

# Matrix Representation of Octonions and Generalizations

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## Abstract

We define a special matrix multiplication among a special subset of  $2N \times 2N$  matrices, and study the resulting (non-associative) algebras and their subalgebras. We derive the conditions under which these algebras become alternative non-associative and when they become associative. In particular, these algebras yield special matrix representations of octonions and complex numbers; they naturally lead to the Cayley-Dickson doubling process. Our matrix representation of octonions also yields elegant insights into Dirac's equation for a free particle. A few other results and remarks arise as byproducts.

## I. Introduction

Complex numbers and functions have played a pivotal role in physics for three centuries. On the other hand, their generalization to other Hurwitz algebras does not seem to have fired the interest of physicists to the same extent, because there is still no *compelling* application of them. Thus, despite the fascination of quaternions and octonions for over a century, it is fair to say that they still await universal acceptance. This is not to say that there have not been valiant attempts to find appropriate uses for them. One can point to their possible impact on

- Quantum mechanics and Hilbert space [1]
- Relativity and the conformal group [2]
- Field theory and functional integrals [3]
- Internal symmetries in particle physics [4]
- Colour field theories [5]
- Formulations of wave equations [6]

In all these cases, there is nothing that stands out and commands our attention; rather, the attempts to describe relativistic physics in terms of quaternions and octonions look rather contrived if not forced, especially for the case of octonions. In this paper we describe a generalization of octonions that allows for Lie algebras beyond the obvious  $SU(2)$  structure that is connected with quaternions. We do not presume that they will lead to new physics, but we do think they will at least provide a new avenue for investigation.

Since octonions are not associative, they cannot be represented by matrices with the usual multiplication rules. In this note, we give representations of octonions and other non-associative algebras by special matrices, which are endowed with very special multiplication rules; these rules

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can be regarded as an adaptation and generalization of Zorn's multiplication rule [7]. These matrix representations suggest generalizations of octonions to other non-associative algebras, which in turn lead one almost automatically to a construction of new algebras from old ones, with double the number of elements; we have called these 'double algebras'. Closer inspection reveals that our procedure can be made to correspond to the Cayley-Dickson construction method [8], except that in our case the procedure seems rather natural, once one accepts the multiplication rule, whereas the Cayley-Dickson rule looks *ad hoc* at first sight.

## II. Definitions, notations and a review of the octonion algebra $\mathcal{O}$

The Cayley or the octonion algebra  $\mathcal{O}$  is an 8-dimensional non-associative algebra, which is defined in terms of the basis elements  $e_\mu$  ( $\mu = 0, 1, 2, \dots, 7$ ) and their multiplication table.  $e_0$  stands for the unit element. We can efficiently summarize the table by introducing the following notation [in general, we shall use Greek indices ( $\mu, \nu, \dots$ ) to include the 0 and latin indices ( $i, j, k, \dots$ ) when we exclude the 0]:

$$\hat{e}_k \equiv e_{4+k} , \quad \text{for } k = 1, 2, 3 . \quad (1)$$

The multiplication rules among the basis elements of octonions  $e_\mu$  can be expressed in the form:

$$-e_4 e_i = e_i e_4 = \hat{e}_i , \quad e_4 \hat{e}_i = -\hat{e}_i e_4 = e_i , \quad e_4 e_4 = -e_0 , \quad (2)$$

$$e_i e_j = -\delta_{ij} e_0 + \epsilon_{ijk} e_k , \quad (3)$$

$$\hat{e}_i \hat{e}_j = -\delta_{ij} e_0 - \epsilon_{ijk} e_k , \quad (i, j, k = 1, 2, 3) \quad (4)$$

$$-\hat{e}_j e_i = e_i \hat{e}_j = -\delta_{ij} e_4 - \epsilon_{ijk} \hat{e}_k . \quad (5)$$

We can formally summarize the rules above by

$$e_\mu e_\nu = g_{\mu\nu} e_0 + \sum_{k=1}^7 \gamma_{\mu\nu}^k e_k , \quad g_{\mu\nu} := \text{diag} (1, -1, \dots, -1) , \quad \gamma_{ij}^k = -\gamma_{ji}^k , \quad (6)$$

where  $\mu, \nu = 0, 1, \dots, 7$  , and  $i, j, k = 1, \dots, 7$  . The multiplication properties are sometimes displayed graphically by a circle surrounded by a triangle, but we shall not bother to exhibit that.

The multiplication law (3) shows that the first four elements form a closed *associative* subalgebra of  $\mathcal{O}$ , which is known as the *quaternion algebra*,

$$\mathcal{Q} \equiv \langle e_0, e_1, e_2, e_3 \rangle_{\mathbb{R}} . \quad (7)$$

while the other rules (2), (4) and (5) show that  $\mathcal{O}$  can be graded as follows:

$$\mathcal{O} = \mathcal{Q} \oplus \hat{\mathcal{Q}} , \quad \text{where } \hat{\mathcal{Q}} := e_4 \mathcal{Q} . \quad (8)$$

$\mathcal{O}$  is a non-associative algebra. Now a measure of the non-associativity in any algebra  $\mathcal{A}$  is provided by the *associator*, which is defined for any 3 elements, as follows

$$(x, y, z) := (xy)z - x(yz) , \quad \text{for } x, y, z \in \mathcal{A} . \quad (9)$$

In particular, the associators for the octonion basis are

$$(e_i, e_j, e_k) = 2\epsilon_{ijkl} e_l , \quad (10)$$

where  $\epsilon_{ijkl}$  are *totally* antisymmetric [9] and equal to unity for the following 7 combinations [5]:

$$1247, 1265, 2345, 2376, 3146, 3157 \text{ and } 4576 . \quad (11)$$

## The quaternionic subalgebra $\mathcal{Q}$

It is very well-known that the quaternions form an associative subalgebra  $\mathcal{Q}$ , which can be represented by the Pauli matrices:

$$e_0 \rightarrow \sigma_0 = 1, \quad \text{and} \quad e_j \rightarrow -i\sigma_j \quad (j = 1, 2, 3), \quad (12)$$

where, as usual,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (13)$$

It is trivial to check that the above map is an isomorphism:

$$e_i e_j \iff -\sigma_i \sigma_j = -(\delta_{ij} + i\epsilon_{ijk} \sigma_k) \iff -\delta_{ij} + \epsilon_{ijk} e_k. \quad (14)$$

## III. Non-associative multiplication

In contrast to  $\mathcal{Q}$ , the Cayley algebra  $\mathcal{O}$  cannot be represented by matrices with the usual multiplication rules, because  $\mathcal{O}$  is not associative. However, as we demonstrate below, it is possible to represent octonions by matrices, provided one defines a special multiplication rule among them.

### Zorn's representation of octonions

Zorn [7] gave a representation of the octonions [8] in terms of  $2 \times 2$  matrices  $M$ , whose diagonal elements are scalars and whose off-diagonal elements are 3-dimensional vectors:

$$\mathcal{O} \ni x \longrightarrow \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}, \quad (15)$$

and invoked a peculiar multiplication rule for these matrices [7]. With slight modification of the rule adopted by Humphreys [10] p. 105 our rule is:

$$\begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} * \begin{pmatrix} \alpha' & \mathbf{a}' \\ \mathbf{b}' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \mathbf{a} \cdot \mathbf{b}' & \alpha\mathbf{a}' + \beta'\mathbf{a} - \mathbf{b} \times \mathbf{b}' \\ \alpha'\mathbf{b} + \beta\mathbf{b}' + \mathbf{a} \times \mathbf{a}' & \beta\beta' + \mathbf{b} \cdot \mathbf{a}' \end{pmatrix}. \quad (16)$$

We propose to adapt this multiplication law to octonions and also replace the necessary 3-dimensional basis vectors  $\hat{v}_k$  by Pauli matrices  $\sigma_k$  ( $k = 1, 2, 3$ ), so that the octonions can be represented by the following ordinary  $4 \times 4$  matrices:

$$\begin{aligned} e_0 &\iff \Omega_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & e_k &\iff \Omega_k \equiv \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, & (k = 1, 2, 3) \\ e_4 &\iff \Omega_4 \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \hat{e}_k &\iff \hat{\Omega}_k \equiv \begin{pmatrix} 0 & i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}. \end{aligned} \quad (17)$$

(Note the equality of  $\Omega_k$  ( $k = 1, 2, 3$ ) to the Dirac matrices  $\gamma_k$ , and  $\Omega_4$  to  $i\gamma_0$  in the Pauli-Dirac representation.) It can be shown by explicit multiplication, that the above map (17) becomes an isomorphism, provided we define the modified product rule, which we denote by  $\heartsuit$ :

$$\begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & A' \\ B' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \frac{1}{2}\text{Tr}(AB') & \alpha A' + \beta' A + \frac{i}{2}[B, B'] \\ \alpha' B + \beta B' - \frac{i}{2}[A, A'] & \beta\beta' + \frac{1}{2}\text{Tr}(BA') \end{pmatrix}. \quad (18)$$

where  $[A, B] \equiv AB - BA$  is the commutator of  $A$  and  $B$ . Of course,  $A = \mathbf{a} \cdot \boldsymbol{\sigma}$  and  $B = \mathbf{b} \cdot \boldsymbol{\sigma}$  are traceless:  $\text{Tr } A = \text{Tr } B = 0$ .

In particular, the above multiplication rule yields the following relations

$$\begin{aligned} \begin{pmatrix} 0 & \eta\sigma_i \\ \xi\sigma_i & 0 \end{pmatrix} \heartsuit \begin{pmatrix} 0 & \eta'\sigma_j \\ \xi'\sigma_j & 0 \end{pmatrix} &= \begin{pmatrix} \eta\xi'\delta_{ij} & \xi\xi'\frac{i}{2}[\sigma_i, \sigma_j] \\ -\eta\eta'\frac{i}{2}[\sigma_i, \sigma_j] & \xi\eta'\delta_{ij} \end{pmatrix} \\ &= \delta_{ij} \begin{pmatrix} \eta\xi' & 0 \\ 0 & \xi\eta' \end{pmatrix} + \epsilon_{ijk} \begin{pmatrix} 0 & -\xi\xi'\sigma_k \\ \eta\eta'\sigma_k & 0 \end{pmatrix}, \end{aligned} \quad (19)$$

which are helpful for checking the multiplication rules (2)-(5), by substituting the appropriate coefficients,  $\eta$  and  $\xi$ .

### The standard conjugate of octonions

Usually, octonions are studied over the field of real numbers  $\mathbb{R}$ ,

$$x = \sum_{\mu=0}^7 x_\mu e_\mu \equiv x_0 + \mathbf{x}, \quad \text{for } x_\mu \in \mathbb{R}, \quad (20)$$

although later we will find it interesting to deal with their complex extension. The standard conjugate  $\bar{x}$  of an octonion over  $\mathbb{R}$  is defined by

$$\bar{x} := x_0 e_0 - \sum_{i=1}^7 x_i e_i \equiv x_0 - \mathbf{x}. \quad (21)$$

The reason for this definition is that the product of  $\bar{x}$  with  $x$  yields a positive definite norm:

$$n(x) = x\bar{x} = \bar{x}x = \sum_{\mu=0}^7 x_\mu^2 \geq 0. \quad (22)$$

Moreover, this norm obeys the composition law,

$$n(xy) = n(x)n(y). \quad (23)$$

However with complex octonions (real  $x \rightarrow$  complex  $z$  in (20)) we shall still formally define the conjugate  $\bar{z}$  of  $z$ , to be

$$\bar{z} := z_0 e_0 - \sum_{i=1}^7 z_i e_i, \quad \text{for } z_\mu \in \mathbb{C}. \quad (24)$$

It follows that the product  $z\bar{z}$  is again proportional to unity :

$$n(z) = z\bar{z} = \bar{z}z = \sum_{\mu=0}^7 z_\mu^2 \in \mathbb{C}, \quad (25)$$

but  $n(z)$  ceases to be real in general; therefore  $n(z)$  should simply be regarded as a scalar function, but not a norm.

It is interesting to calculate  $n(z)$  by using the matrix representation (26): Firstly, we note that if  $z$  is mapped into the matrix  $Z$ , then  $\bar{z}$  will be mapped into  $\bar{Z}$ , as follows:

$$z \longrightarrow Z \equiv \sum_{\mu=0}^7 z_\mu \Omega_\mu = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad \bar{z} \longrightarrow \bar{Z} \equiv \begin{pmatrix} \beta & -A \\ -B & \alpha \end{pmatrix}, \quad (26)$$

where  $A = \mathbf{a} \cdot \boldsymbol{\sigma}$  and  $B = \mathbf{b} \cdot \boldsymbol{\sigma}$ , with

$$\alpha = z_0 + iz_4, \quad \beta = z_0 - iz_4, \quad a_k = -z_k + iz_{4+k}, \quad b_k = z_k + iz_{4+k} \quad (k = 1, 2, 3). \quad (27)$$

Secondly,

$$z\bar{z} \leftrightarrow Z \heartsuit \bar{Z} = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \heartsuit \begin{pmatrix} \beta & -A \\ -B & \alpha \end{pmatrix} = (\alpha\beta - \frac{1}{2}\text{Tr } AB) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = n(z) I_{4 \times 4} \quad (28)$$

Therefore, we reproduce the expression (25), as expected:

$$n(z) := \frac{1}{4}\text{Tr } (Z \heartsuit \bar{Z}) = \alpha\beta - \frac{1}{2}\text{Tr } AB = \alpha\beta - \mathbf{a} \cdot \mathbf{b} = \sum_{\mu=0}^7 z_\mu^2. \quad (29)$$

### Hermitian conjugate of octonions

Since  $\sigma_i$  are Hermitian matrices, *all our representation matrices  $\Omega_k$  are anti-hermitian, with the exception of the identity  $\Omega_0$  (which is Hermitian of course)* :

$$\Omega_k^\dagger = -\Omega_k, \quad k = 1, 2, \dots, 7. \quad (30)$$

This fact enables us to prove that the following ‘hermiticity’ property also holds for the  $\heartsuit$  products:

$$(\Omega_\mu \heartsuit \Omega_\nu)^\dagger = \Omega_\nu^\dagger \heartsuit \Omega_\mu^\dagger, \quad \text{for } \mu, \nu = 0, 1, \dots, 7. \quad (31)$$

First, we note that this equality holds trivially for  $(\Omega_0 \heartsuit \Omega_\mu)^\dagger = \Omega_\mu^\dagger = \Omega_\mu^\dagger \heartsuit \Omega_0^\dagger$ . Second, we prove (31) for  $j, k \neq 0$  by using (6) and noting that  $\gamma_{ij}^k$  are real and antisymmetric in  $j, k$ , so that

$$(\Omega_j \heartsuit \Omega_k)^\dagger = -\delta_{jk} \Omega_0 + \sum_{i=1}^7 \gamma_{jk}^i \Omega_i^\dagger = -\delta_{kj} \Omega_0 + \sum_{i=1}^7 \gamma_{kj}^i \Omega_i = \Omega_k \heartsuit \Omega_j = \Omega_k^\dagger \heartsuit \Omega_j^\dagger, \quad j, k = 1, \dots, 7. \quad (32)$$

The conjugation property (31) of  $\Omega_\mu$  suggests the following formal definition for the *Hermitian conjugate* of the octonionic basis:

$$e_0^\dagger = e_0, \quad e_j^\dagger = -e_j \quad (j = 1, 2, \dots, 7), \quad (33)$$

whereupon the ‘*number operators*’ become equal to the identity element:

$$N_\mu := e_\mu^\dagger e_\mu = e_\mu e_\mu^\dagger = e_0 = 1, \quad (\text{no summation}) \quad (\mu = 0, 1, \dots, 7). \quad (34)$$

We can now define the *Hermitian conjugate* of the complex octonions  $z$  in a natural way, by

$$z^\dagger := \sum_{\mu=0}^7 \bar{z}_\mu e_\mu^\dagger = \bar{z}_0 e_0 - \sum_{i=1}^7 \bar{z}_i e_i \equiv \bar{z}_0 - \bar{\mathbf{z}}, \text{ where } z_i \in \mathcal{C}. \quad (35)$$

We then calculate

$$\begin{aligned} zz^\dagger &\equiv (z_0 + \mathbf{z})(\bar{z}_0 - \bar{\mathbf{z}}) = |z_0|^2 + (\bar{z}_0 \mathbf{z} - z_0 \bar{\mathbf{z}}) - \mathbf{z} \bar{\mathbf{z}} \\ &= \sum_{\mu=0}^7 |z_\mu|^2 - \sum_{k=1}^7 (z_0 \bar{z}_k - z_k \bar{z}_0) e_k + \sum_{1 \leq i < j}^7 (z_i \bar{z}_j - z_j \bar{z}_i) e_i e_j \\ &= N(z) + 2i \sum_{k=1}^7 \Im \left( \bar{z}_0 z_k + \sum_{1 \leq i < j \leq 7} z_i \bar{z}_j \gamma_{ij}^k \right) e_k, \end{aligned} \quad (36)$$

where

$$N(z) = \sum_{\mu=0}^7 |z_\mu|^2 . \quad (37)$$

The definition  $N(z)$  is perfectly reasonable for a norm although the decomposition law (23) is *not* satisfied. We see that the ‘space components’  $(zz^\dagger)_i$  of  $zz^\dagger$  are pure imaginary. To understand why this is expected on general grounds, it is useful to introduce the concept of a *Hermitian octonion*:  $y^\dagger = y$ , which signifies that

$$\bar{y}_0 = y_0 , \quad \bar{y}_i = -y_i \quad (i = 1, \dots, 7) , \quad (38)$$

so that  $y_0$  must be real and all the ‘space components’ must be pure imaginary.

Since  $zz^\dagger$  is Hermitian by (31), we see that its space components can only be pure imaginary. If we wish to get rid of these components and retain only the zero component, we must add the standard conjugate. Thus, we may define the *Hermitian norm* by

$$N(z) = (zz^\dagger + \overline{zz^\dagger})/2 . \quad (39)$$

Hence, if  $z$  is mapped into  $Z$ , then  $z^\dagger$  will be mapped into  $Z^\dagger$ , which is obtained by the standard Hermitian conjugation of the matrix  $Z$ .

One of the main insights gained by using the matrix representation is when we calculate the Hermitian norm. If

$$z \longrightarrow Z = z^\mu \Omega_\mu = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} , \quad \text{then } z^\dagger \longrightarrow Z^\dagger = \begin{pmatrix} \bar{\alpha} & B^\dagger \\ A^\dagger & \bar{\beta} \end{pmatrix} . \quad (40)$$

The product

$$\begin{aligned} Z \heartsuit Z^\dagger &= \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \heartsuit \begin{pmatrix} \bar{\alpha} & B^\dagger \\ A^\dagger & \bar{\beta} \end{pmatrix} \\ &= \begin{pmatrix} \alpha\bar{\alpha} + \frac{1}{2}\text{Tr } AA^\dagger & \alpha B^\dagger + \bar{\beta}A + \frac{i}{2}[B, A^\dagger] \\ \bar{\alpha}B + \beta A^\dagger - \frac{i}{2}[A, B^\dagger] & \beta\bar{\beta} + \frac{1}{2}\text{Tr } BB^\dagger \end{pmatrix} . \end{aligned} \quad (41)$$

The zero component of  $zz^\dagger$  is proportional to the trace of  $Z \heartsuit Z^\dagger$ , so that the new *Hermitian norm* can be expressed in terms of the representation matrices  $Z$  as follows:

$$\begin{aligned} N(z) &= \frac{1}{4}\text{Tr } (Z \heartsuit Z^\dagger) = \frac{1}{2}(|\alpha|^2 + |\beta|^2) + \frac{1}{4}(\text{Tr } (AA^\dagger) + \text{Tr } (BB^\dagger)) \\ &= \frac{1}{2} \left( |\alpha|^2 + |\beta|^2 + \sum_{k=1}^3 (|a_k|^2 + |b_k|^2) \right) = \sum_{\mu=0}^7 |z_\mu|^2 , \end{aligned} \quad (42)$$

in accordance with (37). For real  $z_\mu$  we get  $\beta \rightarrow \bar{\alpha}$ ,  $B \rightarrow -A^\dagger$  in (26). Therefore,  $AB \rightarrow -AA^\dagger = -\mathbf{a} \cdot \boldsymbol{\sigma} \ \bar{\mathbf{a}} \cdot \boldsymbol{\sigma} = -\mathbf{a} \cdot \bar{\mathbf{a}}$ . Thus, the formally defined scalar reduces to a conventional norm:

$$N(z) = \alpha\beta - A \cdot B = |\alpha|^2 + \mathbf{a} \cdot \bar{\mathbf{a}} \rightarrow |\alpha|^2 + \sum_{k=1}^3 |a_k|^2 = \sum_{\mu=0}^7 x_\mu^2 \equiv n(z) \geq 0 . \quad (43)$$

## Non-associative algebras from Lie algebras

The main advantage of our matrix representation over the Zorn vector representation, is that our multiplication rule can be generalized to *any number  $n$  of dimensions*, whereas the Zorn rule is restricted, since it is defined in terms of vector product  $\mathbf{a} \times \mathbf{b}$ , which only applies to 3-vectors!

In particular, given *any* representation of an  $n$ -dimensional Lie algebra  $\mathfrak{g}$  in terms of Hermitian  $N \times N$  matrices  $\lambda_k$  ( $k = 1, 2, \dots, n$ ), we can then define  $2n + 2$  different  $2N$ -dimensional matrices,

$$\begin{aligned} e_0 &\Longleftrightarrow \Omega_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & e_k &\Longleftrightarrow \Omega_k \equiv \begin{pmatrix} 0 & -\lambda_k \\ \lambda_k & 0 \end{pmatrix}, & (k = 1, \dots, n) \\ \hat{e}_0 &\Longleftrightarrow \Omega_{n+2} \equiv \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & \hat{e}_k &\Longleftrightarrow \hat{\Omega}_k \equiv \begin{pmatrix} 0 & i\lambda_k \\ i\lambda_k & 0 \end{pmatrix}. \end{aligned} \quad (44)$$

If we multiply these matrices using the  $\heartsuit$  rule, we end up with a closed algebra, which we shall call the *double algebra*  $\mathfrak{g}^D$ , with the following product rules for their basis elements ( $\hat{\Omega}_0 \equiv \Omega_{n+2}$ ):

$$\begin{aligned} -\hat{\Omega}_0 \Omega_k &= \Omega_k \hat{\Omega}_0 = \hat{\Omega}_k, & \hat{\Omega}_0 \hat{\Omega}_k &= -\hat{\Omega}_k \hat{\Omega}_0 = \Omega_k, & \hat{\Omega}_0 \hat{\Omega}_0 &= -\Omega_0, \\ \Omega_i \Omega_j &= -\delta_{ij} \Omega_0 + f_{ijk} \Omega_k, \\ \hat{\Omega}_i \hat{\Omega}_j &= -\delta_{ij} \Omega_0 - f_{ijk} \Omega_k, \\ -\hat{\Omega}_j \Omega_i &= \Omega_i \hat{\Omega}_j = -\delta_{ij} \hat{\Omega}_0 - f_{ijk} \hat{\Omega}_k. \end{aligned} \quad (45)$$

Above, the  $f_{ijk}$  are the structure constants of the Lie algebra  $\mathfrak{g}$ , defined as usual by

$$[L_i, L_j] = if_{ijk} L_k. \quad (46)$$

The matrices (44) can be regarded as the  $\heartsuit$ -matrix representation of the following (non-associative) abstract algebra:

$$-\hat{e}_0 e_k = e_k \hat{e}_0 = \hat{e}_k, \quad \hat{e}_0 \hat{e}_k = -\hat{e}_k \hat{e}_0 = e_k, \quad \hat{e}_0 \hat{e}_0 = -e_0, \quad \text{where } \hat{e}_0 \equiv e_{n+2}, \quad (47)$$

$$e_i e_j = -\delta_{ij} e_0 + f_{ijk} e_k, \quad (48)$$

$$\hat{e}_i \hat{e}_j = -\delta_{ij} e_0 - f_{ijk} e_k, \quad (49)$$

$$-\hat{e}_j e_i = e_i \hat{e}_j = -\delta_{ij} \hat{e}_0 - f_{ijk} \hat{e}_k. \quad (50)$$

These rules (47)-(50) can all be summarized by ( $\mu, \nu = 0, 1, 2, \dots, 2n + 2$ )

$$e_\mu e_\nu = g_{\mu\nu} e_0 + \sum_{k=1}^{2n+2} \gamma_{\mu\nu}^k e_k, \quad g_{\mu\nu} := \text{diag}(1, -1, \dots, -1), \quad \gamma_{ij}^k = -\gamma_{ji}^k. \quad (51)$$

We note from (44) that the  $e_\mu$ ,  $\mu = 0, 1, \dots, n$  correspond to a *subalgebra*  $\mathfrak{g}_+$  of  $\mathfrak{g}^D$ . The rules (47) show that the double algebra  $\mathfrak{g}^D$  is obtained from  $\mathfrak{g}_+$  simply by adding a new element, called  $\hat{e}_0$ , and defining the other  $\hat{e}_k$ . This Lie algebra example then automatically leads us to a more general doubling procedure, which can be applied to *any algebra* and not just to those constructed from Lie algebras. In fact this doubling idea is exactly the procedure which is known as the *Cayley-Dickson process*, as we shall see below.

## IV. Deformed multiplication and the $\mathcal{A}^\heartsuit$ algebra

Begin with the following subset of  $2N \times 2N$ -matrices:

$$\mathcal{A} := \left\{ \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mid A, B \in M_{N \times N} \right\}, \quad (52)$$

where the  $N \times N$  matrices  $\alpha$  and  $\beta$  in the 1st and 4th quadrants are proportional to unit matrices.

Among these matrices we may define a more general [11] multiplication rule than that given in (18). We shall still denote it by  $\heartsuit$  since it only introduces two complex *deformation parameters*  $\lambda_0$  and  $\lambda$  (their values will be restricted as we impose further conditions on the subalgebras) :

$$X \heartsuit X' \equiv \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & A' \\ B' & \beta' \end{pmatrix} := \begin{pmatrix} \alpha\alpha' + \lambda_0 A \cdot B' & \alpha A' + \beta' A - \lambda[B, B'] \\ \alpha' B + \beta B' + \lambda[A, A'] & \beta\beta' + \lambda_0 B \cdot A' \end{pmatrix}. \quad (53)$$

As before,  $[A, B] \equiv AB - BA$  denotes the commutator, but  $A \cdot B$  may now be chosen to be any suitable *bilinear map* into an appropriate field  $F$ . For example, one might define  $A \cdot B$  by  $A \cdot B \equiv \text{Tr}(AB)/N$ , or if  $A$  and  $B$  belong to a Lie algebra, then one could take  $A \cdot B$  to be the adjoint trace:  $A \cdot B := \text{Tr}(\text{ad } A \text{ ad } B)$ , where  $\text{ad}$  denotes the adjoint representation [12]

When  $\lambda = 0$  and  $\lambda_0 = 1$  the multiplication rule (53) looks *almost* like the usual one for matrices. However, it still yields non-associativity, since we are replacing matrix products, such as  $AB$ , by  $A \cdot B$  times the unit matrix. But in any case, it is evident that with the  $\heartsuit$  product the set  $\mathcal{A}$  becomes a closed algebra, which we denote by  $\mathcal{A}^\heartsuit$  [13].

### Complex numbers from real

Before continuing, let us consider the simplest example of the above matrices, namely the case  $N = 1$ . In this circumstance, the matrices  $A$  and  $B$  become simple *commuting* numbers,  $a$  and  $b$ . If we specialize further, and choose  $\beta = \alpha$  and  $b = -a$  to be real, we end up with 2-parameter matrices. Their products are

$$X \heartsuit X' \equiv \begin{pmatrix} \alpha & a \\ -a & \alpha \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & a' \\ -a' & \alpha' \end{pmatrix} := \begin{pmatrix} \alpha\alpha' - \lambda_0 aa' & \alpha a' + \alpha' a \\ \alpha' a + \alpha a' & \alpha\alpha' - \lambda_0 aa' \end{pmatrix}, \quad (54)$$

and this is nothing but the multiplication rule of two complex numbers  $z$  and  $z'$ , provided that we set  $\lambda_0 = 1$  and identify  $\alpha$  and  $a$  with the real and imaginary parts of  $z$ . Thus, a *subalgebra* of  $\mathcal{A}^\heartsuit$  for  $N = \lambda_0 = 1$  becomes isomorphic to the complex numbers  $\mathbb{C}$ :

$$\mathcal{A}^\heartsuit \ni X = \begin{pmatrix} \alpha & a \\ -a & \alpha \end{pmatrix} \iff z \equiv \alpha + ia \in \mathbb{C} \quad (55)$$

.

### Simple and Hermitian conjugates

The attractive feature of the generalization (53) is that most results and definitions needed for octonions apply almost automatically to  $\mathcal{A}^\heartsuit$ . For example, for every element  $X \in \mathcal{A}^\heartsuit$  we can define a conjugate element  $\overline{X}$ , as follows:

$$\overline{X} = \overline{\begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}} := \begin{pmatrix} \beta & -A \\ -B & \alpha \end{pmatrix}. \quad (56)$$



By substituting  $A' \rightarrow -A$ ,  $B' \rightarrow -B$ ,  $\alpha' \rightarrow \beta$  and  $\beta' \rightarrow \alpha$  in (53), we get immediately

$$X \heartsuit \overline{X} = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \heartsuit \begin{pmatrix} \beta & -A \\ -B & \alpha \end{pmatrix} = (\alpha\beta - \lambda_0 A \cdot B) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv n(X) I_{2N \times 2N}, \quad (57)$$

where  $n(X) \in \mathcal{C}$ . In the meantime, we should again look upon  $n(X)$  simply as a scalar function, defined by the map  $\mathcal{A}^\heartsuit \rightarrow \mathcal{C}$  in (58). Later we shall study the conditions on  $\mathcal{A}^\heartsuit$  under which  $n(X)$  becomes a norm.

## V. Subalgebras of $\mathcal{A}^\heartsuit$

The algebra  $\mathcal{A}^\heartsuit$  has several interesting subalgebras :

- An obvious subalgebra is the one obtained by choosing both matrices  $A$  and  $B$  to be traceless:

$$\mathcal{A}^{\heartsuit}_0 := \left\{ \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mid \text{Tr } A = \text{Tr } B = 0 \right\}, \quad (58)$$

- This subalgebra has in turn another subalgebra  $\mathcal{A}^{\heartsuit}_A \subset \mathcal{A}^{\heartsuit}_0$ , in which  $A$  and  $B$  become antisymmetric matrices.
- A third subalgebra, which we denote by  $\mathcal{A}^{\heartsuit}_+$ , is obtained by choosing  $\beta = \alpha$  and  $B = -A$  :

$$\mathcal{A}^{\heartsuit}_+ := \left\{ \begin{pmatrix} \alpha & A \\ -A & \alpha \end{pmatrix} \right\}. \quad (59)$$

It is easily verified that products of such matrices stay in the same class:

$$\begin{aligned} X \heartsuit X' &= \begin{pmatrix} \alpha & A \\ -A & \alpha \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & A' \\ -A' & \alpha' \end{pmatrix} \\ &= \begin{pmatrix} \alpha\alpha' - \lambda_0 A \cdot A' & \alpha A' + \alpha' A - \lambda[A, A'] \\ -\alpha A' - \alpha' A + \lambda[A, A'] & \alpha\alpha' - \lambda_0 A \cdot A' \end{pmatrix} \in \mathcal{A}^{\heartsuit}_+. \end{aligned} \quad (60)$$

Moreover,  $\mathcal{A}^{\heartsuit}_+$  has the interesting property:

**Proposition 1 :** *The subalgebra  $\mathcal{A}^{\heartsuit}_+$  is flexible for all matrices  $A$ .*

To put this result into perspective, we note that all abelian or anti-commutative algebras are flexible; thus if  $yx = \pm xy$ , then  $x(yx) = \pm(yx)x = (xy)x$ , so that  $(x, y, x) = 0$ . Therefore, it is of interest to show that  $\mathcal{A}^{\heartsuit}_+$ , which is neither abelian nor anti-commutative, is also flexible.

*Proof :* We shall prove the above assertion by explicit multiplication. However, to simplify the calculations we first note that the multiples of unity added to each element do not affect the associators:

$$(X + \alpha 1, Y + \beta 1, Z + \gamma 1) = (X, Y, Z), \quad (61)$$

where 1 is the identity matrix. This follows immediately from the linearity of associators:

$$(X + \alpha 1, Y, Z) = (X, Y, Z) + \alpha(1, Y, Z) = (X, Y, Z). \quad (62)$$

The property (61) is helpful for calculating associators of the subalgebra  $\mathcal{A}_+^\heartsuit$ , since we can set the  $\alpha$ 's equal to zero, when calculating the associators.

We now calculate explicitly the associator  $(X_1, X_2, X_3)$  for general matrices from  $\mathcal{A}_+^\heartsuit$ , but using only those with  $\alpha_i = 0$ , *i.e.*

$$X_i = \begin{pmatrix} 0 & A_i \\ -A_i & 0 \end{pmatrix} \in \mathcal{A}_+^\heartsuit, \quad \text{for } i = 1, 2, 3. \quad (63)$$

We get

$$(X_1, X_2, X_3) \equiv (X_1 X_2) X_3 - X_1 (X_2 X_3) = \begin{pmatrix} p & P \\ -P & p \end{pmatrix} - \begin{pmatrix} q & Q \\ -Q & q \end{pmatrix} = \begin{pmatrix} p - q & P - Q \\ Q - P & p - q \end{pmatrix}, \quad (64)$$

where

$$p = \lambda \lambda_0 ([A_1, A_2] \cdot A_3), \quad (65)$$

$$P = -\lambda_0 (A_1 \cdot A_2) A_3 + \lambda^2 [[A_1, A_2], A_3]. \quad (66)$$

and

$$q = \lambda \lambda_0 (A_1 \cdot [A_2, A_3]), \quad (67)$$

$$Q = -\lambda_0 (A_2 \cdot A_3) A_1 + \lambda^2 [A_1, [A_2, A_3]]. \quad (68)$$

Therefore, the elements of the associator  $(X_1, X_2, X_3)$  are

$$p - q = \lambda \lambda_0 ([A_1, A_2] \cdot A_3 - A_1 \cdot [A_2, A_3]) = 0, \quad (69)$$

$$\begin{aligned} P - Q &= -\lambda_0 ((A_1 \cdot A_2) A_3 - (A_2 \cdot A_3) A_1) + \lambda^2 ([ [A_1, A_2], A_3] - [A_1, [A_2, A_3]] ) \\ &= -\lambda_0 ((A_1 \cdot A_2) A_3 - (A_2 \cdot A_3) A_1) + \lambda^2 [A_2, [A_3, A_1]]. \end{aligned} \quad (70)$$

In other words, *the associator  $(X, Y, X)$  vanishes identically, for any  $\lambda, \lambda_0, A_1 = A_3$  and  $A_2$ ,*

$$(X, Y, X) = 0, \quad \text{for } X, Y \in \mathcal{A}_+^\heartsuit. \quad (71)$$

□

- As a fourth subalgebra, let  $\mathfrak{g}$  be a given Lie algebra of dimension  $n$ , and let  $V_{\mathfrak{g}}$  be the algebra spanned by the representation matrices of  $\mathfrak{g}$ . Then, we can define a subalgebra of  $\mathcal{A}^\heartsuit$  via

$$\mathfrak{g}^D := \left\{ \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mid A, B \in V_{\mathfrak{g}} \right\}. \quad (72)$$

Clearly the off-diagonal elements, such as  $\alpha A' + \beta' A + \lambda [B, B']$ , of the products  $X \heartsuit X'$  belong to  $V_{\mathfrak{g}}$ . Hence,  $\mathfrak{g}^D$  are subalgebras of  $\mathcal{A}_0$ . Moreover, half of  $\mathfrak{g}^D$ , obtained by the intersection of  $\mathfrak{g}^D$  with  $\mathcal{A}_+^\heartsuit$ , will be a subalgebra of  $\mathfrak{g}^D$ :

$$\mathfrak{g}_0^D := \left\{ \begin{pmatrix} \alpha & A \\ -A & \alpha \end{pmatrix} \mid A \in V_{\mathfrak{g}} \right\} \subset \mathfrak{g}^D \subset \mathcal{A}_+^\heartsuit. \quad (73)$$

The commutators of the elements of  $\mathfrak{g}_0^D$  constitute a Lie algebra, which is isomorphic to the original algebra  $\mathfrak{g}$ .

## VI. Grading of $\mathcal{A}^\heartsuit$

**Proposition 2 :** *The algebra  $\mathcal{A}^\heartsuit$  can be graded, as follows:*

$$\mathcal{A}^\heartsuit = \mathcal{A}_+^\heartsuit \oplus \mathcal{A}_-^\heartsuit = \mathcal{A}_+^\heartsuit \oplus K\mathcal{A}_+^\heartsuit = \mathcal{A}_+^\heartsuit \oplus K\heartsuit\mathcal{A}_+^\heartsuit, \quad (74)$$

where the ‘grading matrix’ is

$$K \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (75)$$

Observe that  $K\heartsuit X = KX$  for any  $X \in \mathcal{A}^\heartsuit$ . Also of course

$$\mathcal{A}_\eta^\heartsuit \heartsuit \mathcal{A}_{\eta'}^\heartsuit \subseteq \mathcal{A}_{\eta\eta'}^\heartsuit. \quad (76)$$

*Proof :* Every matrix  $X \in \mathcal{A}^\heartsuit$  can be decomposed, as follows:

$$X \equiv \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} = \begin{pmatrix} \alpha_+ & A_+ \\ -A_+ & \alpha_+ \end{pmatrix} + \begin{pmatrix} \alpha_- & A_- \\ A_- & -\alpha_- \end{pmatrix} \equiv X_+ + \hat{X}_- \quad (77)$$

$$\equiv X_+ + KX_- \quad (78)$$

where

$$\alpha_\pm \equiv \frac{1}{2}(\alpha \pm \beta), \quad A_\mp \equiv \frac{1}{2}(A \pm B), \quad K \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (79)$$

The first set of matrices (with  $\beta = \alpha$  and  $B = -A$ ) constitutes the subalgebra  $\mathcal{A}_+^\heartsuit$ , which we defined earlier in (59). The second set of matrices (with  $\beta = -\alpha$  and  $B = A$ ) will be called  $\mathcal{A}_-^\heartsuit$ . Since  $\mathcal{A}_+^\heartsuit$  is a subalgebra of  $\mathcal{A}^\heartsuit$ , clearly  $\mathcal{A}_+^\heartsuit\mathcal{A}_+^\heartsuit = \mathcal{A}_+^\heartsuit$ . The rest of the inclusion relations (76), namely

$$\hat{X}\heartsuit\hat{X}' \in \mathcal{A}_+^\heartsuit, \quad X\heartsuit\hat{X}' \in \mathcal{A}_-^\heartsuit, \quad \hat{X}\heartsuit X' \in \mathcal{A}_-^\heartsuit. \quad (80)$$

follow immediately from the equalities (85) - (87) which we shall prove below.  $\square$

**Proposition 3 :** *The following equalities hold for any  $X, X' \in \mathcal{A}_+^\heartsuit$ :*

$$KXK = \bar{X}, \quad (81)$$

$$(KX)\heartsuit(KX') = X'\heartsuit\bar{X}, \quad (82)$$

$$X\heartsuit(KX') = K(\bar{X}\heartsuit X'), \quad (83)$$

$$(KX)\heartsuit X' = K(X'\heartsuit X). \quad (84)$$

*Proof :* The proof follows simply by explicit matrix multiplication, using (60):

$$\begin{aligned} KX \heartsuit KX' &\equiv \begin{pmatrix} \alpha & A \\ A & -\alpha \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & A' \\ A' & -\alpha' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \lambda_0 A \cdot A' & \alpha A' - \alpha' A - \lambda[A, A'] \\ -\alpha A' + \alpha' A + \lambda[A, A'] & \alpha\alpha' + \lambda_0 A \cdot A' \end{pmatrix} \\ &= \begin{pmatrix} \alpha' & A' \\ -A' & \alpha' \end{pmatrix} \heartsuit \begin{pmatrix} \alpha & -A \\ A & \alpha \end{pmatrix} \equiv X' \heartsuit \bar{X} \in \mathcal{A}_+^\heartsuit. \end{aligned} \quad (85)$$

$$\begin{aligned} X\heartsuit(KX') &= \begin{pmatrix} \alpha & A \\ -A & \alpha \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & A' \\ A' & -\alpha' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \lambda_0 A \cdot A' & \alpha A' - \alpha' A + \lambda[A, A'] \\ \alpha A' - \alpha' A + \lambda[A, A'] & -\alpha\alpha' - \lambda_0 A \cdot A' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left\{ \begin{pmatrix} \alpha & -A \\ A & \alpha \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & A' \\ -A' & \alpha' \end{pmatrix} \right\} = K(\bar{X} \heartsuit X') \equiv \bar{X} \widehat{\heartsuit} X' \in \mathcal{A}_-^\heartsuit, \end{aligned} \quad (86)$$

$$\begin{aligned}
KX \heartsuit X' &= \begin{pmatrix} \alpha & A \\ A & -\alpha \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & A' \\ -A' & \alpha' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' - \lambda_0 A \cdot A' & \alpha A' - \alpha' A + \lambda[A, A'] \\ \alpha A' - \alpha' A + \lambda[A, A'] & -\alpha\alpha' - \lambda_0 A \cdot A' \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left\{ \begin{pmatrix} \alpha' & A' \\ -A' & \alpha' \end{pmatrix} \heartsuit \begin{pmatrix} \alpha & A \\ -A & \alpha \end{pmatrix} \right\} = K(X' \heartsuit X) \in \mathcal{A}_-^\heartsuit. \tag{87}
\end{aligned}$$

### Matrix representation of the Cayley-Dickson process

If we multiply the grading matrix  $K$  by a real or complex scalar  $v$ , and let  $\mu \equiv v^2$ , we get

$$v_{op} := vK, \quad v_{op}v_{op} := v^2 1 = \mu 1. \tag{88}$$

Therefore, using the relations (81) - (84), we get the multiplication rule

$$(X_1 + v_{op}X_2) \heartsuit (X_3 + v_{op}X_4) = (X_1 \heartsuit X_3 + \mu X_4 \heartsuit \overline{X_2}) + v_{op}(\overline{X_1} \heartsuit X_4 + X_3 \heartsuit X_2), \quad \forall X_i \in \mathcal{A}_+^\heartsuit. \tag{89}$$

This is exactly the multiplication rule given by Cayley and Dickson where one starts with an abstract algebra  $\mathcal{B}$  and defines an abstract operator  $v_{op}$ , and essentially *postulates* the following multiplication rule [8]:

$$(b_1 + v_{op}b_2)(b_3 + v_{op}b_4) = (b_1b_3 + \mu b_4\bar{b}_2) + v_{op}(\bar{b}_1b_4 + b_3b_2), \quad b_i \in \mathcal{B}. \tag{90}$$

where  $\bar{b}_i \in \mathcal{B}$  is the conjugate of  $b_i$ , and  $v_{op} \notin \mathcal{B}$ , such that  $v_{op}^2 = \mu \cdot 1$ .

Observe that the  $\heartsuit$  multiplication rule provides an explicit matrix representation of the Cayley-Dickson process [8], provided that the original algebra  $\mathcal{B}$  can be represented by  $\mathcal{A}_+^\heartsuit$ .

### Composition algebras from $2 \times 2$ matrices

One may wonder what happens if we allow the rudimentary  $2 \times 2$  matrices to contain arbitrary complex elements. Since

$$X \heartsuit X' \equiv \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \heartsuit \begin{pmatrix} \alpha' & a' \\ b' & \beta' \end{pmatrix} := \begin{pmatrix} \alpha\alpha' + \lambda_0 ab' & \alpha a' + \beta' a \\ \alpha' b + \beta b' & \beta\beta' + \lambda_0 ba' \end{pmatrix}, \tag{91}$$

when  $X' = \overline{X}$  this product yields a ‘norm’,

$$X * \overline{X} = n(X) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } n(x) = \alpha\beta - \lambda_0 ab. \tag{92}$$

It is easy to check, by explicit multiplication, that the following identity holds for any  $\lambda_0 \in \mathbb{C}$ :

$$n(x)n(x') = (\alpha\beta - \lambda_0 ab)(\alpha'\beta' - \lambda_0 a'b') = (\alpha\alpha' + \lambda_0 ab')(\beta\beta' + \lambda_0 ba') - \lambda_0(\alpha a' + \beta' a)(\alpha' b + \beta b') = n(xx') \tag{93}$$

which informs us that the standard norm (92) for  $N = 1$  obeys the composition law.

Clearly, the norm (92) is degenerate for any  $\lambda_0 \in \mathbb{C}$ , if we allow  $X$  to be any  $2 \times 2$  matrix. (For example, simply choosing  $\beta = a = b = 1$  and  $\alpha = \lambda_0$  will yield an  $x \neq 0$  with  $n(x) = \alpha\beta - \lambda_0 ab = 0$ .) The question then arises, when is the norm  $n(x)$  in (92) nondegenerate? We can certainly guarantee that  $n(x)$  is nondegenerate, if we restrict  $X$  to have the following special form:

$$X = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, \quad \text{and } \Re \lambda_0 > 0. \tag{94}$$

whence

$$n(x) = |z|^2 + \lambda_0 |w|^2 \neq 0, \quad \text{for } \Re \lambda_0 \neq 0. \quad (95)$$

[Of course, there exist a few equivalent variations of the conditions (94). For instance, we can replace  $-\bar{w}$  by  $\bar{w}$ , but demand that  $\Re \lambda_0 < 0$ .]

Anyhow, this means that we are dealing with a division algebra, which must therefore be one of the 4 possibilities. Because  $X \in M_{2 \times 2}$ , we may expand it in terms of Pauli matrices, getting

$$\begin{aligned} X &= \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = z_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + iw_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + iw_1 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + iz_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\equiv x_0 \sigma_0 - i \mathbf{x} \cdot \boldsymbol{\sigma} \leftrightarrow x_0 e_0 + \sum_{k=1}^3 x_k e_k, \quad \text{where } w_i, z_i \in \mathbb{R} \end{aligned} \quad (96)$$

Since  $e_0 \rightarrow \sigma_0$  and  $e_j \rightarrow -i\sigma_j$   $j = 1, 2, 3$  are known representations of the quaternions, we conclude that this is the quaternion algebra over the *real* field  $\mathbb{R}$ , as expected. Indeed the matrix (96) is the usual representation of  $\mathcal{Q}$  in terms of standard matrices. Later on we shall describe another representation by nonstandard matrices.

## VII. Conditions on deformation parameters

Previously we showed that  $\mathcal{A}_+^\heartsuit$  is flexible. We now ask under what conditions  $\mathcal{A}_+^\heartsuit$  can become alternative.

For this, we must have  $(X_1, X_1, X_3) = 0$ . By setting  $X_2 = X_1$  and noting Eq.(70), we get the condition

$$[A_1, [A_3, A_1]] = \frac{\lambda^2}{\lambda_0} = ((A_1 \cdot A_1) A_3 - (A_1 \cdot A_3) A_1). \quad (97)$$

This condition can be satisfied if  $\lambda^2 = \lambda_0/4$  and the  $A_i = \mathbf{a}_i \cdot \boldsymbol{\sigma}$ : Indeed, for such  $A_i$ , we get

$$\begin{aligned} [A_2, [A_3, A_1]] &= [\mathbf{a}_2 \cdot \boldsymbol{\sigma}, 2i[\mathbf{a}_3 \times \mathbf{a}_1] \cdot \boldsymbol{\sigma}] = -4[\mathbf{a}_2 \times [\mathbf{a}_3 \times \mathbf{a}_1]] \cdot \boldsymbol{\sigma} \\ &= 4((\mathbf{a}_2 \cdot \mathbf{a}_3)\mathbf{a}_1 - (\mathbf{a}_2 \cdot \mathbf{a}_1)\mathbf{a}_3) \cdot \boldsymbol{\sigma} = 4((A_1 \cdot A_2) A_3 - (A_2 \cdot A_3) A_1) \end{aligned} \quad (98)$$

where we used  $(A \cdot B) = \frac{1}{2} \text{Tr}(AB)$ . Hence, we can have  $\lambda = \pm \frac{1}{2} \sqrt{\lambda_0}$ . For the special choice  $\lambda_0 = -1$ , we get  $\lambda = \pm \frac{i}{2}$ .

This sign ambiguity is the origin of the non-uniqueness of the  $\heartsuit$  product [14].

## VIII. Summary

Our algebras provide concrete matrix representation of a big class of non-associative algebras. They may suggest new constructions in the future. One such possibility, which leads to notions of triality, is described in the Appendix A, but anyhow the formulation (18) and its deformations (53) permit generalizations that have obvious affinity to higher symmetry groups, rather than the simple case of  $\text{SU}(2)$ . We also believe that our treatment of hermiticity and norm for the complex case is reasonable; we illustrate their utility with reference to the Dirac equation in Appendix B.

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## Appendix A. The $\diamond$ product and triality

In this appendix we try another type of product, which we denote by  $\diamond$ , where the commutators  $[B, B']$  in the  $\heartsuit$  product are now replaced by the standard matrix products  $BB'$ .

Let us first consider the simplest case,  $N = 1$ , where the matrices  $A$  and  $B$  become scalars, so that we shall first deal with  $2 \times 2$  matrices:

$$X \equiv \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}. \quad (99)$$

We define the new matrix product, as follows

$$X \diamond X' \equiv \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \diamond \begin{pmatrix} \alpha' & a' \\ b' & \beta' \end{pmatrix} := \begin{pmatrix} \alpha\alpha' + \lambda_0 ab' & \alpha a' + \beta' a + \lambda b b' \\ \alpha' b + \beta b' + \eta \lambda a a' & \beta \beta' + \lambda_0 b a' \end{pmatrix}, \quad (100)$$

where  $\eta, \lambda, \lambda_0$  are arbitrary complex numbers. We now ask the question, whether for such a product, we can define for every  $X$  a conjugate  $\overline{X}$ , such that  $X \diamond \overline{X} = n(X) \cdot 1$ , where  $n(X)$  is some quadratic form of  $X$ , i.e.  $n(sX) = s^2 n(X)$ .

Let us try the following ansatz:

$$\overline{X} \equiv \begin{pmatrix} \beta + \gamma & -a \\ -b & \alpha + \delta \end{pmatrix}. \quad (101)$$

We want to determine  $\gamma$  and  $\delta$ , and derive conditions on  $\alpha, \beta, a, b$ , by demanding that  $X \diamond \overline{X} \propto 1$ :

$$\begin{aligned} X \diamond \overline{X} &= \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \diamond \begin{pmatrix} \beta + \gamma & -a \\ -b & \alpha + \delta \end{pmatrix} = \begin{pmatrix} \alpha(\beta + \gamma) - \lambda_0 ab & \delta a - \lambda b^2 \\ \gamma b - \eta \lambda a^2 & \beta(\alpha + \delta) - \lambda_0 ab \end{pmatrix} \\ &= n(X) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } n(X) = \alpha(\beta + \gamma) - \lambda_0 ab. \end{aligned} \quad (102)$$

This condition is obeyed, if

$$\delta a - \lambda b^2 = 0, \quad \gamma b - \eta \lambda a^2 = 0, \quad \text{and} \quad \alpha \gamma = \beta \delta, \quad (103)$$

We cannot satisfy these conditions for general  $a$  and  $b$ . (For example, if  $a = 0$  and  $b \neq 0$ , then  $\delta$  would be infinite.) Thus, for  $\eta \neq 0$  we must assume that either both  $a$  and  $b$  are zero or both are unequal to zero. But for  $\eta = 0$  we must demand  $b = 0$ . With these restrictions, we get

$$\delta(X) = \lambda \frac{b^2}{a}, \quad \gamma(X) = \eta \lambda \frac{a^2}{b}, \quad \text{and} \quad \frac{\beta}{\alpha} = \frac{\gamma}{\delta} = \eta \frac{a^3}{b^3}. \quad (104)$$

An additional condition can be obtained by demanding that the adjoint operation is an involution, so that  $\overline{\overline{X}} = X$ , whence

$$\begin{aligned} \overline{\overline{X}} &= \begin{pmatrix} \alpha + \delta(X) + \gamma(\overline{X}) & a \\ b & \beta + \gamma(X) + \delta(\overline{X}) \end{pmatrix} \\ &= \begin{pmatrix} \alpha + \lambda(\frac{b^2}{a} - \eta \frac{a^2}{b}) & a \\ b & \beta + \lambda(\eta \frac{a^2}{b} + \frac{b^2}{-a}) \end{pmatrix} = \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} = X. \end{aligned} \quad (105)$$

Thus, we get the new condition

$$b^3 = \eta a^3, \quad \text{or} \quad b = \xi a, \quad \text{where} \quad \xi = \eta^{1/3}, \quad (106)$$

Note that for each  $\eta$  we have three cubic roots  $\xi$ . Substituting (106) into the third equality in (104), we get

$$\alpha = \beta, \quad \text{and} \quad \gamma = \delta. \quad (107)$$

Hence,  $a$  can be any complex number as long as  $b = \xi a$ . Finally, by noting all the above conditions, we get for every  $\eta \in \mathbb{C}$  three sets of  $2 \times 2$  matrices, which are closed algebras under the product  $\diamond$ :

$$X(\xi) = \left\{ \begin{pmatrix} \alpha & a \\ \xi a & \alpha \end{pmatrix} \right\}, \quad \xi = \eta^{1/3} \in \mathbb{C}, \quad (108)$$

For these matrices, the adjoint and the corresponding quadratic form are

$$\overline{X} \equiv \begin{pmatrix} \alpha + \xi^2 \lambda a & -a \\ -\xi a & \alpha + \xi^2 \lambda a \end{pmatrix}. \quad (109)$$

and

$$\begin{aligned} n(X) &= \alpha(\beta + \gamma) - \lambda_0 ab = \alpha^2 + \eta \lambda \alpha \frac{a^2}{b} - \lambda_0 ab \\ &= \alpha^2 + \xi^2 \lambda \alpha a - \xi \lambda_0 a^2. \end{aligned} \quad (110)$$

Note that  $n(X)$  is quadratic in  $X$ , i.e.  $N(sX) = s^2 N(X)$ ,  $s \in \mathbb{C}$ .

For the special case  $b = -a$ , or  $\xi = -1$ , we obtain a known quadratic form [11]

$$n(X) = \alpha^2 + \lambda \alpha a + \lambda_0 a^2. \quad (111)$$

Proceeding to larger matrices, let

$$X \diamond X' \equiv \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \diamond \begin{pmatrix} \alpha' & A' \\ B' & \beta' \end{pmatrix} := \begin{pmatrix} \alpha\alpha' + \lambda_0 A \cdot B' & \alpha A' + \beta' A + \lambda B B' \\ \alpha' B + \beta B' + \eta \lambda A A' & \beta\beta' + \lambda_0 B \cdot A' \end{pmatrix}, \quad \eta \in \mathbb{C}. \quad (112)$$

One can readily check that such matrices yield closed algebras with respect to the above  $\diamond$  product. By restricting  $B$  to be  $\xi A$  we get the following subalgebra:

$$X(\xi) = \left\{ \begin{pmatrix} \alpha & A \\ \xi A & \alpha \end{pmatrix} \right\}, \quad \xi = \eta^{1/3} \in \mathbb{C}. \quad (113)$$

We can check, using (112), that products of two such matrices yield a matrix of the same type:

$$X \diamond X' \equiv \begin{pmatrix} \alpha & A \\ \xi A & \alpha \end{pmatrix} \diamond \begin{pmatrix} \alpha' & A' \\ \xi A' & \alpha' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \xi\lambda_0 A \cdot A' & \alpha A' + \alpha' A + \xi^2 \lambda A A' \\ \xi(\alpha' A + \alpha A') + \eta \lambda A A' & \alpha\alpha' + \xi\lambda_0 A \cdot A' \end{pmatrix}. \quad (114)$$

However, if we replace the *scalar*  $a$  in eqs. (109) and (110) by a *matrix*  $A$ , we do not get an adjoint nor a bilinear form, since the appropriate items do not stay scalar, as they should.

Finally we note that if we replace the simple products  $AA'$  in (114) by anticommutators  $\{A, A'\}/2$  and if we make the scalar products  $A \cdot A'$  symmetric, i.e.

$$\begin{pmatrix} \alpha & A \\ \xi A & \alpha \end{pmatrix} \diamond_S \begin{pmatrix} \alpha' & A' \\ \xi A' & \alpha' \end{pmatrix} := \begin{pmatrix} \alpha\alpha' + \xi\lambda_0 A \cdot A' & \alpha A' + \alpha' A + \xi^2 \lambda \{A, A'\}/2 \\ \xi(\alpha' A + \alpha A') + \eta \lambda \{A, A'\}/2 & \alpha\alpha' + \xi\lambda_0 A \cdot A' \end{pmatrix},$$

then the product becomes abelian. Therefore the new algebra will automatically become *flexible*.

One can similarly symmetrize the more general product (112) by defining,  $X \diamond_S X' := (X \diamond X' + X' \diamond X)/2$ , and also get a flexible algebra.

## Appendix B. Application to the Dirac equation

In momentum space, the free Dirac equation reads

$$P\Psi \equiv (p_0 - \mathbf{p} \cdot \boldsymbol{\alpha} - m\beta)\Psi = \begin{pmatrix} p_0 - m & -\mathbf{p} \cdot \boldsymbol{\sigma} \\ -\mathbf{p} \cdot \boldsymbol{\sigma} & p_0 + m \end{pmatrix} \Psi = 0. \quad (115)$$

By noting that the standard matrix product and the  $\heartsuit$  products between the Dirac operator  $P$  and its conjugate  $\bar{P}$  are equal, we immediately obtain

$$P\bar{P} = P\heartsuit\bar{P} = n(P) I = \left( (p_0 - m)(p_0 + m) - \mathbf{p}^2 \right) I = 0, \quad (116)$$

where we used Eq. (28) to calculate the norm  $n(P)$ . Therefore each of the 4 columns of  $\bar{P}$  will be a solution of the Dirac equation (115). Hence, we write

$$\Psi = \bar{P} = \begin{pmatrix} p_0 + m & \mathbf{p} \cdot \boldsymbol{\sigma} \\ \mathbf{p} \cdot \boldsymbol{\sigma} & p_0 - m \end{pmatrix}. \quad (117)$$

The first and second columns of  $\Psi$  are proportional to the positive energy solutions  $u^1(p)$  and  $u^2(p)$ , while the third and fourth columns yield the negative energy solutions  $v^1(p)$  and  $v^2(p)$ , if we replace  $m$  by  $-m$ , because

$$(p_0 - \mathbf{p} \cdot \boldsymbol{\alpha} - m\beta)u^i(p) = 0, \quad (p_0 - \mathbf{p} \cdot \boldsymbol{\alpha} + m\beta)v^i(p) = 0, \quad (i = 1, 2). \quad (118)$$

Therefore the *physical and normalizable* solutions can be expressed as

$$\Psi_{ph} \equiv (u^1|u^2|v^1|v^2) = \frac{1}{\sqrt{2m(p_0 + m)}} \begin{pmatrix} p_0 + m & \mathbf{p} \cdot \boldsymbol{\sigma} \\ \mathbf{p} \cdot \boldsymbol{\sigma} & p_0 + m \end{pmatrix}. \quad (119)$$

The orthogonality and normalization relations among these solutions

$$\begin{aligned} \bar{u}^i(p)v^j(p) &\equiv u^{i\dagger}(p)\beta v^j(p) = 0, & \text{and} & & \bar{v}^i(p)u^j(p) &\equiv v^{i\dagger}(p)\beta u^j(p) = 0, & (i, j = 1, 2), \\ \bar{u}^i(p)u^j(p) &\equiv u^{i\dagger}(p)\beta u^j(p) = \delta_{ij}, & \text{and} & & \bar{v}^i(p)v^j(p) &\equiv v^{i\dagger}(p)\beta v^j(p) = -\delta_{ij} \end{aligned}$$

can also be elegantly summarized and proved using the matrix representations of octonions, as follows

$$\Psi_{ph}^\dagger \beta \Psi_{ph} = \Psi_{ph} \beta \Psi_{ph} = \Psi_{ph} \bar{\Psi}_{ph} \beta = (\Psi_{ph} \heartsuit \bar{\Psi}_{ph}) \beta = n(\Psi_{ph}) \beta = \beta. \quad (120)$$

Finally, it is interesting to note that the Dirac operator  $P$  is equal to our matrix representation of the following octonion:

$$P \equiv (p_0 - \mathbf{p} \cdot \boldsymbol{\alpha} - m\beta) = \begin{pmatrix} p_0 - m & -\mathbf{p} \cdot \boldsymbol{\sigma} \\ -\mathbf{p} \cdot \boldsymbol{\sigma} & p_0 + m \end{pmatrix} \Leftrightarrow p \equiv (p_0 + i\mathbf{p} \cdot \hat{\mathbf{e}} + ime_4), \quad (121)$$

where we have used the correspondence

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} = -i\hat{\Omega}_k \Leftrightarrow -i\hat{e}_k, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\Omega_4 \Leftrightarrow -ie_4. \quad (122)$$

Notice that we cannot write the Dirac equation in terms of quaternions alone, since we require five different basis elements:  $(e_0, e_4, \hat{e}_k; k = 1, 2, 3)$  or  $(e_0, e_4, e_k; k = 1, 2, 3)$ . Also note that the octonion  $p$  is nicely hermitian ( $p = p^\dagger$ ).



## References

- [1] D. Finkelstein, J.M. Jauch, S. Shiminovitch and D. Speiser, J. Math. Phys. **3**, 207 (1962); ibid. **4**, 136 (1963); ibid. **4**, 788 (1963); G.G. Emch, Helv. Phys. Acta **36**, 739 (1963); L.P. Horwitz and L.C. Biedenharn, Helv. Phys. Acta **38**, 385 (1965); ibid. Annals of Physics **157**, 432 (1984); M. Gunaydin, C. Piron and H. Ruegg, Comm. Math. Phys. **61**, 69 (1978); C.G. Nash and G.C. Joshi, J. Math. Phys. **28**, 2883 (1987); ibid. 2886 (1987); S. de Leo and P. Rotelli, Prog. Theor. Phys. **92**, 917 (1994).
- [2] F. Gursey, Proceedings of ‘Symmetries in Physics (1600-1980)’, Sant Feliu de Guixols (1992); S. de Leo, J. Math. Phys. **37**, 2955 (1996). These articles contain reviews of this topic.
- [3] F. Gursey, ‘Quaternion Methods in Field Theory’, John Hopkins Workshop, QCD 161, J55 (1980); K. Morita, Prog. Theor. Phys. **67**, 1860 (1982); S.L. Adler, Phys. Rev. Lett. **55**, 783 (1985); S.L. Adler, Comm. Math. Phys. **104**, 611 (1986); C. Nash and G.C. Joshi, J. Math. Phys. **28**, 463 (1987); C. Nash and G.C. Joshi, Int. J. Theor. Phys. **27**, 409 (1988); S.L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields*, (Oxford Univ. Press, 1995).
- [4] F. Gursey, ‘Algebraic methods and Quark Structure’, in Proceedings of the Kyoto Symposium (Jan. 1975); F. Gursey, ‘Octonionic structure in Particle Physics’, Proceedings of ICGTMP78, p.509 (Berlin, 1979); T. Kugo and P. Townsend, Nucl. Phys. **B221**, 357 (1983); A. Sudbery, J. Phys. **12A**, 939 (1984); P.S. Howe and P.C. West, Int. J. Mod. Phys. **7A**, 6639 (1992); S. de Leo, Int. J. Theor. Phys. **35**, 1821 (1996); F. Gursey and C.H. Tze, *On the Role of Division, Jordan and Related Algebras in Particle Physics* (World Scientific, Singapore, 1996).
- [5] F. Gursey, ‘Color, Quarks and Octonions’, John Hopkins Workshop, Jan (1974); S.L. Adler, Phys. Rev. **D21**, 2903 (1980); S. Catto and F. Gursey, Nuovo Cim. **99A**, 685 (1988); S. de Leo and P. Rotelli, J. Phys. **G22**, 1137 (1996).
- [6] A.J. Davies and G.C. Joshi, J. Math. Phys. **27**, 3036 (1986); P. Rotelli, Mod. Phys. Lett. **A4**, 933 (1989); S. de Leo and P. Rotelli, Mod. Phys. Lett. **A11**, 357 (1996); ibid. Int. J. Theor. Phys. **37**, 1511 (1998).
- [7] M. Zorn, Abh. Mat. Sem. Hamburg **9**, 395 (1933); R.D. Schafer, Amer. Journ. Math. **76**, 435 (1954); G. Moreno, “The zero divisors of the Cayley-Dickson algebras over the real numbers”, Boletin de la Sociedad Matematicas Mexicana **4**, 13 (1998).
- [8] R.D. Schafer, *An Introduction to Non-associative Algebras* (Academic Press, New York, 1966); H.P. Peterson, Abh. Mat. Sem. Hamburg **9**, 35 (1971); S. Okubo, *Introduction to Octonions and Other Non-associative Algebras in Physics* (Cambridge University Press, Cambridge 1995).
- [9] The complete antisymmetry of  $\epsilon_{ijkl}$  in its first three indices is an immediate consequence of the fact that  $\mathcal{O}$  is an *alternative* algebra, such algebras being defined by the two ‘*alternative conditions*’ :

$$x^2y = x(xy) , \quad \text{and} \quad (yx)x = yx^2 , \quad \forall x, y \in \mathcal{A} .$$

These conditions can be expressed in terms of associators as  $(x, x, y) = (y, x, x) = 0$  ,  $\forall x, y \in \mathcal{A}$  . They enable one to prove that the associator  $(x, y, z)$  of an alternative algebra is totally antisymmetric in its three arguments. The proof is based on the ‘*linearization technique*’, which consists of replacing one or more of the arguments, say  $x$ , by  $x + \lambda w$ . By this means we get

$$(x + \lambda w, x + \lambda w, y) = (x, x, y) + \lambda^2(w, w, y) + \lambda(w, x, y) + \lambda(x, w, y) = \lambda((w, x, y) + (x, w, y)) = 0 ,$$

so that  $(w, x, y) = -(x, w, y)$ . One can similarly prove the antisymmetry among the other arguments. In particular, the total antisymmetry yields

$$(x, y, x) = 0, \quad \text{or} \quad (z, y, x) + (x, y, z) = 0,$$

where the second equality follows the first, again by linearization. Algebras which obey these last conditions are called *flexible*. These conditions are much less restrictive than the alternative conditions. In fact, most of the interesting non-associative algebras are flexible. For example, both the Lie and Jordan algebras are flexible, the *Jordan algebras*  $\mathcal{J}$  being defined by the (Jordan) identities:

$$(x^2, y, x) = (x, y, x^2) = 0, \quad \forall x, y \in \mathcal{J}.$$

See N. Jacobson, *Structure and Representations of Jordan Algebras* (American Math. Soc., 1968). We shall find another class of flexible algebras below.

- [10] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer Verlag, New York, 1972).
- [11] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov, A.I. Shirshov, *Rings That Are Nearly Associative* (Academic Press, New York, 1982).
- [12] Note, that although  $X_i \cdot X_j = -2\delta_{ij}$  ( $i, j = 1, 2, 3$ ) for  $X_i \in so(3)$ , the diagonal elements of  $X_i \cdot X_j$  can have different signs, as in  $so(2, 1)$ , where  $(X_i \cdot X_j) = \text{diag}(-2, -2, 2)$ .
- [13] Actually we can generalize  $\mathcal{A}^\heartsuit$  further, by replacing the product of the complex numbers  $\alpha\beta$  with a bilinear product  $\alpha\beta \longleftrightarrow \alpha \cdot \beta = \alpha_+^2 + \lambda_1 \alpha_-^2$ , where  $\alpha_\pm = (\alpha \pm \beta)/\sqrt{2}$ . We have refrained from doing so in the text.
- [14] Thus if we reverse the sign in front of the  $\sigma$  in (17) we only need to change the sign of the commutators in (18) to fix the multiplication rule.