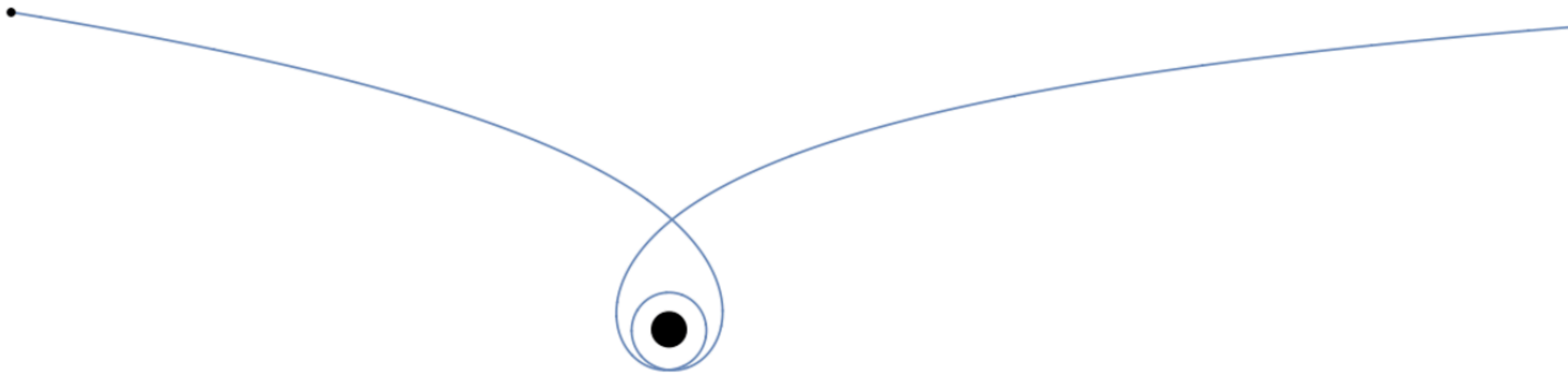
Towards a self-force calculation of the scatter angle in hyperbolic encounters



Oliver Long

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BritGrav21, UCD

Supervisor: Leor Barack

UNIVERSITY OF Southampton

STAG RESEARCH CENTRE

EMRIs: Expanding in the mass ratio

EMRI: compact object orbiting a supermassive BH with mass ratio $\eta < 10^{-4}$

Expand quantities in the mass ratio

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \eta h^{(1)}_{\alpha\beta} + \eta^2 h^{(2)}_{\alpha\beta} + \ldots$$

Schwarzschild/Kerr

1st order sufficient for detection but 2nd order needed for parameter estimation

Metric reconstruction in vacuum

$$\underset{\Psi}{\text{Weyl scalar}} \xrightarrow{\partial^4} \underset{\Phi}{\text{Hertz potential}} \xrightarrow{\partial^2} \underset{h}{\text{Metric perturbation}}$$

$$\hat{\mathcal{O}}_\pm \Psi_\pm = 0 \qquad \hat{\mathcal{O}}^\dagger_\pm \Phi_\pm = \hat{\mathcal{O}}_\mp \Phi_\pm = 0$$

Teukolsky equation

1+1D Teukolsky equation

Master Teukolsky equation

$$\hat{\mathcal{O}}_\pm \Psi_\pm = -\frac{r^4}{\Delta}\frac{\partial^2 \Psi_s}{\partial t^2} + \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial \Psi_s}{\partial r}\right) + 2s\left(\frac{Mr^2}{\Delta} - r\right)\frac{\partial \Psi_s}{\partial t} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial \Psi_s}{\partial\theta}\right)$$
$$+ \frac{1}{\sin^2\theta}\frac{\partial^2 \Psi_s}{\partial\varphi^2} + \frac{2is\cos\theta}{\sin^2\theta}\frac{\partial \Psi_s}{\partial\varphi} - (s^2\cot^2\theta - s)\Psi_s = 0$$

where $\Delta := r^2 - 2Mr$ and $s = \pm 2$

$s = \pm 2$ fields modally decomposed into $s = \pm 2$ harmonics

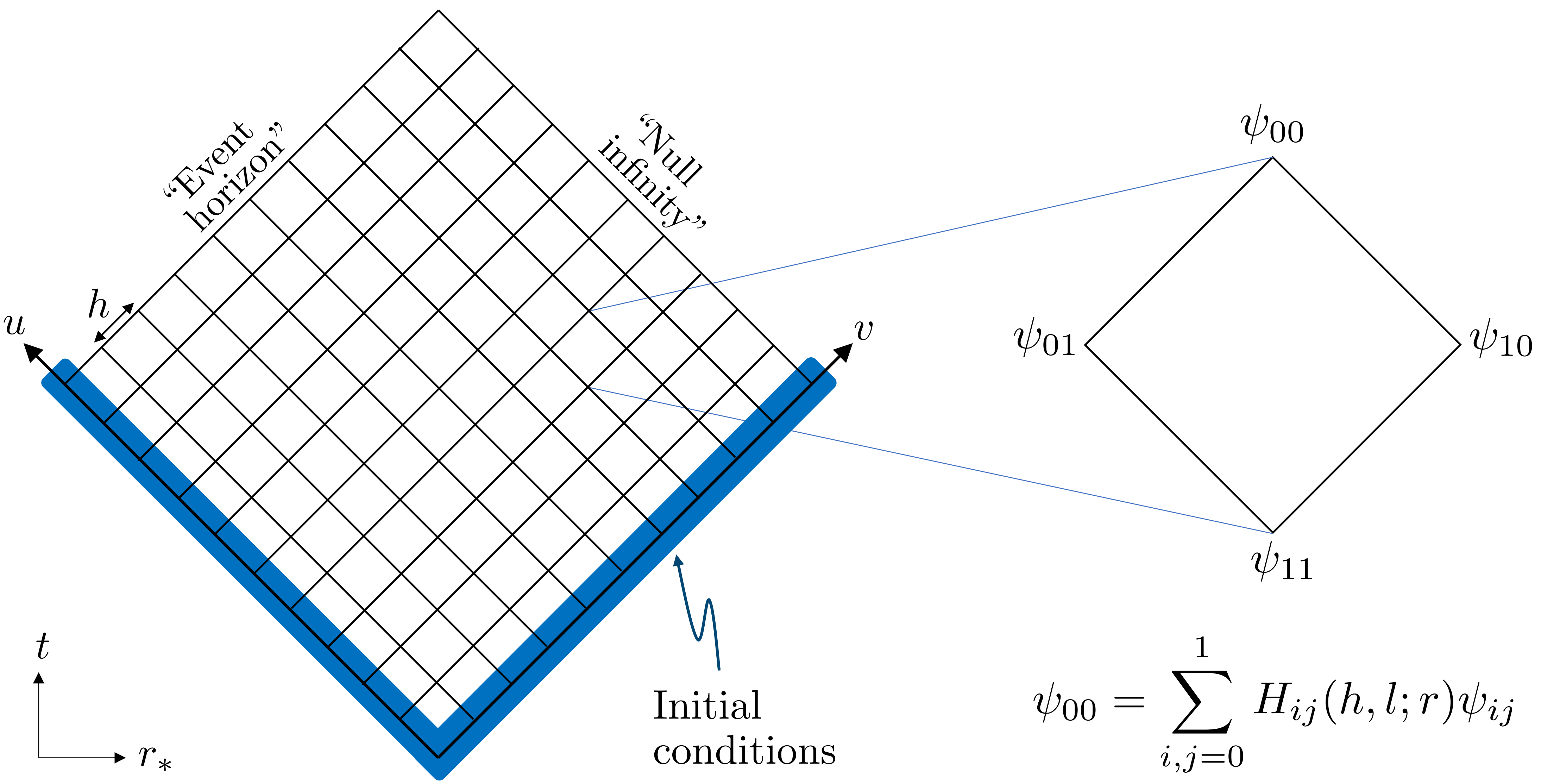
$$\Psi_\pm = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} {}_{\pm 2}\psi_{lm}(t,r)\, {}_{\pm 2}Y_{lm}(\theta,\varphi) \qquad \Phi_\pm = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} {}_{\pm 2}\phi_{lm}(t,r)\, {}_{\pm 2}Y_{lm}(\theta,\varphi)$$

Teukolsky equation split in modes

$$\boxed{\psi_{,uv} + U(r)\psi_{,u} + V(r)\psi_{,v} + W(l;r)\psi = 0}$$

Eddington-Finkelstein coordinates $u = t - r_*$ and $v = t + r_*$

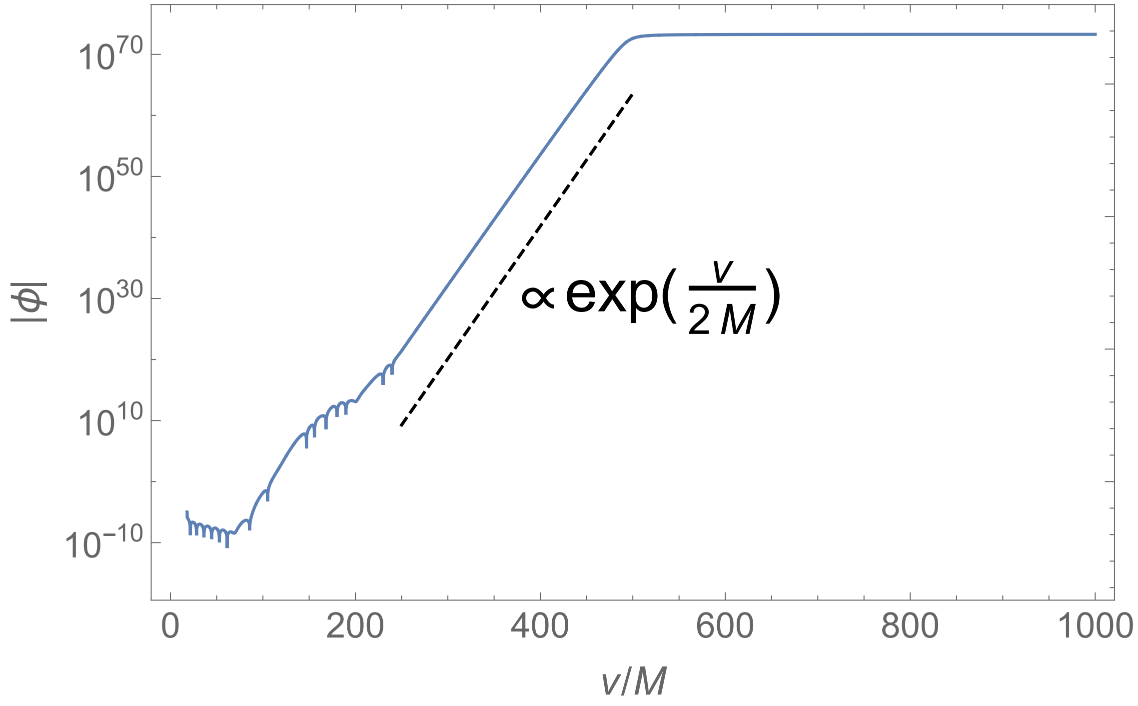
1+1D vacuum evolution code



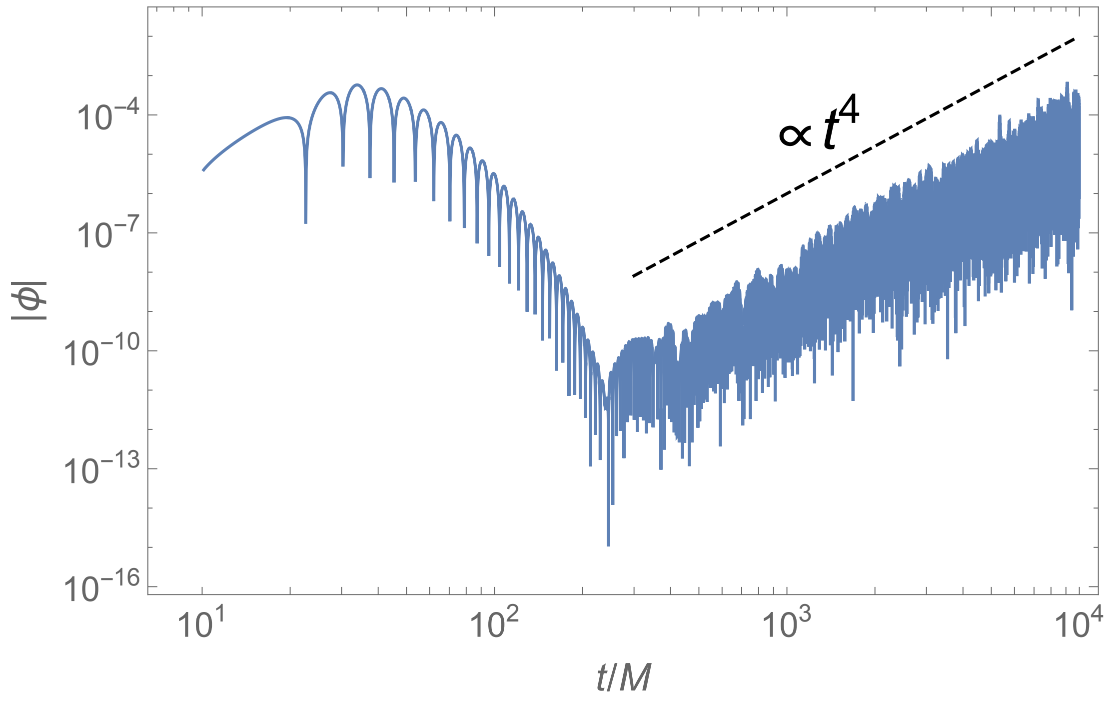
$$\psi_{00} = \sum_{i,j=0}^{1} H_{ij}(h, l; r)\psi_{ij}$$

Problem of divergent modes

$s = +2$



$s = -2$



There exist non-physical asymptotic solutions:

$$\psi_{\mathcal{H}^+} \sim C_1(u)\exp(v\, s/4M) + C_2(v) \qquad \psi_{\mathcal{I}^+} \sim (t+r)^{-2s} + C_4(u)$$

Non-physical terms usually removed by physical boundary conditions

Both terms originate from the $\phi_{,t}$ term of the Teukolsky equation

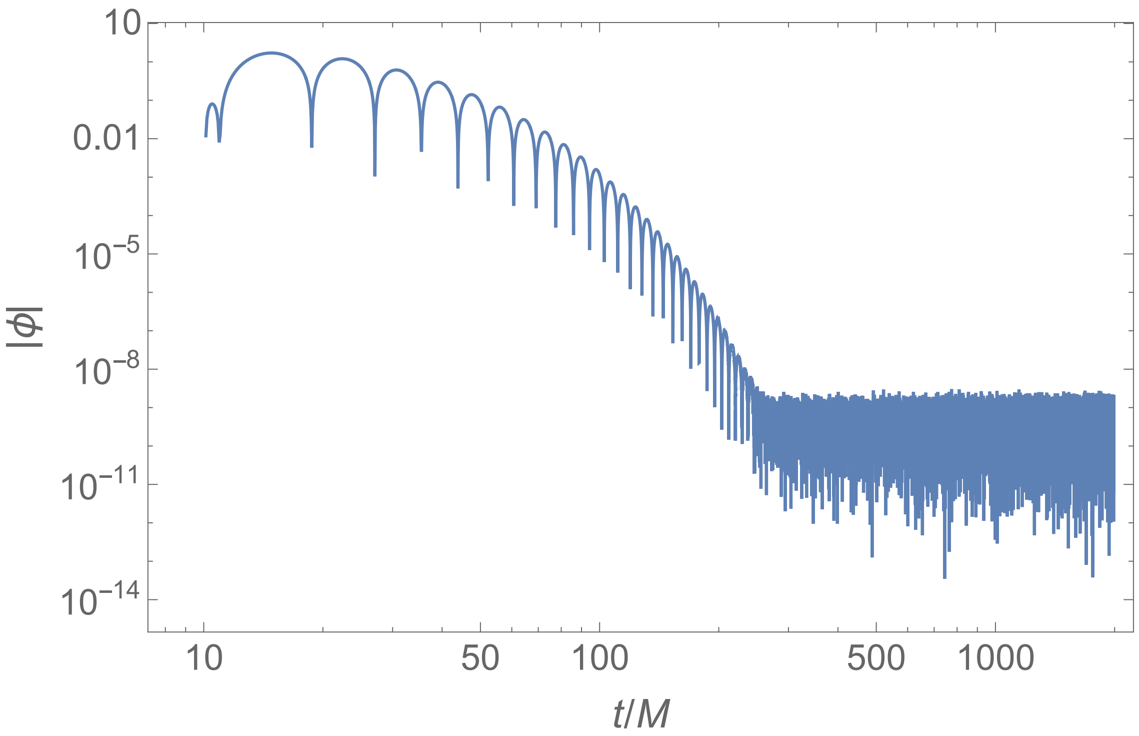
Transformation to Regge-Wheeler form

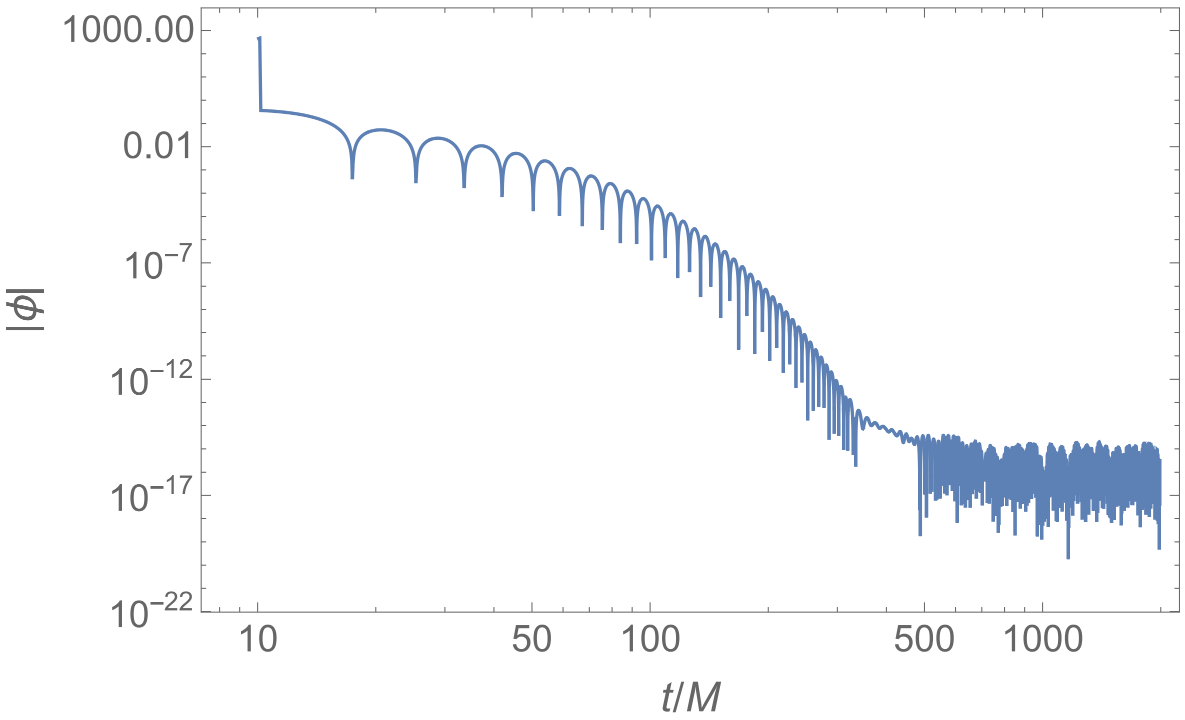
Regge-Wheeler equation

$$\boxed{X_{,uv} + \mathcal{W}(l;r)X = 0}$$

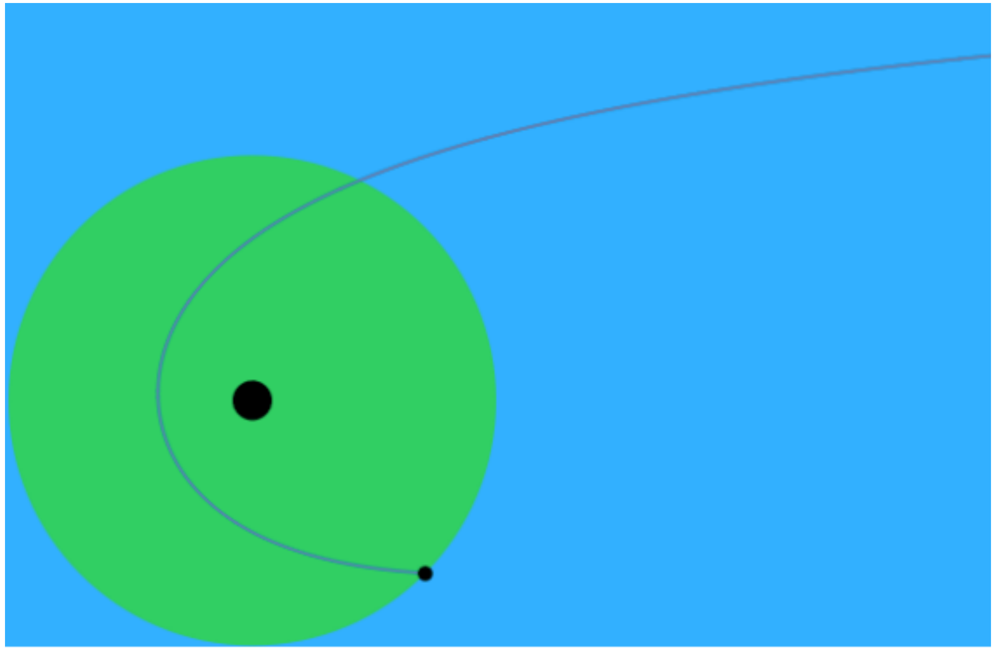
$$\phi_{s=+2} = r^4 \left(X_{,vv} + \frac{r-3M}{r^2} X_{,v} \right)$$

$$\phi_{s=-2} = \frac{r^4}{\Delta^2} \left(X_{,uu} - \frac{r-3M}{r^2} X_{,u} \right)$$





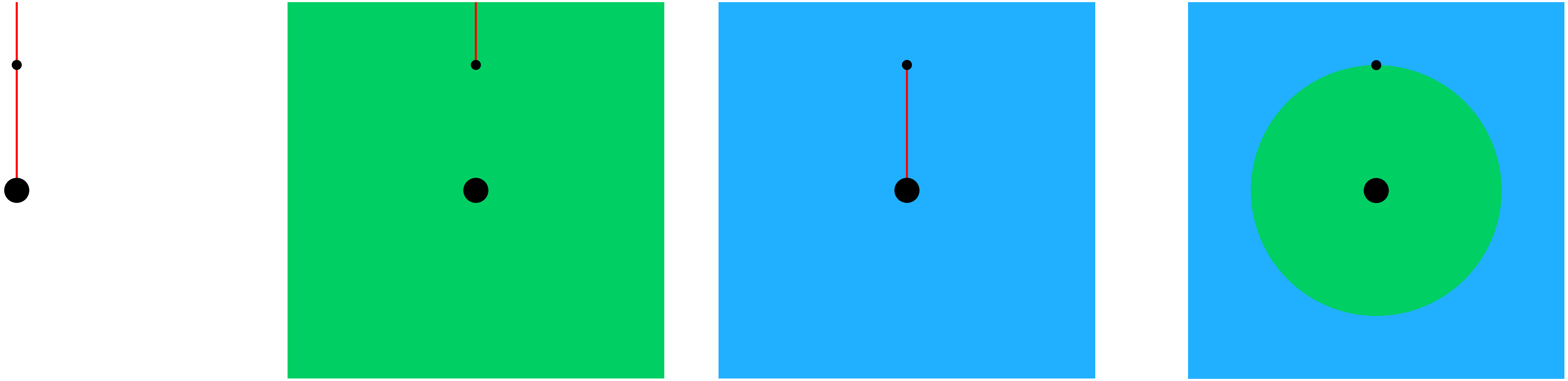
Point-particle metric reconstruction



Sourced Teukolsky equation

$$\hat{\mathcal{O}}_{\pm}\psi_{\pm} = T_{\pm} \propto \delta(r - r_{\mathrm{P}})$$

Point particle solutions for ϕ and $h_{\alpha\beta}$

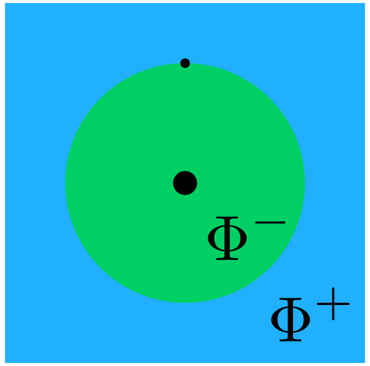


Jumps in the field

Jump in a field

$$[\Phi] := \lim_{\epsilon\to 0}\left[\Phi^+(\tau, r_{\mathrm{P}}(\tau)+\epsilon) - \Phi^-(\tau, r_{\mathrm{P}}(\tau)-\epsilon)\right]$$

$[\psi]$ obtained from the sourced Teukolsky equation



Jumps in the Hertz potential from ODE

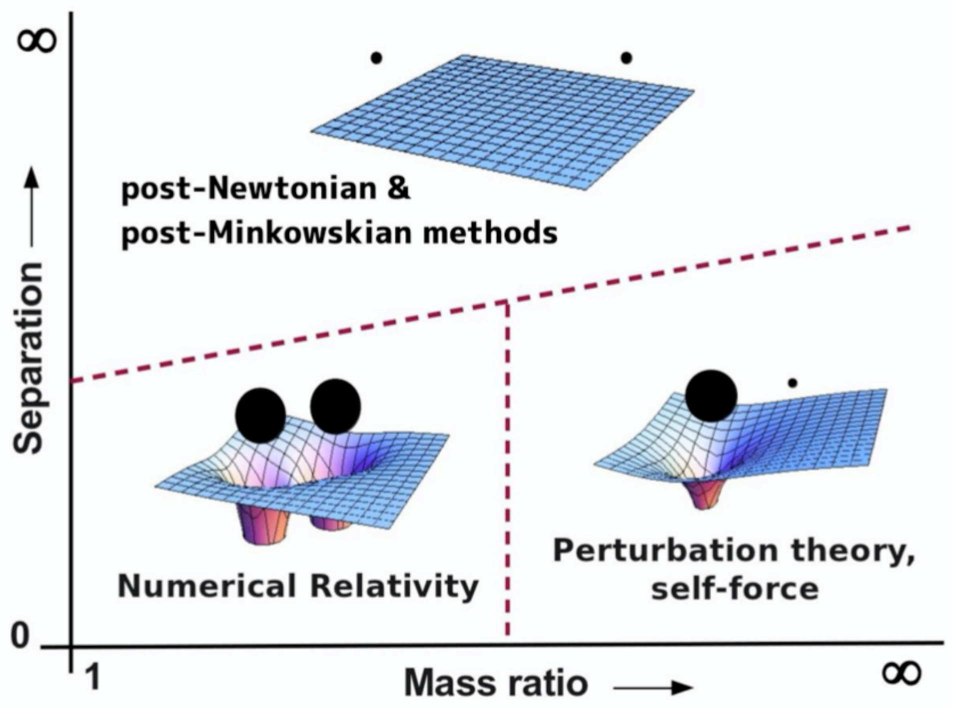
$$[\dot{\phi}] + \mathcal{A}(\tau)[\phi] = \mathcal{B}\left(\tau, [\psi]\right)$$

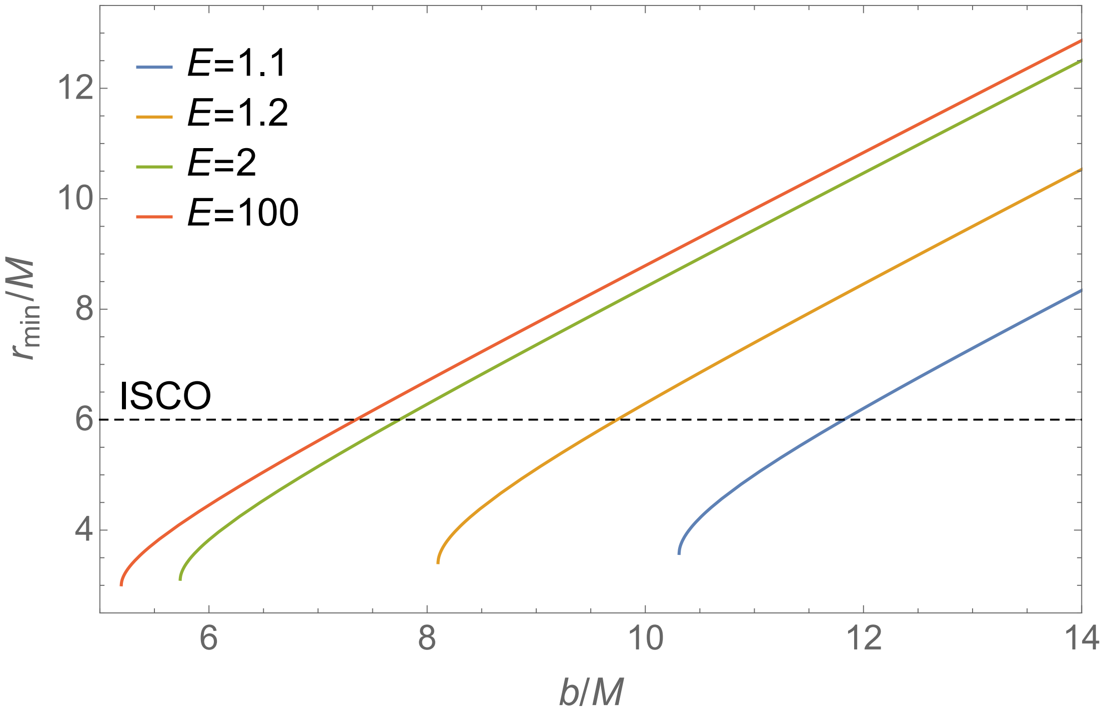
Jumps in the Regge-Wheeler variable from ODE

$$[\dot{X}] + \mathcal{C}(\tau)[X] = \mathcal{D}\left(\tau, [\phi]\right)$$

Motivation: EMRI scatter orbits

- **Exact** post-Minkowskian calculations [Damour '19; Bini, Damour, Geralico '20]
  - 1^{st} order self-force exactly determines 2 body Hamiltonian up to 4PM
- Comparison with quantum-like scatter amplitude calculations
- Calibration of Effective One Body (EOB) exact at $O(\eta)$ (no PN/PM expansions)





Scatter geodesics in Schwarzschild

Energy and angular momentum

$$E > 1 \qquad\qquad L > L_{\text{crit}}(E)$$

Eccentricity and semi-latus rectum

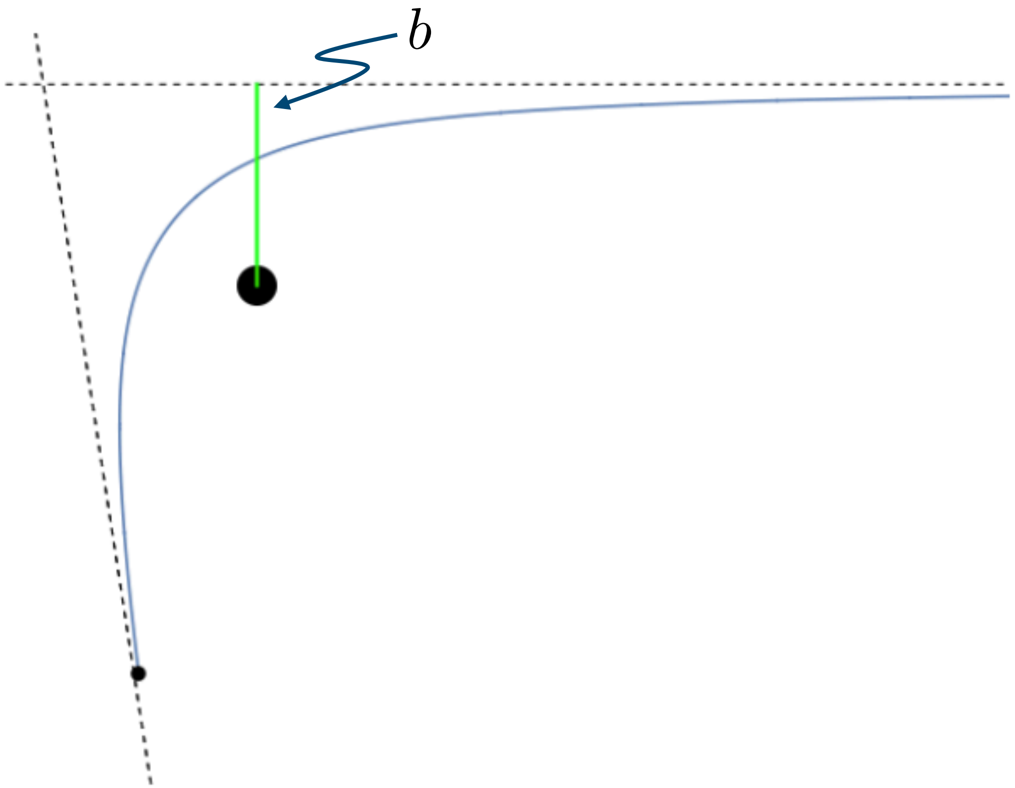
$$e \geq 1 \qquad\qquad p > p_{\text{crit}}(e)$$

$$r = \frac{Mp}{1 + e\cos\chi}$$

$$-\chi_\infty \leq \chi \leq \chi_\infty \text{ where } \chi_\infty := \arccos(-1/e)$$

Velocity at infinity and impact parameter

$$v_\infty := \left.\frac{dr}{dt}\right|_{r\to\infty} \qquad\qquad b := \lim_{r\to\infty} r \sin|\varphi(r) - \varphi(\infty)|$$



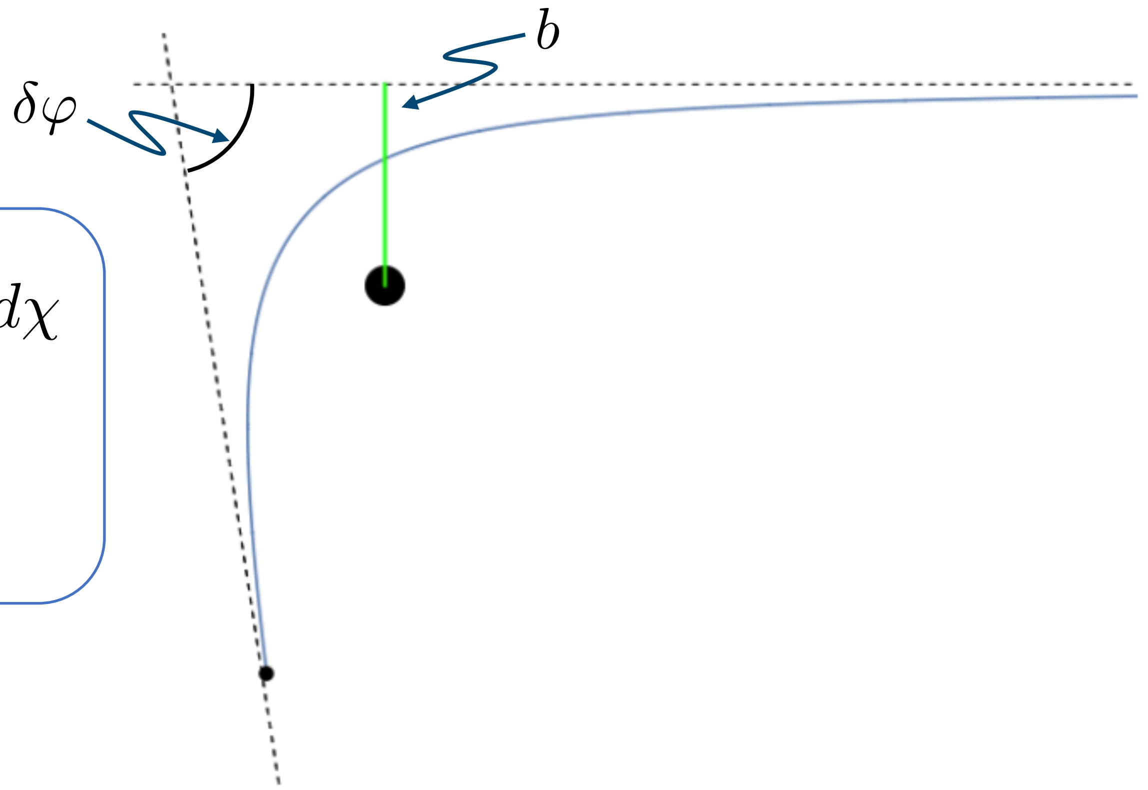
Scatter angle

Definition

$$\delta\varphi := \int_{-\chi_\infty}^{\chi_\infty} \frac{d\varphi}{d\chi} d\chi - \pi = \delta\varphi^{(0)}(v_\infty, b) + \eta\, \delta\varphi^{(1)}(v_\infty, b)$$

geodesic

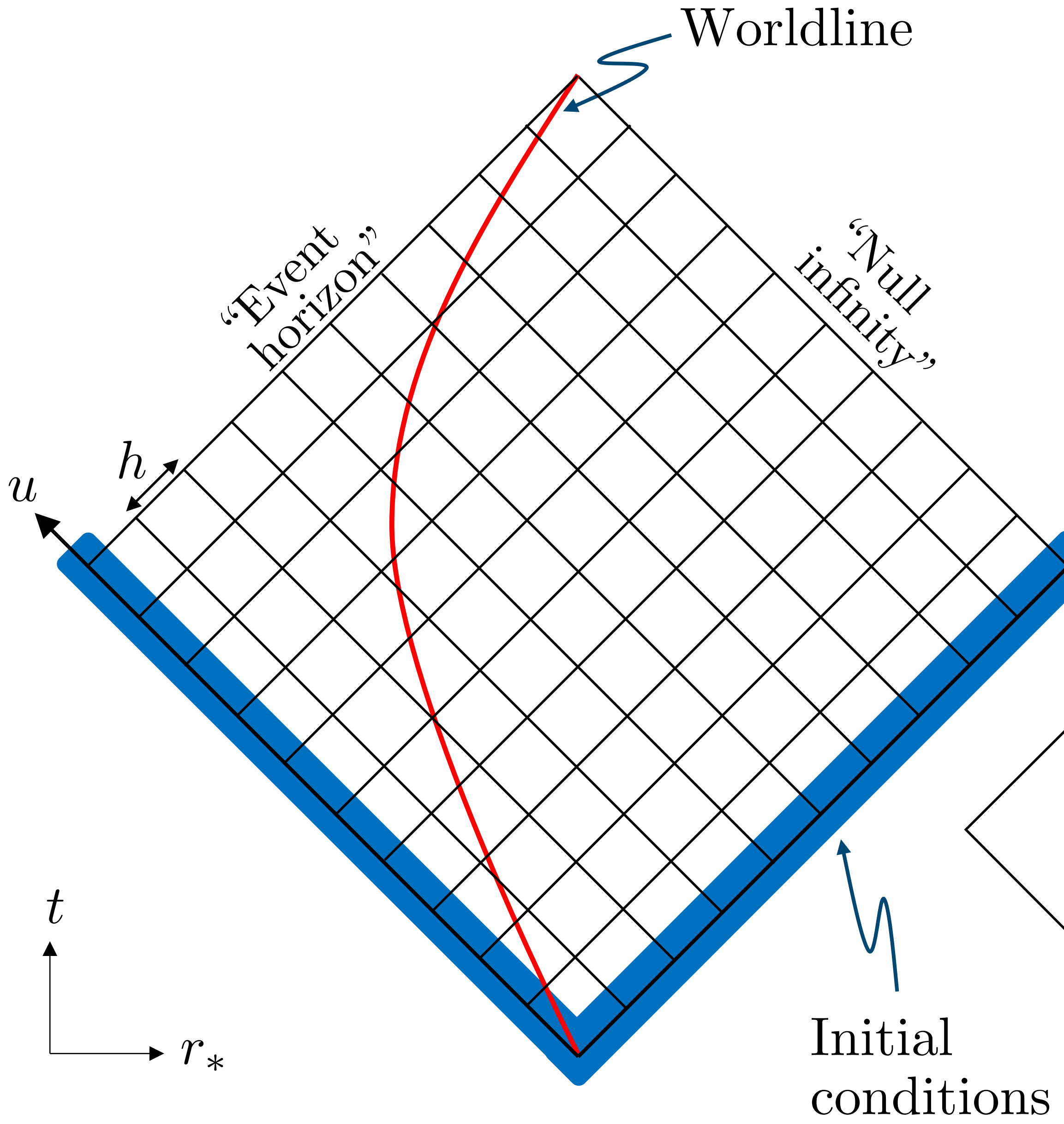
Solve 1st order self-force equations of motion at fixed v_∞ and b



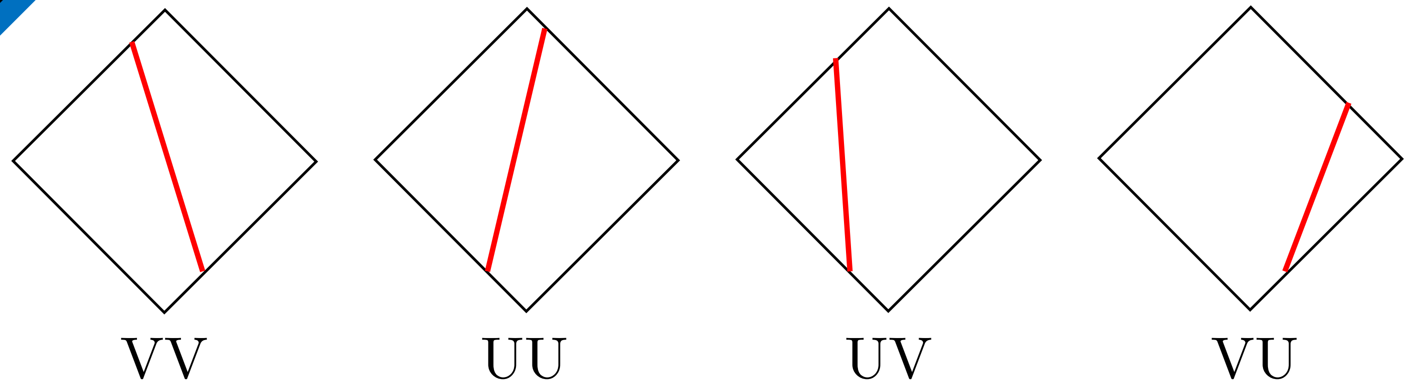
$$\boxed{\begin{aligned}\delta\varphi^{(1)} =& \mathcal{A}(e,p) \int_{-\chi_\infty}^{\chi_\infty} F_t\, d\chi + \mathcal{B}(e,p) \int_{-\chi_\infty}^{\chi_\infty} F_\varphi\, d\chi \\ &+ \int_{-\chi_\infty}^{\chi_\infty} \left[a(\chi;e,p) F_t + b(\chi;e,p) F_\varphi\right] d\chi\end{aligned}}$$

$$\frac{dE}{d\tau} = -\eta F_t \qquad \frac{dL}{d\tau} = \eta F_\varphi$$

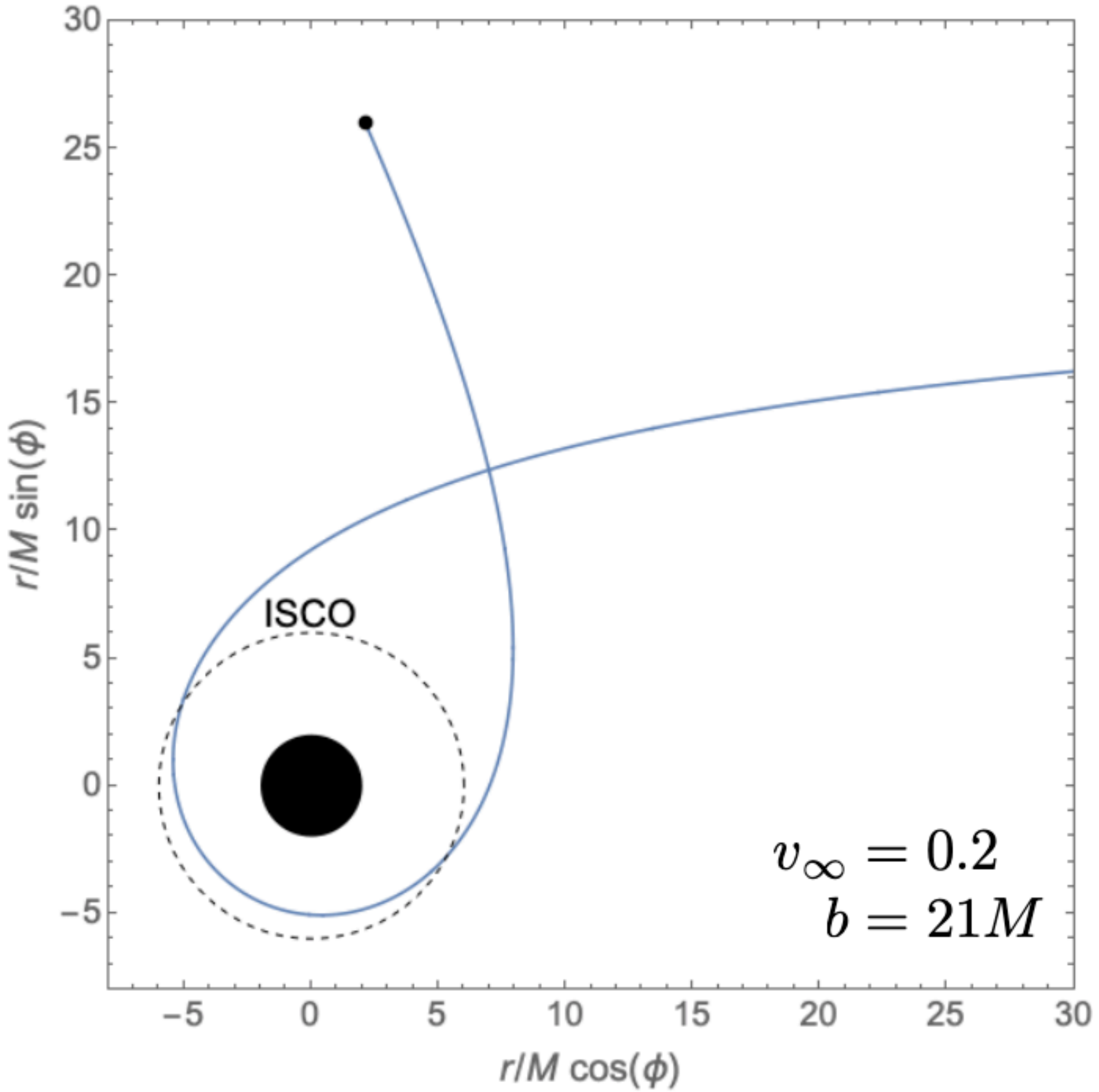
1+1D scatter evolution code



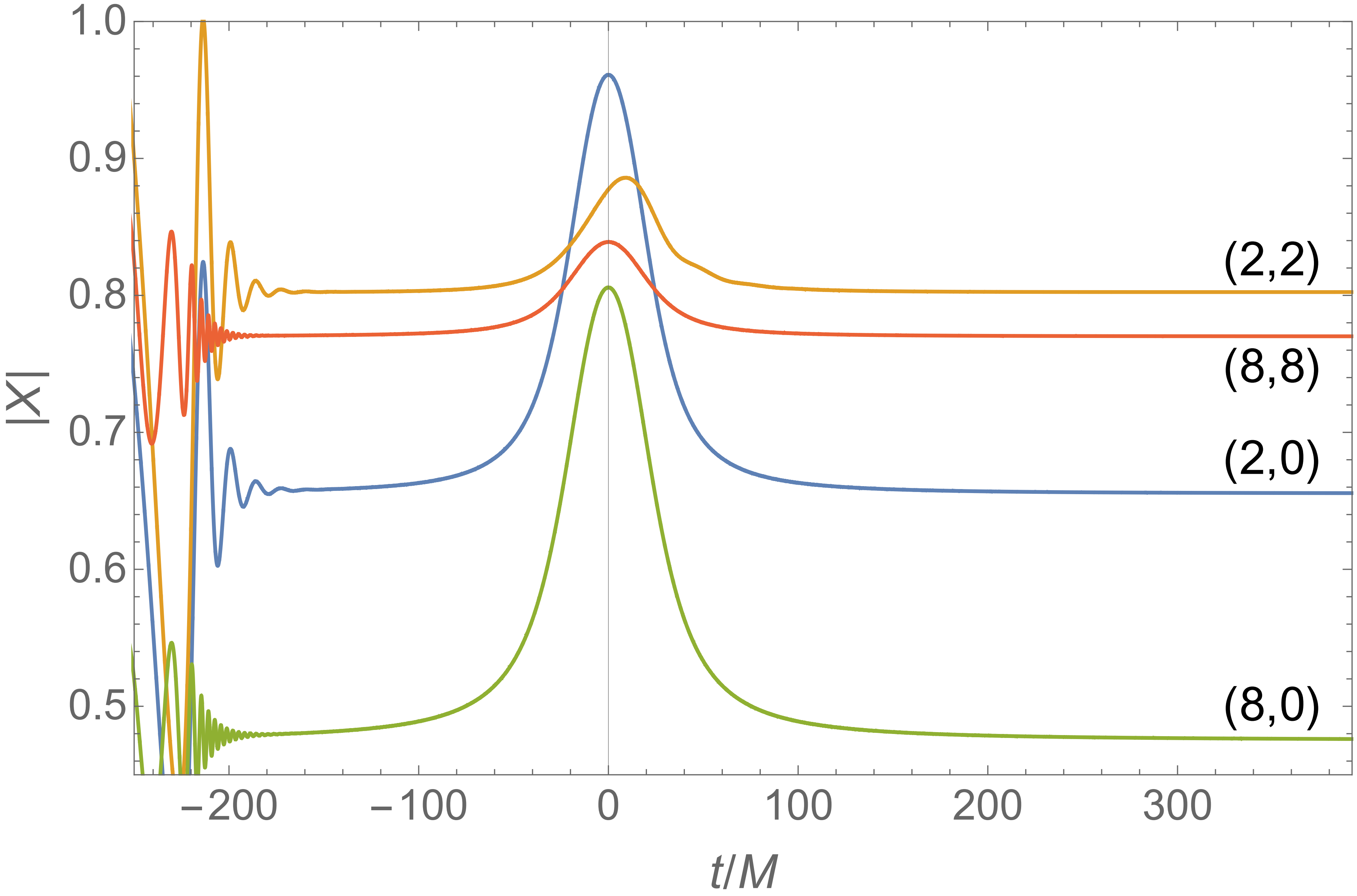
$$X_{00} = \sum_{i,j=0}^{1} H_{ij}(h, l; r) X_{ij} + \mathcal{J}([X], h, l; r)$$



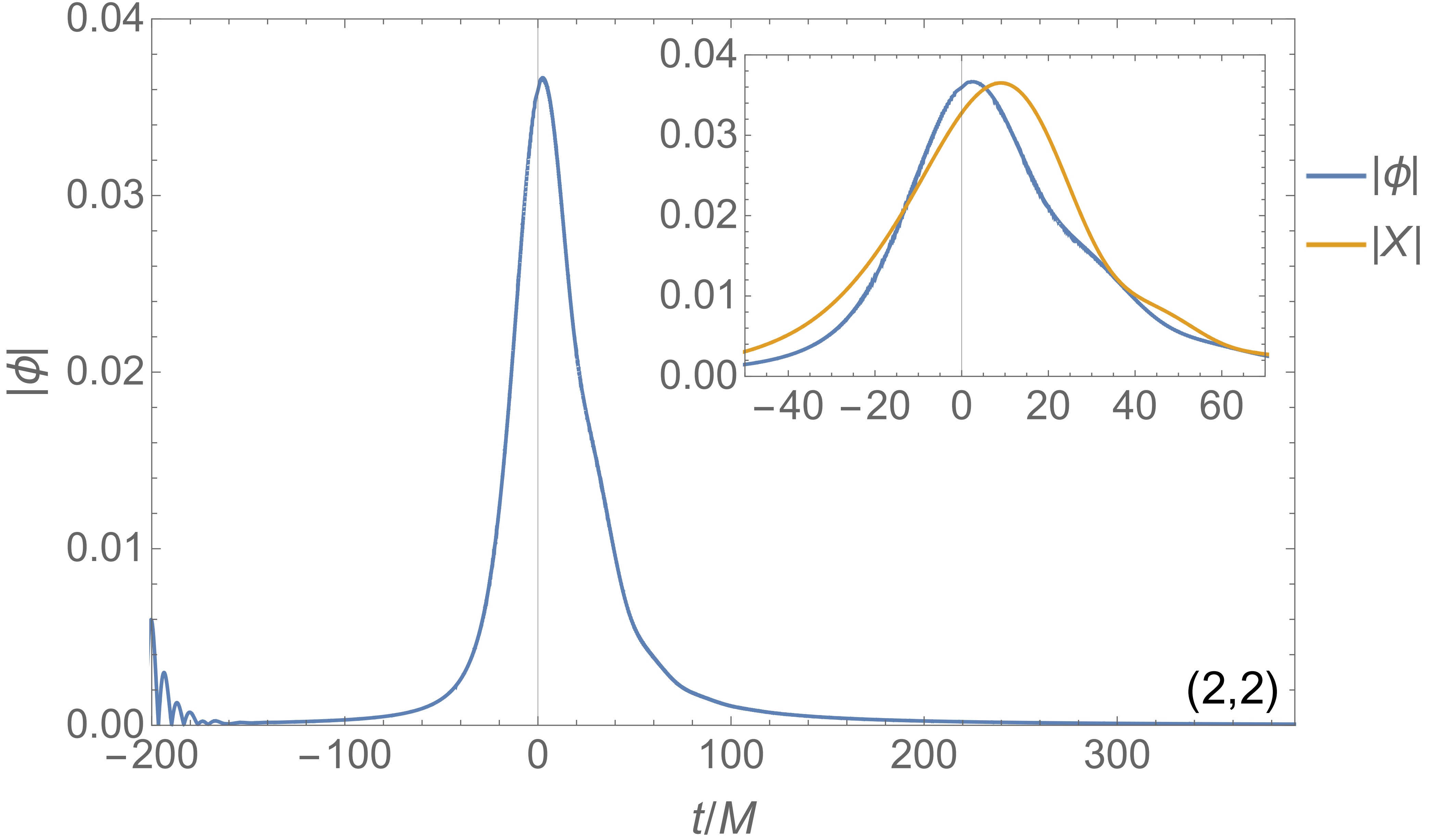
Hyperbolic orbit



Regge-Wheeler field along the worldline



Hertz potential along the worldline



Summary

- The 1+1D Teukolsky equation is numerically unstable without boundary conditions
- Transforming to Regge-Wheeler removes the numerical instability
- Self-force scattering results can be used to inform other modelling methods outside of the EMRI regime
- Future work: calculate metric perturbation and obtain self-force data