A General Theory of Liquidity Provisioning for Automated Market Makers

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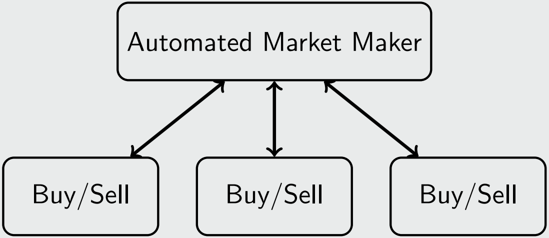
1. Motivation

Automated Market Makers



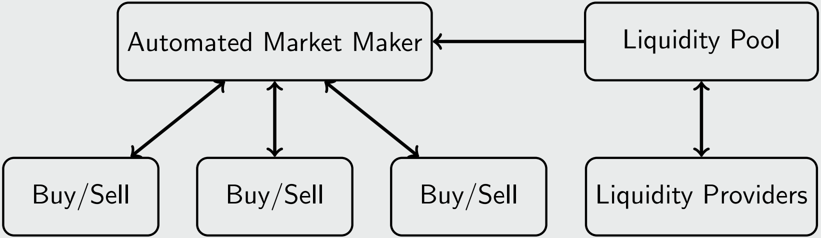
Hanson [2003] introduced automated market makers (AMMs) to solve thin market problems in prediction markets.

AMMs are always willing to offer a price for any bundle of assets.

AMMs ideally ensure that prices are stable, and we quantify the degree of price stability with the term liquidity.

Traditionally, market creators both fostered the trade of assets and bore the risk of providing liquidity to keep prices stable.

Liquidity Provisioning



Recently, AMMs have been used as a easy way to trade assets on chain in decentralized finance (DeFi), [Bartoletti et al., 2022, Xu et al., 2023, Mohan, 2022].

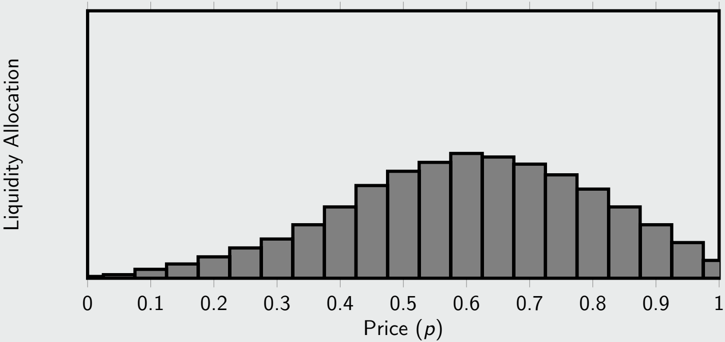
We see a decoupling of roles: the market creator may outsource liquidity provison to external liquidity providers (LPs).

LPs are compensated for taking on risk by getting a cut of fees skimmed off trades.

The Design of Liquidity Provisioning Protocols

Where does the current state of the design space of liquidity provision protocols lie?

Consider Uniswap V3, where LPs allocate liquidity by depositing assets in various discrete "buckets" that partition the price space, while traders exchange assets.



How do we implement Uniswap V3?

function ModifyLiquidity($i \in \mathbb{N}, \alpha' \geq 0, j \in \{0, \ldots, m\}$)

$p = \text{price}(\mathbf{x})$

request $\mathbf{x}' = \begin{cases} \left((\alpha' - \alpha^{ij})\left(\sqrt{\frac{1-a_j}{a_j}} - \sqrt{\frac{1-b_j}{b_j}}\right), 0\right) & \text{if } p < a_j \\ \left(0, (\alpha' - \alpha^{ij})\left(\sqrt{\frac{b_j}{1-b_j}} - \sqrt{\frac{a_j}{1-a_j}}\right)\right) & \text{if } p > b_j \\ \left((\alpha' - \alpha^{ij})\left(\sqrt{\frac{1-p}{p}} - \sqrt{\frac{1-b_j}{b_j}}\right), (\alpha' - \alpha^{ij})\left(\sqrt{\frac{p}{1-p}} - \sqrt{\frac{a_j}{1-a_j}}\right)\right) & \text{if } p \in [a_j, b_j] \end{cases}$

$(\mathbf{x}, \alpha^{ij}) \leftarrow (\mathbf{x} + \mathbf{x}', \alpha')$

function ExecuteTrade($\mathbf{r} \in \mathbb{R}^2$)

Let $p = \text{price}(\mathbf{x})$, $\quad p' = \text{price}(\mathbf{x} - \mathbf{r})$

Let l, u be such that $a_l \leq p \leq b_l$ and $a_u \leq p' \leq b_u$.

check

$$\frac{1}{(\sum_{i=0}^{k} \alpha^{il})^2}\left(x_1 + \sum_{i=0}^{k} \alpha^{il}\sqrt{\frac{1-b_l}{b_l}}\right)\left(x_2 + \sqrt{\frac{a_l}{1-a_l}}\sum_{i=0}^{k} \alpha^{il}\right) = \frac{1}{(\sum_{i=0}^{k} \alpha^{iu})^2}\left(x_1 - r_1 + \sum_{i=0}^{k} \alpha^{iu}\sqrt{\frac{1-b_u}{b_u}}\right)\left(x_2 - r_2 + \sqrt{\frac{a_u}{1-a_u}}\sum_{i=0}^{k} \alpha^{iu}\right)$$

pay $\beta \frac{\sum_j \alpha^{ij}}{\sum_j \sum_o \alpha^{oj}}(-\mathbf{r})_+$ to each LP i where j sums over buckets in $[B^l, B^u]$.

$\triangleright$ WLOG the B^u bucket comes later than the B^l bucket.

$\mathbf{x} \leftarrow \mathbf{x} - \mathbf{r}$

Well, . . . , that's a bit scary.

Expanding the Design Space of Liquidity Provisioning Protocols

Questions

1. *What is a principled way to come up with liquidity provisioning protocols like this?*
2. *How can we characterize the full design space of liquidity provisioning protocols?*

Answer (Our Framework)

Think about liquidity provisioning as LPs running their own automated market makers operating in parallel.

The Rest of This Talk

1. Our New Framework
2. Examples
3. Recap & Future Directions

2. Our New Framework

Overview of the Framework

In traditional markets, liquidity is provided by many market makers quoting prices in parallel.

This narrows the bid-ask spread, leading to price stability.

Question: What would running CFMMs $\varphi^1, \ldots, \varphi^n$ in parallel look like?
Claim: A CFMM with $\varphi = \bigwedge_i \varphi^i$, where $\bigwedge$ is the *infimal convolution* can be thought of as doing this.

Maximally Flexible LP Protocol From Our Framework

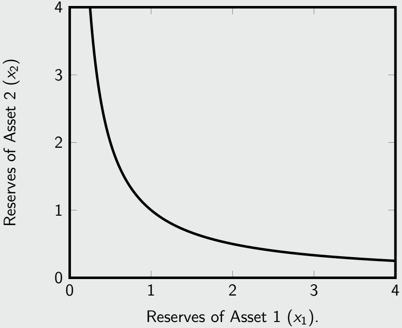
1. Each LP chooses and modifies their own CFMM φ^i arbitrarily.
2. The aggregate market maker uses $\varphi = \bigwedge_i \varphi^i$ to emulate the LPs' choices of CFMMs operating in parallel.

Thm 1. We can recover all existing protocols, like Uniswap V2 and V3.
Thm 2. The general protocol is "reasonable" for any "reasonable" φ^i.

Background: Constant Function Market Makers

A CFMM maintains a reserve vector $\boldsymbol{x} \in \mathbb{R}^n$ and a convex[1] potential function[2] $\varphi : \mathbb{R}^n \to \mathbb{R}$. The trades $\boldsymbol{r} \in \mathbb{R}^n$ available are those satisfying $\varphi(\boldsymbol{x} + \boldsymbol{r}) = \varphi(\boldsymbol{x})$. After a trade, $\boldsymbol{x} \leftarrow \boldsymbol{x} + \boldsymbol{r}$.

For example, take Uniswap V2 with $\varphi(\boldsymbol{x}) = x_1 x_2$:



[1] Conventionally, φ is concave; we use the negation as the potential.

[2] For readers familiar with the prediction market literature, [Chen and Pennock, 2007, Abernethy et al., 2013, Frongillo et al., 2023], φ and φ^i as used in our slides are really cost functions.

Main Results

Definition (Infimal Convolution)

For convex functions $f^i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$*, we define their infimal convolution* $f = \bigwedge_i f^i$ *by* $f(\boldsymbol{x}) = \inf \left\{ \sum_i f^i(\boldsymbol{x}^i) \mid \sum_i \boldsymbol{x}^i = \boldsymbol{x} \right\}$*, where the* $\boldsymbol{x}^i$ *range over* $\mathbb{R}^n$*.*

Alternatively, $\wedge$ *is the conjugate operator of* $+$*.*

Theorem 1 (Existing LP Protocols $\to$ **Aggregate CFMM)**

All existing liquidity provisioning protocols correspond to an aggregate CFMM with potential $\varphi = \bigwedge \varphi^i$ *for certain choices of* φ^i*, where* φ^i *is the CFMM that LP* i *chooses.*

Using our Framework to Create New Protocols

In existing protocols like Uniswap, LPs are constrained to a certain class of φ^i. Can we generalize this to a broader class of functions? Yes!

Theorem 2 (Aggregate CFMM $\rightarrow$ Reasonable LP Protocol)

Any choice of convex $\varphi^1, \ldots, \varphi^n$ corresponds to a reasonable LP protocol.

Maximally Flexible LP Protocol From Our Framework

1. Each LP chooses and modifies their own CFMM φ^i arbitrarily.
   AMM requests assets such that aggregate reserves are nonnegative.
2. The aggregate market maker uses $\varphi = \bigwedge_i \varphi^i$ to emulate the LPs' choices of CFMMs operating in parallel.
3. Valid trades $\boldsymbol{r}$ are those satisfying $\varphi(\boldsymbol{x} + \boldsymbol{r}) = \varphi(\boldsymbol{x})$.

Reasonable? Running CFMMs in parallel, ✓. Liquidity adds, ✓.

Sense Check 1: Running CFMMs in Parallel

Another[3], more “literal,” translation of running CFMMs in parallel:

- Let CFMM i have potential function φ^i. If traders want to make a net trade $\boldsymbol{r}$, they do so by making trades $\boldsymbol{r}^i$ with each CFMM φ^i.
- Rational traders wish to trade at the best prices offered.

Trading with the aggregate CFMM $\varphi = \bigwedge_i \varphi^i$ does exactly that!

As an added benefit, computing the optimal trade split $\boldsymbol{r}$ under the hood allows traders not needing to choose which CFMMs φ^i to trade with for best prices.

[3] 2 other reasonable implementations of what it means for CFMMs to run in parallel, mentioned in our paper, are equivalent!

Sense Check 2: Liquidity Adds!

Let $\varphi = \bigwedge_i \varphi^i$ be our convex aggregate potential. For intuition, suppose $\nabla^2 \varphi \succ \mathbf{0}$.

Intuitively, liquidity is a measure of how insensitive prices are while trades occur. At a price $\boldsymbol{p} = \nabla \varphi(\boldsymbol{x}) \in \Delta_n$, we can define $\ell(\boldsymbol{p}) = \left(\nabla^2 \varphi\right)^{-1}(\boldsymbol{x})$.

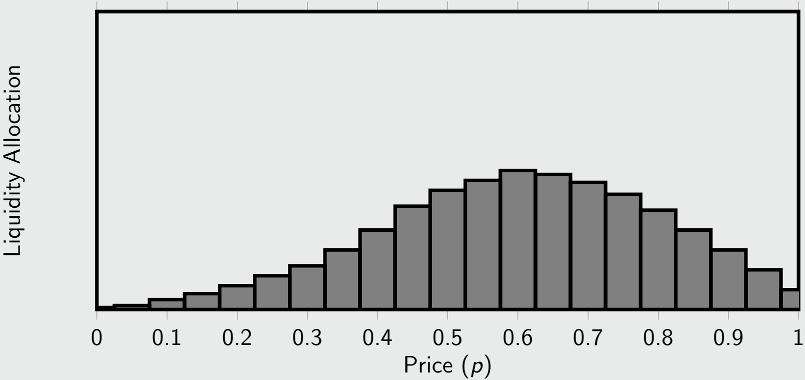
Because ℓ is positive definite, we can say $\ell = \nabla^2 \psi$ for some convex $\psi : \Delta_n \to \mathbb{R}$. This is a special case of convex conjugate duality; $\psi = \varphi^*$.

By results in convex analysis, $\varphi^* = \left(\bigwedge_i \varphi^i\right)^* = \sum_i (\varphi^i)^*$.

So, with $\ell = (\nabla^2 \varphi)^{-1} = \nabla^2 \varphi^*$, the total liquidity is the *sum* of the liquidity from each LP: $\ell = \sum_i \ell^i$.

3. Examples

Remember Uniswap V3's Buckets?



Generalizing the Idea of Buckets From Uniswap V3

Let's say we wanted to use the bucketing mechanism from Uniswap, but with any choice of reserve curve. What's stopping us?

If an LP wanted to deposit liquidity in a certain bucket $[a_j, b_j]$, how much of each asset would they have to provide if the current price was p?

	Uniswap V3	LMSR	Brier
$p \in [a_j, b_j]$	$\begin{pmatrix} \sqrt{\frac{1-b_j}{b_j}} - \sqrt{\frac{1-p}{p}} \\ \sqrt{\frac{a_j}{1-a_j}} - \sqrt{\frac{p}{1-p}} \end{pmatrix}$	$\begin{pmatrix} \log \frac{p}{b_j} \\ \log \frac{1-p}{1-a_j} \end{pmatrix}$	$\begin{pmatrix} (1-b_j)^2 - (1-p)^2 \\ a_j^2 - p^2 \end{pmatrix}$

How did we get this? Some technical details[4] involving the duality between reserve curves and scoring rules.

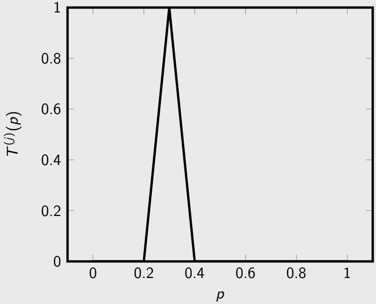
We can do the same for an arbitrary reserve curve!

[4] See Abernethy et al. [2013], Frongillo et al. [2023].

Triangular Liquidity

Bucketing requires LPs to discretize their liquidity allocations. For more expressiveness, let's try "soft" buckets that "fade" in and out around a target price a_j.

Consider a triangular function, $T^{(j)}(p)$.



We can "weight" liquidity functions over "soft" bucket j with $\ell^{(j)}(p) = (\ell T^{(j)})(p)$.

Polynomial Liquidity

Bucketing also requires us to keep track of several parameters.

Operations involving polynomials are efficient, and polynomials can approximate most functions.

Requiring LPs to choose polynomial liquidity functions ℓ of degree k allows expressiveness like bucketing, but only requires k parameters to manage.

This could be especially promising in off-chain settings where the cost of inverting polynomials isn't an issue.

4. Recap & Future Directions

1. Recap: Thinking about LPs running CFMMs operating in parallel gives us a nice way to think about liquidity provisioning.
2. Recap: Within this framework, we can both understand existing protocols more deeply, and we can capture all conceivable protocols.
3. Future: Wrestle with hiccups that arise from more than 2 assets; issues with directional liquidity, [Grugett, 2023], and trading fees may not be budget balanced.
4. Future: Use this framework to create new protocols with considerations like Just-in-Time behavior, impermanent loss, and other desiderata in mind. We'd love to hear your suggestions!

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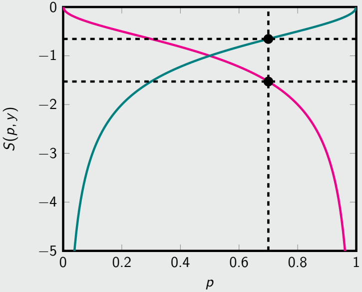
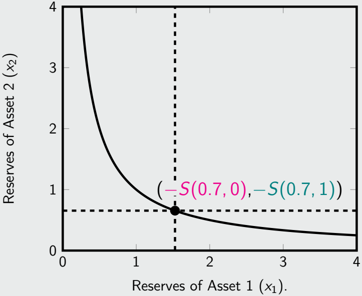
In Case of Questions

Scoring Rule & Reserve Curve Duality





Buckets From Uniswap V3 With Any Scoring Rule

In general, if an LP wanted to deposit liquidity in a certain bucket $[a_j, b_j]$ in a CFMM using a scoring rule $S(p, y)$, they'd need to deposit

	General Scoring Rule $S(p, y)$
$p < a_j$	$\begin{pmatrix} S(a_j, 1) - S(b_j, 1) \\ 0 \end{pmatrix}$
$p \in [a_j, b_j]$	$\begin{pmatrix} S(p, 1) - S(b_j, 1) \\ S(p, 0) - S(a_j, 0) \end{pmatrix}$
$p > b_j$	$\begin{pmatrix} 0 \\ S(b_j, 0) - S(a_j, 0) \end{pmatrix}$