09/11/23. # Garside study of the complex braid group $B(G_{31})$ Part 1

IMJ-PRG

I. Complex braid groups
II. Glimpse of Garside theory
III. Garside groupoid for $B(G_{31})$.

I) 1) Complex reflection groups

Def. Complex Reflection group = $W \subseteq GL_n(\mathbb{C})$ finite + generated by Reflections
CRG

Reflections $\longrightarrow$ [$s \in GL_n(\mathbb{C})$, Order$(s) < \infty$, $\operatorname{codim} \underbrace{\operatorname{Ker}(s-1)}_{H_s} = 1$]

"straightforward" generalization of finite Coxeter groups.

Prop: CRG behave "semi-simply"
↳ interest in irreducible CRG
↳ The representation $W \hookrightarrow GL_n(\mathbb{C})$ is irreducible.

Thm: (Shephard Todd 54)
The irreducible CRG are:
- $G(de,e,n)$ (monomial matrices)
- 34 exceptional groups $G_4, \dots, G_{37}$
(19 excep in Rank 2)

	CRG	Coxeter
eg	$G(2,2,n)$	D_n
eg	$G_{35,36,37}$	$E_{6,7,8}$

2) Complex braid groups.

Let $R \subseteq W$ be the reflections of W

$$X := \{v \in \mathbb{C}^n \,/\, \forall s \in R,\ v \notin \operatorname{Ker}(s-1)\} = \{v \in \mathbb{C}^n \,/\, \forall s \in R,\ s.v \neq v\}.$$

Thm: (Steinberg 64)
The action of W on X is free ($\forall v \in V$, W_v is gen by $R \cap W_v$).

Def: $P(W) := \pi_1(X)$ the pure braid group
$B(W) := \pi_1(X/W)$ the braid group

$1 \to P(W) \to B(W) \to W \to 1$.

→ Understand irreducible $B(W)$. (Broué, Malle, Rouquier 98)
$B(de,e,n)$ $B(G_4), \dots, B(G_{37})$.

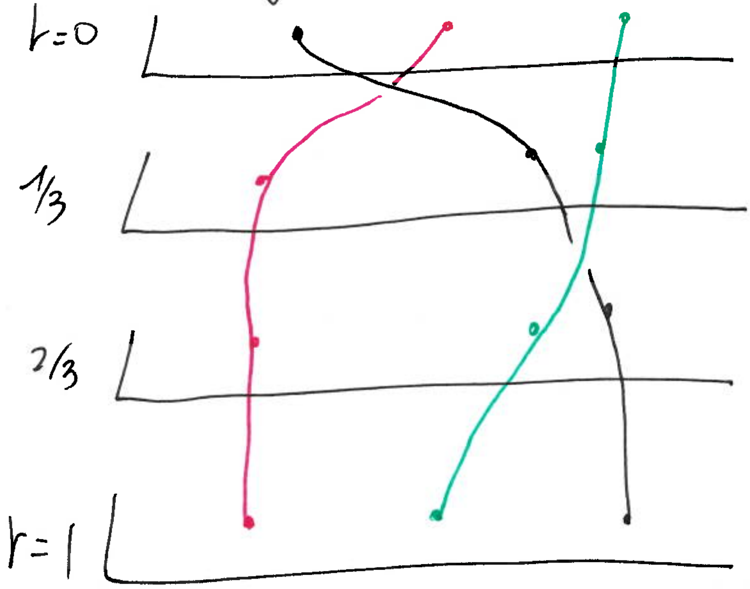
3) Why "braid"

The symmetric group $\mathfrak{S}_n$ is a CRG ($G(1,1,n)$, permutation matrices).
Reflections = transpositions,

$\mathrm{Ker}((i\,j)-1) = \{x \in \mathbb{C}^n \mid x_i = x_j\}$

↳ $X = \{x \in \mathbb{C}^n \mid \forall i \neq j,\ x_i \neq x_j\}$.

↳ $X/W = \mathrm{Conf}_n(\mathbb{C})$ $\pi_1(X/W) = \mathcal{B}r_n$ "usual braid group".



$\mathrm{Conf}_3(\mathbb{C})$

$\mathrm{Conf}_3(\mathbb{C})$.

"To what extend are $\mathcal{B}r_n$ and $B(W)$ similar in general"
↳ often!

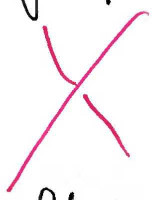
Prop: (Broué Malle Rouquier)

$B(W)$ is generated by "braided reflections" (generators of the monodromy) just like Br_n ($+\exists$ infinite number of braided reflections)

Find presentation with only a finite number of braided reflections.

II) 1) Positive braid monoid

In Br_n, let $M \subseteq Br_n$ be the monoid of "positive braids"

 ok  not ok

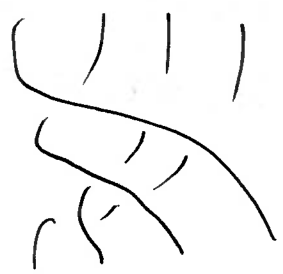
M is a monoid $\rightarrow$ left / right divisibility $\quad a \leq c \Leftrightarrow \exists b \mid ab = c$

$c \geq a \Leftrightarrow \exists b \; c = ba$.

M has very good properties:

- $\rightarrow$ length function additive (# of crossings)
- $\rightarrow$ lcms and gcds
- $\rightarrow$ M generates Br_n
- $\rightarrow$ Special element Δ (half twist, $\sim w_0 \in \mathfrak{S}_n$)

Garside 65, 69



Thm (Garside 65, Adyan 66, Thurston 88, EC Rifai Morton 94)

All $x \in Br_n$ has a unique normal form $\quad k \in \mathbb{Z}, \quad s_1 \neq \Delta$

$$x = \Delta^k s_1 \dots s_r \qquad s_i = (s_i s_{i+1}) \gcd \Delta.$$

+ k, r are well defined and useful for studying conjugacy.

Dehornoy, Paris, Squier, etc...

↳ Abstract the properties of Π into the notion of **Garside monoid**

<u>Theo</u> Brieskorn Saito, Picantin, Bessis, Corran Picantin.
Deligne

Every irreducible $B(W)$ is generated by a Garside monoid except!

$B(de, e, n)$ $d>1, e>1$ not too bad

$B(G_{31})$ $\boxed{\text{too bad}}$

Garside monoids give good presentations

↳ $B(de, e, n)$, use Reidemeister Schreier with $B(de, 1, n)$

↳ $B(G_{31})$, brutal computations in GAP3? no explicit proof with this method
Bessis Michel.

<u>Thm</u> (Digne, Marin, Michel) 11

If $U \subseteq B(W)$ has finite index, then $Z(U) \subseteq Z(B(W))$
(W irred)

Shown using Garside (+ $B(de, e, n) \to B(de, 1, n)$).
$B(G_{31})$ is handled with representation theory.

III 1). Definition

<u>Thm</u> (Bessis 15)

$B(G_{31})$ is equivalent to a groupoid $\mathcal{B}_{31}$, wich contains a <u>Garside Category</u> C_{31}

↳ Krammer, Bessis, Digne Michel $\rightsquigarrow$ DDGKM 15.
05 - 10

Cor: In $\mathcal{B}_{31}$, maps are written uniquely

$f: u \to v$ $\quad f = \underbrace{\Delta^k(u)\, s_1 \dots s_r}_{\text{a path}}$ $\quad$ Same conditions as before.

Can we see braided reflections in $\mathcal{B}_{31}$?. $\quad B(G_{31}) = \mathcal{B}_{31}(u_0,u_0)$.

2) Atomic loops

Because of the length functor on C_{31} (Garside). There are no nontrivial isomorphisms in C_{31}.

\+ elements of length 1 = atoms

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "a"'] & & \cdot \end{tikzcd} \Rightarrow f = a \text{ or } g = a$$

Prop (G23) For all atom $a: u \to v$ in C_{31}, $\exists!\, a^\#: v \to u$ atom. The composition $a a^\# = \lambda(a) \in \mathcal{B}_{31}(u,u)$ is an atomic loop

Theo (G23) $\sigma \in B(G_{31}) \simeq \mathcal{B}_{31}(u_0,u_0)$ is a braided refl if and only if $\exists f: u_0 \to v$ in $\mathcal{B}_{31}$, and $\lambda \in \mathcal{B}_{31}(v,v)$ atomic loop such that $\sigma f = \lambda$.

"braided refl. are the conjugates in $\mathcal{B}_{31}$ of atomic loops".

660 atom loops.

Theo (G23). Let $\lambda \in \mathcal{B}_{31}(u,u)$ be an atomic loop, $f: u \to v$. If $\lambda^n f = f g$ for some $g \in C_{31}(v,v)$, then $g = (\lambda')^n$ where λ' is an atomic loop and $\lambda f = f \lambda'$.

Same res as DMM.

Cor $\forall n \in \mathbb{N}$, $\sigma \in B(G_{31})$ braided refl, $C_{B(G_{31})}(\sigma^n) = C_{B(G_{31})}(\sigma)$.

Cor: $U \subseteq B(G_{31})$ finite index $\Rightarrow Z(U) \subseteq Z(B(G_{31}))$

2] Presentations

C_{31} and $\mathcal{B}_{31}$ are defined by (groupoid) presentations.

$$\langle S \mid R \rangle$$

oriented graph

$\hookrightarrow p_1 = p_2$ where p_1, p_2 are paths in S.

How to deduce a presentation of $B(G_{31})$?

Def: A Schreier transversal (rooted in u_0) is a family $T = \{ r_v \mid v \in \mathrm{Ob} \}$ of paths $r_v : u_0 \to v$, stable under prefix

it always exist but it is not unique (good choice?)

$$\begin{tikzcd} v \arrow[rr, "s"] & & v' \\ & u_0 \arrow[ul, "r_v"] \arrow[ur, "r_{v'}"'] & \end{tikzcd}$$

For $s : v \to v'$ in S, we can define

$\gamma(s) = r_v s r_{v'}^{-1} \in \mathcal{B}_{31}(u_0, u_0)$.

+ if $p_1 = p_2$ is a relation $x_1 \dots x_n = y_1 \dots y_m$

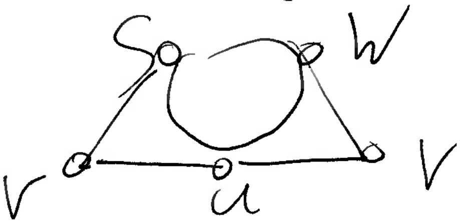
we define a relation $\gamma(x_1) \dots \gamma(x_n) = \gamma(y_1) \dots \gamma(y_m)$

R^* the set of such relation

Thm (G23) $\langle \gamma(S) \mid R^* \rangle \simeq \mathcal{B}_{31}(u_0, u_0) \simeq B(G_{31})$.

This is a general method to give presentations of a group from a groupoid

for $B(G_{31})$



Part 2

0) Motivation: parabolic subgroups

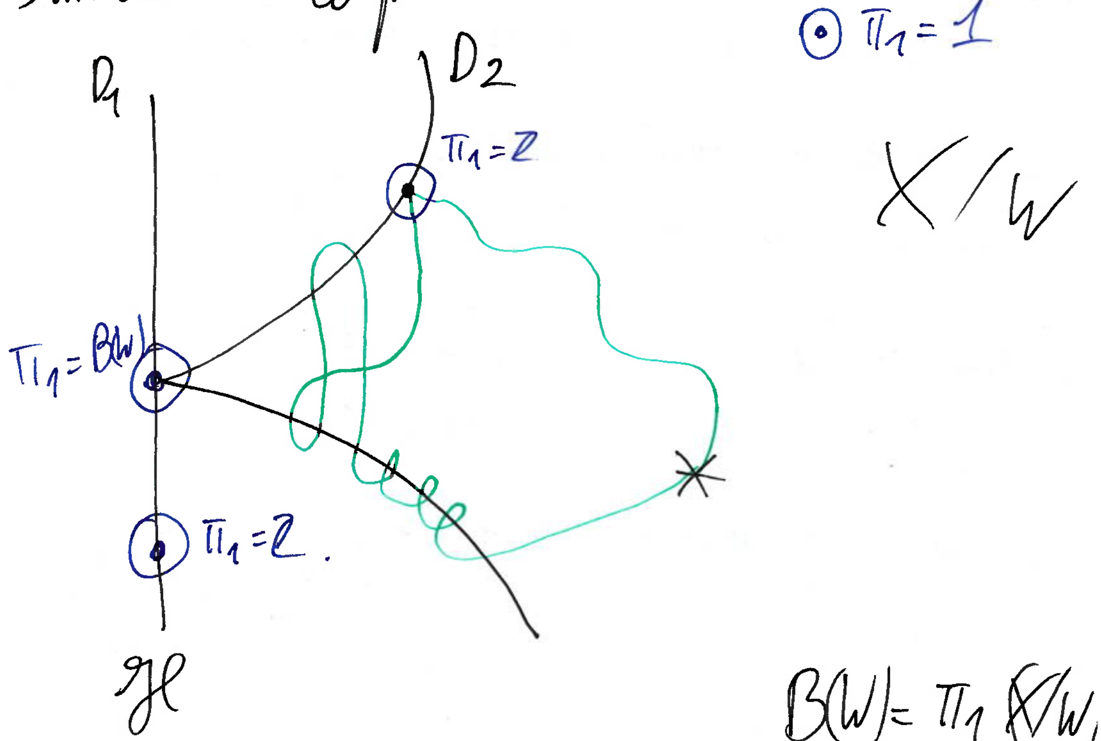
I). Topological/combinatorial description of X/W (for G_{31})

II). Definition of the groupoid $\mathcal{B}_{31}$

III). Parabolics in $\mathcal{G} \simeq \mathcal{B}_{31}$

0). $X/W \hookrightarrow V/W$ is the complement of an algebraic hypersurface $\mathcal{H}$. (discriminant hypersurface).

$\mathcal{H} = \bigcup D \rightarrow$ irreducible component



$B(W) = \pi_1(X/W, *)$.

The local π_1 of X/W around x_0 depends only on the irred. comp of $\mathcal{H}$ to wich x_0 belongs!

We use "normal rays" to see such a local π_1 in $B(W)$.

Theo (Marin, González Meneses)

The image in $B(W)$ of any local π_1 using a normal ray depends, up to conjugacy, only on the irred. comp. of $\mathcal{H}$ to wich the endpoint belongs

We call them parabolic subgroups of $B(W)$.

Thm (Cupdinasiny) Two parabolics $B_0, B_1 \leq B(W)$ are conjugate in $B(W)$ iff their images are conjugate in W.

Thm: If $W \neq G_{31}$, parabolic subgroups of $B(W)$ are stable under intersection [Marin, González-Meneses]

↳ uses a lot of Garside theory.

↳ adapt the arguments to $B(G_{31})$.

II) $W = G_{37} = E_8$ $\qquad$ $W' = G_{31}$

$V = \mathbb{C}^8$ $\qquad$ $V' = \mathbb{C}^4$.

Thm (Springer, Lehrer, Deng Losen)

$(V/W)^{\mu_4} \simeq V'/W'$ $\qquad$ $(X/W)^{\mu_4} \simeq X'/W'$ (homeo)

↳ $B(G_{31}) \simeq \pi_1\left((X/W)^{\mu_4}\right)$

Let R = reflections of W

$\ell(w)$ = minimal lenght of a word in R expressing $w \in W$

(⚠ ≠ Coxeter length).

invariant under conjugacy

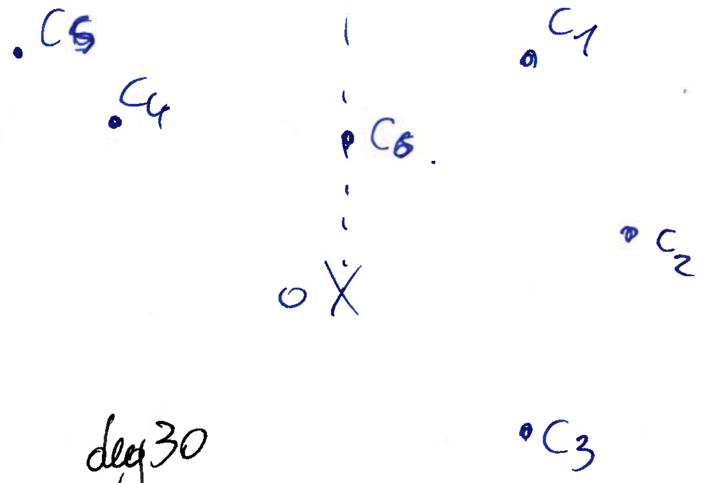
Let also c be a Coxeter element of W.

Def $D_\cdot(c)$ is the set of length additive decompositions of c.

$\{(c_1 \dots c_m) \mid c_1 \cdots c_m = c,\ \sum \ell(c_i) = \ell(c) = 8\}$

Thm (Bessis 15).

$x \in (X/W) \iff \begin{cases} \bullet \text{ a multiset } LL(x) \subseteq \mathbb{C}\setminus 0. \\ \bullet\ \mathrm{clbl}(x) := (c_1 \dots c_m) \in D_m(c) \end{cases}$



$\ell(c_1)$ is the multip of the point in $LL(x)$.

Prop: LL is a homogeneous ramified covering (deg 37968750).

deg 30

$+\ D_m(c)$ is equipped with $\tau : (c_1, \dots, c_m) \longmapsto (c_2, \dots, c_m, c_1^{\,c})$

Prop: $x \in (X/W)^{\mu_4} \iff \begin{cases} LL(x) = -LL(x) \to LL(x) = (x_1 \dots x_{2m}) \\ \tau^{15m}\,\mathrm{clbl}(x) = \mathrm{clbl}\,x \end{cases}$

↳ Good desc of $(X/W)^{\mu_4}$.

Cor: If $LL(x)$ has 2 $\neq$ pts only, $\mathrm{clbl}(x) = (u, v)$.

$x \in (X/W)^{\mu_4} \iff -LL(x) = LL(x)$ and $(u, v) = (v^{c^7}, u^{c^8})$

$\iff (u, v) = (u, u^{c^8})$ $c^{15} \in Z(W)$

+ general : $\mathrm{clbl}(x) = (c_1 \dots c_m, c_1^{c^8}, \dots, c_m^{c^8})$

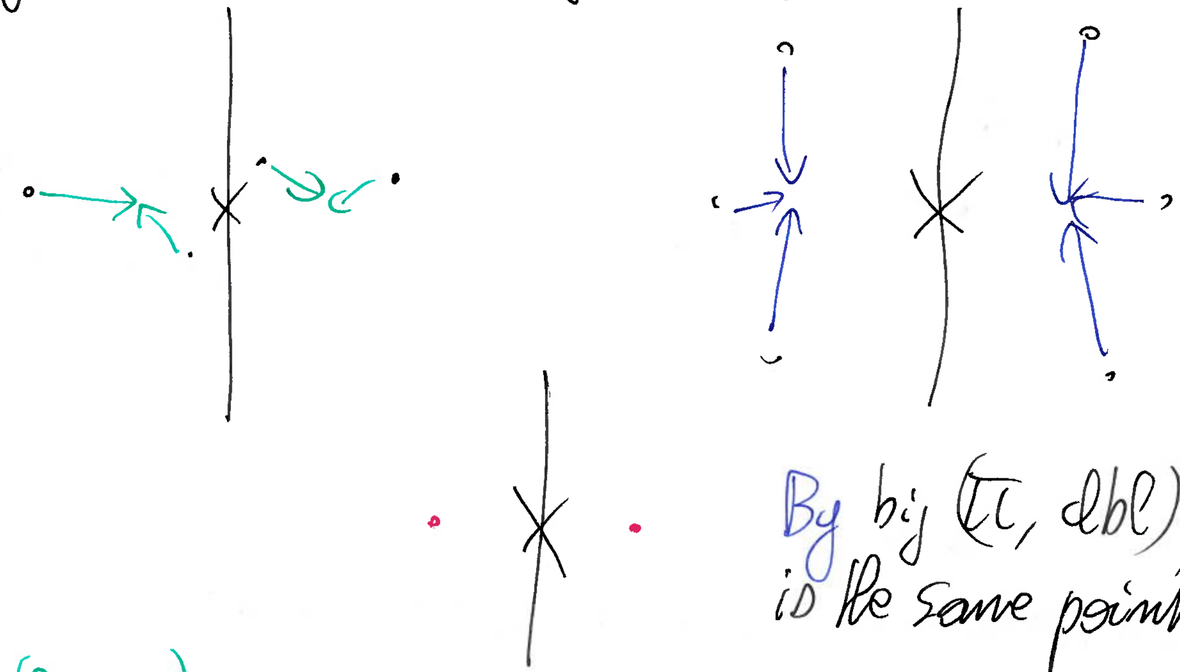
II). Def $\mathcal{U} = \{x \in (X/W)^{\mu_4} \mid TC(x) \cap i\mathbb{R} = \emptyset\}$.
dense, open subset of $(X/W)^{\mu_4}$.

Utopological

For $x \in \mathcal{U}$, $dbl(x) = (c_1 \dots c_m, c_1^{c\ell}, \dots, c_m^{c\ell})$

Def: The cyclic content of x is def as $cc(x) = c_1 \dots c_m$

if x, y are in the same cc of $\mathcal{U}$, they have same cc



By bij (TC, dbl) this is the same point!

Theo (Bonis).
The cyclic content induces a bijection between $\pi_0(\mathcal{U})$ and the set $\mathcal{O} := \{u \mid u u^{c\ell} = c \text{ and } \ell(u) = 4\}$
+ the connected components are Contractible.

Def: $\mathcal{B} := \pi_1((X/W)^{\mu_4}, \mathcal{U})$
- Objects $= \pi_0(\mathcal{U}) = \mathcal{O}$
- Map = htopy classes of paths between two points in $\mathcal{U}$

2) Combinatorial

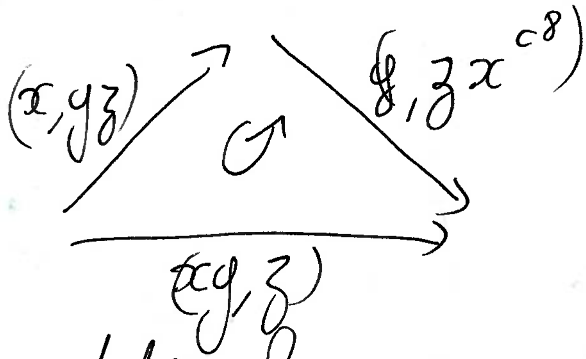
Let S be the following oriented graph.

$Ob(S) = \mathcal{O}$ $\quad S = \{(a,b) \mid ab \in Ob(S),\ \ell(a)+\ell(b)=4\}$.

$(a,b) : \underset{\mathcal{O}}{ab} \longrightarrow ba^{c8} \in \mathcal{O}$.

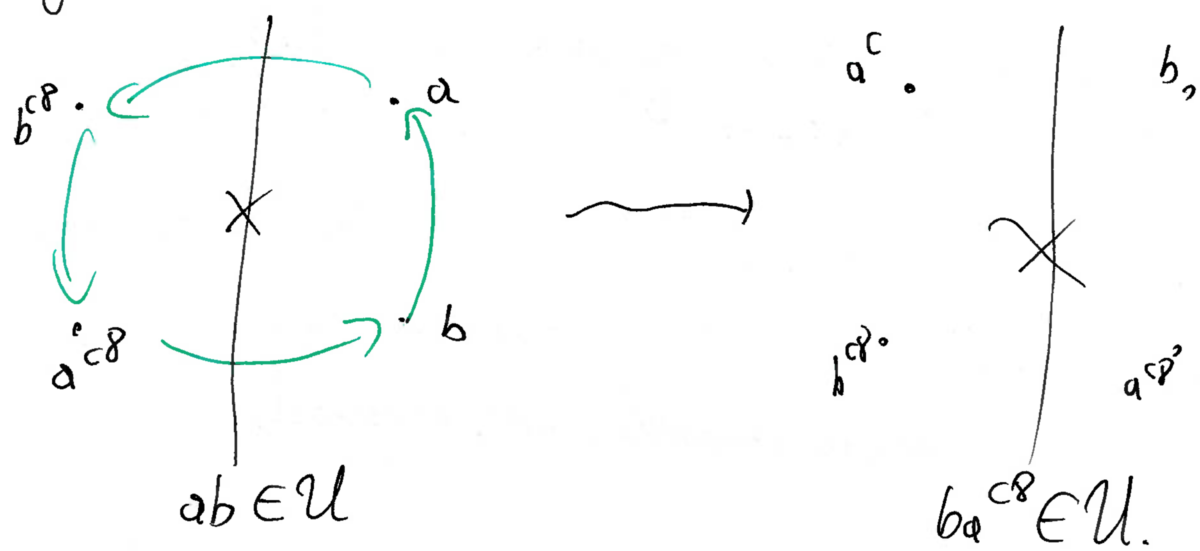
<u>Relations</u>: $\{(x,y,z) \mid xyz \in Ob(S) + \ell(x) \dots\}$.

models



This is a presentation for a groupoid $\mathcal{G}$.

For $(a,b) \in S$. define a rotational motion



Theo (Bemis)

This induces a functor $\mathcal{G} \to \mathcal{B}$ wich is an isomorphism of Groupoids

This is $\mathcal{B}_{31}$

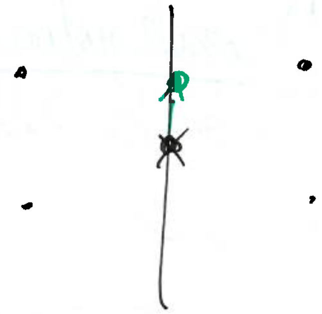
IV. If $x_0 \in \mathcal{H}$, then $\Pi(x_0) \ni 0$.

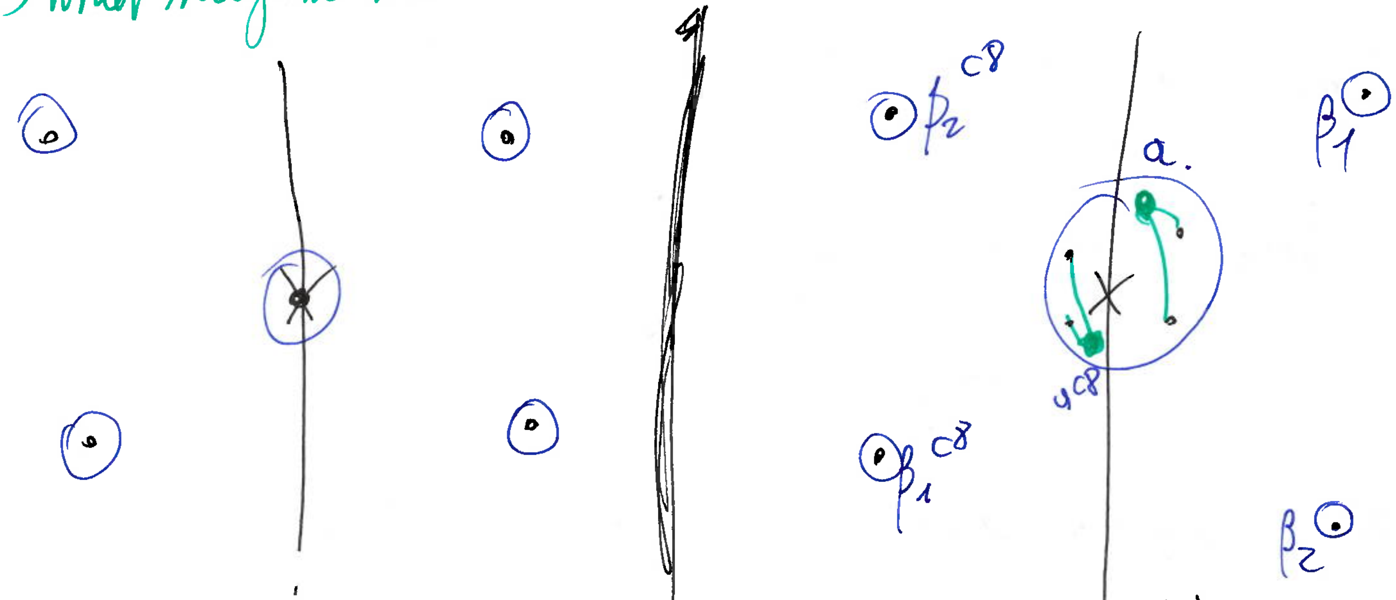
↳ redefine dbl by desingularizing.

↳ notion of outer cable dbl.

→ description of (V/W) and $(V/W)^{\mu_4}$.

→ what neighborhood should we take.





Prop (G23). c_ς induces a bijection b/w $\pi_0(U \cap \mathcal{U})$ and

$\{u \in O \mid u = x^\beta$ for some x, $\ell(u)=\ell(x)+\ell(\beta)\}$ $\beta = \beta_1\beta_2$

Thm (G23) $\pi_1((U \cap X/W)^{\mu_4}, U \cap \mathcal{U})$ is well defined and isomorphic to the groupoid $\mathcal{G}_\beta \subset \mathcal{G}$ generated by $\{(a,b) \in S \mid \exists x \text{ with } b = x^\beta\}$.

Standard parabolic subgroupoid.

Thm (G23). Let $B_0 \subseteq \mathcal{B}_{31}(u_0,u_0)$ be a subgroup. It is parabolic if and only if $\exists f : u \to v$ in $\mathcal{B}_{31}$ with $f^{-1} B_0 f = \mathcal{G}_\beta(v,v)$ for a parabolic subgroupoid $\mathcal{G}_\beta$ of $\mathcal{G} = \mathcal{B}_{31}$