A General Theory of Liquidity Provisioning for Automated Market Makers

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July 8, 2024

Work supported by the National Science Foundation and Ethereum Foundation

Collaborators

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A, Rafael Frongillo, and Maneesha Papireddygari. A general theory of liquidity provisioning for automated market makers, 2023. URL https://arxiv.org/abs/2311.08725.

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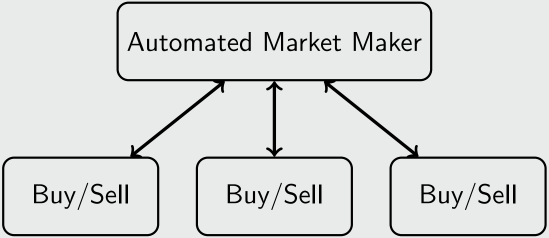
1. Motivation

Automated Market Makers



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AMMs are always willing to offer a price for any bundle of assets.

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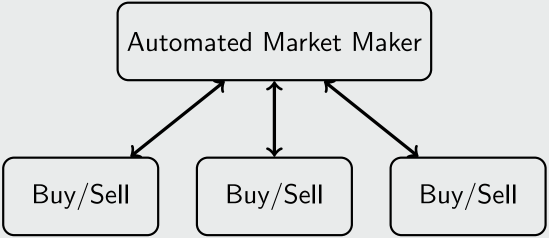

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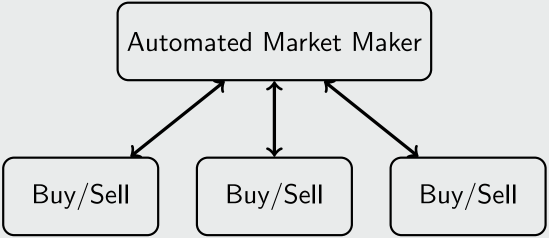

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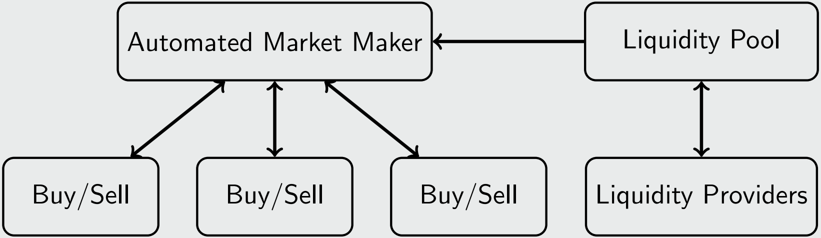
Traditionally, market creators both fostered the trade of assets and bore the risk of providing liquidity to keep prices stable.

Liquidity Provisioning



Recently, AMMs have been used as a easy way to trade assets on chain in decentralized finance (DeFi), [Bartoletti et al., 2022, Xu et al., 2023, Mohan, 2022].

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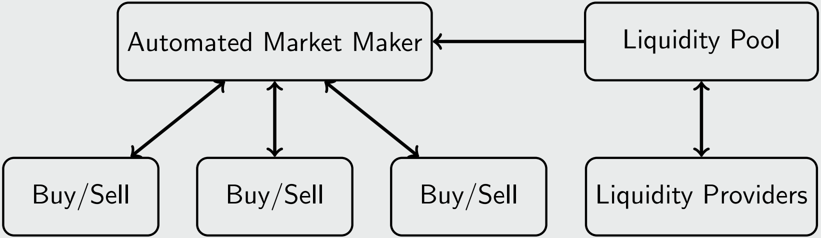

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We see a decoupling of roles: the market creator may outsource liquidity provison to external liquidity providers (LPs).

LPs are compensated for taking on risk by getting a cut of fees skimmed off trades.

The Design of Liquidity Provisioning Protocols

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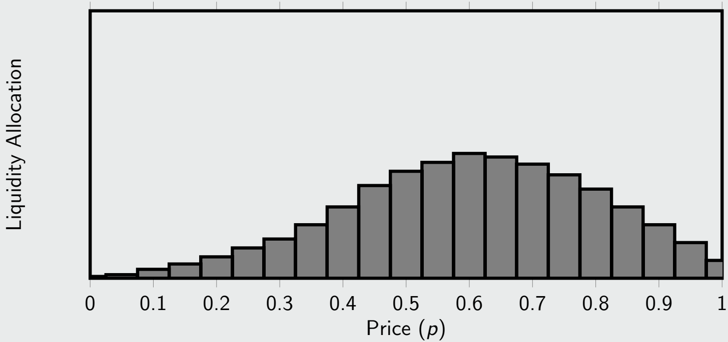

The Design of Liquidity Provisioning Protocols

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function ModifyLiquidity($i \in \mathbb{N}, \alpha' \geq 0, j \in \{0, \ldots, m\}$)

  $p = \text{price}(\mathbf{x})$

  request $\mathbf{x}' = \begin{cases} \left((\alpha' - \alpha^{ij})\left(\sqrt{\frac{1-a_j}{a_j}} - \sqrt{\frac{1-b_j}{b_j}}\right), 0\right) & \text{if } p < a_j \\ \left(0, (\alpha' - \alpha^{ij})\left(\sqrt{\frac{b_j}{1-b_j}} - \sqrt{\frac{a_j}{1-a_j}}\right)\right) & \text{if } p > b_j \\ \left((\alpha' - \alpha^{ij})\left(\sqrt{\frac{1-p}{p}} - \sqrt{\frac{1-b_j}{b_j}}\right), (\alpha' - \alpha^{ij})\left(\sqrt{\frac{p}{1-p}} - \sqrt{\frac{a_j}{1-a_j}}\right)\right) & \text{if } p \in [a_j, b_j] \end{cases}$

  $(\mathbf{x}, \alpha^{ij}) \leftarrow (\mathbf{x} + \mathbf{x}', \alpha')$

function ExecuteTrade($\mathbf{r} \in \mathbb{R}^2$)

  Let $p = \text{price}(\mathbf{x})$, $p' = \text{price}(\mathbf{x} - \mathbf{r})$

  Let l, u be such that $a_l \leq p \leq b_l$ and $a_u \leq p' \leq b_u$.

  check

$$\frac{1}{(\sum_{i=0}^{k} \alpha^{il})^2}\left(x_1 + \sum_{i=0}^{k} \alpha^{il}\sqrt{\frac{1-b_l}{b_l}}\right)\left(x_2 + \sqrt{\frac{a_l}{1-a_l}}\sum_{i=0}^{k} \alpha^{il}\right) = \frac{1}{(\sum_{i=0}^{k} \alpha^{iu})^2}\left(x_1 - r_1 + \sum_{i=0}^{k} \alpha^{iu}\sqrt{\frac{1-b_u}{b_u}}\right)\left(x_2 - r_2 + \sqrt{\frac{a_u}{1-a_u}}\sum_{i=0}^{k} \alpha^{iu}\right)$$

  pay $\beta \frac{\sum_j \alpha^{ij}}{\sum_j \sum_o \alpha^{oj}}(-\mathbf{r})_+$ to each LP i where j sums over buckets in $[B^l, B^u]$.

  $\triangleright$ WLOG the B^u bucket comes later than the B^l bucket.

  $\mathbf{x} \leftarrow \mathbf{x} - \mathbf{r}$

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  ▷ WLOG the B^u bucket comes later than the B^l bucket.

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Well, . . . , that's a bit scary.

Questions

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Answer (Our Framework)

Think about liquidity provisioning as LPs running their own automated market makers operating in parallel.

The Rest of This Talk

1. Our New Framework
2. Examples
3. Recap & Future Directions

2. Our New Framework

Overview of the Framework

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Thm 1. We can recover all existing protocols, like Uniswap V2 and V3.
Thm 2. The general protocol is "reasonable" for any "reasonable" φ^i.

Background: Constant Function Market Makers

A CFMM maintains a reserve vector $\boldsymbol{x} \in \mathbb{R}^n$ and a convex[1] potential function[2] $\varphi : \mathbb{R}^n \to \mathbb{R}$. The trades $\boldsymbol{r} \in \mathbb{R}^n$ available are those satisfying $\varphi(\boldsymbol{x} + \boldsymbol{r}) = \varphi(\boldsymbol{x})$. After a trade, $\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{r}$.

[1] Conventionally, φ is concave; we use the negation as the potential.

[2] For readers familiar with the prediction market literature, [Chen and Pennock, 2007, Abernethy et al., 2013, Frongillo et al., 2023], φ and φ^i as used in our slides are really cost functions.

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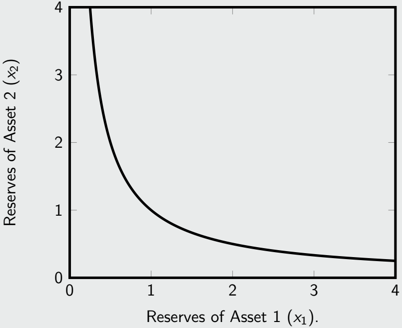
For example, take Uniswap V2 with $\varphi(\boldsymbol{x}) = x_1 x_2$:



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Definition (Infimal Convolution)

For convex functions $f^i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$*, we define their infimal convolution* $f = \bigwedge_i f^i$ *by* $f(\boldsymbol{x}) = \inf \left\{ \sum_i f^i(\boldsymbol{x}^i) \mid \sum_i \boldsymbol{x}^i = \boldsymbol{x} \right\}$*, where the* $\boldsymbol{x}^i$ *range over* $\mathbb{R}^n$*.*

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Main Results

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Theorem 1 (Existing LP Protocols $\to$ Aggregate CFMM)

All existing liquidity provisioning protocols correspond to an aggregate CFMM with potential $\varphi = \bigwedge \varphi^i$ *for certain choices of* φ^i*, where* φ^i *is the CFMM that LP* i *chooses.*

Using our Framework to Create New Protocols

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Reasonable? Running CFMMs in parallel, ✓. Liquidity adds, ✓.

Sense Check 1: Running CFMMs in Parallel

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As an added benefit, computing the optimal trade split $\boldsymbol{r}$ under the hood allows traders not needing to choose which CFMMs φ^i to trade with for best prices.

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Intuitively, liquidity is a measure of how insensitive prices are while trades occur. At a price $\boldsymbol{p} = \nabla \varphi(\boldsymbol{x}) \in \Delta_n$, we can define $\ell(\boldsymbol{p}) = \left(\nabla^2 \varphi\right)^{-1}(\boldsymbol{x})$.

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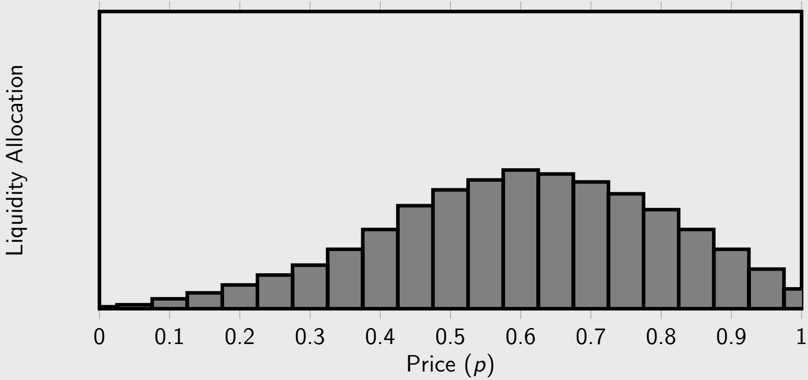
So, with $\ell = (\nabla^2 \varphi)^{-1} = \nabla^2 \varphi^*$, the total liquidity is the *sum* of the liquidity from each LP: $\ell = \sum_i \ell^i$.

3. Examples

Remember Uniswap V3's Buckets?



Generalizing the Idea of Buckets From Uniswap V3

Let's say we wanted to use the bucketing mechanism from Uniswap, but with any choice of reserve curve. What's stopping us?

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We can do the same for an arbitrary reserve curve!

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Triangular Liquidity

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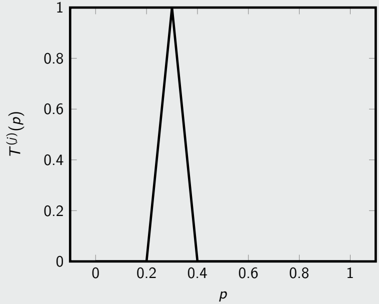

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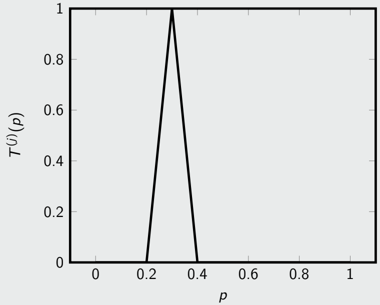

We can "weight" liquidity functions over "soft" bucket j with $\ell^{(j)}(p) = (\ell T^{(j)})(p)$.

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Requiring LPs to choose polynomial liquidity functions ℓ of degree k allows expressiveness like bucketing, but only requires k parameters to manage.

This could be especially promising in off-chain settings where the cost of inverting polynomials isn't an issue.

4. Recap & Future Directions

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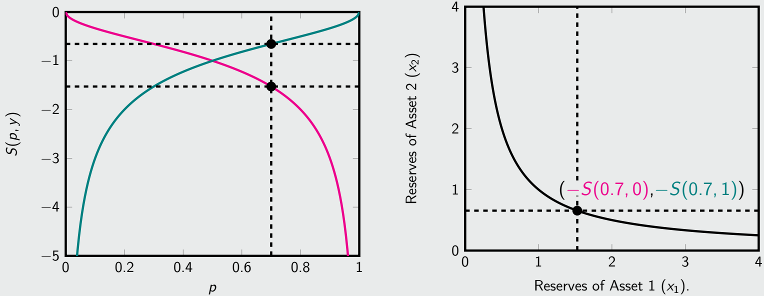
In Case of Questions

Scoring Rule & Reserve Curve Duality



Buckets From Uniswap V3 With Any Scoring Rule

In general, if an LP wanted to deposit liquidity in a certain bucket $[a_j, b_j]$ in a CFMM using a scoring rule $S(p, y)$, they'd need to deposit

	General Scoring Rule $S(p, y)$
$p < a_j$	$\begin{pmatrix} S(a_j, 1) - S(b_j, 1) \\ 0 \end{pmatrix}$
$p \in [a_j, b_j]$	$\begin{pmatrix} S(p, 1) - S(b_j, 1) \\ S(p, 0) - S(a_j, 0) \end{pmatrix}$
$p > b_j$	$\begin{pmatrix} 0 \\ S(b_j, 0) - S(a_j, 0) \end{pmatrix}$