

Extraction of high-order post-Minkowskian results from self-force calculations



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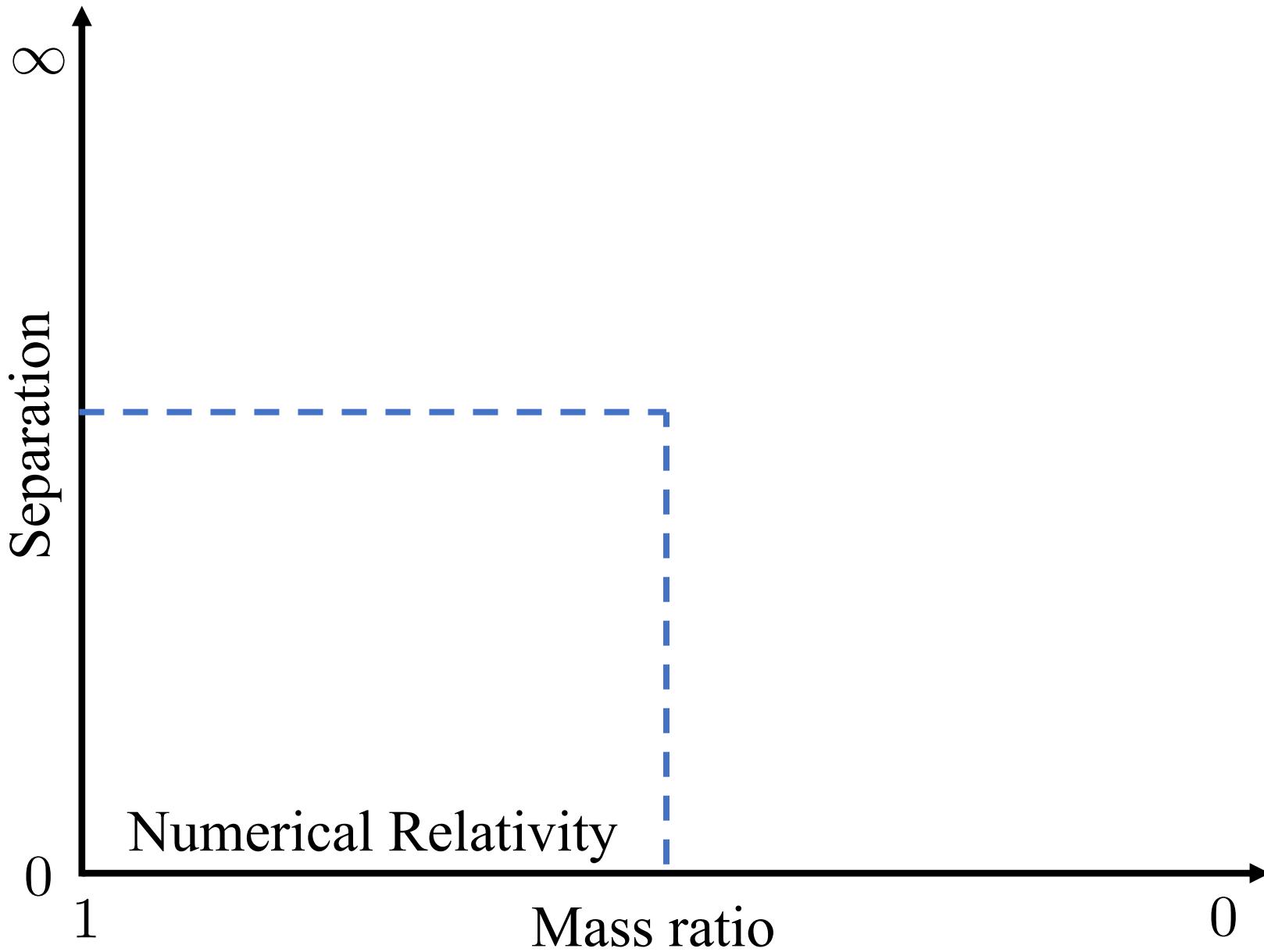


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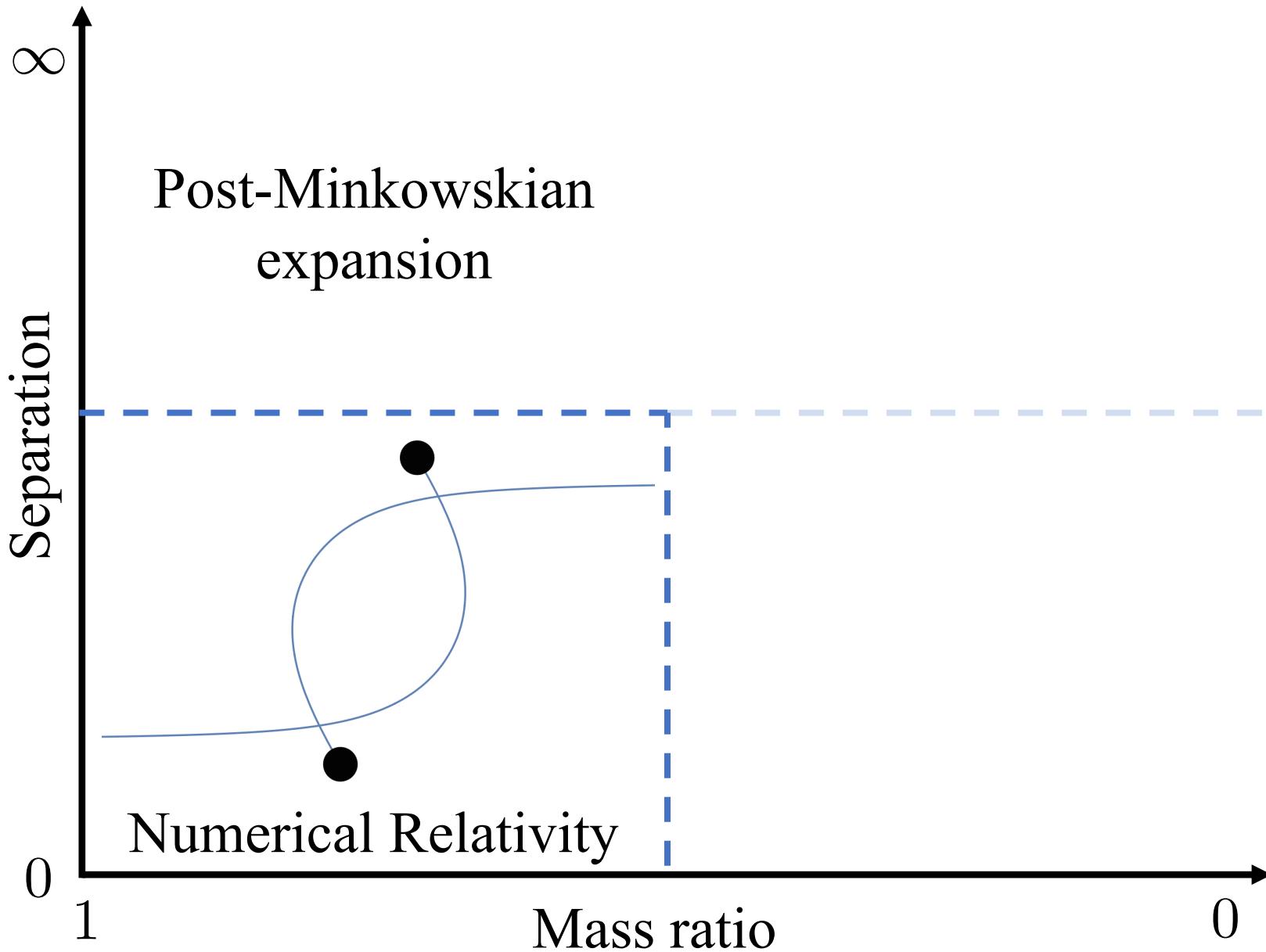
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- Motivation for extreme-mass ratio scattering.
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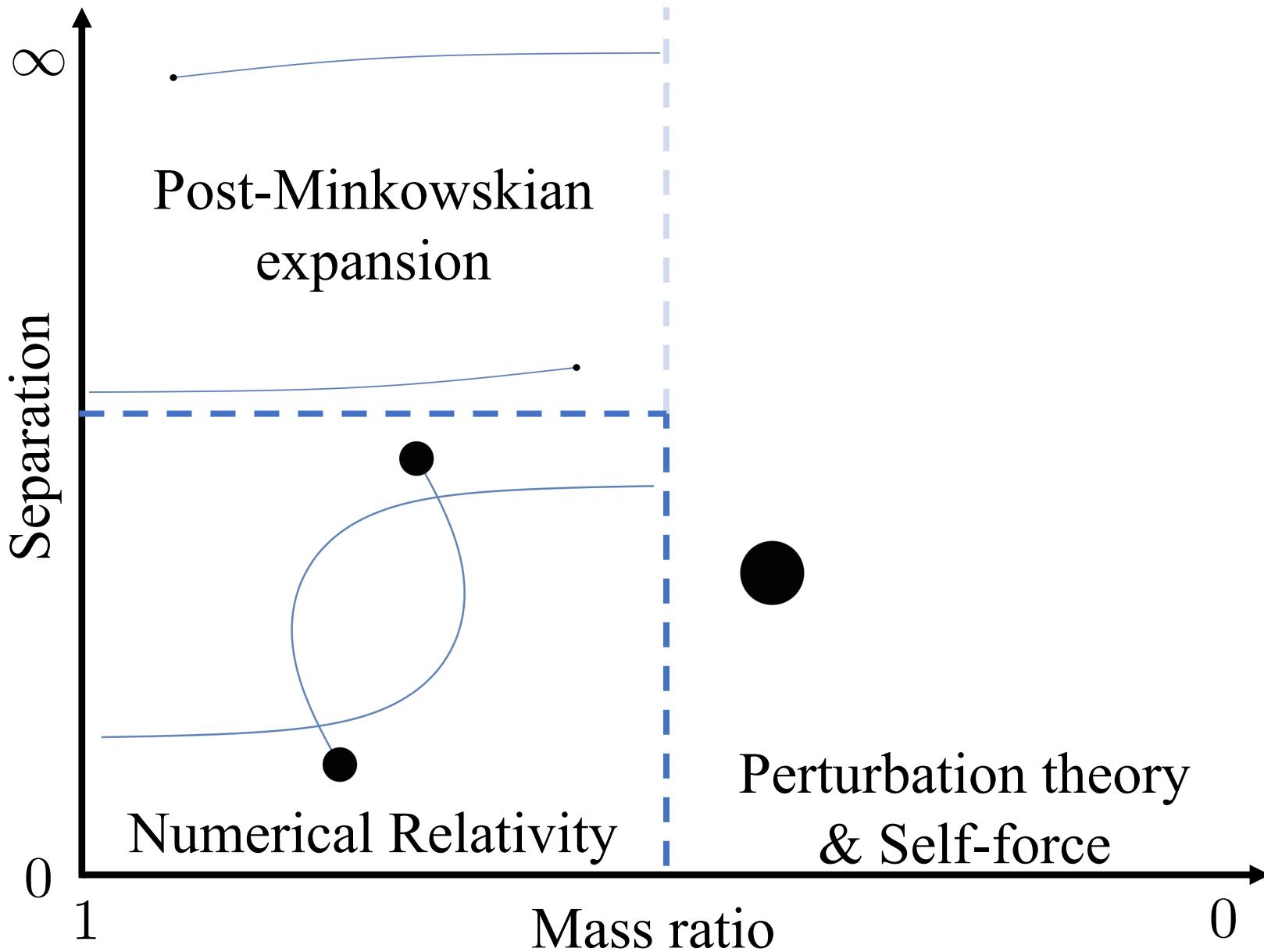
Two-body parameter space for scattering



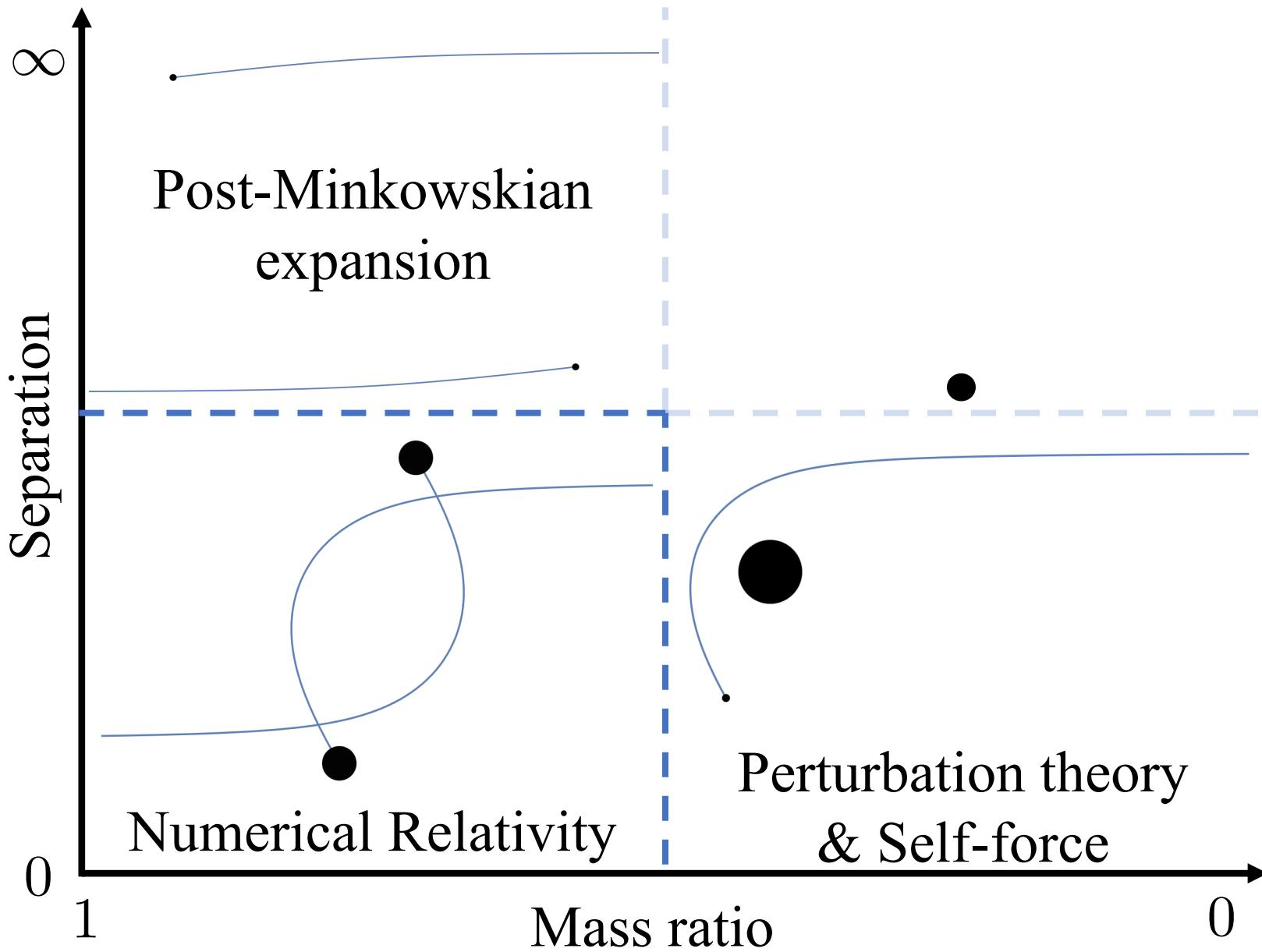
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Motivation: Extreme-mass ratio scatter orbits

- Clean environment with **well-defined** asymptotic states.
- **Exact** post-Minkowskian calculations for bound orbits
[Damour '19; Bini, Damour & Geralico '20].

1SF \rightarrow 4PM

2SF \rightarrow 6PM

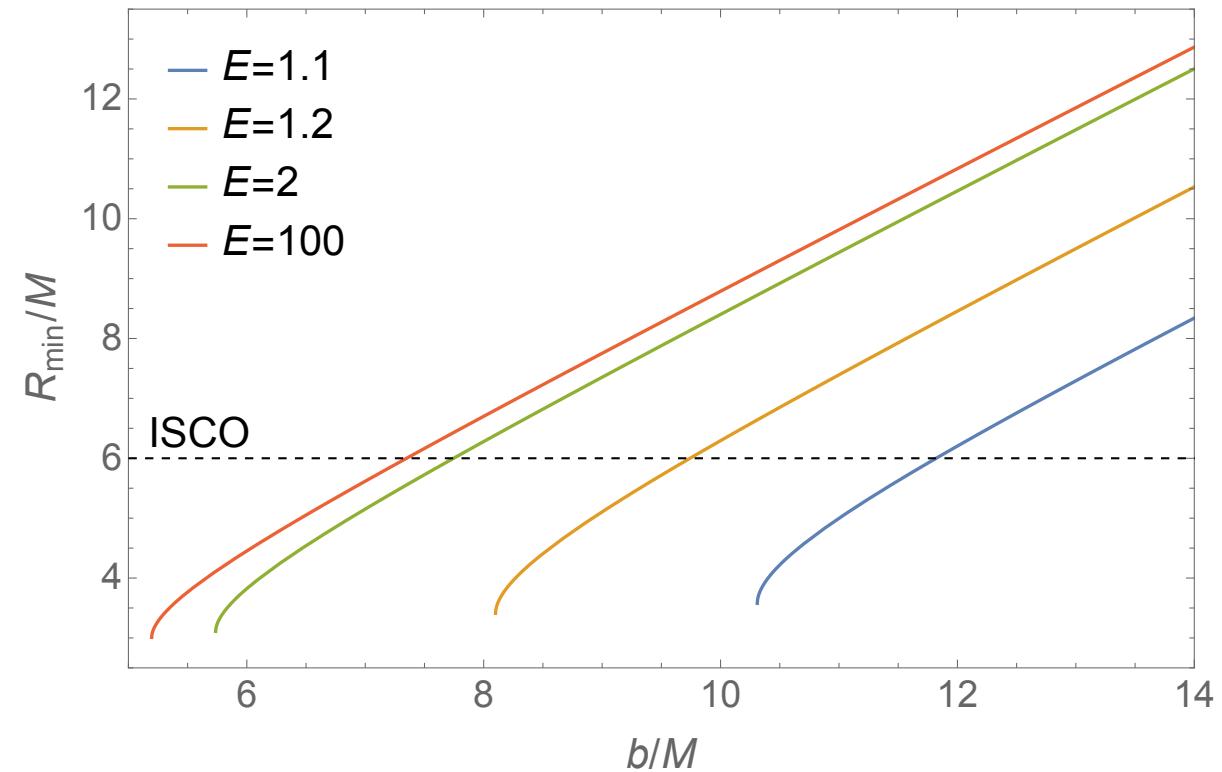
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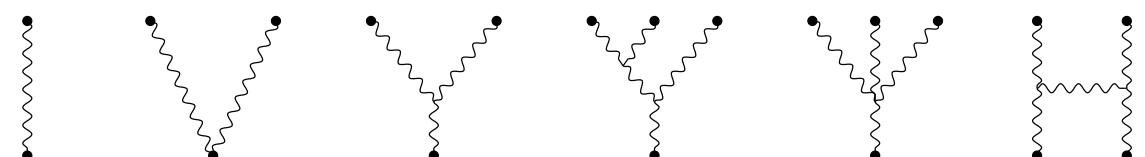
- Calibration of Effective-One-Body (EOB) in the **ultra-strong** field.
- Comparisons with **scatter amplitude** calculations:
 - QFT and EFT [Bern et al. '21].



General Relativity



High-energy physics



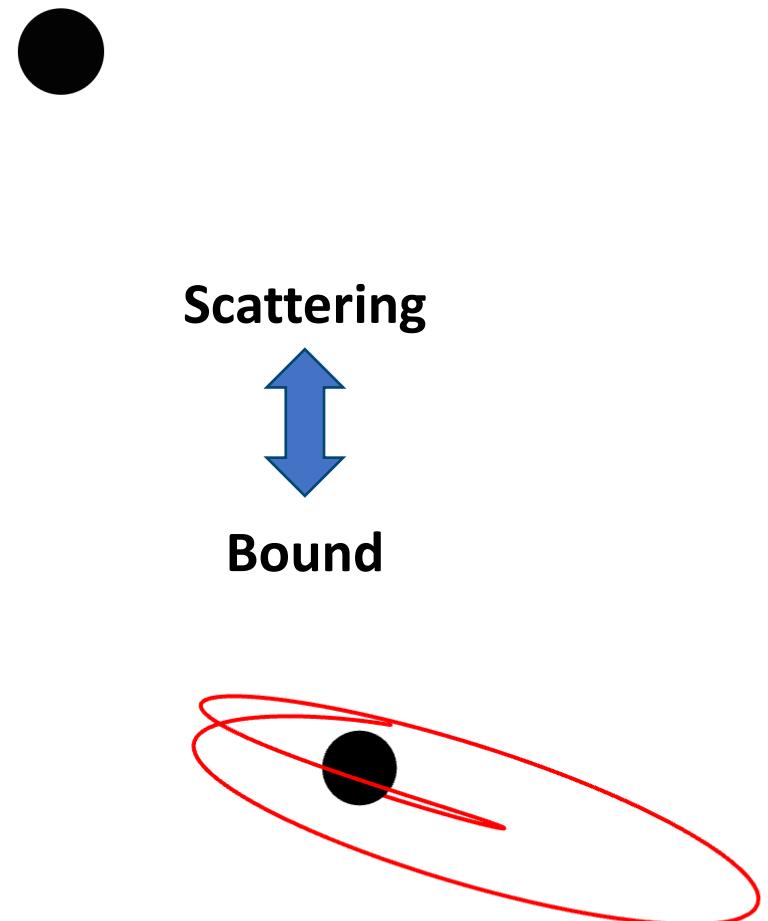
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- Clean environment with **well-defined** asymptotic states.
- **Exact** post-Minkowskian calculations for bound orbits [Damour '19; Bini, Damour & Geralico '20].

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- Calibration of Effective-One-Body (EOB) in the **ultra-strong** field.
- Comparisons with **scatter amplitude** calculations:
 - QFT and EFT [Bern et al. '21].
- Dictionary between scatter and bound [Cho et al. '21].
 - Scattering angle \leftrightarrow periastron advance.



Scattering geodesics



Energy and angular momentum:

$$E > 1 \quad L > L_{\text{crit}}(E)$$

Scattering geodesics



Energy and angular momentum:

$$E > 1 \quad L > L_{\text{crit}}(E)$$

Velocity at infinity and impact parameter:

$$v := \frac{dr}{dt} \Big|_{r \rightarrow \infty} \quad b := \lim_{r \rightarrow \infty} r \sin |\varphi(r) - \varphi(\infty)|$$

Geodesics:

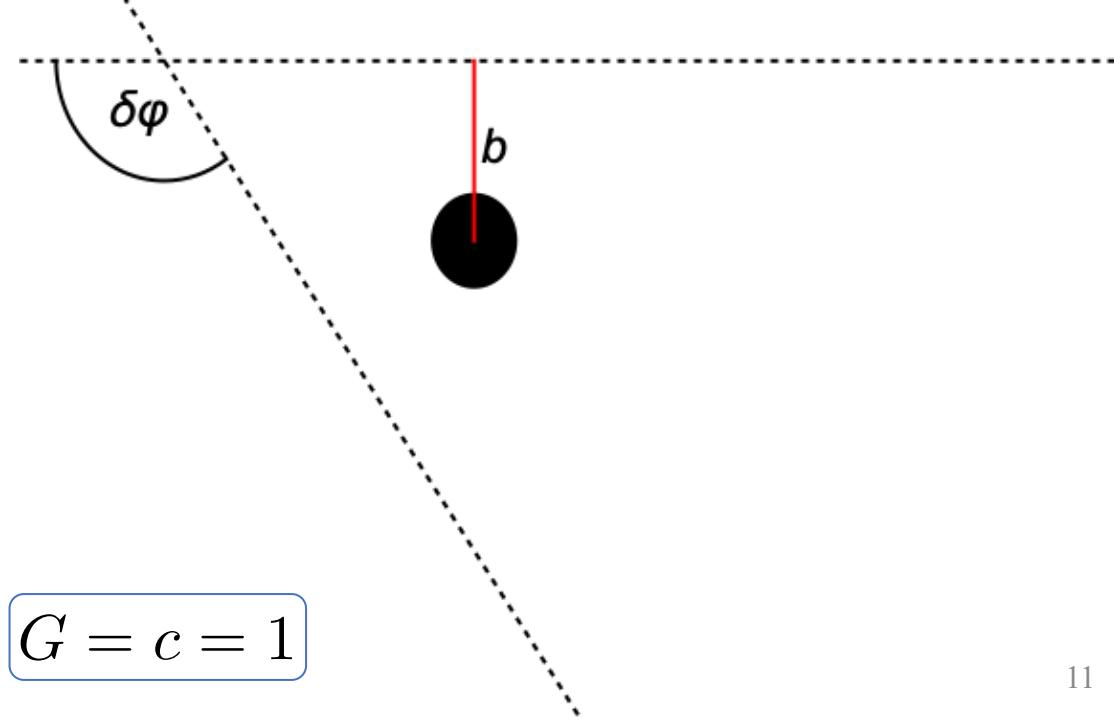
$$\frac{dt}{d\tau} = \frac{E r}{r - 2M}$$

$$\frac{d\varphi}{d\tau} = \frac{L}{r^2}$$

$$\left(\frac{dr}{d\tau} \right)^2 = E^2 - V(L; r)$$

Scattering angle:

$$\delta\varphi := \int_{-\infty}^{\infty} \frac{d\varphi}{dt} dt - \pi$$



$$G = c = 1$$

Self-forced equations of motion



Expansion in the mass ratio:

$$g_{\alpha\beta} = g_{\alpha\beta} + \eta h_{\alpha\beta}^{(1)} + \eta^2 h_{\alpha\beta}^{(2)} + \dots$$

$$\eta := \frac{\mu}{M}$$

Perturbed equations of motion:

Schwarzschild/Kerr

$$E = E_\infty - \eta \int_{-\infty}^{\tau} F_t d\tau$$

$$L = L_\infty + \eta \int_{-\infty}^{\tau} F_\varphi d\tau$$

Geodesic

Can split self-force into **conservative** and **dissipative** pieces:

$$F_\alpha^{\text{cons}}(r, \dot{r}) = -F_\alpha^{\text{cons}}(r, -\dot{r})$$

$$F_\alpha^{\text{diss}}(r, \dot{r}) = F_\alpha^{\text{diss}}(r, -\dot{r})$$

$$\alpha = t, \varphi$$

Dissipative self-force **removes energy and angular momentum from the system.**

Self-force correction to the scattering angle

Scattering angle as a radial integral:

$$\delta\varphi = \sum_{\pm} \int_{r_p^{\pm}}^{\infty} \frac{\dot{\varphi}^{\pm}}{\dot{r}^{\pm}} dr - \pi = \sum_{\pm} \int_{r_p^{\pm}}^{\infty} \frac{H^{\pm}(r; E, L)}{\sqrt{r - r_p^{\pm}}} dr - \pi$$

Perturb equation [Barack & OL '22]:

$$\delta\varphi = \delta\varphi^{(0)} + \eta\delta\varphi^{(1)}$$

$$\delta\varphi^{(1)} = \sum_{\pm} \int_{R_{\min}}^{\infty} [\mathcal{G}_E^{\pm}(r; E_{\infty}, L_{\infty}) \boxed{F_t^{\pm}} - \mathcal{G}_L^{\pm}(r; E_{\infty}, L_{\infty}) \boxed{F_{\varphi}^{\pm}}] dr$$

Can split into **conservative** and **dissipative** pieces on outgoing leg:

$$\delta\varphi_{\text{cons}}^{(1)} = \int_{R_{\min}}^{\infty} [\mathcal{G}_E^{\text{cons}} F_t^{\text{cons}} - \mathcal{G}_L^{\text{cons}} F_{\varphi}^{\text{cons}}] dr \quad \delta\varphi_{\text{diss}}^{(1)} = \int_{R_{\min}}^{\infty} [\mathcal{G}_E^{\text{diss}} F_t^{\text{diss}} - \mathcal{G}_L^{\text{diss}} F_{\varphi}^{\text{diss}}] dr$$

Scalar self-force model



Endow particle with a spin-0 scalar charge q .

Keep inertial mass but ignore gravitational mass.

New small expansion parameter:

$$\eta := \frac{q^2}{\mu M}$$

Scalar field Φ obeys the Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} \partial^\alpha \Phi) = -4\pi q \int_{-\infty}^{\infty} \frac{1}{\sqrt{-g}} \delta^4 (x^\mu - x_p^\mu(\tau)) d\tau$$

Scalar self-force:

$$F_\mu := q \nabla_\mu \Phi \Big|_{x_p}$$

Scalar self-force in terms of amplitudes

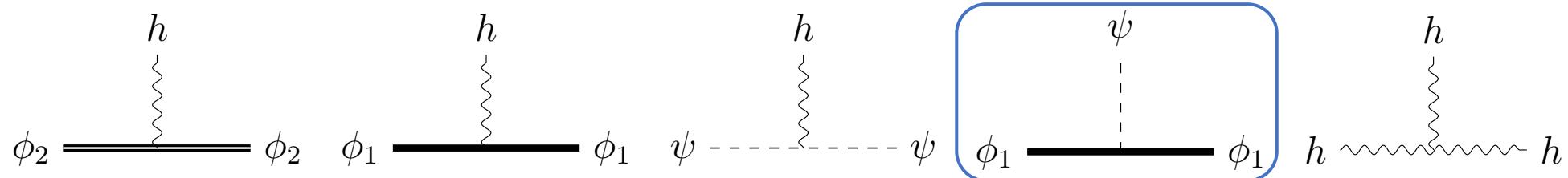
Lagrangian:

$$S = \int d^D x \sqrt{-g} \left[-\frac{2}{\kappa^2} R + \frac{1}{2} \phi_1 (\square + m_1^2) \phi_1 + \frac{1}{2} \phi_2 (\square + m_2^2) \phi_2 + \frac{1}{2} \psi \square \psi + \frac{1}{2} q \psi \phi_1^2 \right]$$

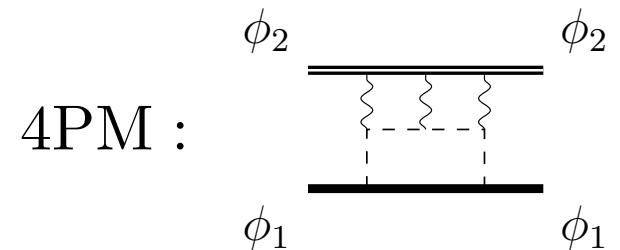
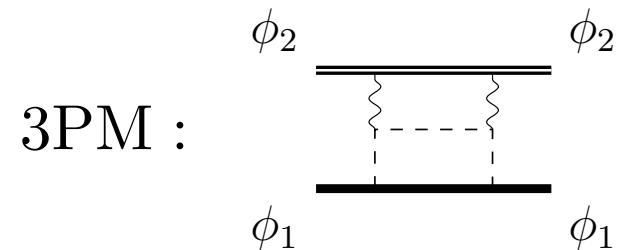
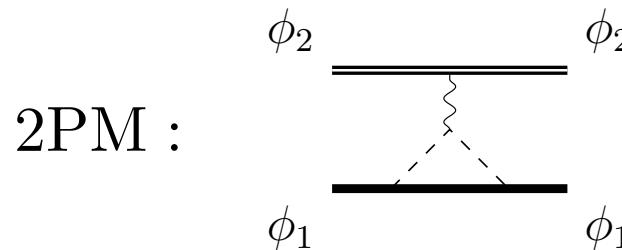
$\phi_{1,2}$: black holes

ψ : scalar field

Three-point interaction vertices:



Only keep terms which are **linear** in mass-ratio and proportional to q^2 :



Scattering angle correction: PM expansion

2PM [Gralla & Lobo '22]:

$$\delta\varphi_{\text{cons}}^{\text{2PM}} = -\frac{\pi}{4} \left(\frac{M}{b} \right)^2 \quad \delta\varphi_{\text{diss}}^{\text{2PM}} = 0$$

3PM:

$$\delta\varphi_{\text{cons}}^{\text{3PM}} = -\frac{4(3-v^2)}{3v^2\sqrt{1-v^2}} \left(\frac{M}{b} \right)^3 \quad \delta\varphi_{\text{diss}}^{\text{3PM}} = \frac{2(v^2+1)^2}{3v^3\sqrt{1-v^2}} \left(\frac{M}{b} \right)^3$$

4PM dissipative:

$$\delta\varphi_{\text{diss}}^{\text{4PM}} = \left(r_1 + r_2 \operatorname{arcsech} \left(\sqrt{1-v^2} \right) + r_3 \log \left[\frac{1}{2} \left(\frac{1}{\sqrt{1-v^2}} + 1 \right) \right] \right) \left(\frac{M}{b} \right)^4$$

r_i = rational coefficients

Scalar field evolution scheme

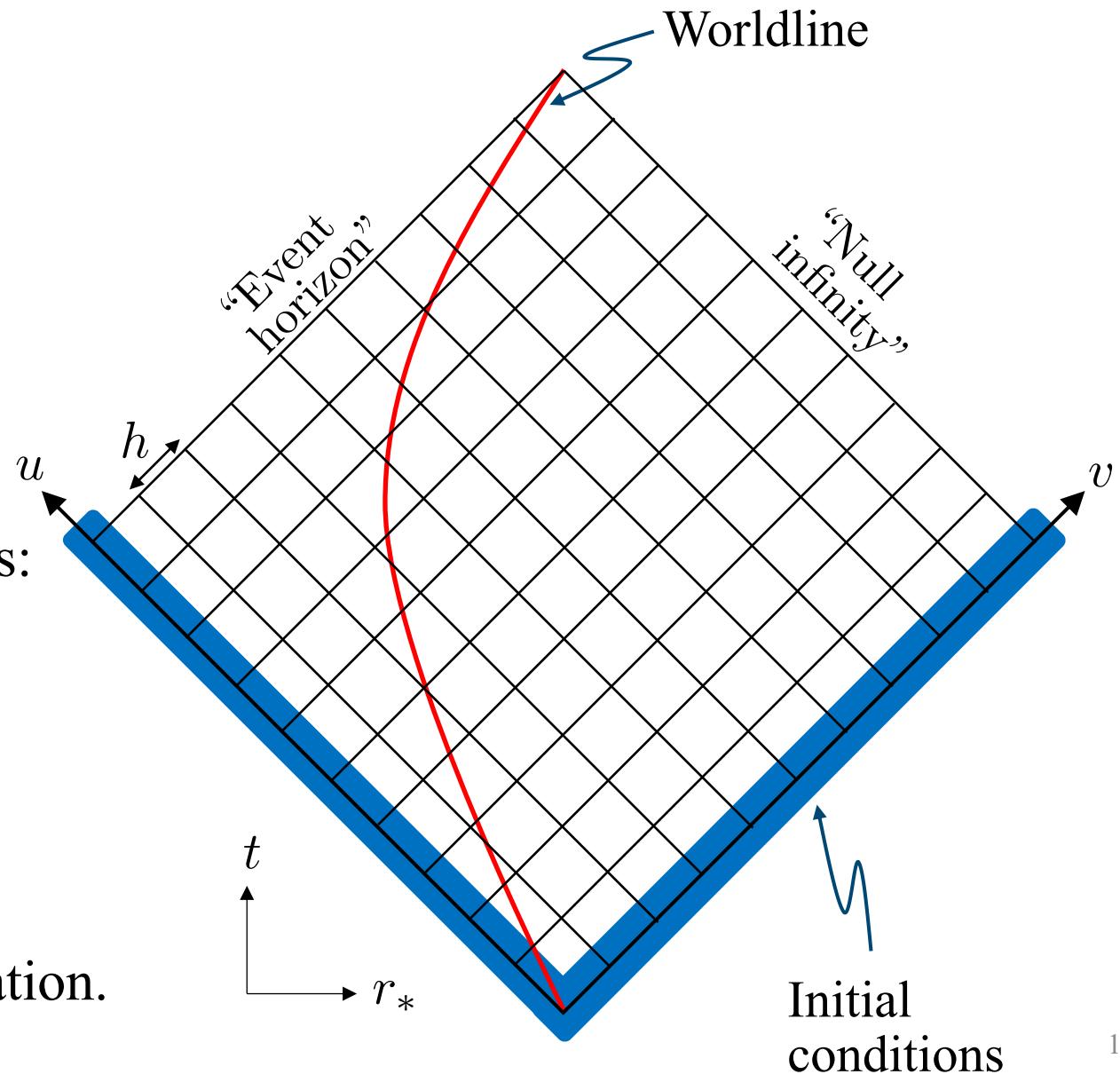
Decompose scalar field in the time-domain:

$$\Phi = \frac{2\pi q}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \psi_{\ell m}(t, r) Y_{\ell m}(\theta, \varphi)$$

1+1D scalar wave equation in null-coordinates:

$$\psi_{,uv} + V(\ell; r)\psi = S_\psi(\ell; x_p^\mu) \delta(r - R)$$

Evolve finite-difference version of 1+1D equation.



Post-processing: Truncation at finite radius

Can only numerically determine the self-force up to a **finite radius** $R = R_{\text{final}}$:

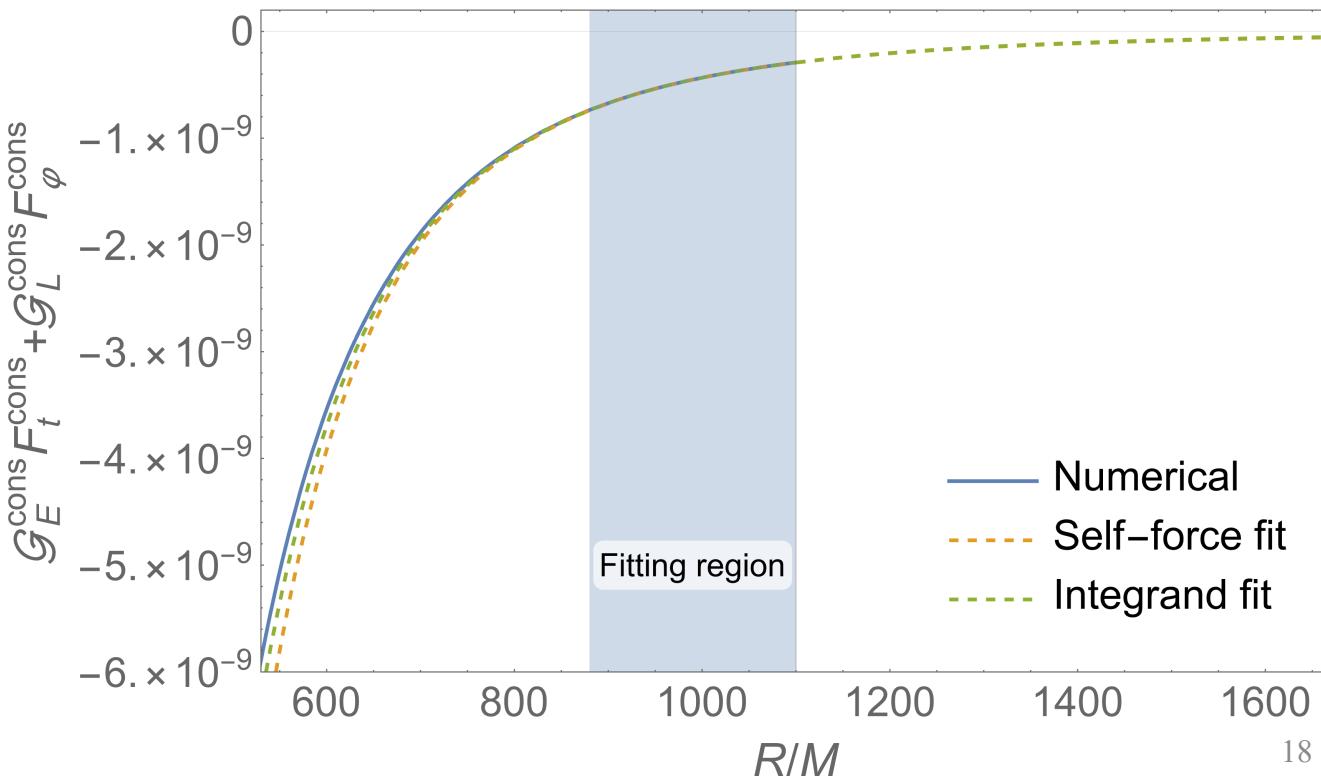
$$\delta\varphi^{(1)} = \int_{R_{\min}}^{R_{\text{final}}} [\mathcal{G}_E F_t - \mathcal{G}_L F_\varphi] dr + \int_{R_{\text{final}}}^{\infty} [\mathcal{G}_E F_t - \mathcal{G}_L F_\varphi] dr$$

Numerical Error ($\sim 1\%$)

Form an analytic tail by fitting to the data:

- Fit the **self-force** data.
- Fit the **integrand** directly.

Tail contributes an error $\sim 0.01\%$.



Post-processing: Richardson extrapolation

Next dominant error due to **finite resolution** $\sim 0.1\%$.

Can increase the convergence from quadratic to **cubic** using Richardson extrapolation.

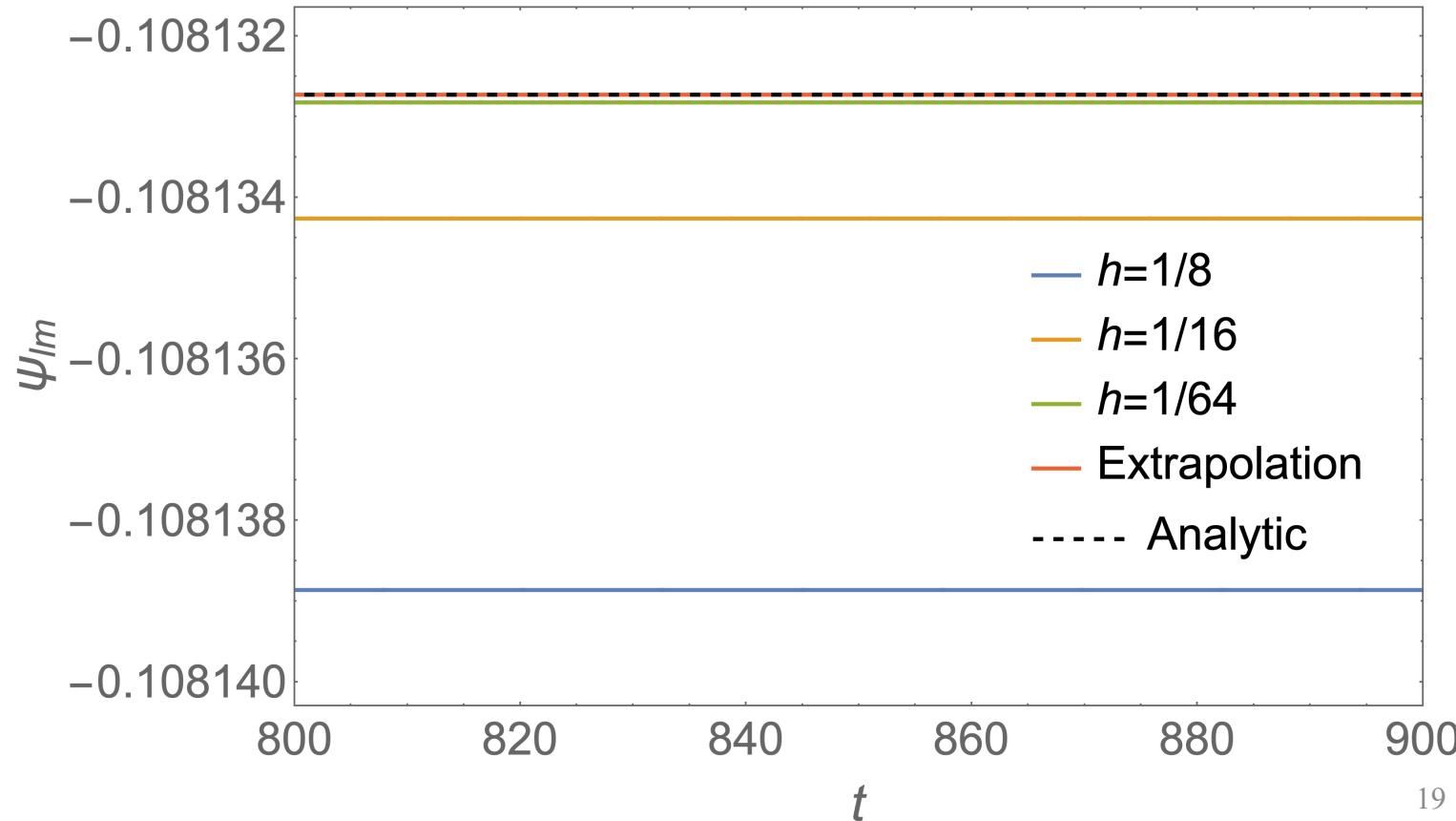
Model:

$$A(h) = A_{\text{exact}} + Ch^n + O(h^{n+1})$$

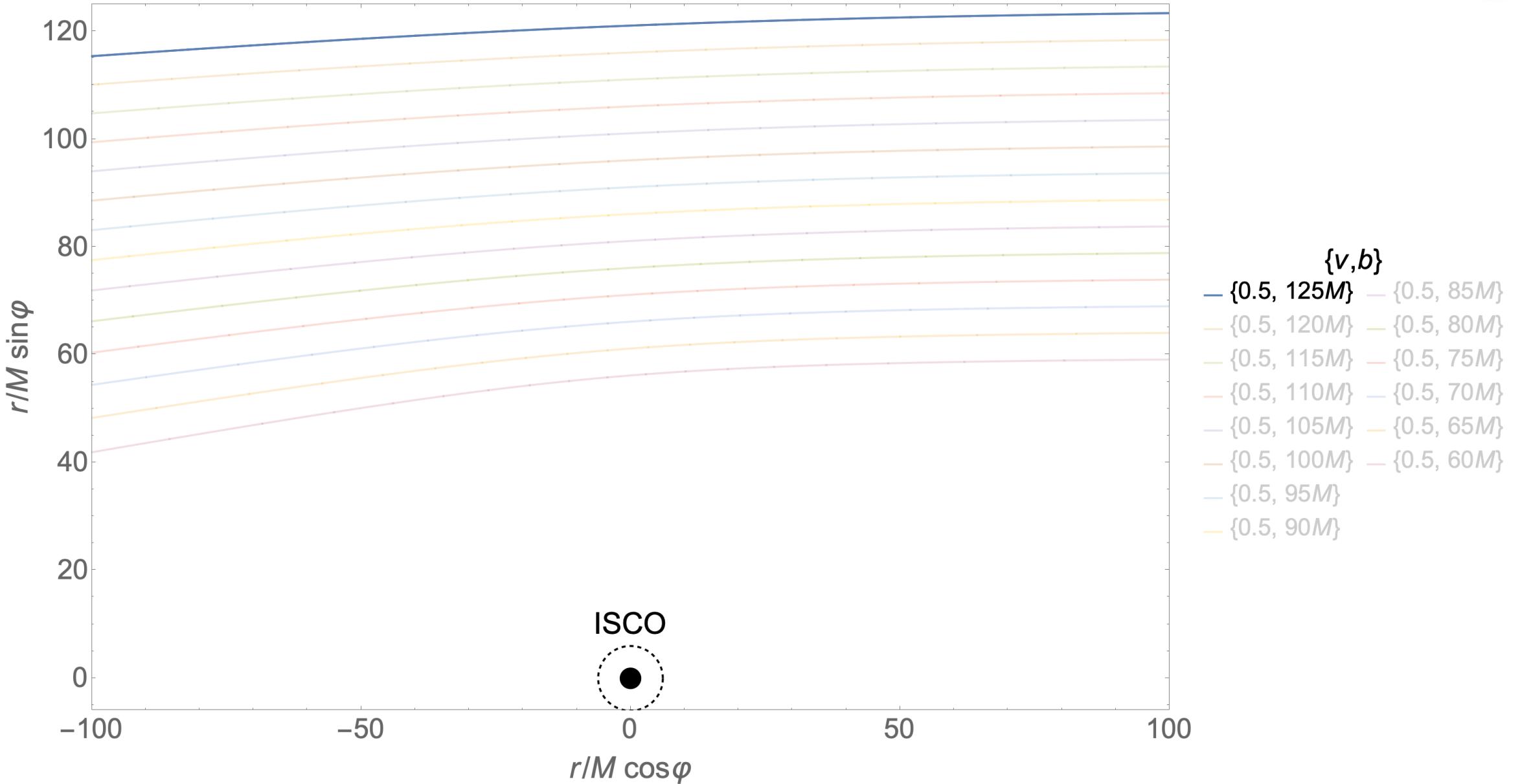
Extrapolation:

$$\begin{aligned} A_{\text{Extr}} &= \frac{t^n A\left(\frac{h}{t}\right) - A(h)}{t^n - 1} \\ &= A_{\text{exact}} + O(h^{n+1}) \end{aligned}$$

Error in extrapolation $< 0.001\%$.



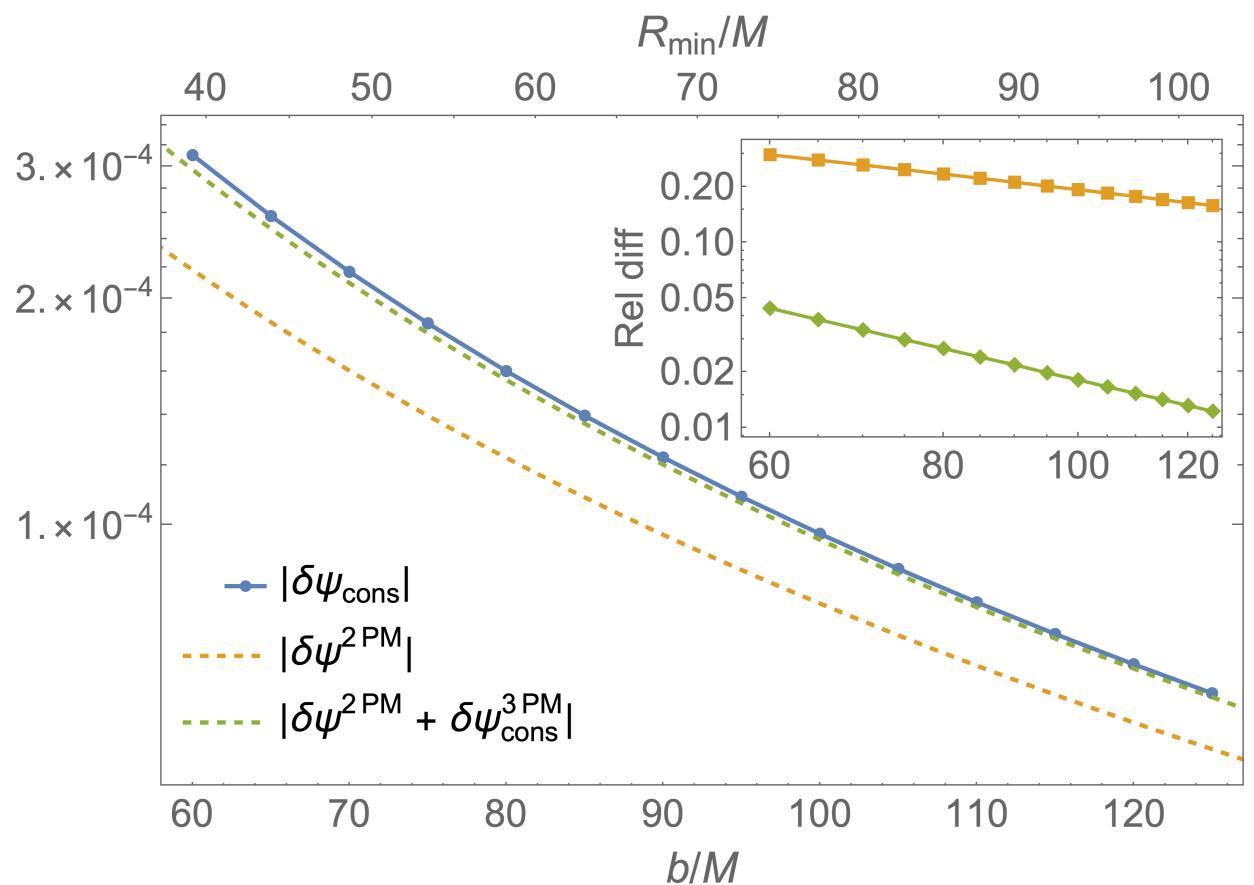
Sample orbits



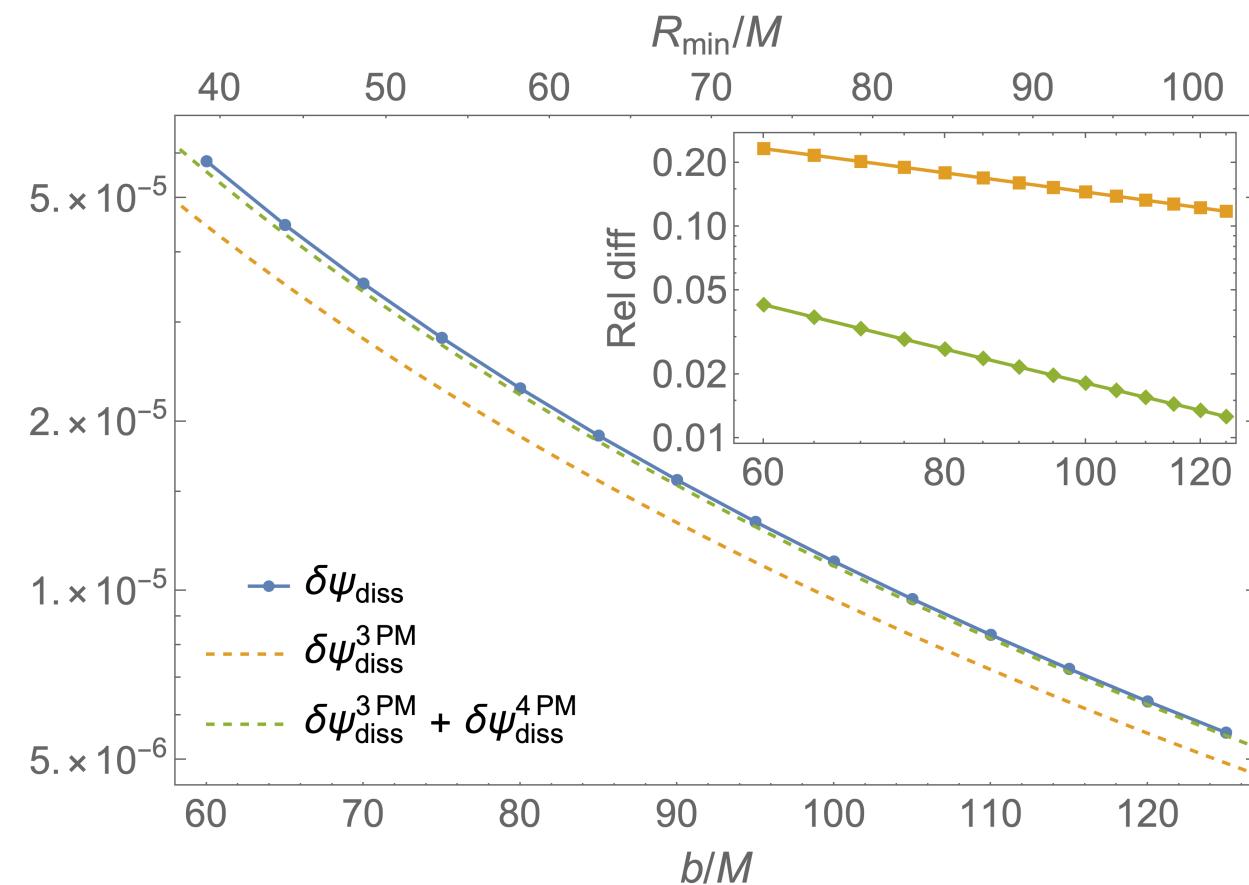
PM comparison: $v = 0.5$



Conservative



Dissipative



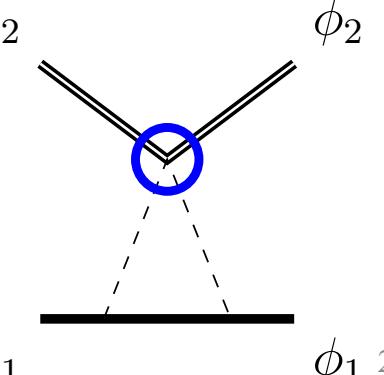
Scattering angle correction: 4PM conservative



$$\begin{aligned}
 \delta\varphi_{\text{cons}}^{\text{4PM}} = & \left(r_1 + r_2 \operatorname{arccosh} \left(\frac{1}{\sqrt{1-v^2}} \right) + r_3 \operatorname{arccosh} \left(\frac{1}{\sqrt{1-v^2}} \right)^2 + r_4 E \left(-\frac{v^2 + 2\sqrt{1-v^2} - 2}{v^2} \right)^2 \right. \\
 & \left. + r_5 K \left(-\frac{v^2 + 2\sqrt{1-v^2} - 2}{v^2} \right) E \left(-\frac{v^2 + 2\sqrt{1-v^2} - 2}{v^2} \right) + r_6 K \left(-\frac{v^2 + 2\sqrt{1-v^2} - 2}{v^2} \right)^2 \right. \\
 & \left. + r_7 \log \left(\frac{v}{2\sqrt{1-v^2}} \right) + r_8 \log \left(\frac{v}{2\sqrt{1-v^2}} \right) \operatorname{arccosh} \left(\frac{1}{\sqrt{1-v^2}} \right) \right. \\
 & \left. + r_9 \log \left(\frac{v}{2\sqrt{1-v^2}} \right) \log \left(\frac{1}{2} \left(\frac{1}{\sqrt{1-v^2}} + 1 \right) \right) + r_{10} \log \left(\frac{1}{2} \left(\frac{1}{\sqrt{1-v^2}} + 1 \right) \right) \right. \\
 & \left. + r_{11} \log^2 \left(\frac{1}{2} \left(\frac{1}{\sqrt{1-v^2}} + 1 \right) \right) + r_{12} \alpha + r_{13} \frac{\beta}{v^2} + r_4 \log(b) \right) \left(\frac{M}{b} \right)^4
 \end{aligned}$$

↗ Elliptic integrals
↗ Free coefficients
↗ Log term

r_i = rational coefficients

ϕ_1 

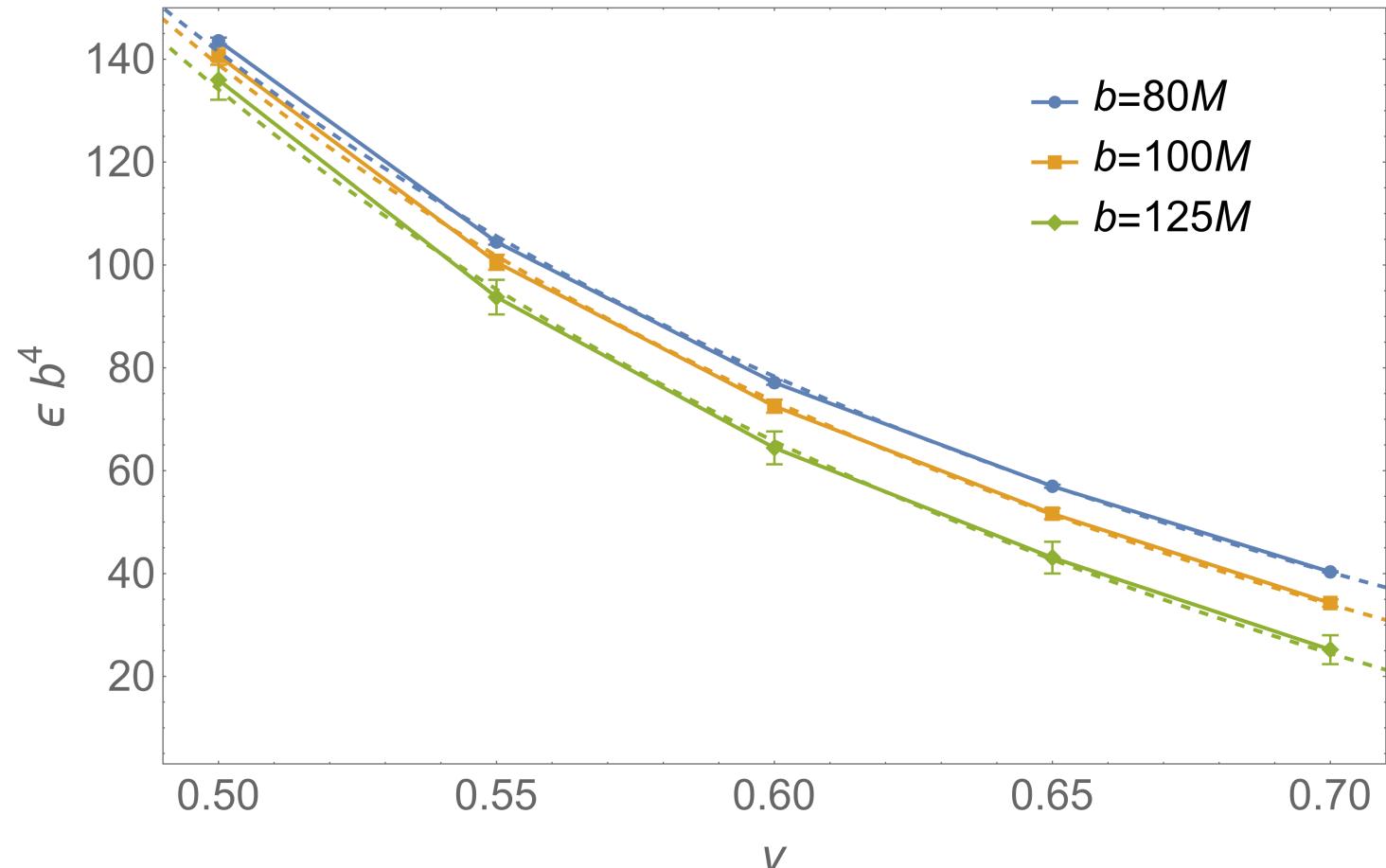
Preliminary

Extraction of high-order conservative PM results

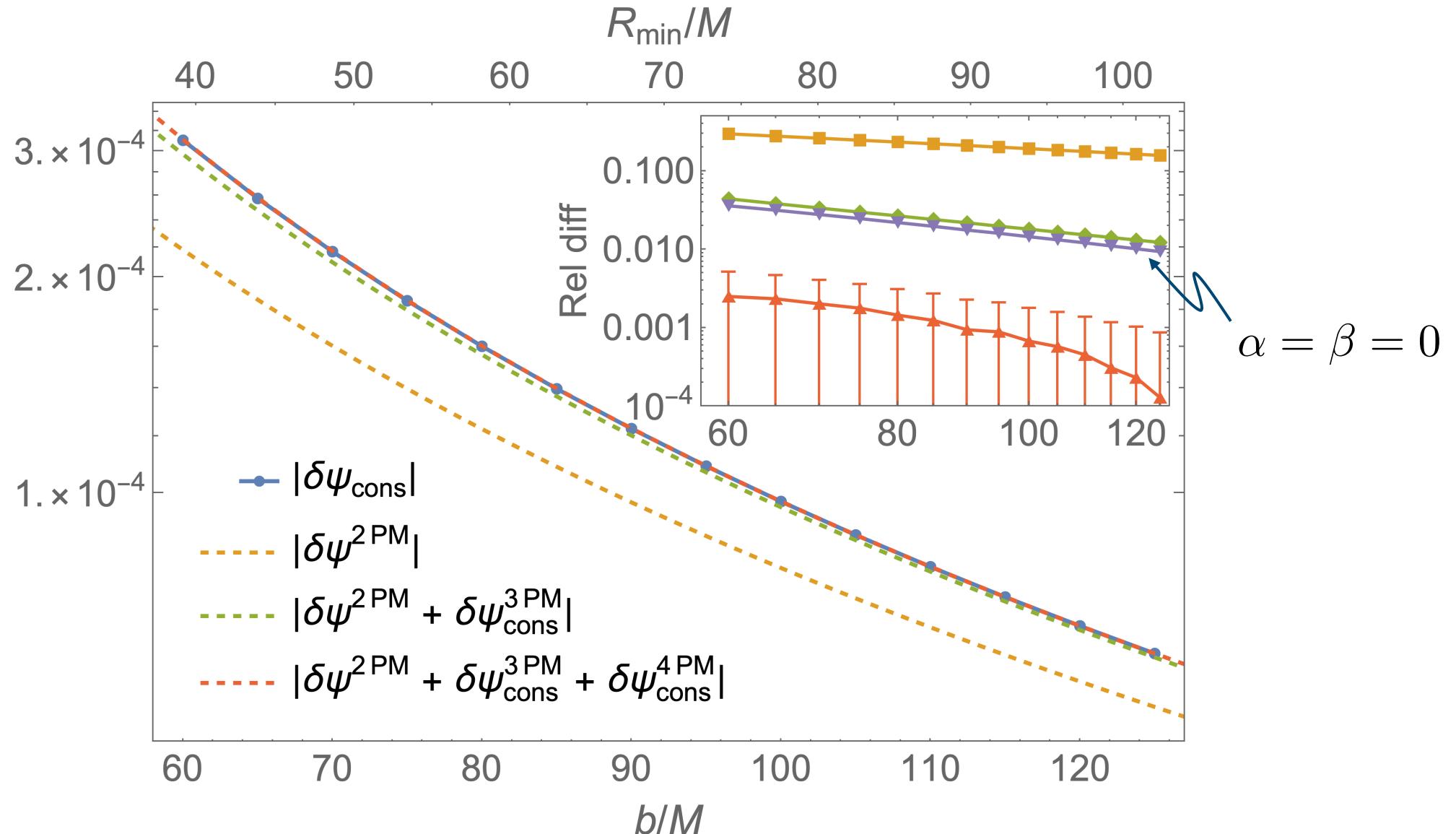
Subtract known [analytic](#) parts:

$$\epsilon := \delta\phi_{\text{cons}} - \delta\phi_{\text{cons}}^{\text{2PM}} - \delta\phi_{\text{cons}}^{\text{3PM}} - \frac{A(v) + B \log(b)}{b^4} = \frac{\pi}{32} \left(\alpha + \frac{\beta}{v^2} \right) \frac{1}{b^4} + O\left(\frac{1}{b^5}\right)$$

b/M	α	β
80	-202.6 ± 1.7	20.07 ± 0.09
100	-96.8 ± 2.7	8.52 ± 0.13
125	-114 ± 8	8.91 ± 0.35



PM comparison: Conservative $v = 0.5$



Extraction of high-order dissipative PM results

PM expansion with free parameters:

$$\delta\varphi_{\text{diss}}^{\text{PM}} = \frac{a_3}{b^3} + \frac{a_4}{b^4} + \frac{a_5}{b^5} + \frac{a_6}{b^6} + \dots$$

Up to 4PM can fit value or use
analytic value.

a_3	a_4	a_5	a_6
11.2 ± 0.1			
9.439 ± 0.01	184 ± 1		
9.632 ± 0.002	143.8 ± 0.4		
9.593 ± 0.002	152.7 ± 0.3		
9.6225	166.7 ± 1.3		
9.6225	146.82 ± 0.09	1707 ± 7	
9.6225	148.0 ± 0.02	1476 ± 34	-62100 ± 8700
9.6225	143.344	1987 ± 17	
9.6225	143.344	2372 ± 29	-25500 ± 2100

< 1% $\sim 1\%$ $\sim 1600(?)$???

Summary and future work

Compared numerical **scalar self-force** results with analytic PM results.

Good **agreement** for both conservative and dissipative:

- $\sim 10\%$ for LO.
- $\sim 1\%$ for NLO.

Have investigated extractions of higher-order components from numerics:

- Fixed **free parameters** in 4PM conservative: agreement with numerics to $< 0.5\%$.
- Extraction of 5PM dissipative coefficient.

Future work:

- Calculate the **gravitational self-force** correction to the scattering angle.
- Compare to PM(/NR/EOB) in the gravitational case.

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