Contents

Chapter 1

Preliminaries

1.1 sigma-algebra, Probability space, and Random Variables

From [3]. If Ω is a given set, a **σ-algebra** $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties

1. $\emptyset \in \mathcal{F}$
2. for a set F, if $F \in \mathcal{F}$ then $F^C \in \mathcal{F}$, for $F^C = \Omega \backslash \mathcal{F}$
3. for sets $A_1, A_2, A_3, ... \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$

Why is it important to have a σ-algebra in a probability space? The σ-algebra represents all events that we can assign probabilities to. Property 1 together with property 2 states that in the space the probability of all events occurring, should be measurable and equal 1, whereas it's complement, no events occurred should be measurable and assigned probability 0. For other events, the probability of an event occurring and its complement should both be measurable. Property 3 assigns some "continuity" in the space of events and allows us to "group" elementary events without risking the creation of an event which is not included in the event space.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. A probability measure P is a function $P : \mathcal{F} \to [0, 1]$ which assigns each elementary event $A \in \mathcal{F}$ with a number representing its likelihood.

The triplet $(\Omega, \mathcal{F}, P)$ is a **probability space**. All probability spaces can be made complete, hence we refer to them as complete from now on.

A function $Y : \Omega \to \mathbb{R}^n$ is said to be $\mathcal{F}$-measurable if

$$Y^{-1}(U) := \{\omega \in \Omega; Y(\omega) \in \mathcal{F}\}$$

For all open sets $U \in \mathbb{R}^n$. We call ω an elementary event. It is, in fact, a continuous trajectory in the space of continuous functions (but non-differentiable).

A **random variable** X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$. Every random variable induces a probability measure μ_X on $\mathbb{R}^n$, defined by

$$\mu_X(B) = P(X^{-1}(B))$$

The function

$$(\omega, t) \to X(\omega, t)$$

represent a two-dimensional process, where fixing t gives us all the values of the random process X at a particular point in time (assuming t represents time), whereas fixing ω gives us one trajectory over time (one realization of the process).The process X is referred to as a **stochastic process**.

1.2 Time Correlation Function and Response Function

Taken from Doi & Edwards. One characterization of a Brownian path is the time correlation function. This function is defined as follows, for a quantity A of a system of Brownian particles for many samples in the *equilibrium* state. Let $A(t)$ be the measured values of A at time t. The time correlation function $C_{AA}(t)$ is the average of $A(t)A(0)$

$$C_{AA}(t) = \langle A(t)A(0)\rangle$$

Averaging is performed over samples (not time). $C_{AA}(t)$ usually decreases with time since $A(t)$ and $A(0)$ become uncorrelated with time, and becomes $\langle A(t)\rangle \langle A(0)\rangle$. the time in which these quantities become uncorrelated is called the *correlation time*.

The time correlation is also defined for different quantities

$$C_{AB}(t) = \langle A(t)B(0)\rangle$$

which is the cross correlation function.

1.3 Noise

From [3]. While trying to incorporate noise into a differential equation representing a dynamical system, one needs to define the characteristics of the noise in the system. For an equation of the form

$$dX = a(t)dt + b(t) \cdot noise$$

It is reasonable to look for some stochastic process W_t to represent the noise term. The noise term W_t has to have the following characteristics

1. $t_1 \neq t_2 \Rightarrow W_{t_1} \neq W_{t_2}$
2. W_t is stationary, i.e. the joint distribution of $W_{t1+t}, ...W_{t_k+t}$ does not depend on t.
3. the expectation $E[W_t] = 0$ for all t

One other requirement might be $E[W_t^2] = t$. It turns out that there is no "reasonable" process satisfying the first two requirements. However, it is possible to describe W_t as a generalized stochastic process called white noise process. This means that it can be constructed as a measure on the space of distribution in $[0, \infty)$.

From the requirement that $E[W_t] = 0, \quad E[W_t^2] = t$, we get the first and second moment of the generating distribution function for the process. This means that the process must be normally distributed.

White noise is a process for which there is no correlation between any values at different times. As an idealization of such a process there exist the **Orenstein-Uhlenbeck process** (to be defined) for which produces an almost uncorrelated noise. For this model, the second order correlation function can, up to a constant factor, be described as

$$\langle x(t), x(t') \rangle = \frac{\gamma}{2} \exp(-\gamma |t - t'|)$$

1.4 The Space of Elementary Events

From [4]. The space of elementary events for a Brownian motion is the space of all continuous real functions

$$\Omega = \{\omega(t) : \mathbb{R}_+ \to \mathbb{R}\}$$

1.5 Cylinder Sets

From [4].A cylinder set of Brownian trajectories is defined by a sequence of times $0 \leq t_1 < t_2 < ... < t_n$, and real intervals $I_k = (a_k, b_k)$, $k = 1...n$, as

$$C(t_1, t_2, ..., t_n; I_1, I_2, ...I_n) = \{\omega \in \Omega | x(\omega, t_k) \in I_k\}, \forall 1 \leq k \leq n$$

For a cylinder not to contain a trajectory, it is enough that in one time point the trajectory is not in the interval. The cylinder C contains entire trajectory, which fulfill the inclusion demand.

The joint PDF of $w(t_1, \omega), w(t_2, \omega), w(t_3, \omega)$ is the Weiner measure of the cylinder $C(t_1, t_2, t_3; I^x, I^y, I^z)$. These points belong to the same process ω at three different times.

1.6 Filtration

The σ-algebra $\mathcal{F}_t$ of Brownian event is defined by cylinder sets confined to times $0 \leq t_i < t$, for some fixed t. Obviously, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ if $0 \leq s < t < \infty$. The family of σ algebras $\mathcal{F}_t$ for $t \geq 0$ is called the Brownian filtration and is said to be generated by the Brownian events up to time t

In simple words, filtration is an increasing sequence of σ-algebras on a measurable space.

1.7 Adapted Process

The process $x(t, \omega)$ is said to be adapted to the Brownian filtration $\mathcal{F}_t$ if $\{\omega \in \Omega | x(t, \omega) \leq y\} \in F_t$ for every $t \geq 0$ and $y \in R$. In that case we also say that $x(t, \omega)$ is $\mathcal{F}_t$-measurable.

Thus an adapted process does not depend on the future behavior of the Brownian trajectory from time t on.

1.8 The Weiner Measure

The probability measure Pr defined on all the Brownian trajectories (events in the set Ω) are defined on cylinder sets and then extended to all events in the space by elementary properties of probability sets.

For a cylinder set $C(t;I)$, where $t>0$ and $I=(a,b)$, the Weiner measure is

$$Pr\{C(t;I)\}=\frac{1}{\sqrt{2\pi t}}\int_a^b \exp^{-x^2/2t}\,dx$$

The d-dimensional Weiner measure of a cylinder is defined as

$$Pr\{C(t_1,t_2,...,t_k;I_1,I_2,...,I_k)\}=\int_{I_1,..,I_k}\prod_{j=1}^{k}\frac{\exp\left(-\frac{|x_j-x_{j-1}|^2}{2(t_j-t_{j-1}))}\right)}{[2\pi(t_j-t_{j-1})]^{n/2}}dx_j$$

The PDF of the d-dimensional mathematical Brownian motion is defined in terms of the probability density of the cylinder sets

$$F_w(x,t)=Pr\{w\in\Omega|w(t,w)\leq x\}=\frac{1}{(2\pi t)^{n/2}}\int_{-\infty}^{x_1,x_2,..,x_d} e^{-|y|^2/2t}dy_1dy_2,..dy_d$$

1.9 Martingale

A martingale is a stochastic process $x(t)$ with $\mathbb{E}|x(t)|<\infty$,$\forall t$, and for $t_1<t_2<...<t_n$

$$\mathbb{E}[x(t)|x(t_1)=x1,x(t_2)=x_2,...x(t_n)=x_n]=x_n$$

Chapter 2

Brownian Motion

2.1 Equivalence transformation

Several transformations allows us to calculate the statistical properties of one path in terms of the other. For every $c > 0$

1. $w(t) = \sqrt{c}w(t/c)$
2. $w(t) = tw(1/t)$

2.2 Recurrence of Brownian motion

In 1D, the law of iterated logarithms shows that every point is visited infinitely many times as $t \to \infty$. For any dimension bigger than 1 there is no point recurrence at all, and only neighborhood recurrence is seen.

2.3 Stochastic Integration

For an arbitrary function of time $G(t')$ and a Weiner process $W(t)$, the stochastic integral

$$\int_{t_0}^{t} G(t')\mathrm{d}W(t')$$

is defined as a kind of Riemann integral, that is, we divide the interval $[t_0, t]$ into n subintervals $t_0 \leq t_1 \leq ... \leq t$, and define intermediate points

$t_{i-1} \leq \tau_i \leq t_i$. The integral is then defined as a limit of the partial sums

$$S_n = \sum_{i=1}^{n} G(\tau_i)[W(t_i) - W(t_{i-1})]$$

Note that the Weiner process in the partial sums is contributing the values at the edges of the interval, whereas the function G contributes the values somewhere in its middle. The choice of τ_i affects the value of the integral. It can be seen that f we choose $\tau_i = \alpha t_{i-1} + (1-\alpha)t_i$, with $0 \leq \alpha \leq 1$, than the mean value of the integral can take any value between 0 and $(t - t_0)$.

for the **Itô integral**, we choose $\tau_i = t_{i-1}$, which is the left boundary of the subinterval and thus define the stochastic integral as

$$\int_{t_0}^{t} G(t')\mathrm{d}W(t') = ms - lim_{n\to\infty}\left\{\sum_{i=1}^{n} G(t_{i-1})[W(t_i) - W(t_{i-1})]\right\}$$

by $ms - lim$, we mean the mean-square limit, which is defined by

$$ms - lim_{n\to\infty}\int p(\omega)[X_n(\omega) - X(\omega)]^2 dw = lim_{n\to\infty} < (X_n - X)^2 >$$

If indeed $lim_{n\to\infty} < (X_n - X)^2 >$ we say that $ms - lim_{n\to\infty} X_n = X$.
One major consequence of the choice $\tau_i = t_{i-1}$ is that

$$\int_{t_0}^{t} W(t')dW(t') = 0$$

That is, the stochastic integral of the Weiner process over any interval is zero. Because the left point of the subinterval is chosen, the value of the function $G(t'_{i-1})$ is independent of the increments $[W(t_i) - W(t_{i-1})]$, since G is F_t adapted process.

2.4 Properties of Itô integral

We define the class $H_2[0, T]$ of F_t-adapted stochastic processes such that

$$\int_0^T \mathbb{E}f^2(s, \omega)ds < \infty$$

and list the following properties

1. **linearity**. For $f(t), g(t) \in H_2[0,T]$, and α, β real, then $\alpha f(t) + \beta g(t) \in H_2[0,T]$, and
$$\int_0^t \alpha f(s) + \beta g(s) d\omega(s) = \alpha \int_0^t f(s) d\omega(s) + \beta \int_0^t g(s) d\omega(s)$$
2. **additivity**. If $f(t) \in H_2[0,T_1]$ and $f(t) \in H_2[T_1,T]$ for $0 < T_1 < T$, then
$$\int_0^T f(s) d\omega(s) = \int_0^{T_1} f(s) d\omega(s) + \int_{T_1}^T f(s) d\omega(s)$$
3. for a deterministic and integrable $f(t)$
$$\int_0^t f(s) d\omega(s) \sim N\left(0, \int_0^t f^2(s) ds\right)$$
4. for $f(t) \in H_2[0,T]$
$$\mathbb{E} \int_0^t f(s) d\omega(s) = 0$$
5. for $0 < \tau < t < T$
$$\mathbb{E}\left[\int_0^t f(s) d\omega(s) | \int_0^\tau f(s) d\omega(s) = x\right] = x$$
6. for $f(t), g(t) \in H_2[0,T]$
$$\mathbb{E}\left[\int_0^T f(s) d\omega(s) \int_0^T g(s) d\omega(s)\right] = \int_0^T \mathbb{E}[f(s)g(s)] ds$$
note that the right-hand-side is integrated with respect to s.

We can see by property 5 that the Itô integral is a martingale (see subsection 1.9).

2.5 Integration with Respect to a Measure

A measure μ_A defines an integral of a nonnegative measurable function $f(\omega)$ by
$$\int_\Omega f(\omega) d\mu(\omega) = \lim_{h \to 0} \lim_{N \to \infty} \sum_{n=0}^N nh\mu\{\omega : nh \leq f(\omega) \leq (n+1)h\}$$
If the limit exists we say the $f(\omega)$ is a measurable function.

2.6 The Itô Calculus

From [4] p.82. Consider two processes $a(t), b(t)$ of class $H_2[0,T]$ (defined in 2.4) and define the stochastic process

$$x(t) = x_0 + \int_0^t a(s)ds + \int_0^t b(s)d\omega(s)$$

where x_0 is a random variable independent on $\omega(t), \forall t > 0$, then for $0 \leq t_1 \leq t_2 \leq T$

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} a(s)ds + \int_{t_1}^{t_2} b(s)d\omega(s)$$

We abbreviate the this notation by

$$dx = a(t)dt + b(t)d\omega(t)$$

Example 1: We calculate the differential of $\omega^2(t)$. By the definition of the Ito integral we know that

$$\omega^2(t_2) - \omega^2(t_1) = (t_2 - t_1) + 2\int_{t_1}^{t_2} \omega(s)d\omega(s) = \int_{t_1}^{t_2} 1dt + 2\int_{t_1}^{t_2} \omega(s)d\omega(s)$$

. In our abbreviation we can write

$$d\omega^2(t) = 1dt + 2\omega(t)d\omega(t)$$

with $a(t) = 1$ and $b(t) = 2\omega(t)$. So the Ito calculus does not obey the normal chain rule of calculus. ∂R **Example 2:** We now calculate the differential of $f(t)\omega(t)$. If $f(t)$ is a deterministic smooth function then we can integrate by parts

$$\int_{t_1}^{t_2} f(t)d\omega(t) = f(t_2)d\omega(t_2) - f(t_1)d\omega(t_1) - \int_{t_1}^{t_2} f'(t)\omega(t)dt$$

we now set $x(t) = f(t)\omega(t)$ so

$$dx(t) = f'(t)d\omega(t)dt + f(t)d\omega(t) = \omega(t)df(t) + f(t)d\omega(t)$$

as in the classical calculus.

Example 3: the differential of the product of functions. If $x_1(t)$ and $x_2(t)$ have the Itô differentials

$$dx_1 dt = a_1(t)dt + b_1(t)d\omega(t)$$

$$dx_2(t) = a_2(t)dt + b_2(t)d\omega(t)$$

where $a_1, b_1, a_2, b_2 \in H_2[0,T]$ (see 2.4), then

$$d[x_1(t)x_2(t)] = x_1(t)dx_2(t) + x_2(t)dx_1(t) + b_1(t)b_2(t)dt$$

2.6.1 The Wong-Zakai Correction

From [4]. To see the relationship between the Ito integral and the Stratonovich integral we have the Wong-Zakai correction. If $f(x,t)$ has a second continuous derivative such that $|f_{xx}(x,t)| < A(t)e^{\alpha(t)|x|}$ for some positive continuous function $\alpha(t)$ and $A(t)$ for all $a \leq t \leq b$ then

$$\int_a^b f(\omega(t),t)d_S\omega(t) = \int_a^b f(\omega(t),t)d\omega(t) + \frac{1}{2}\int_a^b \frac{\partial}{\partial x}f(\omega(t),t)dt \qquad (2.1)$$

where $\omega_S(t)$ represents the Stratonovich integral.

Example 1: We now use the Wong-Zakai correction to show that the Stratonovich differential satisfies the classical rule

$$d_S x_1(t)x_2(t) = x_1(t)dx_2(t) + x_2(t)dx_1(t)$$

The Statonovich differential is defined as

$$d_S x(t) = a(t)dt + b(t)d_S\omega(t)$$

[Unfinished]

2.6.2 The Itô Equation

Equation of the form

$$\mathrm{d}X_t = \mu(X_t,t)\,\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}B_t$$

is an informal way of expressing the more appropriate integral equation

$$X_{t+s} - X_t = \int_t^{t+s} \mu(X_u,u)\mathrm{d}u + \int_t^{t+s} \sigma(X_u,u)\,\mathrm{d}B_u$$

where the dynamic of the particle X is given as a sum of two integrals, the first one is a standard Lebesgue integral (see 3.0.1), and the second is an Itô integral (2.3). The functions $\mu(X_t, t)$ and $\sigma(X_t, t)$ are well defined continuous functions, and B_t is a Brownian motion (a Weiner process, or white noise). An informal way of interpreting the equation above is to say that at each small time interval of size δ the process X_t is changed by a normally distributed value with expectation $\mu(X_t, t)\delta$ and variance $\sigma(X_t, t)^2\delta$. The function $\mu(X_t, t)$ is referred to a s the drift coefficient, and the function $\sigma(X_t, t)$ as the diffusion coefficient.

2.6.3 Itô's formula

The chain rule of differentiation of composite function does not apply for stochastic processes in the regular sense. Consider n Itô-differentiable processes

$$dx^i = a^i dt + \sum_{j=1}^{m} b^{ij} dw^j, \quad i = 1, 2, .., n.$$

and define the function $f(x1, x2, ..., x_n)$, with $x = (x_1(t), x_2(t), ..., x_n(t)$, if $a^i(t)$ and $b^{ij}(t)$ are continuous functions, such that $f(x, t)$ is twice continuously differentiable and $|f_{xx}(x, t)| \leq A(t) \exp(-\alpha(t)|x|)$ for $A(t), \alpha(t) > 0$ then

$$df(x,t) = \left[\frac{\partial f(x,t)}{\partial t} + L_x^* f(x,t)\right] dt + \sum_i^n \sum_j^m b^{ij}(t) \frac{\partial f(x,t)}{\partial x^i} dw^j$$

where

$$L *_x f(x,t) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma^{ij}(t) \frac{\partial^2 f(x,t)}{\partial x^i \partial x^j} + \sum_{i=1}^{n} a^i(t) \frac{\partial f(x,t)}{\partial x^i}$$

with

$$\sigma^{ij}(t) = \frac{1}{2} \sum_{k=1}^{m} b^{ik}(t) b^{jk}(t)$$

The function $n \times n$ matrix $\{\sigma^{ij}(t)\}$ is the diffusion matrix. The operator $L *_x (f(x(t), t))$ is called the *backward Kolmogorov operator* and comes out naturally in the derivation of the Itô's formula. This operator will be discussed next.

2.7 Stochastic Differential Equations

Taken from [4]. Throughout this section $x_{x,s}(t)$ with $t > s$ will denote the solution of the Itô system

$$dx(t) = a(x(t), t)dt + B(X(t), t)dwt, \quad x(s) = x \tag{2.2}$$

where $a(x(t), t) : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d, B(x, t) : \mathbb{R}^d \times [0, T] \to \mathbb{M}_{n,m}$, and $w(t)$ is m-dimensional Brownian motion. We assume that $a(x(t), t), B(x, t)$ satisfy the conditions of existence and uniqueness theorem 2.10. To repeat these conditions using the current symbol system: $a(x(t), t), B(x, t)$ are measurable functions, for which there exist constants C and D such that

$$|a(x, t)| + |B(x, t)| \leq C(1 + |x|)$$
$$|a(x, t) - a(y, t)| + |B(x, t) - B(y, t)| \leq D|x - y|$$

We further use the notion

$$B(t) = \{b^{ij}(t)\}_{n\times m}$$

for the diffusion matrix.

Consider the Itô system and the corresponding *backward Kolmogorov operator*

$$L^*_x u(x, s) = a(x, s) \cdot \nabla u(x, s) + \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma^{ij}(x, s) \frac{\partial^2 u(x, s)}{\partial x^i x^j} \tag{2.3}$$

We further assume that the function $u(x, s, t)$ satisfies the backward parabolic equation

$$\frac{\partial u(x, s, t)}{\partial s} + L^*_x u(x, s, t) = 0, \quad t > 0 \tag{2.4}$$

with the terminal value

$$\lim_{s\to t} u(x, s, t) = f(x)$$

for sufficiently regular function $f(x)$.

According to the **Kolmogorov's representation formula** the solution of the terminal value problem 2.4, has the representation

$$u(x, s, t) = \mathbb{E}[f(x(t))|x(s) = x]$$

Note again that $u(x, s, t)$ is a solution to equation 2.4 and not the Itô system. It will be shown that according to the representation formula, we can associate this solution with modes of the distributions of the Itô solution

2.8 The backward Kolmogorov equation

The backward Kolmogorov equation is a partial differential operator derived from continuous-time continuous-state Markov processes. If we have information about the state of a system x at a terminal time s, which is the probability distribution $p_s(x)$, we would like to know the probability distribution of the system at previous time $t < s$. If the system state $x(t)$ evolves according to the Itô equation

$$dx = a(x,t)dt + B(x,t)dw$$

then the Kolmogorov backward equation is given by

$$-\frac{\partial}{\partial t}p(x,t) = a(x,t)\frac{\partial}{\partial t}p(x,t) + \frac{1}{2}B^2(x,t)\frac{\partial^2}{\partial x^2}p(x,t) \tag{2.5}$$

for $t < s$, and subject to the final condition $p(x,s) = u_s(x)$, with $u_s(x)$ being the indicator function on the set of states x at time s (being 1 if the path belongs to x at time s and 0 otherwise).

2.9 Andronov-Vitt-Pontryagin equation

The transition probability density function $p(y,t|x,s)$ of being at position y at time t given that the process where at x at time s is the solution of the backward Kolmogorov differential equation in a bounded domain $D \subset \mathbb{R}^d$ with regular boundary ∂D

$$\frac{\partial p(y,t|x,s)}{\partial t} + L_x^* p(y,t|x,s) = 0$$

for $x, y \in D$, $s < t$ and boundary condition

$$p(y,t|x,s) = 0$$

for $x \in \partial D$, $y \in D$, s, t, and terminal condition

$$p(y,t|x,s) = \delta(y-x)$$

for $x, y \in D$, $t \in \mathbb{R}$. the value $p(y,t|x,s)dy = P\{x(s) \in y + dy | x(t) = x\}$ is the transition probability of the Itô process in D with trajectories terminated at ∂D given that they have started at a point inside D

2.10 Existence and Uniqueness of Itô SDE solutions

For an Ito SDE taking values in n-dimensional Euclidean space, if for $T > 0$

$$\begin{aligned} \mu: &\ \mathbb{R}^2 \times [0,T] \to \mathbb{R}^n \\ \sigma: &\ \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m} \end{aligned}$$

are measurable functions, for which there exist constants C and D such that

$$\begin{aligned} &|\mu(x,t)| + |\sigma(x,t)| \le C(1+|x|) \\ &|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le D|x-y| \end{aligned}$$

for all $t \in [0,T]$ and all $x, y \in \mathbb{R}^n$, and $|\sigma|^2 = \sum_{i,j=1} |\sigma_{i,j}|^2$. If Z is a random variable, independent of the σ-algebra generated by B_s, $s \ge 0$, and with finite second moment, then the SDE, with initial condition $X_0 = Z$ has an almost surely unique solution in $t \in [0,T]$, $X_t(\omega)$, such that X is adapted to the filtration $\mathcal{F}_t$ generated by Z and B_s, $s \le t$.

2.11 The Langevin Equation

[Unfinished] An equation of the type

$$\frac{dx}{dt} = a(x,t) + b(x,t)\xi(t)$$

2.12 Non-anticipating Functions

[Unfinished]

2.13 The Fokker-Planck equation

For an Îto process given by the stochastic equation,

$$dX_t = \mu(X_t,t)dt + \sqrt{2D(X_t,t)}dW_t$$

with drift $\mu(X_t, t)$ and diffusion $D(X_t, t)$, and a Weiner process W_t, the Fokker Planck-equation for the probability density $f(x, t)$ of X_t is

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x}[\mu(x,t) f(x,t)] + \frac{\partial^2}{\partial x^2}[D(x,t) f(x,t)] = \mathcal{L} f(x,t)$$

2.13.1 Eigenvalues of the Fokker-Planck operator

To find the eigenvalues of the Fokker Planck operator, we solve $\mathcal{L}f = \lambda f$. rearranging the terms in this equation we arrive at

$$\frac{\partial^2}{\partial x^2} f(x,t) + p(x,t) \frac{\partial}{\partial x} f(x,t) + q(x,t) f(x,t) = 0$$

where

$$p(x,t) = \frac{2\frac{\partial}{\partial x} D(x,t) - \mu(x,t)}{D(x,t)}$$

$$q(x,t) = \frac{\frac{\partial^2}{\partial x^2} D(x,t) - \lambda - \frac{\partial}{\partial x}\mu(x,t)}{D(x,t)}$$

which is a second order, homogeneous, ordinary differential equation with variable coefficients.

2.14 First passage problems

We first consider the discrete Markov process, described by the "forward" equation

$$\frac{\partial P_{ki}}{\partial t} = M_{kj} P_{ji}$$

where P_{kj} is the probability that the system is in state k at time t given that it started at state i at $t = 0$. The absorbing states, $\mathcal{A}$, of the system gives transition rates of zero in the transition matrix M. The *survival probability*, defined as $S_i(t) = \sum_{k \notin \mathcal{A}} P_{ki}(t)$ will vanish as $t \to \infty$.

The *first passage time distribution* can be derived from $S_i(t)$. For this end, it is convenient to consider the adjoint equation that is also obeyed by $P_{ki}(t)$. If the transition matrix M_{kj} is time-independent, then the "backward" equation:

$$\frac{\partial P_{ki}}{\partial t} = P_{kj} M_{ji}$$

This equation does not operate on the final state k, so we can calculate the sum $S_i(t) = \sum_{k \notin \mathcal{A}} P_{ki}(t)$ to find the survival probability

$$\frac{dS_i}{dt} = S_j(t) M_{ji} = -J_i(t)$$

The boundary conditions $S_k(0) = 1$, for $k \notin \mathcal{A}$ and $S_k(t) = 0$ for $k \in \mathcal{A}$. The survival probability defines the probability that the system has not reached any absorbing configuration up to time t, given that it started at configuration i at $t = 0$. J is the time dependent probability flux into the absorbing states, given that the system started at configuration i.

The *lifetime distribution function* is the sum over all absorbing states: $F_i(t) := \sum_{k \in \mathcal{A}} P_{ki}(t)$. From the lifetime distribution $F_i(t)$ one can find the probability that the system reached any absorbing configuration between time t and $t + dt$, as $F_i(t + dt) - F_i(t) = S_i(t) - S_i(t + dt)$. Therefore the first passage distribution can be found from

$$w_i(t)dt = \frac{dF_i(t)}{dt}dt = -\frac{dS_i(t)}{dt}dt$$

All n moments of the first passage time are

$$\langle T^n \rangle = \int_0^\infty w_i(t) t^n dt$$

for which, the first moment, $n = 1$, gives the mean first passage time

$$\langle T \rangle = \int_0^\infty S_i(t) dt$$

For the continuous representation we now define $P(y_j, t|x_j, 0)$ as the probability that all particles j are located between y_j and $y_j + dy_j$ at time t given that they were at position x_j at time $t = 0$. Expanding the probability flux equation as a Taylor series, we reach the Fokker-Planck equation

$$\frac{\partial P(y_j, t|x_j, 0)}{\partial t} = \sum_{k=1}^{N} \nabla_k \cdot (V_k P) + \sum_{k=1}^{N} \nabla_k^2 (D(y_k) P) = \mathcal{L} P(y_j, t|x_j, 0)$$

were ∇_k is the gradient with respect to the k^{th} particle, N is the total number of particles, and $\mathcal{L}$ is the Fokker Planck operator. The density $P(y_j, t|x_j, 0)$ obeys the backward Kolmogorov equation

$$\partial_t P(y_j, t|x_j, 0) = \mathcal{L}^\dagger P(y_j, t|x_j, 0)$$

with $\mathcal{L}^\dagger = \sum_k^N V_k \cdot \nabla_k + \sum_k^N D(x_j)\nabla_k^2$ is the operator adjoint to $\mathcal{L}$.

$\mathcal{L}^\dagger$ operates on the initial position x_j, so we can integrate over the y_j within the domain, excluding the absorbing surface and get the survival probability

$$\partial_t S(x_j;t) = \mathcal{L}^\dagger(S(x_j;t))$$

with the boundary conditions $S(x_j;t) = 0, \forall x_j \in \partial\Omega_{\mathcal{A}}$, and $S(x_j;0) = 1, \forall x_j \notin \partial\Omega_{\mathcal{A}}$. where $\partial\Omega_{\mathcal{A}}$ is the absorbing boundary. From the survival probability, all moments of the first passage time to an absorbing boundary $\partial\Omega_{\mathcal{A}}$ can be calculated. The following explicit relation holds

$$\mathcal{L}^\dagger \langle T^n(x_j)\rangle = -n \left\langle T^{n-1}(x_j)\right\rangle$$

2.15 Diffusion on a sphere

2.16 Approximation to the stochastic path as $L^2[0,T]$ expansion

From [1]. It $X(t)$ is a random trajectory of the process $X(\omega,t)$ for a given ω, the Weiner process $W(t)$ has trajectories belonging to $L^2[0,T]$ for almost all ω's, and the **Karhunen-Loeve** expansion for it takes the form

$$W(t) = W(\omega,t) = \sum_{i=1}^{\infty} Z_i(\omega)\phi_i(t),$$

with $0 \le t \le T$, and

$$\phi_i(t) = \frac{2\sqrt{2T}}{(2i+1)\pi} \sin\left(\frac{(2i+1)\pi t}{2T}\right)$$

Chapter 3

Appendix

3.0.1 The Lebesgue Integral

When integration on a space more general than the real line and when irregular function such that arouse from limiting processes were analyzed, more delicate integration methods needed to be introduced (e.g taking the limit of a sequence of functions). The Lebesgue integral generalizes the Reimannian integration to those spaces and enables dealing with irregularity of limiting processes functions.

3.1 From characteristic function to distribution

The characteristic function of a random variable completely defines its probability distribution function. The characteristic function is the inverse Fourier transform of the probability density function. the function is defined as

$$\varphi_X(t) = E[e^{itX}]$$

with X the random variable.

Taking the Laplace transform of a random variable (e.g. the first passage time) allows us to calculate the characteristic function of this variable and hence obtain its distribution. For example, (see [2], Lemma 2.11) if $T(x) = inf\{t > 0; w(t) = x\}$ is the first passage time of a Brownian motion $w(t)$ to

a level set x, then the Laplace transform of $T(x)$ is given by

$$E \exp(-\lambda T(x)) = \exp(-\sqrt{2\lambda}x)$$

and the characteristic function is

$$\varphi(s) = \exp(-(1 \mp i)\sqrt{\pm s}x)$$

This characteristic function is the one corresponding to the Cauchy distribution.

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