PhD students seminar
IMB
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A stroll in Group Theory with circular groups.

0 Introduction, presentation
I. Circular groups
II. Abelianization
III. Center and periodic elements
IV. Final classification.

0). Fix E a set of letters (eg $E=\{s,t\}$)
Word on E = finite sequence of letters (eg $stsssts$) + empty word
concatenation of words: $sts \cdot tstt = ststt$.

Def: The free monoid $F(E)$ on E is the set of words on E, endowed with the concatenation operation.

ex: $\mathbb{N} = F(\{1\})$.

The next step is to add relations: words that we want to be equal
eg: (ab, ba), written $ab=ba$.

Prop: If R is a set of relations, then there is a smallest congruence relation $\equiv$ on $F(E)$, containing R. The quotient $F(E)/_{\equiv}$ is a monoid written $\langle E, R\rangle^{+}$.

Eg: $\langle a,b \mid a^n = (), ab=ba\rangle \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{N}$.
$\langle s,t \mid \emptyset\rangle \not\simeq \mathbb{N}\times\mathbb{N} \to s,t$ do not commute we have to require them to: $\langle s,t \mid st=ts\rangle \simeq \mathbb{N}\times\mathbb{N}$

Presented group: Same thing, but add a formal copy $\bar{a}$ of each $a\in E$, along with relations $a\bar{a} = () = \bar{a}a$ to get inverses.

I. 1] Définitions, exemples.

Let $\{a_0, \dots, a_{m-1}\}$ be an alphabet (+ convention $a_m = a_0 \dots$)

For $i \in [\![0, m-1]\!]$, $p \geq 0$, $s(i,p) = a_i \cdots a_{i+p-1}$ is the product of p consecutive letters

Def: Let m, ℓ be positive integers. The <u>circular group</u> $G(m,\ell)$ is defined by $G(m,\ell) = \langle a_0 \dots a_{m-1} \mid s(i,\ell) = s(i+1,\ell) \ \forall i \in [\![0, m-1]\!] \rangle$.
We also denote $\Delta := s(0,\ell)$ $(= s(1,\ell) \dots)$ $M(m,\ell)$ the underlying monoid

Eg: $G(1,\ell) = G(1,\ell) = G(m,1) = \mathbb{Z}$, $\mathbb{Z}^2 \simeq G(2,2) = \langle a, b \mid ab = ba \rangle$

$G(3,3) = \langle a, b, c \mid abc = bca = cab \rangle$ $\quad$ $\mathrm{Artin}(I_2(e)) = G(2,e)$.

Rq: All complex braid groups of rank 2 are isomorphic to circular groups.

If $\mathbb{Z} = \langle z \rangle$, $F_2 = \langle x, y \rangle$ then we have a group morphism

$$\begin{aligned} \mathbb{Z} \times F_2 &\longrightarrow G(3,3) \\ z &\longmapsto \Delta = abc \\ x &\longmapsto a \\ y &\longmapsto b. \end{aligned}$$

which is an isomorphism, of inverse $\begin{cases} a \longmapsto x \\ b \longmapsto y \\ c \longmapsto (xy)^{-1} z. \end{cases}$ In general we have

Lemma: For $m \geq 1$, $G(m,m) \simeq \mathbb{Z} \times F_{m-1}$. In part $G(m,m) \not\simeq G(m',m')$ for $m' \neq m$.

How can we generalize?

Question: What are the pairs $(m,\ell), (m',\ell')$ such that $G(m,\ell) \simeq G(m',\ell')$.

We already know $G(1,\ell) \simeq G(m,1) = \mathbb{Z}$, this is the easy case.
Artin groups enthusiast know that $\underbrace{G(2,e)}_{\text{Artin Tits}} \simeq \underbrace{G(e,2)}_{\text{dual braid monoid}}$

2) Description of elements

Def: An element $x \in G(m,\ell)$ is simple if $x = s(i,p)$ for $0 \leq p \leq \ell$.

(The simple elements generate $G(m,\ell)$ since $a_i = s(i,1)$)

Consider a product of two simples $s(i,p)\,s(i',p')$, the last letter of $s(i,p)$ is a_{i+p-1}, the first one of $s(i',p')$ is $a_{i'}$.

Def: A product $s(i,p)\,s(i',p')$ is normal if a_{i+p-1} and $a_{i'}$ are not consecutive, i.e if $i+p \not\equiv i' \,[m]$. (or if $s(i,p) = \Delta$ by convention).

Theo: Every $x \in G(m,\ell)$ can be written uniquely as a product

$$x = s(i_1,p_1) \cdots s(i_r,p_r)$$

where each product $s(i_k,p_k)\,s(i_{k+1},p_{k+1})$ is normal. $\to$ greedy normal form.

Eg: If $p < \ell$, then $s(i,p)\Delta$ is not normal: $s(i,p)\Delta = s(i,p)\,s(i+p,\ell) = s(i,p+\ell)$
$= s(i,\ell)\,s(i+\ell,p)$.
$= \Delta\, s(i+\ell,p)$

This is the "conjugation by Δ".

Since conjugation by Δ permutes the simple elements, it has finite order:

Prop: Some power of Δ is central in $G(m,\ell)$ (exercise: it is $\Delta^{\frac{m}{m \wedge \ell}}$).

Cor: $Z(G(m,\ell))$ is not trivial. (complicated question in general).

II) Abelianization of a group G = "biggest quotient of G which is abelian"

This exists and it is given by $G/D(G) \to$ commutator subgroup $[x,y] = xyx^{-1}y^{-1}$

kill that, and $xy = yx$

If $G \simeq H$, then $G^{ab} \simeq H^{ab} \to$ it is a group theoretic invariant

G finitely generated $\Rightarrow G^{ab}$ is a finitely generated abelian group

Prop: If $G = \langle S, R \rangle$, then $G^{ab} = \langle S \mid R \sqcup \{ab = ba \ \forall a,b \in S\} \rangle$.

In $G(m,\ell)^{ab}$, we have $\Delta s(i,p) = s(i,p)\Delta = \Delta s(i,p+\ell)$ thus $s(i,p) = s(i,p+\ell)$ in the quotient. In part, we have $a_i = a_{i+\ell}$ in the quotient.

Theo: This is enough: $G(m,\ell)^{ab} = G(m,\ell)/(a_i = a_{i+\ell}\ \forall i) \simeq \mathbb{Z}^{m\wedge\ell}$

Cor: $G(m,\ell) \simeq G(m',\ell') \Rightarrow \mathbb{Z}^{m\wedge\ell} \simeq \mathbb{Z}^{m'\wedge\ell'} \Rightarrow m\wedge\ell = m'\wedge\ell'$.

This is a first good result: one can recover $m\wedge\ell$ from the group $G(m,\ell)$.
But this is not enough: $G(2,2) \not\simeq G(2,4)$
abelian / not abelian

II.)

Def: Let $a,b \geq 0$, an element $x \in G(m,\ell)$ is (a,b)-periodic if $x^a = \Delta^b$. (periodic if $\exists a\,b$ such that x is (a,b)-periodic).

This notion is not obviously preserved under isomorphism, since Δ has no reason to be, but bear with me.

Ex: In $G(3,4)$, $s(0,3)$ is $(4,3)$ periodic.

$abc\,abc\,abc\,abc = abca\,bcab\,cabc = \Delta^3$.

Theo If $d = a\wedge b$ and $a' = \frac{a}{d}$, $b' = \frac{b}{d}$, then every (a,b)-periodic element is conjugate to a (a',b') periodic element of the form $\Delta^k s(0,p)$
And such an element is periodic iff $p + k\ell \equiv 0\ [m]$

Garside theory

Cor: The periodic elements are exactly the conjugates of powers of either $s(0,m)$ or $s(0,\ell) = \Delta$

If $m \mid \ell$, then Δ is a power of $s(0,m)$, and $s(0,m)$ is (up to conjugacy) the only periodic element of $G(m,\ell)$ without root

If $m \nmid \ell$, then $s(0,m)$, $s(0,\ell)$ are (up to conjugacy) the only periodic elements of $G(m,\ell)$ without roots, and they are not conjugate.

• If $\ell | m$, then $s(0,m)$ is a power of Δ and Δ is (up to conjugacy) the only periodic element of $G(m\ell)$ without roots.

Rq: Having roots is a group theoretic property!

By studying centralisers of Δ, $s(0,m)$, we obtain

Theo: If $G(m\ell)$ is not abelian, then $Z(G(m\ell)) = \langle \Delta^{\frac{m}{m \wedge \ell}} \rangle$.

Abelian: $G(1m) = G(\ell,1) = \mathbb{Z}$, and $G(22) = \mathbb{Z}^2$.

Since the center (wich is a group theoretic invariant) is generated by Δ, We get

Cor: An element of $G(m\ell)$ is periodic iff it has a power wich belongs to $Z G(m\ell)$)

This is group theoretic! An isomorphism $G(m\ell) \simeq G(m'\ell')$ must send rootless periodic elements to rootless periodic elements, and we just classified those!

IV.

Theo: If $G(m\ell)$, $G(m',\ell')$ are non abelian, then
$G(m\ell) \simeq G(m',\ell') \iff (m\ell) = (m',\ell')$ or $(m,\ell) = (\ell',m')$

+ abelian case is easy.

dem: ($\Rightarrow$) by studying rootless periodic elements.

($\Leftarrow$). Let $G(m\ell) = \langle a_0, \dots, a_{m-1} \rangle$ $G(\ell,m) = \langle b_0, \dots, b_{\ell-1} \rangle$.

$$f: \begin{cases} a_0 \longmapsto b_{m-1} \\ a_1 \longmapsto b_{m-2}\, b_{m-1} \\ a_2 \longmapsto b_{m-3}\, b_{m-2}\, b_{m-1} \\ \quad \vdots \\ a_{m-1} \longmapsto b_0\, b_1\, b_2 \cdots b_{m-1} \end{cases}$$

is an isomorphism

inverse

$$\begin{cases} b_0 \longmapsto a_{m-1}\, s(0,m-1)^{-1} \\ b_1 \longmapsto a_{m-2}\, s(0,m-2)^{-1} \\ \quad \vdots \\ b_{m-1} \longmapsto a_0 \end{cases}$$