!

Victor (Vector) Perkins

1
Vectors and Matrices

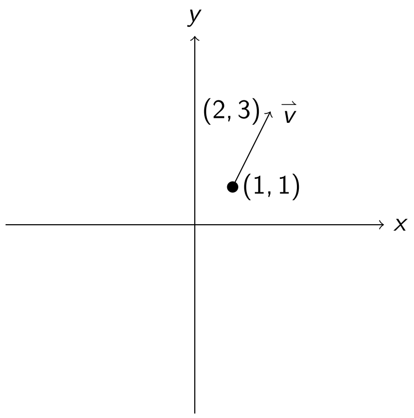
1.1 Lecture 1: August 22, 2022

1.1.1 Notations, Definitions, and Conventions

Consider the following table for a very basic review of fundamental sets.

Symbol	Explanation
$\mathbb{N}$	Natural Numbers ($\mathbb{N} = \{1, 2, ...\}$)
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rationals
$\mathbb{R}$	Reals
$\mathbb{R}^n$	$\{[v_1, v_2, \dots, v_n] : v_1, \dots, v_n \in \mathbb{R}\}$

We will also note that there is an important distinction between vectors and points. Vectors describe "movement," whereas points describe location. For example, the vector $[1, 2]$ starting at the point $(1, 1)$ ends at the point $(2, 3)$. Consider the following diagram.



Consider the following definitions and statements.

Definition 1.1.1: The Zero Vector

We define

$$\vec{0} = [0, 0, \dots, 0].$$

Definition 1.1.2: Vector Equality

Two vectors $\vec{v} = [v_1, v_2, \dots, v_n]$ and $\vec{w} = [w_1, w_2, \dots, w_n]$ are equal if and only if

$$v_1 = w_1, v_2 = w_2, \dots, v_n = w_n.$$

Definition 1.1.3: Vector Magnitude

Given a $\vec{v} = [v_1, \dots, v_n] \in \mathbb{R}^n$, we define $||\vec{v}||$, the magnitude, or norm, of $\vec{v}$, as

$$||\vec{v}|| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Definition 1.1.4: Scalar Multiplication

Given a $\vec{v} = [v_1, \dots v_n] \in \mathbb{R}^n$, and a scalar $a \in \mathbb{R}$, we define scalar multiplication as

$$a\vec{v} = [av_1, \dots, av_n].$$

Consider the following theorem.

Theorem 1.1.1: Scalar Multiplication and Magnitude

Given a scalar $a \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$,

$$||a\vec{v}|| = |a|||\vec{v}||.$$

Proof.

$$\begin{aligned} ||a\vec{v}|| &= \sqrt{(av_1)^2 + \cdots + (av_n)^2} \\ &= \sqrt{a^2v_1^2 + \cdots + a^2v_n^2} \\ &= \sqrt{a^2}\sqrt{v_1^2 + \cdots + v_n^2} \\ &= |a|||\vec{v}||. \end{aligned}$$

The theorem is hence proved. □

Consider the following definitions.

Definition 1.1.5: Vector Direction

Two nonzero vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are

1. in the same direction if there exists $c > 0$ such that $\vec{v} = c\vec{w}$.
2. in opposite directions if there exists $c < 0$ such that $\vec{v} = c\vec{w}$.

Definition 1.1.6: Unit Vectors

The vector $\vec{v} \in \mathbb{R}^n$ is a unit vector if and only if $||\vec{v}|| = 1$.

Consider the following theorem.

Theorem 1.1.2: Unit Vectors Represent Direction

Given a nonzero $\vec{v} \in \mathbb{R}^n$, there exists a unique unit vector in the same direction as $\vec{v}$.

Proof. Since $\vec{v} \neq \vec{0}$, $||\vec{v}|| \neq 0$. Let $\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$, meaning that

$$\begin{aligned} ||\vec{u}|| &= \left|\left|\frac{1}{||\vec{v}||}\vec{v}\right|\right| \\ &= \left|\frac{1}{||\vec{v}||}\right| ||\vec{v}|| \\ &= 1. \end{aligned}$$

This means that $\vec{u}$ is a unit vector and, because $\frac{1}{||\vec{v}||}$ is a scalar, is in the same direction as $\vec{v}$. We have now shown existence. For uniqueness, assume $\vec{w}$ is a unit vector in the same direction as $\vec{v}$. Then, for a positive $c \in \mathbb{R}$, $\vec{w} = c\vec{v}$ and $||\vec{w}|| = 1$. That is,

$$||\vec{w}|| = |c|||\vec{v}|| = 1,$$

meaning that $|c| = \frac{1}{||\vec{v}||}$. Because $c > 0$, $|c| = c$, so $\vec{w} = \vec{u}$. $\square$

In Theorem 1.1.2, we used the fact that for some $\vec{v} \in \mathbb{R}^n$, $\vec{v} \neq \vec{0} \implies ||\vec{v}|| \neq 0$. We will now provide a proof.

Theorem 1.1.3: Nonzero Vector Implies Nonzero Magnitudes

For some $\vec{v} = [v_1, \dots, v_n] \in \mathbb{R}^n$,

$$\vec{v} \neq \vec{0} \implies ||\vec{v}|| \neq 0.$$

Proof. We will proceed by proving the contrapositive. Suppose that $||\vec{v}|| = 0$, meaning that

$$||\vec{v}|| = \sqrt{v_1^2 + \cdots + v_n^2} = 0.$$

The only way this is true is if all of $v_1, \dots, v_n$ are zero, which, by definition, means that $\vec{v} = \vec{0}$. $\square$

1.2 Lecture 2: August 24, 2022

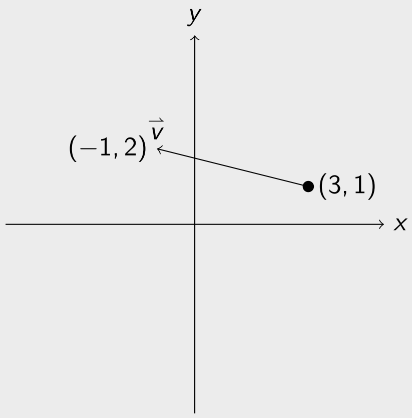
1.2.1 Vector Operations

As a review from the previous lecture, consider the following exercises.

Exercise 1.2.1: ☢☢ Vector Representing Movement

Find the vector representing the movement from $(3, 1)$ to $(-1, 2)$.

We simply find the change in the y coordinates and the change in the x coordinates to find the vector $[-4, 1]$, *starting from* $(3, 1)$. This is illustrated below.



Exercise 1.2.2: ☢☢ Finding a Unit Vector

Find the unit vector in the direction $[3, -1, -\pi]$.

We construct the unit vector by normalization. This produces $\frac{[3,-1,-\pi]}{\sqrt{3^2+(-1)^2+(-\pi)^2}}$.

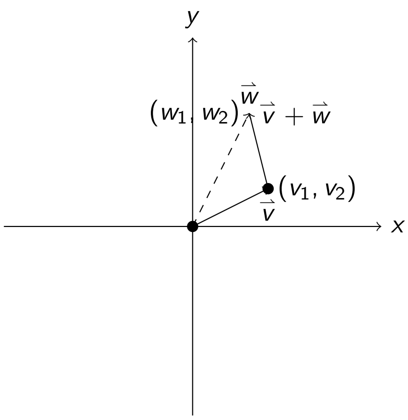
We will now define addition and subtraction with vectors and provide a few properties.

Definition 1.2.1: Addition and Subtraction With Vectors

Let $\vec{v} = [v_1, \dots, v_n]$ and $\vec{w} = [w_1, \dots, w_n]$ be vectors in $\mathbb{R}^n$. Then,

$$\vec{v} \pm \vec{w} = [v_1 \pm w_1, \dots, v_n \pm w_n].$$

Geometrically, we may visualize vector addition as



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Theorem 1.2.1: Properties of Vector Addition and Scalar Multiplication

Let $\vec{u} = [u_1, \dots, u_n]$, $\vec{v} = [v_1, \dots, v_n]$, and $\vec{w} = [w_1, \dots, w_n]$ be vectors in $\mathbb{R}^n$. Let c and d be scalars. Then,

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
3. $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $(cd)\vec{u} = c(d\vec{u})$
8. $1\vec{u} = \vec{u}$

Note that $\vec{0}$ is called the *identity element* for addition and $-\vec{u}$ is the *additive inverse element of* $\vec{u}$.

We present the proof of one of the components of Theorem 1.2.1.

Example 1.2.1: ☢☢ Commutativity of Addition

Let $\vec{v} = [v_1, \dots, v_n]$ and $\vec{w} = [w_1, \dots, w_n]$ be vectors in $\mathbb{R}^n$. Prove that

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}.$$

Proof. Consider the following.

$$\begin{aligned}\vec{v} + \vec{w} &= [v_1, \dots, v_n] + [w_1, \dots, w_n] \\ &= [v_1 + w_1, \dots, v_n + w_n] \\ &= [w_1 + v_1, \dots w_n + v_n] \\ &= \vec{w} + \vec{v}.\end{aligned}$$

The proposition is hence proved. □

Theorem 1.2.2: 🛑 Scalar Multiplication Producing the Zero Vector

Let $\vec{v} \in \mathbb{R}^n$ and let c be a scalar. Then,

$$(c = 0 \vee \vec{v} = \vec{0}) \iff c\vec{v} = \vec{0}.$$

Proof. First, we wish to show that

$$(c = 0 \vee \vec{v} = \vec{0}) \implies c\vec{v} = \vec{0}.$$

Suppose $c = 0$. We have

$$\vec{0} = c\vec{v} = [cv_1, \dots, cv_n].$$

We wish to show that all of $cv_1, \dots, cv_n$ must be zero, no matter the components of $\vec{v}$. By basic arithmetic, zero multiplied by any other number is also zero, so if $c = 0$, $c\vec{v}$ must be $\vec{0}$, as all the components of $c\vec{v}$ are zero. Suppose $\vec{v} = \vec{0}$. We again have

$$\vec{0} = c\vec{v} = [cv_1, \dots, cv_n]$$

and wish to show that all of $cv_1, \dots, cv_n$ must be zero, no matter the value of c. If $\vec{v} = 0$, all components of $\vec{v}$ are zero, and again, zero multiplied by any other number is zero, so all components of $c\vec{v}$ are zero. We have now shown that, indeed, $(c = 0 \vee \vec{v} = \vec{0}) \implies c\vec{v} = \vec{0}$. Now, we wish to show that

$$c\vec{v} = \vec{0} \implies (c = 0 \vee \vec{v} = \vec{0}).$$

Suppose that $\vec{v} \neq \vec{0}$. Again, we have

$$\vec{0} = c\vec{v} = [cv_1, \dots, cv_n].$$

There must be some nonzero $v_1, \dots, v_n$. Let this nonzero number be n. The only way $cn = 0$ is if $c = 0$. □

1.3 Lecture 3: August 26, 2022

1.3.1 Matrices

Consider the following warm-up exercise.

Exercise 1.3.1: General Vector Movement

What is the formula for the vector $\vec{v}$ representing the movement from $A = (a_1, \ldots, a_n)$ to $B = (b_1, \ldots, b_n)$. Then, find the magnitude of $\vec{v}$.

We see that

$$\vec{v} = [b_1 - a_1, \ldots, b_n - a_n],$$

starting from A. Then,

$$||\vec{v}|| = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}.$$

Consider the following definitions.

Definition 1.3.1: Matrices

Let $m, n \in \mathbb{N}$. An $m \times n$ matrix is a rectangular array of real numbers with m rows and n columns. Matrices are often denoted with capital letters. The elements are called entries, and are usually written as

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

The main diagonal of A consists of $a_{11}, a_{22}, a_{33}, \ldots$.

Definition 1.3.2: Square Matrices

A matrix is square if and only if $m = n$.

Definition 1.3.3: Diagonal Matrices

A matrix is diagonal if and only if it is square and $a_{ij} = 0$ whenever $i \neq j$. That is, all elements not on the main diagonal are zero.

Definition 1.3.4: The Identity Matrix

The identity matrix A is an $n \times n$ matrix where

$$a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Definition 1.3.5: The Zero Matrix

The zero matrix A is an $m \times n$ matrix where $a_{ij} = 0$ for all i and j.

Definition 1.3.6: Upper Triangular Matrices

An upper triangular matrix is a matrix such that $a_{ij} = 0$ for all $i > j$.

Definition 1.3.7: Lower Triangular Matrices

An lower triangular matrix is a matrix such that $a_{ij} = 0$ for all $i < j$.

Definition 1.3.8: The Set of All $m \times n$ Matrices

The set $\mathcal{M}_{mn}$ is the set of all $m \times n$ matrices.

Definition 1.3.9: Matrix Addition

Given $A, B \in \mathcal{M}_{mn}$, we define $A \pm B$ to be the matrix in $\mathcal{M}_{mn}$ with entries $a_{ij} \pm b_{ij}$.

Consider the following example.

Example 1.3.1: Matrix Addition

Consider the following addition.

$$\begin{bmatrix} 3 & 1 & e \\ 0 & \pi & 5 \end{bmatrix} + \begin{bmatrix} -1 & 3 & e \\ 5 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2e \\ 5 & \pi & 3 \end{bmatrix}.$$

Consider the following definition.

Definition 1.3.10: Scalar Multiplication With Matrices

Given $A \in \mathcal{M}_{mn}$ and $c \in \mathbb{R}$, we define cA as the matrix with elements ca_{ij}.

Consider the following example.

Example 1.3.2: Scalar Multiplication With Matrices

Consider the following scalar multiplication.

$$5 \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 5 \\ -5 & 0 \end{bmatrix}.$$

Consider the following definition.

Definition 1.3.11: Matrix Transpose

Given $A \in \mathcal{M}_{mn}$, we define $A^T \in \mathcal{M}_{nm}$ to be the matrix with the (i, j) entry equal to the (j, i) entry of A.

Consider the following example.

Example 1.3.3: Matrix Transpose

Consider the following transpose.

$$\begin{bmatrix} 2 & -5 \\ 3 & -1 \\ \pi & -\pi \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & \pi \\ -5 & -1 & -\pi \end{bmatrix}.$$

Consider the following definition.

Definition 1.3.12: Symmetric and Skew-Symmetric Matrices

Suppose A is a matrix. Then,

1. A is symmetric if and only if $A = A^T$
2. A is skew-symmetric if $A = -A^T$.

Note that A being square is necessary for the above conditions, but this condition is not sufficient.

Example 1.3.4: A Symmetric Matrix

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Is A symmetric?

We see that

$$A^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

so A is symmetric.

Example 1.3.5: A Skew-Symmetric Matrix

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Is A symmetric?

We see that

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A,$$

so A is skew-symmetric.

Consider the following theorems.

Theorem 1.3.1: Transpose Properties

Suppose $A, B \in \mathcal{M}_{mn}$ and $c \in \mathbb{R}$. Then,

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$.

Proof. Consider the matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

We see that

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}.$$

We will take the transpose of A^T, that is $(A^T)^T$, to yield

$$\begin{aligned} (A^T)^T &= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \\ &= A. \end{aligned}$$

We have proved the first property. Let the matrix

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix},$$

and therefore,

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}.$$

Then,

$$\begin{aligned}(A+B)^T &= \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{m1}+b_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n}+b_{1n} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{m1} \\ \vdots & \ddots & \vdots \\ b_{1n} & \cdots & b_{mn} \end{bmatrix} \\ &= A^T + B^T.\end{aligned}$$

Thus, we have proved the second property. Finally, consider the matrix

$$cA = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}.$$

Then,

$$\begin{aligned}(cA)^T &= \begin{bmatrix} ca_{11} & \cdots & ca_{m1} \\ \vdots & \ddots & \vdots \\ ca_{1n} & \cdots & ca_{mn} \end{bmatrix} \\ &= c\begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix} \\ &= cA^T.\end{aligned}$$

The final property is hence proved. □

Consider the following definition.

Definition 1.3.13: Trace

Let $A \in \mathcal{M}_{nn}$. Then,

$$\operatorname{trace} A = \sum_{i=1}^{n} a_{ii}.$$

That is, the sum of the elements on the main diagonal.

Theorem 1.3.2: Sum and Difference of Matrices and Their Transpose

Suppose A is an $n \times n$ matrix. Then,

1. $A + A^T$ is symmetric.
2. $A - A^T$ is skew-symmetric.

Proof. Consider the transpose of $A + A^T$. That is,

$$\begin{aligned}(A + A^T)^T &= A^T + (A^T)^T \\ &= A^T + A \\ &= A + A^T.\end{aligned}$$

This means that $A + A^T$ is symmetric. Similarly, Consider the transpose of $A - A^T$. That is,

$$\begin{aligned}(A - A^T)^T &= A^T - (A^T)^T \\ &= A^T - A \\ &= -(A - A^T).\end{aligned}$$

This means that $A - A^T$ is skew-symmetric. □

Theorem 1.3.3: The Relation Between Square, Symmetric, and Skew-Symmetric

Suppose A is a square matrix. There exists a symmetric matrix S and a skew-symmetric matrix R such that

$$A = S + R.$$

Proof. Note that

$$2A = (A + A^T) + (A - A^T).$$

Dividing both sides by 2 produces

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$

meaning $S = \frac{1}{2}(A+A^T)$ and $R = \frac{1}{2}(A-A^T)$. By Theorem 1.3.2, and basic properties of the transpose operation, S is symmetric and R is skew-symmetric, as desired. We have now shown existence, and will now show uniqueness. Suppose P is a symmetric matrix and Q is a skew-symmetric matrix, and $A = P + Q$. Then, $A^T = P^T + Q^T = P - Q$. This must mean that $P = \frac{1}{2}(A + A^T)$ and $Q = \frac{1}{2}(A - A^T)$. □

1.4 Lecture 4: August 29, 2022

1.4.1 The Dot Product

We will now define another vector operation: the dot product. We will also provide some important properties.

Definition 1.4.1: The Dot Product

Let $\vec{v} = [v_1, \dots, v_n]$ and $\vec{w} = [w_1, \dots, w_n]$ be vectors in $\mathbb{R}^n$. The dot product, or inner product, of $\vec{v}$ and $\vec{w}$ is given by

$$\vec{v} \cdot \vec{w} = \sum_{k=1}^{n} v_k w_k.$$

Note that $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v} \cdot \vec{w} = 0$. Consider the following examples.

Example 1.4.1: Dot Product 1

Find $[1, \pi] \cdot [-1, \pi]$.

We see that $[1, \pi] \cdot [-1, \pi] = -1 + \pi^2$.

Example 1.4.2: Dot Product 2

Find $[1, 0] \cdot [0, 1]$.

We see that $[1, 0] \cdot [0, 1] = 0$.

Theorem 1.4.1: Properties of the Dot Product

Let $\vec{u} = [u_1, \dots, u_n]$, $\vec{v} = [v_1, \dots, v_n]$, and $\vec{w} = [w_1, \dots, w_n]$ be vectors in $\mathbb{R}^n$. Let c be a scalar. Then,

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot \vec{u} = ||\vec{u}||^2 \geq 0$
3. $\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = \vec{0}$
4. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$
6. $(\vec{u} + \vec{v}) \cdot \vec{w} = (\vec{u} \cdot \vec{w}) + (\vec{v} \cdot \vec{w})$

Consider the following example.

Example 1.4.3: ☢☢ Proving a Dot Product Property

Let $\vec{v} = [v_1, \ldots, v_n]$ and $\vec{w} = [w_1, \ldots, w_n]$ be vectors in $\mathbb{R}^n$. Let c be a scalar. Prove that

$$c(\vec{v} \cdot \vec{w}) = (c\vec{v}) \cdot \vec{w}.$$

Proof. Consider the following transitive chain of equality:

$$\begin{aligned} c(\vec{v} \cdot \vec{w}) &= c(v_1 w_1 + \cdots + v_n w_n) \\ &= (cv_1 w_1 + \cdots + cv_n w_n) \\ &= ((cv_1)w_1 + \cdots + (cv_n)w_n) \\ &= (c\vec{v}) \cdot \vec{w}. \end{aligned}$$

The proposition is hence proved. □

Consider the following theorem regarding the angle between two vectors.

Theorem 1.4.2: The Angle Between Two Vectors

Given nonzero $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos\theta,$$

where θ is the angle between the two vectors.

Proof. Consider two unit vectors $\hat{v}$ and $\hat{w}$ where

$$\hat{v} = [\cos\alpha, \sin\alpha], \quad \hat{w} = [\cos\beta, \sin\beta].$$

The angle between the two vectors is $\theta = \beta - \alpha$. The dot product between the two vectors is

$$\begin{aligned} \hat{v} \cdot \hat{w} &= \cos\alpha\cos\beta + \sin\beta\sin\alpha \\ &= \cos(\beta - \alpha) \\ &= \cos\theta. \end{aligned}$$

Now, if we consider $\vec{v} = c_1\hat{v}$ and $\vec{w} = c_2\hat{w}$, meaning that $||\vec{v}|| = c_1$ and $||\vec{w}|| = c_2$, we simply scale the above result with $\hat{v}$ and $\hat{v}$ by the magnitudes of $\vec{v}$ and $\vec{w}$ to produce

$$\vec{v} \cdot \vec{w} = ||\vec{v}||||\vec{w}|| \cos\theta,$$

The proposition is hence proved, but it is also of note to realize that the same proposition can be proved with the law of cosines. □

We will now state the famous Cauchy-Schwarz Inequality.

Theorem 1.4.3: The Cauchy-Schwarz Inequality

If $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$|\vec{v} \cdot \vec{w}| \leq ||\vec{v}||||\vec{w}||.$$

Proof. Consider the following lemma. If $\vec{v}, \vec{w} \in \mathbb{R}^n$, with $||\vec{v}|| = ||\vec{w}|| = 1$,

$$-1 \leq \vec{v} \cdot \vec{w} \leq 1.$$

Proof. We will start by showing that $-1 \leq \vec{v} \cdot \vec{w}$. Consider $||\vec{v} + \vec{w}||^2 \geq 0$. We have that

$$\begin{aligned} ||\vec{v} + \vec{w}||^2 &= (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) \\ &= \vec{v} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w} \\ &= ||\vec{v}||^2 + 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2 \\ &= 1 + 2\vec{v} \cdot \vec{w} + 1 \\ &= 2 + 2\vec{v} \cdot \vec{w} \\ &\geq 0. \end{aligned}$$

We then solve the resulting inequality which yields

$$2\vec{v} \cdot \vec{w} \geq -2 \implies \vec{v} \cdot \vec{w} \geq -1.$$

Similarly, consider that $||\vec{v} - \vec{w}||^2 \geq 0$. We then have

$$\begin{aligned} ||\vec{v} - \vec{w}||^2 &= (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) \\ &= ||\vec{v}||^2 - 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2 \\ &= 2 - 2\vec{v} \cdot \vec{w} \\ &\geq 0. \end{aligned}$$

Then, we have

$$2 - 2\vec{v} \cdot \vec{w} \geq 0 \implies \vec{v} \cdot \vec{w} \leq 1$$

as desired. □

Consider the following cases. If $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$, then, $\vec{v} \cdot \vec{w} = 0$ and $||\vec{v}||\vec{w}|| = 0$. Thus, in this case, $|\vec{v} \cdot \vec{w}| = ||\vec{v}||||\vec{w}|| = 0$. If $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$, Consider the unit vectors $\frac{1}{||\vec{v}||}\vec{v}$ and $\frac{1}{||\vec{w}||}\vec{w}$. Hence, by the lemma,

$$-1 \leq \frac{1}{||\vec{v}||}\vec{v} \cdot \frac{1}{||\vec{w}||}\vec{w} \leq 1.$$

This implies that

$$-||\vec{v}||||\vec{w}|| \leq \vec{v} \cdot \vec{w} \leq ||\vec{v}||||\vec{w}||.$$

That is,

$$|\vec{v} \cdot \vec{w}| \leq ||\vec{v}||||\vec{w}||,$$

which is precisely the statement of the theorem. □

We will now state the famous Triangle Inequality, or Minkowski's Inequality.

Theorem 1.4.4: The Triangle Inequality

If $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||.$$

Proof. Since $f(x) = \sqrt{x}$ is always increasing, we need only prove

$$||\vec{v} + \vec{w}||^2 \leq (||\vec{v}|| + ||\vec{w}||)^2.$$

We have

$$\begin{aligned} ||\vec{v} + \vec{w}||^2 &= ||\vec{v}||^2 + 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2 \\ &\leq ||\vec{v}||^2 + 2|\vec{v} \cdot \vec{w}| + ||\vec{w}||^2 \\ &\leq ||\vec{v}||^2 + 2||\vec{v}||||\vec{w}|| + ||\vec{w}||^2 \\ &= (||\vec{v}|| + ||\vec{w}||)^2. \end{aligned}$$

The theorem is hence proved. □

We will now define the projection onto a vector.

Definition 1.4.2: The Projection Onto a Vector

Given $\vec{v}, \vec{w} \in \mathbb{R}^n$, where $\vec{w} \neq \vec{0}$, we conclude that the projection of $\vec{v}$ onto $\vec{w}$ is

$$\begin{aligned} \text{proj}_{\vec{w}} \vec{v} &= ||\vec{v}|| \cos\theta \frac{\vec{w}}{||\vec{w}||} \\ &= ||\vec{v}|| \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||} \frac{\vec{w}}{||\vec{w}||} \\ &= \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||^2} \vec{w} \\ &= \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}. \end{aligned}$$

Consider the following example.

Example 1.4.4: Projections

Given $\vec{v} = [x, y]$, find $\text{proj}_{[1,0]} \vec{v}$ and $\text{proj}_{[0,1]} \vec{v}$.

We see that

$$\text{proj}_{[1,0]} \vec{v} = [x, 0], \quad \text{proj}_{[0,1]} \vec{v} = [0, y].$$

Consider the following theorems.

Theorem 1.4.5: Sum of Parallel and Perpendicular Projections

Given $\vec{w} \neq 0$ where $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$\vec{v} = \operatorname{proj}_{\vec{w}} \vec{v} + (\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}).$$

The first addend returns a vector parallel to $\vec{w}$, and the second added returns a vector perpendicular to $\vec{w}$.

Proof. Consider the expansion of the right hand side, that is,

$$\begin{aligned}\operatorname{proj}_{\vec{w}} \vec{v} + (\vec{v} - \operatorname{proj}_{\vec{w}} \vec{v}) &= \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} + \left(\vec{v} - \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w}\right) \\ &= \vec{v}.\end{aligned}$$

We have arrived at the desired result. □

Theorem 1.4.6: Projections Depend on Lines

Given $\vec{w} \neq 0$ where $\vec{v}, \vec{w} \in \mathbb{R}^n$, and $c \in \mathbb{R}$ where $c \neq 0$,

$$\operatorname{proj}_{\vec{w}} \vec{v} = \operatorname{proj}_{c\vec{w}} \vec{v}.$$

Proof. Consider the expansion of the left hand side, that is,

$$\begin{aligned}\operatorname{proj}_{c\vec{w}} \vec{v} &= \frac{\vec{v} \cdot c\vec{w}}{c\vec{w} \cdot c\vec{w}} c\vec{w} \\ &= \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\vec{w} \\ &= \operatorname{proj}_{\vec{w}} \vec{v}.\end{aligned}$$

The proposition is hence proved. □

1.5 Lecture 5: August 31, 2022

1.5.1 Matrix Multiplication

Consider the following definition of matrix multiplication.

Definition 1.5.1: Matrix Multiplication

For $A \in \mathcal{M}_{mn}$ and $B \in \mathcal{M}_{np}$, such that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix},$$

the matrix product $C = AB$ is defined such that $C \in \mathcal{M}_{mp}$ and is given by

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$$

such that $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$. The matrix product is only defined if the number of columns of the matrix on the left is equal to the number of the rows of the matrix on the right. Note that matrix multiplication is not commutative. We may also say that the entry in c_{ij} is the dot product of the ith row of A and the jth column of B.

Consider the following examples.

Example 1.5.1: Matrix Multiplication 1

Find $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

We see that

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Example 1.5.2: Matrix Multiplication 2

Find $\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} \pi \\ -3 \\ 7 \end{bmatrix}$.

We see that

$$\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} \pi \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3\pi - 10 \end{bmatrix}.$$

Example 1.5.3: Matrix Multiplication 3

Find $\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 7 \\ 1 & 3 \end{bmatrix}$.

We see that

$$\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 7 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 13 \end{bmatrix}.$$

Example 1.5.4: Matrix Multiplication 4

Find $\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We see that

$$\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}.$$

Example 1.5.5: Matrix Multiplication 5

Find $\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

We see that

$$\begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consider the following properties of matrix multiplication.

Theorem 1.5.1: Properties of Matrix Multiplication

Suppose A, B, and C are matrices such that matrix multipication is well-defined. Then,

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$.

We now ponder two questions.

1. Does $AB = BA$? No.

2. If $AB = 0$, does $A = 0$ or $B = 0$? No.

Consider the following definition.

Definition 1.5.2: Raising a Matrix to a Power

Consider an $n \times n$ matrix A. Then,

$$A^k = \underbrace{A \cdot A \cdot A \cdot \cdots \cdot A}_{k \text{ times}}$$

and $A^k = I$.

2
Systems of Linear Equations

2.1 Lecture 6: September 2, 2022

2.1.1 Systems of Linear Equations

Consider the following definitions.

Definition 2.1.1: Linear Equations

A linear equation is an equation of the form

$$a_1x_1 + \cdots + a_nx_n = b.$$

Definition 2.1.2: Systems of Linear Equations

A system of linear equations is a system of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

2.2 Lecture 7: September 7, 2022

2.2.1 Systems of Linear Equations as Matrices

We may write systems of linear equations in terms of matrices as

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix},$$

where A is the matrix with entries a_{ij}. Consider the following theorem.

Theorem 2.2.1: Characterizing Solutions of Linear Systems

A system of linear equations can either have

1. No solution.
2. One unique solution.
3. Infinitely many solutions.

2.2.2 Matrix Row Operations

Consider the following operations.

1. Multiplication of a row by a nonzero scalar. Notated as $c\langle r_1\rangle \to \langle r_1\rangle$.
2. Addition of a scalar multiple of one row to another. Notated as $\langle r_1\rangle + (c)\langle r_2\rangle \to \langle r_1\rangle$
3. Switching the elements of two rows. Notated as $\langle r_1\rangle \leftrightarrow \langle r_2\rangle$.

Consider the following examples.

Example 2.2.1: Row Operation 1

Consider the matrix $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 5 \end{bmatrix}$. Find $4\langle 3\rangle \to \langle 3\rangle$.

We obtain

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \\ -4 & 4 & 20 \end{bmatrix}.$$

Example 2.2.2: ☢☢ Row Operation 2

Consider the matrix $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 5 \end{bmatrix}$. Find $\langle 1\rangle + (-3)\langle 2\rangle \to \langle 1\rangle$.

We obtain

$$\begin{bmatrix} 0 & 1 & -4 \\ 1 & 0 & 1 \\ -1 & 1 & 5 \end{bmatrix}.$$

Example 2.2.3: ☢☢ Row Operation 3

Consider the matrix $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 5 \end{bmatrix}$. Find $\langle 2\rangle \leftrightarrow \langle 3\rangle$.

We obtain

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix}.$$

2.3 Lecture 8: September 9, 2022

2.3.1 Solving Linear Systems

Given a linear system of equations, we solve by the following steps.

1. Convert the linear system into the matrix equation $AX = B$, written $[A|B]$.
2. Use the three row operations to reduce $[A|B]$ to one with "lots of zeroes and ones."
3. Perform back substitution and analyze the solution set.

Consider the following examples.

Example 2.3.1: ☢☢ No Solution

Consider the matrix

$$\left[\begin{array}{cccc|c} 3 & -6 & 0 & 3 & 9 \\ -2 & 4 & 2 & -1 & -11 \\ 4 & -8 & 6 & 7 & -5 \end{array}\right].$$

By row operations, we obtain

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & -2 \end{array}\right].$$

Looking at the last row, we see the equation $0 = -2$, which is not true. Hence, the system has no solution.

Example 2.3.2: ☢☢ Infinitely Many Solutions

Consider the matrix

$$\left[\begin{array}{cccc|c} 3 & 1 & 7 & 2 & 13 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{array}\right].$$

By row operations, we obtain

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{7}{3} & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -2 & \frac{1}{2} & 4 \\ 0 & 0 & 0 & 1 & 59 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We see that x_1, x_2, and x_4 are determined because their respective column has a 1 in the correct position. In contrast, x_3 is a free variable. To find the solution set, let $x_3 = c \in \mathbb{R}$ and solve for x_1, x_2, and x_4 in terms of c. We have $x_4 = -2$. Then to find x_2 we have

$$x_2 - 2x_3 + \frac{1}{2}x_4 = 4 \implies x_2 - 2 + \frac{1}{2}(-2) = 4 \implies x_2 = 2c + 5.$$

For x_1, we have

$$x_1 + \frac{1}{3}x_2 + \frac{7}{3}x_3 + \frac{2}{3}x_4 = \frac{13}{3} \implies x_1 = -3c + 4.$$

The solution set is then $\{(-3c + 4, 2c + 5, c, -2) : c \in \mathbb{R}\}$.

We generally agree that back substitution is not much fun. Consider the following example.

Example 2.3.3: No More Back Substitution

Note that the matrix

$$\begin{bmatrix} 3 & -3 & -2 & | & 23 \\ -6 & 4 & 3 & | & -40 \\ -2 & 1 & 1 & | & -12 \end{bmatrix}$$

reduces into

$$\begin{bmatrix} 1 & -1 & -\frac{2}{3} & | & \frac{23}{3} \\ 0 & 1 & \frac{1}{3} & | & -\frac{10}{3} \\ 0 & 0 & 1 & | & 2 \end{bmatrix}.$$

We may now "get rid of" -1, $-\frac{2}{3}$, and $\frac{1}{3}$. We perform $-\frac{1}{3}\langle 3\rangle + \langle 2\rangle \to \langle 2\rangle$ which produces

$$\begin{bmatrix} 1 & -1 & -\frac{2}{3} & | & \frac{23}{3} \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}.$$

Then, we will perform $\langle 1\rangle + \langle 2\rangle \to \langle 1\rangle$, yielding

$$\begin{bmatrix} 1 & 0 & -\frac{2}{3} & | & \frac{23}{3} - 4 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}.$$

Finally, we will perform $\frac{2}{3}\langle 3\rangle + \langle 1\rangle \to 1$, obtaining

$$\begin{bmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

which means $x_1 = 5$, $x_2 = -4$, and $x_3 = 2$.

Consider the following theorem.

Theorem 2.3.1: Row Operations

Suppose $A \in \mathcal{M}_{mn}$ and $B \in \mathcal{M}_{np}$. Then,

1. If R is a row operation, $R(AB) = (R(A))B$.
2. If $R_1, \dots, R_n$ are row operations, $R_n(\dots(R_2(R_1(AB)))\dots) = (R_n(\dots(R_2(R_1(A)))\dots))B$.

Proof. For (1), see Assignment 2, Question 2 for the relationship between matrix multiplication and row operations. For (2), use induction on k. $\square$

Consider the following examples of solving linear systems.

Example 2.3.4: ☢☢ Linear System 1

Solve the following system:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right]$$

We first perform the row operation $\frac{1}{2}\langle 1\rangle \to \langle 1\rangle$, which produces

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 3 & 4 & 0 \end{array}\right].$$

Then, we perform $\langle 1\rangle - \langle 2\rangle \to \langle 2\rangle$. We obtain

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{7}{2} & -\frac{7}{2} & 0 \end{array}\right].$$

Next, we have $-\frac{2}{7}\langle 2\rangle \to \langle 2\rangle$. This yields

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

From here, let $x_3 = c$. Then, $x_2 = -c$. To find x_1, we use the equation

$$x_1 - \frac{1}{2}(-c) + \frac{1}{2}c = 0,$$

which implies that $x_1 = -c$. Thus, the solution set is $\{(-c, -c, c) : c \in \mathbb{R}\}$.

Example 2.3.5: ☢☢ Linear System 2

Solve the following system:

$$\begin{bmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 1 & 1 & -1 & 1 & | & 2 \\ 1 & 7 & -5 & -1 & | & 3 \end{bmatrix}$$

First, we perform $\langle 1 \rangle - \langle 2 \rangle \to \langle 2 \rangle$, yielding

$$\begin{bmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 0 & -3 & 2 & 1 & | & -1 \\ 1 & 7 & -5 & -1 & | & 3 \end{bmatrix}.$$

Then, we perform $\langle 1 \rangle - \langle 3 \rangle \to \langle 3 \rangle$. We obtain

$$\begin{bmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 0 & -3 & 2 & 1 & | & -1 \\ 0 & -9 & 6 & 3 & | & -2 \end{bmatrix}.$$

Then, we have $-\frac{1}{3}\langle 2 \rangle \to \langle 2 \rangle$. This produces

$$\begin{bmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & | & \frac{1}{3} \\ 0 & -9 & 6 & 3 & | & -2 \end{bmatrix}.$$

Our final row operation is $\langle 3 \rangle + 9\langle 2 \rangle \to \langle 3 \rangle$. This provides us with

$$\begin{bmatrix} 1 & -2 & 1 & 2 & | & 1 \\ 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & 0 & | & 1 \end{bmatrix},$$

meaning that there is no solution to the system.

Example 2.3.6: ☢☢ Linear System 3

Solve the following system:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

Our first row operation is $2\langle 1\rangle - \langle 2\rangle \to \langle 2\rangle$. This produces

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -2 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right].$$

Then, we have $\langle 1\rangle - \langle 3\rangle \to \langle 3\rangle$, providing

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & 2 & -2 & -1 \end{array}\right].$$

Next, we perform $\langle 2\rangle + \langle 3\rangle \to \langle 3\rangle$. We obtain

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We perform another row operation, $-\frac{1}{2}\langle 2\rangle \to \langle 2\rangle$. This yields

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Next, we have $\langle 2\rangle + \langle 1\rangle \to \langle 1\rangle$, which gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Let $x_3 = c$. Then, $x_2 = -\frac{1}{2} + c$ and $x_1 = \frac{1}{2} - c$. This means the solution set is $\left\{\left(\frac{1}{2} - c, -\frac{1}{2} + c, c\right) : c \in \mathbb{R}\right\}$.

2.4 Lecture 9: September 12, 2022

2.4.1 Formalizing Previous Notions Part I

Consider the following definitions.

Definition 2.4.1: Row Echelon Form

A matrix A is in row echelon form if and only if

1. All rows consisting of only zeroes are at the bottom.
2. The leading coefficient, or the pivot, of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

Definition 2.4.2: Reduced Row Echelon Form

A matrix A is in reduced row echelon form if and only if

1. The first nonzero entry in each row is one.
2. Each successive row has its first nonzero entry in a later column.
3. All entries above and below the first nonzero entry are zero.
4. All rows consisting of only zeroes are at the bottom.

Note that every matrix has a unique reduced row echelon form.

Consider the following theorems and definitions.

Theorem 2.4.1: Number of Solutions to a Linear System

Let $AX = B$ be a system of linear equations. Let C be the reduced row echelon form augmented matrix obtained by row reducing $[A|B]$. Then,

1. If there is a row of C having all zeroes to the left of the augmentation bar but with its last entry nonzero, $AX = B$ has no solution.
2. If not, and if one of the columns of C to the left of the augmentation bar has no nonzero pivot entry, $AX = B$ has an infinite number of solutions. The nonpivot columns correspond to (independent) variables that can take on any value, and the values of the remaining (dependent) variables are determined from those.
3. Otherwise $AX = B$ has a unique solution.

Consider the following definitions.

Definition 2.4.3: Homogeneous Systems

Given $A \in \mathcal{M}_{mn}$, the homogeneous system associated with A is

$$AX = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Theorem 2.4.2: Solutions to Homogeneous Systems

Given $A \in \mathcal{M}_{mn}$, the homogeneous system always has at least one solution, called the *trivial solution*. Namely,

$$x_1 = 0, \quad x_2 = 0, \quad \dots, \quad x_n = 0.$$

Also, consider the following.

1. If $m < n$, the solution set is infinite.
2. If $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $X_\sim = \begin{bmatrix} x_{\sim 1} \\ \vdots \\ x_{\sim n} \end{bmatrix}$ are solutions,
$$cX + X_\sim$$
is a solution for any $c \in \mathbb{R}$.
3. If $AX = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ and $A\hat{X} = B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$,
$$cX + \hat{X}$$
is a solution to $[A|B]$. Notice that (2) is a special case of (3).

Proof. Consider $A(cX + \hat{X})$. We wish to show that $A(cX + \hat{X}) = B$. We see that

$$\begin{aligned} A(cX + \hat{X}) &= A(cX) + A\hat{X} \\ &= cAX + A\hat{X} \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \\ &= B, \end{aligned}$$

as desired. □

This process is analogous to solving homogeneous differential equations to solve nonhomogeneous differential equations.

Definition 2.4.4: Equivalence of Linear Systems

Two systems $[A|B]$ and $[A_\sim|B_\sim]$ are equivalent if and only if

$$AX = B \wedge A_\sim X = B_\sim.$$

That is, if they have the same solution sets.

Definition 2.4.5: Row Equivalence

A matrix A is row equivalent to a matrix B if B can be obtained by a finite number of row operations conducted on A.

For example, Gaussian Elimination and Gauss-Jordan Elimination produce matrices that are row equivalent to the original matrix.

One may ask: What is the relationship between these relations? We see that row equivalence implies system equivalence. But, two systems can have the same solution set, but have different sizes, making row equivalence impossible. For the latter case, consider two matrices of different sizes, but with an empty solution set. Recall from discrete mathematics,

Definition 2.4.6: Equivalence Relations

A relation $\sim$ on a set S is an equivalence relation on S if and only if $\sim$ is reflexive, symmetric, and transitive. That is, if

1. $\forall a \in S, a \sim a.$
2. $\forall a, b \in S, a \sim b \implies b \sim a$
3. $\forall a, b, c \in S, a \sim b \wedge b \sim c \implies a \sim c.$

Consider the following theorem.

Theorem 2.4.3: System Equivalence and Row Equivalence are Equivalence Relations

First, consider the following table.

Row Operation	Reverse Operation
$c\langle i\rangle \to \langle i\rangle$	$\frac{1}{c}\langle i\rangle \to \langle i\rangle$
$c\langle i\rangle + \langle j\rangle \to \langle j\rangle$	$-c\langle i\rangle + \langle j\rangle \to \langle j\rangle$
$\langle i\rangle \leftrightarrow \langle j\rangle$	$\langle i\rangle \leftrightarrow \langle j\rangle$

Proof. We will consider row equivalence first, and wish to show that row equivalence is reflexive, symmetric, and transitive. Reflexivity is trivial. We can simply not perform any row operations on a matrix A, and we are left with A. The above table can be used to show that row equivalence is symmetric. If a sequence of row operations is carried out on A and produces a matrix B, we can simply carry out the reverse operations on B to lead us back to A. For transitivity, if a sequence of row operations is carried out on A and leads to B, and a second sequence of row operations is performed on B and leads to C, we simply carry out the sequences, in sequence, on A to get us to C.

Now, we consider system equivalence. The system $[A|B]$, of course, has the same solution set as itself. If the system $[A|B]$ has the same solution set as $[C|D]$, $[C|D]$ has the same solution set as $[A|B]$. If $[A|B]$ has the same solution set as $[C|D]$ and $[C|D]$ has the same solution set as the system $[E|F]$, $[A|B]$ has the same solution set as $[E|F]$. $\square$

2.5 Lecture 10: September 14, 2022

2.5.1 Formalizing Previous Notions Part II

Consider the following formalization of our last discoveries.

Theorem 2.5.1: Row Equivalence Implies System Equivalence

If $[A|B]$ is equivalent to $[C|D]$, $[A|B]$ is equivalent to $[C|D]$. We assume that

$$R_n(R_{n-1}(\dots R_1([A|B]))\dots)$$

We will use induction on n. The base case is $R_1([A|B]) = [C|D]$. There are three cases,

1. R_1 is $c\langle i\rangle \to \langle i\rangle$. We have that R_1 is $c\langle i\rangle \to \langle i\rangle$ for $c \neq 0$. The system has the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

   After $c\langle i\rangle \to \langle i\rangle$, the system is

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ ca_{i1}x_1 + \cdots + ca_{in}x_n &= cb_i \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

   We see that the solution sets are the same. The first and third rows are the same, so the solution sets will be the same. For the second row, simply factor by c and divide.

2. R_1 is $\langle c\rangle\langle j\rangle + \langle i\rangle \to \langle i\rangle$.
3. R_1 is $\langle i\rangle \leftrightarrow \langle j\rangle$.

The proposition for $n = k$ is,

$$R_k(R_{k-1}(\dots R_1([A|B]))\dots)$$

has the same solution as $[A|B]$, but

$$R_{k+1}(R_k(R_{k-1}(\dots R_1([A|B]))\dots)$$

has the same solution as

$$R_k(R_{k-1}(\dots R_1([A|B]))\dots).$$

Thus,

$$R_{k+1}(R_k(R_{k-1}(\dots R_1([A|B]))\dots)$$

has the same solution set as $[A|B]$.

Theorem 2.5.2: Uniqueness of Reduced Row Echelon Form

Every matrix is row equivalent to a unique matrix in reduced row echelon form. Two matrices are row equivalent if and only if they have the same reduced row echelon form.

Definition 2.5.1: Rank

Given $A \in \mathcal{M}_{mn}$, $\operatorname{rank} A$ is the number of nonzero rows in the unique matrix that is row equivalent to A and is in reduced row echelon form.

Consider the following example.

Example 2.5.1: Rank 1

Consider

$$A = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & -2 & 12 & -5 \\ 2 & -3 & 22 & -14 \end{bmatrix}.$$

By row reduction, we have the matrix

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and see that $\operatorname{rank} A = 2$.

Consider the following theorem.

Theorem 2.5.3: Number of Solutions to Homogeneous Systems

If $A \in \mathcal{M}_{mn}$,

1. If $\operatorname{rank} A < n$, $AX = 0$ has an infinite solution set.
2. If $\operatorname{rank} A = n$, $AX = 0$ has only the trivial solution.

We will now define linear combinations of vectors.

Definition 2.5.2: Linear Combinations

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$. The vector $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if and only if there are scalars $c_1, c_2, \ldots, c_k$ such that

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k.$$

In general, $\{c\vec{v} : c \in \mathbb{R}\}$ is a line unless $\vec{v} = \vec{0}$. Also, $\{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$ is usually a plane, but could be either a point or a line. This pattern is an introduction to the concept of linear dependence, which will be elaborated on later in the text.

Consider the following example.

Example 2.5.2: ☢☢ Is a Vector a Linear Combination of Others? 1

Let $\vec{v} = [1, 0]$, $\vec{v}_1 = [\pi, 1]$, and $\vec{v}_2 = [2, 1]$. Is $\vec{v}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

Notice that $\vec{v}_1 - \vec{v}_2 = [\pi - 2, 0]$. Then,

$$\frac{1}{\pi - 2}[\pi - 2, 0] = [1, 0] = \vec{v}.$$

We then have that

$$\vec{v} = \frac{1}{\pi - 2}[\pi, 1] - \frac{1}{\pi - 2}[2, 1].$$

Therefore, $\vec{v}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

The above solution used a bit of trickery. Instead, given $\vec{v}_1, \dots, \vec{v}_k$, we form the equation

$$\left[\vec{v}_1, \dots, \vec{v}_k\right] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \left[\vec{v}\right]$$

and solve for the necessary constants.

Consider the following examples.

Example 2.5.3: ☢☢ Is a Vector a Linear Combination of Others? 2

Let $\vec{v} = [1, 0, 0]$, $\vec{v}_1 = [-4, 2, 0]$, and $\vec{v}_2 = [2, 1, 1]$. Is $\vec{v}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

Consider the system

$$\begin{bmatrix} -4 & 2 & | & 1 \\ 2 & 1 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}.$$

We first perform the row operation $-\frac{1}{4}\langle 1\rangle \to \langle 1\rangle$ to obtain

$$\begin{bmatrix} 1 & -\frac{1}{2} & | & -\frac{1}{4} \\ 2 & 1 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}.$$

Then, we have $2\langle 1\rangle - \langle 2\rangle \to \langle 2\rangle$, producing

$$\begin{bmatrix} 1 & -\frac{1}{2} & | & -\frac{1}{4} \\ 0 & -2 & | & -\frac{1}{2} \\ 0 & 1 & | & 0 \end{bmatrix}.$$

Next, we will carry out $-\frac{1}{2}\langle 2\rangle \to \langle 2\rangle$ to yield

$$\begin{bmatrix} 1 & -\frac{1}{2} & | & -\frac{1}{4} \\ 0 & 1 & | & \frac{1}{4} \\ 0 & 1 & | & 0 \end{bmatrix}.$$

We will then compute $\langle 2\rangle - \langle 3\rangle \to \langle 3\rangle$; we have

$$\begin{bmatrix} 1 & -\frac{1}{2} & | & -\frac{1}{4} \\ 0 & 1 & | & \frac{1}{4} \\ 0 & 0 & | & \frac{1}{4} \end{bmatrix}.$$

There is no solution, so $\vec{v}$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Example 2.5.4: ☢☢ Is a Vector a Linear Combination of Others? 3

Let $\vec{v} = [14, -21, 7]$, $\vec{v}_1 = [2, -3, 1]$, and $\vec{v}_2 = [-4, 6, 2]$. Is $\vec{v}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

Consider the system

$$\left[\begin{array}{cc|c} 2 & -4 & 14 \\ -3 & 6 & -21 \\ 1 & 2 & 7 \end{array}\right].$$

We first perform the row operation $\frac{1}{2}\langle 1\rangle \to \langle 1\rangle$ to obtain

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ -3 & 6 & -21 \\ 1 & 2 & 7 \end{array}\right].$$

Then, we have $3\langle 1\rangle + \langle 2\rangle \to \langle 2\rangle$, producing

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 1 & 2 & 7 \end{array}\right].$$

Next, we will carry out $\langle 2\rangle \leftrightarrow \langle 3\rangle$ to yield

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 1 & 2 & 7 \\ 0 & 0 & 0 \end{array}\right].$$

We will then compute $\langle 1\rangle + \langle 2\rangle \to \langle 2\rangle$; we have

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 2 & 0 & 14 \\ 0 & 0 & 0 \end{array}\right].$$

Then, we will execute $2\langle 1\rangle - \langle 2\rangle \to \langle 2\rangle$, and we obtain

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

We then have, by $-\frac{1}{4}\langle 2\rangle \to \langle 2\rangle$,

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Finally, we have the operation $\langle 1\rangle + 2\langle 2\rangle \to \langle 1\rangle$, which produces

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Here, we see that $\vec{v} = 7\vec{v}_1$. We note that it would have been simple to conclude this based on the problem statement, but the method shown is the systematic algorithm for answering such questions.

Example 2.5.5: ☢☢ Is a Vector a Linear Combination of Others? 4

Let $\vec{v} = [14, -21, 7]$, $\vec{v}_1 = [2, -3, 1]$, and $\vec{v}_2 = [-4, 6, -2]$. Is $\vec{v}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

Consider the system

$$\begin{bmatrix} 2 & -4 & | & 14 \\ -3 & 6 & | & -21 \\ 1 & -2 & | & 7 \end{bmatrix}.$$

By row reduction, we finally obtain

$$\begin{bmatrix} 1 & -2 & | & 7 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

Here, the solution set is $\{(2c + 7, c) : c \in \mathbb{R}\}$. There are thus infinitely many ways to express $\vec{v}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

Consider the following definition.

Definition 2.5.3: Row Space

Suppose $A \in \mathcal{M}_{mn}$. The row space of A is the subset of $\mathbb{R}^n$ consisting of the linear combinations of the rows of A.

Consider the following examples.

Example 2.5.6: ☢ Row Space 1

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}.$$

The row space of A is

$$\{c_1[1, 2] + c_2[5, 10] : c_1, c_2 \in \mathbb{R}\}.$$

In this case, the row space of A is a line. Generally, though, with two vectors, the row space will be a plane.

Example 2.5.7: ☢ Row Space 2

Consider

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 10 \end{bmatrix}.$$

The row space of A is

$$\{c_1[1, 3] + c_2[5, 10] : c_1, c_2 \in \mathbb{R}\}.$$

In this case, the row space of A is a plane.

To determine if a vector is in the row space of a matrix A, we consider the system $[A^T|X]$. One may ask: why? Well, considering A instead of A^T would provide the wrong system of equations to solve. All we are doing when determining if a vector is in the row space of A is asking if the vector can be written as a linear combination of the rows of A. Consider the following example.

Example 2.5.8: ☢☢ Are Vectors in the Row Space?

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}$$

and recall that the row space of A is

$$\{c_1[1,2] + c_2[5,10] : c_1, c_2 \in \mathbb{R}\}.$$

Is $[3,6]$ in the row space of A? Is $[1,0]$ in the row space of A? We consider

$$\begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

For $[3,6]$, we have

$$\begin{bmatrix} 1 & 5 & | & 3 \\ 2 & 10 & | & 6 \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} 1 & 5 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

The system has (infinitely many) solutions, so $[3,6]$ is in the row space of A. For $[1,0]$, we have

$$\begin{bmatrix} 1 & 5 & | & 1 \\ 2 & 10 & | & 0 \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} 1 & 5 & | & 1 \\ 0 & 0 & | & -2 \end{bmatrix}.$$

The system has no solution, so $[1,0]$ is not in the row space of A.

Consider the following theorems.

Theorem 2.5.4: Transitivity of Linear Combinations

Suppose that $\vec{x}$ is a linear combination of $\vec{q}_1, \dots, \vec{q}_k$, and suppose also that each of $\vec{q}_1, \dots, \vec{q}_k$ is itself a linear combination of $\vec{r}_1, \dots, \vec{r}_l$. Then, $\vec{x}$ is a linear combination of $\vec{r}_1, \dots, \vec{r}_l$.

Proof. Because $\vec{x}$ is a linear combination of $\vec{q}_1, \dots, \vec{q}_k$,

$$\vec{x} = c_1\vec{q}_1 + \cdots + c_k\vec{q}_k$$

for $c_1, \dots, c_k \in \mathbb{R}$. Then, since each of $\vec{q}_1, \dots, \vec{q}_k$ can be written as a linear combination of $\vec{r}_1, \dots, \vec{r}_l$, there exist scalars $d_{11}, \dots, d_{kl}$ such that

$$\begin{aligned}
\vec{q}_1 &= d_{11}\vec{r}_1 + d_{12}\vec{r}_2 + \cdots + d_{1l}\vec{r}_l \\
\vec{q}_2 &= d_{21}\vec{r}_1 + d_{22}\vec{r}_2 + \cdots + d_{2l}\vec{r}_l \\
&\vdots \\
\vec{q}_k &= d_{k1}\vec{r}_1 + d_{k2}\vec{r}_2 + \cdots + d_{kl}\vec{r}_l
\end{aligned}$$

Then,

$$\begin{aligned}
\vec{x} &= c_1(d_{11}\vec{r}_1 + d_{12}\vec{r}_2 + \cdots + d_{1l}\vec{r}_l) \\
&\quad + c_2(d_{21}\vec{r}_1 + d_{22}\vec{r}_2 + \cdots + d_{2l}\vec{r}_l) \\
&\quad \vdots \\
&\quad + c_k(d_{k1}\vec{r}_1 + d_{k2}\vec{r}_2 + \cdots + d_{kl}\vec{r}_l) \\
&= (c_1d_{11} + c_2d_{21} + \cdots + c_kd_{k1})\vec{r}_1 \\
&\quad + (c_1d_{12} + c_2d_{22} + \cdots + c_kd_{k2})\vec{r}_2 \\
&\quad \vdots \\
&\quad + (c_1d_{1l} + c_2d_{2l} + \cdots + c_kd_{kl})\vec{r}_l.
\end{aligned}$$

We have just written $\vec{x}$ as a linear combination of $\vec{r}_1, \dots, \vec{r}_l$. $\square$

Note that this theorem may be rephrased as follows: If $\vec{x}$ is in the row space of a matrix Q and each row of Q is in the row space of a matrix R, $\vec{x}$ is in the row space of R.

Theorem 2.5.5: Row Equivalence Implies Equal Row Space

Suppose A and B are row equivalent. Then, the row space of A is equal to the row space of B.

2.6 Lecture 11: September 16, 2022

2.6.1 Linear Maps

Consider the following definition.

Definition 2.6.1: Linear Maps

Given $A \in \mathcal{M}_{mn}$, we define

$$\begin{aligned} T_A &: \mathbb{R}^n \to \mathbb{R}^m \\ &: \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}. \end{aligned}$$

Consider the following example.

Example 2.6.1: Some Special Maps in $\mathbb{R}^2$

Consider the following maps, and name them.

1. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the zero map.
2. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map.
3. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the x axis.
4. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto the y axis.

2.7 Lecture 12: September 19, 2022

2.7.1 Inverses of Matrices

Consider the following definitions and theorems.

Definition 2.7.1: Multiplicative Inverse of a Matrix

Let $A \in \mathcal{M}_{nn}$. Then, $B \in \mathcal{M}_{nn}$ is a multipliative inverse of A if and only if

$$AB = BA = I_n.$$

Theorem 2.7.1: Inverse Commutativity

Let $A, B \in \mathcal{M}_{nn}$. If either AB or BA equals I_n, the other product also equals I_n, and A and B are inverses of each other.

Definition 2.7.2: Singularity

A matrix is `singular` if and only if it is square and does not have an inverse. A matrix is *nonsingular* if and only if it is square and has an inverse.

Theorem 2.7.2: Uniqueness of the Inverse

If B and C are both inverses of $A \in \mathcal{M}_{nn}$, $B = C$.

Proof. $B = BI_n = B(AC) = (BA)C = I_nC = C$. □

We denote the unique inverse of A as A^{-1}. We can use the inverse to define negative integral powers of a nonsingular matrix A. Consider the following definition.

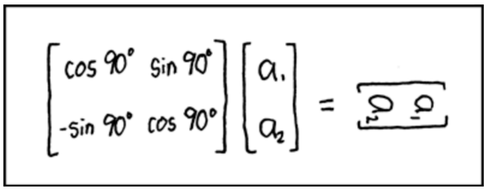
Definition 2.7.3: Negative Integral Powers of a Nonsingular Matrices

Let A be a nonsingular matrix. Then, the negative integral powers of A are given as follows: A^{-1} is the unique inverse of A. For $k \geq 2$, $A^{-k} = (A^{-1})^k$.

3

Determinants and Eigenvalues

4
Finite Dimensional Vector Spaces



5
Linear Transformations

6
Orthogonality