

0 Rewriting system

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Let A be an alphabet, $F(A)$ the free monoid on A
elems = words, product = concatenation, unit = empty word.

Def: Rewriting system = $R \subseteq (F(A) \setminus 1) \times F(A)$. An element $(u, v) \in R$
is a "rule" denoted $u \rightarrow v$.

$\xrightarrow{R} : x_1 \overset{!}{u} x_2 \xrightarrow{R} x_1 v x_2$ where $u \rightarrow v \in R$.

$\xrightarrow{R^*}$: transitive reflexive closure of \xrightarrow{R}
 $\xrightarrow{R^*}$ Sym (congruence).

$M := F(A) / \xrightarrow{R^*}$ is a presentation of M .

Ex: $R = \{bab \rightarrow aba\}$ on $F(\{a, b\})$, $M = \langle a, b \mid aba = bab \rangle$.

Orientation of R (may) give rise to "normal forms" for elements of M .
 $x \in M \rightsquigarrow \hat{x} \in F(A)$ such that $\hat{x} \xrightarrow{R} y$ is impossible.

ex: $bab = aba$.

Thm: (Squier 87, Amich 86, Kobayashi 89)

Def $F(A)$ endowed with a complete rewriting system R . 1. 2
 \hookrightarrow resolution of \mathbb{Z} as a $\mathbb{Z}M$ module, free, recursive differential.

Idea: under combinatorial assumption on a monoid (lcms...)

\hookrightarrow rewriting system \rightarrow complex.

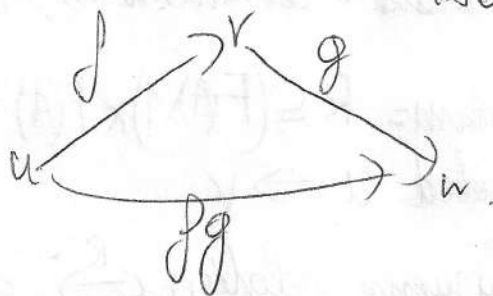
\hookrightarrow Dehornoy-Lafont complex for monoids.

I. Gaunian categories

Let C be a (small) category. $Ob(C)$ its objects. Morphisms from u to v is denoted by $C(u, v)$.

Δ convention for composition:

~~composition of maps~~
product.



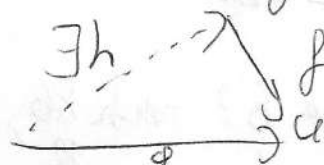
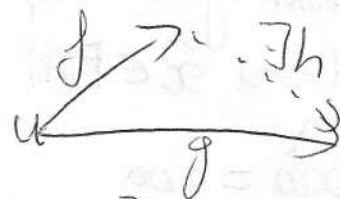
Def. A left Gaunian category is a right cancellative right Noetherian category with admits left-lcms.

Def. C is right cancellative if every morphism is a monomorphism.
 $fg = hg \Rightarrow f = h \quad \forall f, g, h \in C.$

Let $u \in Ob(C)$, sets $C(u, -)$, $C(-, u)$.

On $C(u, -)$ $f \leq g \stackrel{\text{def}}{=} \exists h \mid fh = g$

$C(-, u)$ $g \geq f \stackrel{\text{def}}{=} \exists h \mid hf = g$



We focus on \geq . Reflexive \checkmark , transitive \checkmark , antisymmetric \times .
 if $\varphi \in C(-, u)$ is $\varphi \geq 1_u \geq \varphi \geq 1_u \dots$

Prop. If $C^\times = \{\text{identities}\}$ and C is right cancellative. Then \geq is always a preorder.

proof. Let $f \geq g$ and $g \geq f$. we have $f = hg$ $g = hf$.

Thus $f = hh'f$ $g = h'hg$, $1 = hh'$ $1 = h'h$ and $h = 1 = h'$

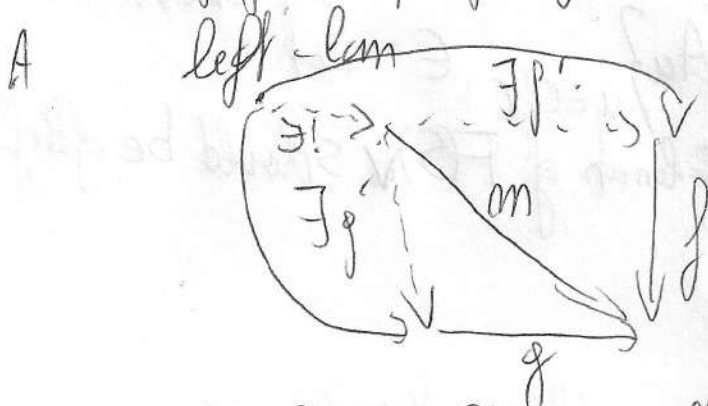
$\hookrightarrow f = g.$

□. (2)

Def: Cis right Noetherian \Leftrightarrow admit no infinite strictly descending chain
 \hookrightarrow necessary for \geq induction.

Def: An atom is a \geq minimal element (no identity).

Lemma: A common left multiple of $f, g \in (E, a)$ is a cumulative square.
A left-lcm is the pullback of f, g .



Solution: $(f/g)g = f \cdot g = (gf)g^0$ + well defined by cancellativity.

Example: $B_+ = \langle a, b \mid aba = bab \rangle$. $a/b = ba$ $b/a = ab$
 $ab = baba$

Rg: $bab \rightarrow aba$ gives a complete rewriting system!

IPC is gaussian, $f \in C$ can be written uniquely as a composition
 $f = a_1 \dots a_m$ of above

with $a_i = \text{md}(a_1, \dots, a_i) \quad \forall i \in [1, m]$. R is $\text{NF}(P)$.

→ adapt the DL context to complete $\frac{H_*(ZC, A)}{???$

II. Homology of a category — 1) Def. free modules

Def: A $\mathbb{Z}C$ module is a (contravariant) functor $C \rightarrow \text{Ab}$.
 Δ caption!

\mathbb{Z} as a trivial $\mathbb{Z}C$ mod: every obj to \mathbb{Z} , maps to 1.

What are free modules? 1

The forgetful functor gives $\{A_u\}_{u \in \text{Ob}(C)} \in \text{Set}^{\text{Ob}(C)}$.

Conversely, let $S = \{S_u\}_{u \in \text{Ob}(C)}$. Elements of $F(S)$ should be of the form

$$\sum_{\substack{m_j \in \mathbb{Z} \\ f \in C(v, u) \\ S \in S_u}} m_j f \cdot s.$$

This is free.

Ex: $u_0 \in \text{Ob}(C)$. $S_u = \begin{cases} \emptyset & u \neq u_0 \\ \{x\} & u = u_0 \end{cases}$ $F(S_u) \cong \mathbb{Z}C(-, u_0)$

and the adjunction is simply Yoneda.

$$\text{Hom}_{\mathbb{Z}C}(\mathbb{Z}C(-, u_0), A) \cong A_{u_0} \cong \text{Hom}_{\text{Set}^{\text{Ob}(C)}}(S_{u_0}, A).$$

$\mathbb{Z}C$ -mod is abelian, we can derive — $\text{Der}_{\mathbb{Z}C} A := (\mathbb{Z}C\text{-mod})^{\text{drt}} \rightarrow \text{Ab}$ in \mathbb{Z} .
 to compute $H_n(C, A) = \text{Tor}_n^{\mathbb{Z}C}(\mathbb{Z}, A)$.

2). Scalar inversion.

For C a cat, there is a universal groupoid $\mathcal{G} := \mathcal{G}(C)$ with

$$\begin{array}{ccc} C & \xrightarrow{\quad} & \mathcal{G} \text{ groupoid} \\ \downarrow \alpha & \searrow & \uparrow \\ \mathcal{G} & & \end{array} \quad \text{!}$$

The functor $C \rightarrow \mathcal{G}$ induces $\mathbb{Z}\mathcal{G}\text{-mod} \rightarrow \mathbb{Z}C\text{-mod}$. "forgetful".

$\mathbb{Z}\mathcal{G}$ is a $\mathcal{G}\text{-}C$ bimodule, thus we have. $\mathbb{Z}\mathcal{G} \otimes_C - : \mathbb{Z}C\text{-mod} \rightarrow \mathbb{Z}\mathcal{G}\text{-mod}$.

The "scalar inversion" $\mathbb{Z}\mathcal{G} \otimes_C -$

Thm (Spier 94, G22) Scalar inversion. + forgetful.

Moreover, if C is left Ore (in part, if C left cancellative). The scalar inversion is exact. Then $H_n(C, A) \simeq H_n(\mathcal{G}, \mathbb{Z}\mathcal{G} \otimes_C A)$. $A \in \mathbb{Z}C\text{-mod}$

Prop: If G is equivalent to \mathcal{G} , $\mathbb{Z}G\text{-mod}$ and $\mathbb{Z}\mathcal{G}\text{-mod}$ are equivalent, so $H_n(G, A) = H_n(\mathcal{G}, A)$

↳ example p.7 about posets.

III. The complex.

Let C be a Gaunian category with atoms A , $<$ a lin order on A .

Def: A m -cell is $[\alpha_1, \dots, \alpha_m]$ a tuple of atoms with the same target with $\alpha_1 < \dots < \alpha_m$ and

$$\forall i \in [1, m], \alpha_i = \text{md}(\text{lcm}(\alpha_1, \dots, \alpha_i))$$

$\text{md}(f) = \text{least right div of } f$

The source of the cell is that of $\text{lcm}(\alpha_1, \dots, \alpha_m)$

C_m is the free $\mathbb{Z}C$ -module associated. Elmts of $(C_m)_u$ are of the form

$$\sum_{f \in C(u,v)} \pi_f [\alpha_1, \dots, \alpha_m]$$

Rq: There is one 0 cell $[\emptyset]_u$ by object u . 1 cells = atoms.

\mathbb{Z} -linear.

∂_m is constructed recursively, along with a cracking $\text{htpy } S_{m+1}$

$$\partial_0([\emptyset]_u) = 1, \quad S_{-1}(1) = [\emptyset], \quad \pi_0 = \partial_0 \circ S_{-1}: f[\emptyset] \mapsto [\emptyset]$$

$$\partial_1([\alpha]) = \alpha[\emptyset] - [\emptyset]$$

$$S_0(f[\emptyset]) = \sum_{i=1}^m [\alpha_1, \dots, \alpha_i, \alpha_i] \text{ on } \alpha_1, \dots, \alpha_m = \text{NF}(f)$$

$$\partial_{m+1}([A, A]) = \alpha/A[A] - \pi_m(\alpha/A[A])$$

$$S_m(f[A]) = \begin{cases} 0 & \text{if } \text{md}(f \text{ lcm } A) = A \cdot 1 \\ g[\alpha/A] + S_m(g(\pi_m(\alpha/A[A]))) & \text{otherwise} \end{cases}$$

$$\alpha = \text{md}(f \text{ lcm } A) \\ g \alpha/A = f$$

$$\pi_{m+1} = S_m \circ \partial_{m+1}$$

\hookrightarrow well def by some induction

Prop: $\partial_{m+1} S_m + S_{m+1} \partial_m = 1 \quad \forall m \in \mathbb{N}$: The complex (C_m, ∂_m) is exact!

$$\begin{array}{ccccccc} C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & \mathbb{Z} \\ \uparrow \pi_2 & \swarrow S_1 & \uparrow \pi_1 & \swarrow S_0 & \uparrow \pi_0 & \swarrow S_{-1} & \\ C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & \mathbb{Z} \quad \textcircled{6} \end{array}$$

Example: $B_4 = \langle a, b \mid aba = bab \rangle$. $a < b$.

0 cell: $[\emptyset]$, 1 cell: $[a], [b]$, 2 cells: $[a, b]$.

$$\partial_1[a] = a[\emptyset] - [\emptyset], \quad \partial_1[b] = (b-1)[\emptyset].$$

$$\begin{aligned} \partial_2[a, b] &= a/b[b] - \pi_1(a/b[b]) \\ &= ba[b] - \pi_1(ba[b]) \end{aligned}$$

$$\begin{aligned} \pi_1(ba[b]) &= S_0 \partial_1(ba[b]) = S_0(bab[\emptyset] - ba[\emptyset]) \\ &= S_0(aba[\emptyset] - ba[\emptyset]) \\ &= ab[a] + a[b] + [a] - b[a] - [b]. \end{aligned}$$

$$\rightarrow \partial_2[a, b] = (ab - b + 1)[a] + (ba - a + 1)[b].$$

If we spe $a=b=1$, we get matrices. $\begin{pmatrix} 0 & 0 \end{pmatrix}$ for ∂_1 , $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for ∂_2 .

Thus $H_n(B_4, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}, 0]$.

Example: Posets. Let (P, \leq) be a poset, C_P its category ($C(x, y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$)
if P is a lattice, C_P is left Gaussian. What is $H_*(C_P, \mathbb{Z})$?

$\dagger C_P$ is an ore category: in groupoid, $G(C_P)(x, x) = \{x \leq x\}$ is trivial.

Thus. $H_n(C_P, \mathbb{Z}) \simeq H_n(G(C_P), \mathbb{Z}) \simeq H_n(\{*\}, \mathbb{Z})$
min useless.

IV. Complex braid groups

Def: A complex reflection group (CRG) is a finite subgroup of $GL_n(\mathbb{C})$ generated by reflection (fixing pointwise a hyperplane).

If $W \leq GL_n(\mathbb{C})$ is a reflection group. Acts freely on

$$X = \{v \in \mathbb{C}^n \mid \forall \text{ reflection } s \in W, s.v \neq v\}.$$

The Braid group of W is defined as $B(W) := \pi_1(X/W)$ \hookrightarrow path connected.

Harder to understand than reflection group, with version...

Thm: Every CBG is equivalent to the enveloping groupoid of a Gannion
Category Dehomog, Paris, Digne, Michel, Broué, Malle, Reiner, Benard...

\rightarrow Case by case, hard.

! A lot of these are actually Gannion groups directly.

The only problematic exception is the group $B_{3,1}$. By using DL on categories, we were able to compute.

$H_n(B_{3,1}, \mathbb{Z})$	0	1	2	3	4	=m
\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z}/6$	\mathbb{Z}	\mathbb{Z}	known hard
$\mathbb{Z} \in$ "sign" rep	\mathbb{Z}_2	0	\mathbb{Z}_6	\mathbb{Z}_{20}	0	
$R \mathbb{Q}[t, t^{-1}]$ mod.	\mathbb{Q}	0	\mathbb{Q}/ϕ_6	$\mathbb{Q}/\frac{t^{10}-1}{t+1} \phi_5$	0	

+ part on \mathbb{F}_2 \mathbb{F}_3 .

$$H_2(\mathbb{F}_2) = \phi_4 \phi_6 = H_2(\mathbb{F}_3).$$

$$H_3(\mathbb{F}_2) = (t^{10}-1) \phi_{15} = \frac{t-1}{t+1} (t^{10}-1) \phi_{15}.$$