4M17 Coursework Assignment 1

Practical Optimisation

2023/24

1. Consider the norm approximation problem

$$\min_{x} \quad \|Ax - b\|, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given, $x \in \mathbb{R}^n$ is the variable and $\|\cdot\|$ is a norm on $\mathbb{R}^m$.

(a) Define the l_1, l_2, and l_∞-norms on $\mathbb{R}^m$ and write down the norm approximation problem in terms of the components of A, x and b for each of these norms.

For the l_2-norm, show that the problem can be expressed as an optimisation problem with a convex quadratic function with an analytic solution, which amounts to solving a linear system of equations.

(b) Show that the norm approximation problems corresponding to the l_1 and l_∞-norms on $\mathbb{R}^m$ can be cast as linear programming (LP) problems of the form

$$\begin{aligned} \min_{\tilde{x}} \quad & \tilde{c}^T\tilde{x} \\ & \tilde{A}\tilde{x} \le \tilde{b}. \end{aligned} \tag{2}$$

Specify the dimensions of all terms and provide expressions for $\tilde{A}$, $\tilde{b}$, $\tilde{c}$ in terms of A and b for each of the two norms.

(c) Derive the dual problem for the above primal problem.

(d) In the coursework data folder, 5 pairs of problem data `(A0, b0)`, $\ldots$, `(A4, b4)` are provided for $m = 2n$ and $n = 16$, 64, 256, 512, 1024. Use a (i) simplex method-based and an (ii) interior point linear programming solver to solve the LP problems for the l_1 and l_∞-norms, and a linear solver for the l_2-norm case.

Produce a table showing the values of the minimised l_1, l_2, and l_∞ norms $\|Ax - b\|$ corresponding to each of the 5 pairs of data and for each LP solver, and the running time of each case.

You may use SciPy or other LP libraries to solve this problem.

(e) Solve the dual problem for the l_1 and l_∞ problems for each data set and compute the duality gap. Report your results.

2. Consider the optimisation problem

$$\begin{aligned} &\min_{x} \ f_0(x) \\ &\text{subject to } f_i(x) \le 0, \quad i = 1, \ldots, m \end{aligned}$$

where f_0 and f_i are convex and twice continuously differentiable functions. The central path is the solution of the unconstrained minimisation

$$\min_{x} \ tf_0(x) + \phi(x) \tag{3}$$

for any $t \ge 0$, where ϕ is the logarithmic barrier function of the feasibility set $S = \{x : f_i(x) \le 0\}$.

(a) Write down the form of eq. (3) for the formulation of the l_1-norm approximation problem from the first part of the coursework assignment. Calculate the expression for the gradient of the cost in eq. (3) corresponding to a fixed $t \geq 0$.

(b) Apply a first-order gradient method with backtracking linesearch to solve the problem in (2a) for $t = 2$ using the pair `(A3,b3)` in the data folder ($n = 256$). Report the computed cost function and compare your results to Question 1d.

(c) Place an outer loop around your method from (2b) and perform three outer loops, starting with $t = 1$ and increasing t by factor of 10 for each outer loop. Report the computed cost function and compare your results to Question 1d, and comment on performance of the algorithm.

3. Consider the l_1-regularised least squares problem

$$\min_x \|Ax - b\|_2^2 + \lambda \|x\|_1 \tag{4}$$

where $\lambda > 0$ is a regularisation parameter.

(a) Show that eq. (4) can be transformed to a convex quadratic problem with linear inequality constraints. Define a suitable logarithmic barrier function Φ that characterises the inequality constraints and show that the central path formulation of eq. (4) takes the form

$$\phi_t(x, u) = t\|Ax - b\|_2^2 + t\lambda \sum_{i=1}^{n} u_i + \Phi(x, u) \tag{5}$$

where the parameter t varies from 0 to ∞.

(b) Derive expressions for the gradient and Hessian of ϕ_t.

(c) We consider a sparse signal recovery problem with a signal $x_0 \in \mathbb{R}^{256}$ which consists of a small number of spikes of amplitude ± 1. The measurement matrix $A \in \mathbb{R}^{60\times 256}$ and x_0 are given in the coursework data folder. The vector of observations is $b = Ax_0 + \epsilon$, where ϵ is uniform random noise in the range $(-0.005, 0.005)$.

From A and b, perform a sparse signal reconstruction by applying a Newton interior-point method to eq. (5) with regularisation parameter $\lambda = 0.01\lambda_{\max}$ with $\lambda_{\max} = \|2A^T b\|_\infty$, and plot the original and reconstructed signals.

(d) Examine the influence of the regularisation parameter λ.

Your Report

- The deadline for submitting your report is given on the course Moodle page. You need to submit an electronic copy via Moodle, including a coursework cover sheet. *Undergraduates should only include their Coursework Candidate Numbers (CCN), and must ensure that their names do NOT appear in the report, or in any of the file names.*
- *Marks will be deducted if you fail to use a coursework coversheet.*
- Include figures and associated explanations as part of your report. Plots without explanations will receive very few marks. Figures should be carefully designed and presented.
- Attach the source code of all programs you have developed as an appendix to your report.
- Your report should not exceed 10 pages in length, excluding source code.
- This assignment counts for 50% of your final grade. Questions on this assignment should be directed to Garth Wells (gnw20).