Module 4M17: Practical Optimisation

Coursework 1 introduction

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Administration

- Due date: 8 December 2023, 16:00
- Submit on Moodle

Problem 1 I

1. Consider the norm approximation problem

$$\min_{x} \quad \|Ax - b\|, \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given, $x \in \mathbb{R}^n$ is the variable and $\|\cdot\|$ is a norm on $\mathbb{R}^m$.

   (a) Define the l_1, l_2, and l_∞-norms on $\mathbb{R}^m$ and write down the norm approximation problem in terms of the components of A, x and b for each of these norms.
   For the l_2-norm, show that the problem can be expressed as an optimisation problem with a convex quadratic function with an analytic solution, which amounts to solving a linear system of equations.

Problem 1 II

   (b) Show that the norm approximation problems corresponding to the l_1 and l_∞-norms on $\mathbb{R}^m$ can be cast as linear programming (LP) problems of the form

$$\begin{aligned} \min_{\tilde{x}} \quad & \tilde{c}^T \tilde{x} \\ & \tilde{A}\tilde{x} \le \tilde{b}. \end{aligned} \tag{2}$$

   Specify the dimensions of all terms and provide expressions for $\tilde{A}$, $\tilde{b}$, $\tilde{c}$ in terms of A and b for each of the two norms.

   (c) Derive the dual problem for the above primal problem.

   (d) In the coursework data folder, 5 pairs of problem data `(A0, b0),...,(A4, b4)` are provided for $m = 2n$ and $n = 16$, 64, 256, 512, 1024. Use a (i) simplex method-based and an (ii) interior point linear programming solver to solve the LP problems for the l_1 and l_∞-norms, and a linear solver for the l_2-norm case.

Problem 1 III

Produce a table showing the values of the minimised l_1, l_2, and l_∞ norms $\|Ax - b\|$ corresponding to each of the 5 pairs of data and for each LP solver, and the running time of each case. *You may use SciPy or other LP libraries to solve this problem.*

(e) Solve the dual problem for the l_1 and l_∞ problems for each data set and compute the duality gap. Report your results.

Problem 2 I

2 Consider the optimisation problem

$$\min_x \; f_0(x)$$
$$\text{subject to } f_i(x) \leq 0, \quad i = 1, \ldots, m$$

where f_0 and f_i are convex and twice continuously differentiable functions. The central path is the solution of the unconstrained minimisation

$$\min_x \; tf_0(x) + \phi(x) \tag{3}$$

for any $t \geq 0$, where ϕ is the logarithmic barrier function of the feasibility set $S = \{x : f_i(x) \leq 0\}$.

(a) Write down the form of eq. (3) for the formulation of the l_1-norm approximation problem from the first part of the coursework assignment. Calculate the expression for the gradient of the cost in eq. (3) corresponding to a fixed $t \geq 0$.

Problem 2 II

(b) Apply a first-order gradient method with backtracking linesearch to solve the problem in (a) for $t = 2$ using the pair (`A3,b3`) in the data folder ($n = 256$). Report the computed cost function and compare your results to Question d.

(c) Place an outer loop around your method from (b) and perform three outer loops, starting with $t = 1$ and increasing t by factor of 10 for each outer loop. Report the computed cost function and compare your results to Question d, and comment on performance of the algorithm.

Problem 3 I

3 Consider the l_1-regularised least squares problem

$$\min_x \|Ax - b\|_2^2 + \lambda \|x\|_1 \tag{4}$$

where $\lambda > 0$ is a regularisation parameter.

(a) Show that eq. (4) can be transformed to a convex quadratic problem with linear inequality constraints. Define a suitable logarithmic barrier function Φ that characterises the inequality constraints and show that the central path formulation of eq. (4) takes the form

$$\phi_t(x, u) = t\|Ax - b\|_2^2 + t\lambda \sum_{i=1}^{n} u_i + \Phi(x, u) \tag{5}$$

where the parameter t varies from 0 to ∞.

(b) Derive expressions for the gradient and Hessian of ϕ_t.

Problem 3 II

(c) We consider a sparse signal recovery problem with a signal $x_0 \in \mathbb{R}^{256}$ which consists of a small number of spikes of amplitude ± 1. The measurement matrix $A \in \mathbb{R}^{60 \times 256}$ and x_0 are given in the coursework data folder. The vector of observations is $b = Ax_0 + \epsilon$, where ϵ is uniform random noise in the range $(-0.005, 0.005)$.
From A and b, perform a sparse signal reconstruction by applying a Newton interior-point method to eq. (5) with regularisation parameter $\lambda = 0.01\lambda_{\text{max}}$ with $\lambda_{\text{max}} = \|2A^T b\|_\infty$, and plot the original and reconstructed signals.

(d) Examine the influence of the regularisation parameter λ.

Tips

- Use short functions to structure your code. Design the functions such that you can pass in problem-specific data or functions. This will allow you yo re-use code and test your code for simple test cases.

Tips: question 1

- Question 1: Scipy has LP solvers that are useful. Read the documentation very carefully to get the parameters right. `https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.linprog.html`

Tips: question 2

- Build up in steps, testing your implementation for simple cases before moving to the question problem. E.g., start with an unconstrained quadratic cost function.
- Typical stopping criteria is $\left\|\nabla(x)\right\|_2 < \epsilon$.
- See lecture notes for backtracking.
- Test for admissibility in your backtracking algorithm.
- Carefully consider how you can find an admissible starting value.

Tips: question 3

- Structure your code for functions that return the Jacobian (∇f), Hession ($\nabla^2 f$).
- Test your Newton implementation for a quadratic cost function – it should converge in one iteration.
- Take care with how you plot sparse, discrete data.

Some useful NumPy/SciPy modules and functions

```
numpy.block
numpy.dot
numpy.isnan
numpy.linalg.norm
scipy.optimize.linprog   # (read docs carefully)
matplotlib.pyplot.bar
matplotlib.pyplot.hist   # (see density option)
```

General points and summary

Strong reports: presentation of data

Well presented and carefully considered tables and graphs.

Strong reports: analysis and commentary

Strong reports will complete what is asked in the coursework sheet, but will also provide careful analysis and insightful commentary.

Strong reports: accuracy and presentation

Strong reports will be mathematically accurate, and professionally presented and well typeset.

Code

Include your code in appendix. Use a fixed-width font for code.

Warning

Marks will be deducted if you fail to use the correct coversheet.