Arrow's & May's Theorems*

Adithya Bhaskara

July 23, 2024

1 Motivation & Preliminaries

In this set of notes, we first examine Arrow's Theorem, a famous impossibility result in social choice theory. Originally stated in [1] by Kenneth J. Arrow in 1951, we have the unfortunate result that all "reasonable" voting rules, involving 3 or more alternatives, must be dictatorial. After our melancholic endeavor of proving Arrow's result, we will consider May's Theorem, which will provide slight relief, for the case of 2 alternatives.

We start with some definitions, following closely the exposition of [2].

Let $N = \{1, \ldots, n\}$ be a finite set of voters, and let A be a finite set of alternatives–or candidates. Consider the following definitions.

Definition 1 (Weak and Linear Orders). *A binary relation on a finite set A is a weak order if it is both complete and transitive. A linear order is a weak order that is additionally antisymmetric. Denote the set of weak orders $\succeq$ on A by $\mathcal{R}(A)$ and the set of linear orders $\succeq$ on A by $\mathcal{L}(A)$. Note that $\succ$ denotes the strict part of $\succeq$.*

Weak orders are used to model preferences permitting ties, and linear orders are used to model strict preferences. The preference of $i \in N$ is denoted by $\succ_i$. We now define social welfare functions, the central object of Arrow's Theorem.

Definition 2 (Social Welfare Functions (SWFs)). *A social welfare function f is a map of the form $f : \mathcal{L}(A)^n \to \mathcal{R}(A)$.*

Think of Definition 2 as a type of voting rule: taking in a profile of voters' preferences $P = (\succeq_1, \ldots, \succeq_n)$, f returns aggregates the preferences into a weak order. Arrow suggested two natural axioms that SWFs should satisfy. We state them here.

Axiom 1 (Weakly Paretian). *An SWF f is weakly Paretian if for all $x_1, x_2 \in A$, $x_1 \succ_f x_2$ whenever $x_1 \succ_i x_2$ for all $i \in N$.*

Axiom 2 (Independence of Irrelevant Alternatives (IIA)). *An SWF f is independent of irrelevant alternatives if for all $x_1, x_2 \in A$, the relative ranking of x_1 and x_2 by f depends only on the relative rankings of x_1 and x_2 provided by the individuals, and not on the individuals' rankings of some irrelevant alternative x_3.*

Intuitively, Axiom 1 specifies that if all voters rank one candidate over another, the SWF must reflect this preference. Axiom 2 ensures that SWF's ranking of two alternatives shouldn't depend on voters' preferences involving a third "irrelevant alternative." Axiom 2 can be thought of as a guard against voters' having an incentive to strategize and misreport their true preferences.

We now define the notion of a dictatorial SWF.

*Originally written to better understand basic ideas in social choice theory.

Definition 3 (Dictatorial SWFs)**.** *An SWF f is dictatorial if there exists $i^* \in N$ such that for all $x_1, x_2 \in A$, $x_1 \succ_{i^*} x_2$ implies $x_1 \succ_f x_2$. We refer to i^* as the dictator under f.*

Now, we define the central object of May's Theorem: social choice functions.

Definition 4 (Social Choice Functions (SCFs))**.** *A social choice function f is a map of the form $f : \mathcal{L}(A)^n \to 2^A \backslash \emptyset$, where 2^A is the power set of A.*

Think of Definition 4 as a type of voting rule: taking in a profile of voters' preferences $P = (\succ_1, \ldots, \succ_n)$, f returns a set of "winners." If $|f(P)| = 1$, we say f is single-valued on P. In this case, we may use the semantics of $f : \mathcal{L}(A)^n \to A$. We say f is resolute if it is single-valued for all profiles. We now state some natural axioms, just as we did for SWFs.

Axiom 3 (Anonymity)**.** *An SCF f is anonymous if each pair of voters are interchangeable. That is, $f(P) = f(P^*)$ for profiles P and P^*, whenever P^* is obtained from P by swapping the ballots cast by two voters i and j. Moreover, we say that f is dictatorial with dictator i^* if $f(P)$ corresponds to the top-ranked alternative of i^* for all profiles P.*

Axiom 4 (Neutrality)**.** *An SCF f is neutral if each pair of alternatives are interchangeable. That is, when P^* is obtained from P by swapping the positions of alternatives x_1 and x_2 in every ballot, $f(P^*)$ is obtained from P by a similar swap. Moreover, we say that f is imposed if there exists an unelectable candidate x; i.e. for no profile P does $f(P) = \{x\}$.*

Axiom 5 (Monotonicity & Positive Responsiveness)**.** *An SCF f is monotone if for a preference P, $x \in f(P)$ and for P^* obtained from P by just having one voter rank x higher in their ballot, $x \in f(P^*)$. We say f is positive responsive if $x \in f(P)$ and for P^* obtained from P by just having one voter rank x higher in their ballot, $\{x\} = f(P^*)$.*

Intuitively, Axiom 3 specifies that the SCF treats all voters equally: a ballot cast by one voter yields the same preference as the same ballot cast by another voter. Note that nondictatoriality is a very weak form of anonymity. Axiom 4 ensures that permuting alternatives' identities on the ballots yields an analogous permutation in the results. Note that nonimposition is a very weak form of neutrality. Finally, Axiom 5 requires that alternatives are not negatively affected by voters ranking them higher.

Remark. *As we will explore soon, Axiom 5 is especially insightful when there are only two alternatives; positive responsiveness, in particular, helps us grapple with ties.*

2 The Heart of Arrow's Theorem

We build up to Arrow's Theorem with some lemmas and definitions. Here, we follow the general argument of [2] with some reorganization and notational differences. For the remainder of this section, $|A| \geq 3$ unless otherwise stated.

Lemma 1 (Dictatorial SWF $\implies$ Weakly Paretian and IIA)**.** *Any dictatorial SWF f is both weakly Paretian and IIA.*

Proof. Let f be a dictatorial SWF with dictator i^*. Consider arbitrary $x_1, x_2 \in A$.

If for all $i \in N$ $x_1 \succ_i x_2$, then it must be the case that $x_1 \succ_{i^*} x_2$, so $x_1 \succ_f x_2$. Therefore, f is weakly Paretian.

Since the ordering $\succ_f$ is equivalent to the ordering $\succ_{i^*}$, it is also immediate that the ranking of x_1 and x_2 under f is equivalent to that of that of i^* and doesn't depend on the preferences that i^* has on a third alternative. ∎

Definition 5 (Coalitions)**.** *A subset $C \subseteq N$ is called a coalition. We say C is decisive over (x_1, x_2) if $x_1 \succ_f x_2$ whenever $x_1 \succ_i x_2$ for all $i \in C$. Additionally, we say C is weakly decisive over (x_1, x_2) if $x_1 \succ_f x_2$ whenever $x_1 \succ_i x_2$ for all $i \in C$ and $x_2 \succ_j x_1$ for all $j \notin C$.*

Lemma 2 (Field Expansion, Weakly Decisive $\implies$ Decisive)**.** *A weakly Paretian and IIA SWF f, with a weakly decisive coalition C over (x_1, x_2), is a decisive for all alternatives.*

Proof. Consider mutually distinct $x_1, x_2, x_1', x_2' \in A$, and let C be weakly decisive over (x_1, x_2). We will show that C is decisive over the alternatives (x_1', x_2').

Let $x_1' \succ_i x_1 \succ_i x_2 \succ_i x_2'$ for all $i \in C$, and for all $j \notin C$, let $x_1' \succ_j x_1$, $x_2 \succ_j x_2'$, and $x_2 \succ_j x_1$, Since C is weakly decisive over (x_1, x_2), we must have $x_1 \succ_f x_2$. Since f is weakly Paretian, we also have $x_1' \succ_f x_1$ and $x_2 \succ_f x_2'$. By transitivity of $\succ_f$, we get $x_1' \succ_f x_2'$. So, C is decisive over (x_1', x_2').

Because the choice of (x_1', x_2') was arbitrary, C is decisive over all pairs of alternatives. ∎

Remark. *To be very explicit, the construction of preferences in Lemma 2 is to exploit two assumptions. Since $x_1 \succ_i x_2$ for $i \in C$ and $x_2 \succ_j x_1$ for $j \notin C$, we can use the weakly decisiveness of C to conclude $x_1 \succ_f x_2$. Since $x_1' \succ_k x_1$ and $x_2 \succ_k x_2'$ for all $k \in N$, we can use the weakly Paretian nature of f to conclude $x_1' \succ_f x_1$ and $x_2 \succ_f x_2'$.*

Importantly, in our proof, we also did not need to consider how voters outside C rank x_1' versus x_2'. We obtained $x_1' \succ_f x_2'$ only from exploiting the weak decisiveness of C, the weakly Paretian SWF f, and the transitivity of $\succ_f$. Also, note that by IIA, we only needed to make sure that all $i \in C$ had $x_1' \succ_i x_2'$. If instead $x_2 \succ_i x_1$, a similar argument would hold.

Lemma 3 (Splitting)**.** *Let $C \subseteq N$ be a decisive coalition, with respect to some pair of alternatives. Additionally, let $|C| \geq 2$. Then, we can write $C = C_1 \cup C_2$ with $C_1 \neq \emptyset$, $C_2 \neq \emptyset$, and $C_1 \cap C_2 = \emptyset$, where either C_1 or C_2 is decisive over all pairs of alternatives.*

Proof. Recalling that $|A| \geq 3$, suppose $x_1 \succ_i x_2 \succ_i x_3$ for all $i \in C_1$, $x_2 \succ_j x_3 \succ_j x_1$ for all $j \in C_2$, and $x_3 \succ_k x_1 \succ_k x_2$ for all $k \notin C_1 \cup C_2$. Because C is decisive, $x_2 \succ_f x_3$. Then, either $x_1 \succ_f x_3$ or $x_3 \succeq_f x_1$.

Case 1 ($x_1 \succ_f x_3$): We see that the preferences in C_1 match those aggregated by f. Since f is IIA, whenever voters in C_1 rank x_1 above x_3, the SWF does the same. So, C_1 is weakly decisive over (x_1, x_3). But by Lemma 2, C_1 is decisive for all pairs of alternatives.

Case 2 ($x_3 \succeq_f x_1$): By transitivity, and $x_2 \succ_f x_3$, we have that $x_2 \succ_f x_1$. So, the preferences in C_2 match those aggregated by f. Since f is IIA, whenever voters in C_2 rank x_2 above x_1, the SWF does the same. So, C_2 is weakly decisive over (x_1, x_2). But by Lemma 2, C_2 is decisive for all pairs of alternatives.

We have shown the desired result. ∎

Theorem 1 (Arrow, Weakly Paretian and IIA $\iff$ Dictatorial SWF)**.** *When $|A| \geq 3$, an SWF is weakly Paretian and IIA if and only if it is dictatorial.*

Proof. By Lemma 1, we need only show that an arbitrary weakly Paretian and IIA SWF f is dictatorial. Note that N is a decisive coalition since f is weakly Paretian. Then, we can apply Lemma 3 repeatedly to obtain smaller and smaller decisive coalitions. Once we obtain a singleton decisive coalition, we are done. The element of the singleton decisive coalition is the dictator under f. Note that the inductive argument is valid since N is finite. ∎

3 May's Theorem

We state and prove May's Theorem, with relative ease compared to the previous section. We again follow [2].

Theorem 2 (May, Majority Rule is Best for 2 Alternatives)**.** *For two alternatives and an odd number of voters, majority rule is the unique resolute, anonymous, neutral, and monotone SCF.*

For two alternatives and any number of voters, it is the unique anonymous, neutral, and positively responsive SCF.

Proof. Let $A = \{x, y\}$. Trivially, majority rule satisfies all the above properties.

For uniqueness, with any other SCF, we choose a profile where x wins, but with fewer votes than y. Suppose we switch enough ballots to reverse the number of votes x and y each have. Monotonicity implies that x still wins; however, neutrality and anonymity implies that y wins. If x and y tie, meaning $\{x, y\} \in f(P)$, but x has fewer votes than y, positive responsiveness similarly contradicts neutrality and anonymity. ∎

References

[1] Arrow, K.J.: Social Choice and Individual Values. Yale University Press (1951)

[2] Brandt, F., Conitzer, V., Endriss, U., Jérôme, L., Procaccia, A.D.: Handbook of Computational Social Choice. Cambridge University Press (2016), https://procaccia.info/wp-content/uploads/2020/03/comsoc.pdf