# Bayesian Inference

Oliver Filla

November 20, 2024

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$\Rightarrow$ Thus, we need probabilities to predict events!

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$p(6) = \frac{1}{6} \Rightarrow$ *I expect that every 6th dice roll shows a 6, on average.*

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- $C(H)$ can be updated with new data

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$p(6) = \frac{1}{6} \Rightarrow$ *I believe the hypothesis H "the die roll shows a 6" with a credibility of* $C(H) = \frac{1}{6}$.

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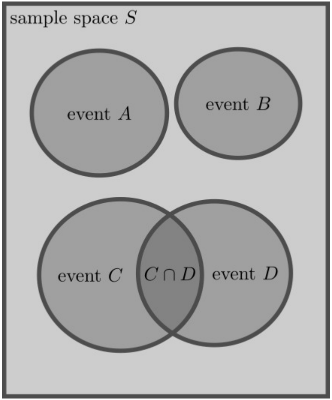

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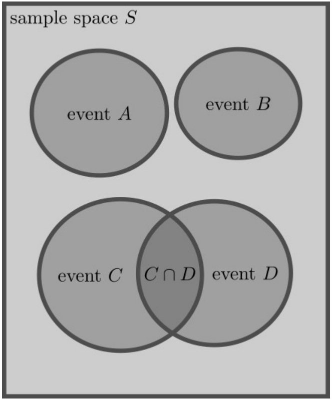
$\Rightarrow$ $A$ and $B$ mutually exclusive: $P(A \cup B) = P(A) + P(B)$



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- equal for given hypothesis $H$ and events $E$
- ▶ but only if no prior knowledge is available
- ▶ Bayes' theorem generalizes this, using prior knowledge

$$L(H|E) \stackrel{?}{=} P(E|H) \tag{1}$$

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$$P(E) = P(E|H)P(H) + P(E|\neg H)P(\neg H) \tag{4}$$

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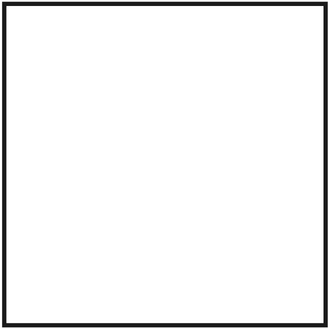
- square: probability 1



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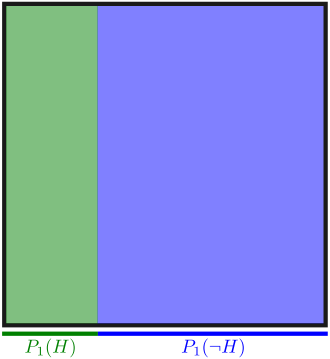


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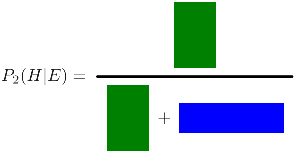


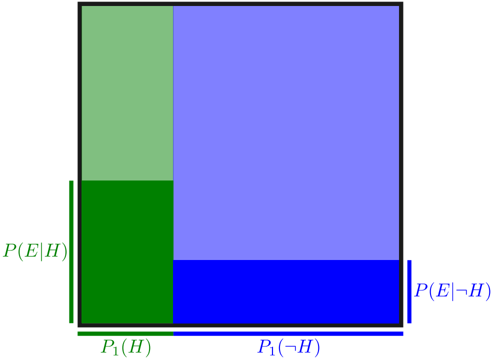


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4. calculate posterior probability $P_2(H|E)$

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- overall likelihood $P(E)$ becomes irrelevant

$$\frac{P_2(H_A|E)}{P_2(H_B|E)} = \frac{P(E|H_A)}{P(E|H_B)} \frac{P_1(H_A)}{P_1(H_B)} \tag{5}$$

# Example

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My rapid test is positive: How likely do I have COVID?

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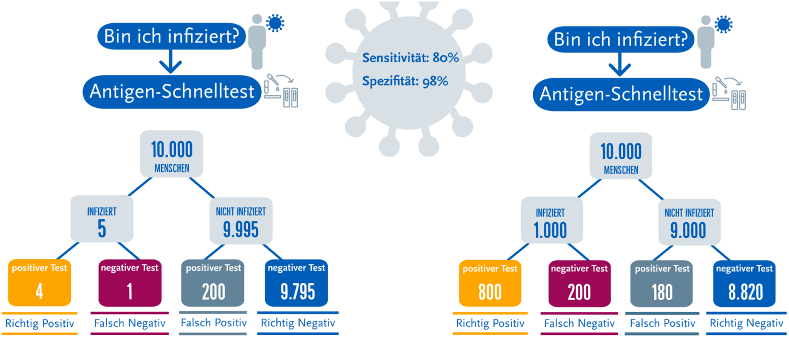


Source: Robert Koch Institute

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# Thank you for your attention!