Contents

Chapter 1

General Probability Theory

1.1 σ-algebra

From [1]. If Ω is a given set, the a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties

(i) $\emptyset \in \mathcal{F}$

(ii) if a set $F \in \mathcal{F}$ then $F^C \in \mathcal{F}$, where $F^C = \Omega \backslash F$

Chapter 2

2.1 Noise

[Unfinished] White noise is a process for which there is no correlation between any values at different times. As an idealization of such a process there exist the Orenstein-Uhlenbeck process (to be defined) for which produces an almost uncorrelated noise. For this model, the second order correlation function can, up to a constant factor, be described as

$$< x(t), x(t') >= \frac{\gamma}{2} \exp(-\gamma|t - t'|)$$

2.2 Cylinder Sets

A cylinder set of Brownian trajectories is defined by a sequence of times $0 \leq t_1 < t_2 < ... < t_n$, and real intervals $I_k = (a_k, b_k)$, $k = 1...n$, as

$$C(t_1, t_2, ..., t_n; I_1, I_2, ...I_n) = \{\omega \in \Omega | x(\omega, t_k) \in I_k\}, \forall 1 \leq k \leq n$$

For a cylinder not to contain a trajectory, it is enough that in one time point the trajectory is not in the interval. The cylinder C contains entire trajectory, which fulfill the inclusion demand.

The joint PDF of $w(t_1, \omega), w(t_2, \omega), w(t_3, \omega)$ is the Weiner measure of the cylinder $C(t_1, t_2, t_3; I^x, I^y, I^z)$. These points belong to the same process ω at three different times.

2.3 Filtration

The σ-algebra F_t of Brownian event is defined by cylinder sets confined to times $0 \leq t_i < t$, for some fixed t. Obviously, $F_s \subset F_t \subset F$ if $0 \leq s < t < \infty$. The family of σ algebras F_t for $t \geq 0$ is called the Brownian filtration and is said to be generated by the Brownian events up to time t

In simple words, filtration is an increasing sequence of σ-algebras on a measurable space.

2.4 Adapted Process

The process $x(t,\omega)$ is said to be adapted to the Brownian filtration F_t if $\{\omega \in \Omega | x(t,\omega) \leq y\} \in F_t$ for every $t \geq 0$ and $y \in R$. In that case we also say that $x(t,\omega)$ is F_t-measurable.

Thus an adapted process does not depend on the future behavior of the Brownian trajectory from time t on.

2.5 The Weiner Measure

The probability measure Pr defined on all the Brownian trajectories (events in the set Ω) are defined on cylinder sets and then extended to all events in the space by elementary properties of probability sets.

For a cylinder set $C(t;I)$, where $t > 0$ and $I = (a,b)$, the Weiner measure is

$$Pr\{C(t;I)\} = \frac{1}{\sqrt{2\pi t}} \int_a^b \exp^{-x^2/2t} dx$$

2.5.1 Martingale

A martingale is a stochastic process $x(t)$ with $\mathbb{E}|x(t)| < \infty$,$\forall t$, and for $t_1 < t_2 < ... < t_n$

$$\mathbb{E}[x(t)|x(t_1) = x1, x(t_2) = x_2, ...x(t_n) = x_n] = x_n$$

2.6 The Langevin Equation

[Unfinished] An equation of the type

$$\frac{dx}{dt} = a(x,t) + b(x,t)\xi(t)$$

2.7 The Itô Equation

Equation of the form

$$\mathrm{d}X_t = \mu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}B_t$$

is an informal way of expressing the more appropriate integral equation

$$X_{t+s} - X_t = \int_t^{t+s} \mu(X_u, u)\mathrm{d}u + \int_t^{t+s} \sigma(X_u, u)\,\mathrm{d}B_u$$

where the dynamic of the particle X is given as a sum of two integrals, the first one is a standard Lebesgue integral, and the second is an Itô integral (to be defined). The functions $\mu(X_t, t)$ and $\sigma(X_t, t)$ are well defined continuous functions, and B_t is a Brownian motion (a Weiner process, or white noise). An informal way of interpreting the equation above is to say that at each small time interval of size δ the process X_t is changed by a normally distributed value with expectation $\mu(X_t, t)\delta$ and variance $\sigma(X_t, t)^2\delta$. the function $\mu(X_t, t)$ is referred to a s the drift coefficient, and the function $\sigma(X_t, t)$ as the diffusion coefficient.

2.8 Existence and Uniqueness of Itô SDE solutions

For an Ito SDE taking values in n-dimensional Euclidean space, if for $T > 0$

$$\begin{aligned} \mu :\ & \mathbb{R}^2 \times [0,T] \to \mathbb{R}^n \\ \sigma :\ & \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m} \end{aligned}$$

are measurable functions, for which there exist constants C and D such that

$$\begin{aligned} &|\mu(x,t)| + |\sigma(x,t)| \leq C(1+|x|) \\ &|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq D|x-y| \end{aligned}$$

for all $t \in [0,T]$ and all $x, y \in \mathbb{R}^n$, and $|\sigma|^2 = \sum_{i,j=1} |\sigma_{i,j}|^2$. If Z is a random variable, independent of the σ-algebra generated by B_s, $s \geq 0$, and with finite second moment, then the SDE, with initial condition $X_0 = Z$ has an almost surely unique solution in $t \in [0,T]$, $X_t(\omega)$, such that X is adapted to the filtration F_t generated by Z and B_s, $s \leq t$.

2.9 Stochastic Integration

For an arbitrary function of time $G(t')$ and a Weiner process $W(t)$, the stochastic integral

$$\int_{t_0}^{t} G(t')\mathrm{d}W(t')$$

is defined as a kind of Riemann integral, that is, we divide the interval $[t_0, t]$ into n subintervals $t_0 \leq t_1 \leq ... \leq t$, and define intermediate points $t_{i-1} \leq \tau_i \leq t_i$. The integral is then defined as a limit of the partial sums

$$S_n = \sum_{i=1}^{n} G(\tau_i)[W(t_i) - W(t_{i-1})]$$

Note that the Weiner process in the partial sums is contributing the values at the edges of the interval, whereas the function G contributes the values somewhere in its middle. The choice of τ_i affects the value of the integral. It can be seen that f we choose $\tau_i = \alpha t_{i-1} + (1-\alpha)t_i$, with $0 \leq \alpha \leq 1$, than the mean value of the integral can take any value between 0 and $(t - t_0)$.

for the **Itô integral**, we choose $\tau_i = t_{i-1}$, which is the left boundary of the subinterval and thus define the stochastic integral as

$$\int_{t_0}^{t} G(t')\mathrm{d}W(t') = ms - lim_{n\to\infty} \left\{ \sum_{i=1}^{n} G(t_{i-1})[W(t_i) - W(t_{i-1})] \right\}$$

by ms-lim, we mean the mean square limit, which is defined by

$$ms - lim_{n\to\infty} \int p(\omega)[X_n(\omega) - X(\omega)]^2 dw = lim_{n\to\infty} < (X_n - X)^2 >$$

if indeed $lim_{n\to\infty} < (X_n - X)^2 >$ we say that $ms - lim_{n\to\infty} X_n = X$.

One major consequence of the choice $\tau_i = t_{i-1}$ is that

$$\int_{t_0}^{t} W(t')dW(t') = 0$$

That is, the stochastic integral of the Weiner process over any interval is zero. Because the left point of the subinterval is chosen, the value of the function $G(t'_{i-1})$ is independent of the increments $[W(t_i) - W(t_{i-1})]$, since G is F_t adapted process.

2.10 Properties of Itô integral

We define the class $H_2[0,T]$ of F_t-adapted stochastic processes such that

$$\int_0^T \mathbb{E} f^2(s,\omega) ds < \infty$$

and list the following properties

1. **linearity**. For $f(t), g(t) \in H_2[0,T]$, and α, β real, then $\alpha f(t) + \beta g(t) \in H_2[0,T]$, and
$$\int_0^t \alpha f(s) + \beta g(s) d\omega(s) = \alpha \int_0^t f(s) d\omega(s) + \beta \int_0^t g(s) d\omega(s)$$
2. **additivity**. If $f(t) \in H_2[0,T_1]$ and $f(t) \in H_2[T_1,T]$ for $0 < T_1 < T$, then
$$\int_0^T f(s) d\omega(s) = \int_0^{T_1} f(s) d\omega(s) + \int_{T_1}^T f(s) d\omega(s)$$
3. for a deterministic and integrable $f(t)$
$$\int_0^t f(s) d\omega(s) \sim N\left(0, \int_0^t f^2(s) ds\right)$$
4. for $f(t) \in H_2[0,T]$
$$\mathbb{E} \int_0^t f(s) d\omega(s) = 0$$
5. for $0 < \tau < t < T$
$$\mathbb{E}\left[\int_0^t f(s) d\omega(s) | \int_0^\tau f(s) d\omega(s) = x\right] = x$$
6. for $f(t), g(t) \in H_2[0,T]$
$$\mathbb{E}\left[\int_0^T f(s) d\omega(s) \int_0^T g(s) d\omega(s)\right] = \int_0^T \mathbb{E}[f(s)g(s)] ds$$
note that the right-hand-side is integrated with respect to s

2.11 Non-anticipating Functions

[Unfinished]

2.12 Stochastic Differential Equations

Bibliography

[1] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications (Universitext)*. Springer, 6th edition, January 2014.