Mathematical Definitions

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1 Quadratic Forms

From Mathai [1](Chapters 1 and 4).For a $p \times 1$ random vector X with mean value $E(X) = \mu$ and $Cov(X) = E[(X - E(X))(X - E(X))'] = \Sigma > 0$, we set

$Y = \Sigma^{-0.5}X \Rightarrow E(Y) = \Sigma^{-0.5}\mu$ and $Cov(Y) = \Sigma^{-0.5}Cov(X)\Sigma^{-0.5} = I$ $Z = (Y - \Sigma^{-0.5}\mu) \Rightarrow E(Z) = 0$ and $Cov(Z) = I$ We can express the quadratic form $Q(X) = X'AX$ with the centralize variable Z and a symmetric positive definite matrix A as

$$Q(X) = X'AX = Y'\Sigma^{0.5}A\Sigma^{0.5}Y = (Z + \Sigma^{-0.5}\mu)'\Sigma^{0.5}A\Sigma^{0.5}(Z + \Sigma^{-0.5}\mu) \quad (1)$$

As an example, if we set $\mu = 0$, $A = I$, $Cov(X) = \Sigma = \sigma I$ and get $Q(X) = X'X$, $Y = \frac{X}{\sqrt{\sigma}}$, $Z = (\frac{X}{\sqrt{\sigma}} - 0)$, then

$$Q(X) = X'X = (\frac{X}{\sqrt{\sigma}})'(\sigma I)(\frac{X}{\sqrt{\sigma}}) = XX' \quad (2)$$

which is the normal form.

If P is a $p \times p$ orthogonal matrix which diagonalize $\Sigma^{0.5}A\Sigma^{0.5}$, that is

$$P'\Sigma^{0.5}A\Sigma^{0.5}P = diag(\lambda_1, \lambda_2, ..., \lambda_p)$$

with $P'P = I$, and λ_i are the eigenvalues of $\Sigma^{0.5}A\Sigma^{0.5}$. Then if $U = P'Z$, we have

$$Z = PU, \qquad E(U) = 0 \qquad Cov(U) = I$$

Then we can express the quadratic form of X by

$$Q(X) = (U + b)'diag(\lambda_1, \lambda_2, ..., \lambda_p)(U + b) \quad (3)$$

with $b = (P'\Sigma^{-0.5}\mu)'$

References

[1] Arakaparampil M Mathai and Serge B Provost. *Quadratic forms in random variables: theory and applications*. M. Dekker New York, 1992.