Hypercomplex numbers and their matrix representations

A short guide for engineers and scientists

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Abstract

Hypercomplex numbers are composite numbers that sometimes allow to simplify computations. In this article, the multiplication table, matrix representation and useful formulas are compiled for eight hypercomplex number systems.

Contents

1 Introduction

Hypercomplex numbers are composite numbers that allow to simplify the mathematical description of certain problems. They are used e. g. in signal processing, computer graphics, relativistic kinematics, orbital mechanics, air and space flight. The author came across hypercomplex numbers in accelerator physics, where they can be used to describe symplectic transformations. Hypercomplex numbers come along with matrix representations, that reproduce the addition and multiplication law.

This article is aimed at engineers and scientists, and presents 8 hypercomplex number systems. The multiplication table, matrix representation, determinant, inverse, Euler formulas, polar decomposition and singular value decomposition are given. Some mathematical background is required. An occasional look at www.wikipedia.org/ (preferably the english version) or mathworld.wolfram.com/ should allow the reader to fill in gaps in comprehension.

In the appendix, four Octave [1] / Matlab demonstration programs are provided.

2 Hypercomplex numbers

2.1 History and basic properties

[2] A mathematicians basic stock-in-trade are the real numbers. In the 18th century, a combination of 2 real numbers X and Y into a new object $\mathbf{Z} = (X, Y)$, the complex number, became popular. Addition of two complex numbers was defined component-wise, but multiplication had to be defined in a different way, such that the "law of the moduli" was satisfied. The modulus $|\mathbf{Z}| = \sqrt{X^2 + Y^2}$ is the length of the vector (X, Y). The law of the moduli requires that the modulus of the product of any 2 complex numbers $\mathbf{A}$ and $\mathbf{B}$ is equal to the product of their moduli: $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

In the 19th century, mathematicians made up more composite numbers $\mathbf{Z} = (Z_1, \dots Z_N)$ consisting of $N = 4$ real numbers (quaternions), of 8 real numbers (bi-quaternions, octonions), and even of 16 real numbers (sedenions). The collective term for these new numbers was "hypercomplex numbers". Like real and complex numbers, hypercomplex (hc) numbers can be added and subtracted, multiplied and (barring accidents) divided, i. e. they form a "ring". To include hc number systems with non-associative multiplication (such as Caley numbers and octonions), mathematicians prefer to talk of "algebras" (which is a much broader term than ring, and too general for our purposes.)

Addition of two hc numbers $\mathbf{A}$ and $\mathbf{B}$ is always defined component-wise: $\mathbf{A} + \mathbf{B} = (A_1 + B_1, \dots A_N + B_N)$. Multiplication is defined such that the law of the moduli holds, with the modulus, or "euclidean norm", defined as the length of the vector $(Z_1, \dots Z_N)$: $|\mathbf{Z}| = \sqrt{(Z_1)^2 + \ldots + (Z_N)^2}$. If this is not possible, then some other norm (a real-valued function) $|\mathbf{Z}|$ with the property $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$, must be found.

Because of the special definition of multiplication, the multiplicative attributes of hc rings vary. When dealing with a hc ring, it is good to know these attributes. One should determine:

- A: Is multiplication **a**ssociative?
- C: Is multiplication **c**ommutative?
- D: Is **d**ivision by all numbers other than 0 possible?
- E: Is the norm of the numbers **e**uclidean ?

Here is the answer to these questions for the hc rings (or algebras) listed in this article:

HC ring	dimension	A	C	D	E
Real numbers	1	+	+	+	+
Bi-real numbers	2	+	+	−	−
Complex numbers	2	+	+	+	+
Cockle quaternions	4	+	−	−	−
Hamilton quaternions	4	+	−	+	+
Hamilton Bi-quaternions	8	+	−	−	−
Anonymous-3	8	+	−	−	−
Clifford Bi-quaternions	8	+	−	−	−
Space-Time algebra	16	+	−	−	−
Anonymous-4	16	+	−	−	−
Tessarines	4	+	+	−	−

Rings are associative, and therefore have a square matrix representation.

Only the real numbers, complex numbers and Hamilton quaternions have the properties A, D and E. They are "euclidean division rings".

Only the real and complex numbers have all 4 properties A, C, D, E. They are "fields".

2.2 Matrix representations

Square matrices share the structural properties of hypercomplex numbers, and can be used to represent them. While hc numbers are always made up of real numbers, we will admit representative matrices with real, complex or quaternionic elements. (Matrices with complex or quaternionic elements are nested hypercomplex numbers.)

Matrix multiplication is associative, but in general not commutative. Division by a hc number corresponds to multiplication with the inverse of the representative matrix. The norm of a hc number is the absolute value of the determinant of the representative matrix.

To find a representation, let us first write the hc number $\mathbf{Z} = (Z_1, \ldots Z_N)$ in the old-fashioned way as $\mathbf{Z} = Z_0 + iZ_1 + jZ_2 + kZ_3 + \ldots$ This number is composed of a real part (Z_0) and an imaginary part. By definition, the "imaginary units" $i, j, k \ldots$ square to either $+1$, 0 or -1. It is helpful to distinguish the three types of imaginary numbers in the notation. I will do this as follows:

$$\begin{aligned}
\mathbf{Z} \;=\; & \mathbf{1}Z_0 + && \text{real part} \\
& \left.\begin{aligned} &\boldsymbol{\beta}_1 Z_1 + \ldots + \boldsymbol{\beta}_K Z_K + && \text{bireal} \\ &\boldsymbol{\gamma}_1 Z_{K+1} + \ldots + \boldsymbol{\gamma}_L Z_{K+L} + && \text{complex} \\ &\boldsymbol{\delta}_1 Z_{K+L+1} + \ldots + \boldsymbol{\delta}_M Z_{K+L+M} && \text{dual} \end{aligned}\right\} \text{imaginary part}
\end{aligned}$$

- $\mathbf{1}$ is the multiplicative identity, or real unit. Its representation is the unit matrix.
- $\boldsymbol{\beta}$ is a **b**ireal unit and squares to $+1$: $\boldsymbol{\beta}^2 = \mathbf{1}$.
  Its real representation is a symmetric, traceless, orthogonal matrix:
  $\boldsymbol{\beta} = \boldsymbol{\beta}^\intercal$, $\mathrm{Tr}(\boldsymbol{\beta}) = 0$, $\boldsymbol{\beta}\boldsymbol{\beta}^\intercal = \mathbf{1}$.
  Its complex or quaternionic representation is a hermitian, unitary matrix with imaginary trace: $\boldsymbol{\beta} = \boldsymbol{\beta}^*$, $\Re\,\mathrm{Tr}(\boldsymbol{\beta}) = 0$, $\boldsymbol{\beta}\boldsymbol{\beta}^* = \mathbf{1}$.
- $\boldsymbol{\gamma}$ is a **c**omplex unit and squares to -1: $\boldsymbol{\gamma}^2 = -\mathbf{1}$.
  Its real representation is a skew-symmetric, orthogonal matrix:
  $\boldsymbol{\gamma} = -\boldsymbol{\gamma}^\intercal$. $\boldsymbol{\gamma}\boldsymbol{\gamma}^\intercal = \mathbf{1}$.
  Its complex or quaternionic representation is a skew-hermitian, unitary matrix:
  $\boldsymbol{\gamma} = -\boldsymbol{\gamma}^*$, $\boldsymbol{\gamma}\boldsymbol{\gamma}^* = \mathbf{1}$
- $\boldsymbol{\delta}$ is a **d**ual unit and squares to 0: $\boldsymbol{\delta}^2 = 0$.

Dual units have their use, but will be omitted in this article.

I already mentioned that addition of two hc numbers is defined component-wise as the usual addition of real numbers. The multiplication of two hc numbers is specified with a multiplication table of the units $\mathbf{1}, \boldsymbol{\beta}_k, \boldsymbol{\gamma}_l$. The matrix representation of the units together with matrix multiplication must reproduce this table.

For example, take the complex numbers $\mathbf{Z} = \mathbf{1}X + \boldsymbol{\gamma}Y$, with X and Y real.

The multiplication table is

$\cdot$	$\mathbf{1}$	$\boldsymbol{\gamma}$
$\mathbf{1}$	$\mathbf{1}$	$\boldsymbol{\gamma}$
$\boldsymbol{\gamma}$	$\boldsymbol{\gamma}$	$-\mathbf{1}$

To save space, let's truncate this to

$\mathbf{1}$	$\boldsymbol{\gamma}$
$\boldsymbol{\gamma}$	$-\mathbf{1}$

The multiplication of two complex numbers is now the familiar

$$\mathbf{A} \cdot \mathbf{B} = (\mathbf{1}A_0 + \boldsymbol{\gamma}A_1)(\mathbf{1}B_0 + \boldsymbol{\gamma}B_1) = \mathbf{1}(A_0B_0 - A_1B_1) + \boldsymbol{\gamma}(A_0B_1 + A_1B_0)$$

The real representation of $\mathbf{Z}$ is a 2×2 matrix, e. g.

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \mathbf{Z} = \mathbf{1}X + \boldsymbol{\gamma}Y \cong \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$$

Matrix multiplication now reproduces the abstract multiplication law.

2.3 Scalar product

Let

$$\bar{\mathbf{Z}} = \mathbf{1}Z_0 - \boldsymbol{\beta}_1 Z_1 \ldots - \boldsymbol{\beta}_K Z_K - \boldsymbol{\gamma}_1 Z_{K+1} \ldots - \boldsymbol{\gamma}_L Z_{K+L}$$

the conjugate of $\mathbf{Z}$, and

$$\mathbf{Z}^* = \mathbf{1}Z_0 + \boldsymbol{\beta}_1 Z_1 \ldots + \boldsymbol{\beta}_K Z_K - \boldsymbol{\gamma}_1 Z_{K+1} \ldots - \boldsymbol{\gamma}_L Z_{K+L}$$

the complex conjugate of $\mathbf{Z}$. This second definition corresponds to the conjugate transpose matrix (assuming real, complex or quaternionic matrix elements).

Also, let

$$\Re \mathbf{Z} = Z_0$$

the real part of $\mathbf{Z}$.

If $\mathbf{Z}$ is a real $d \times d$ matrix, then

$$\Re \mathbf{Z} = \tfrac{1}{d} \operatorname{Tr} \mathbf{Z}$$

since all imaginary (bireal and complex) units have trace 0.

If $\mathbf{Z}$ is a complex or quaternionic $d \times d$ matrix, then the trace is complex or quaternionic, too, and the above formula must be amended to

$$\Re \mathbf{Z} = \Re \left(\tfrac{1}{d} \operatorname{Tr} \mathbf{Z} \right)$$

It is easy to verify that

$$(\mathbf{A}|\mathbf{B}) \equiv \Re \left(\mathbf{A}^* \mathbf{B} \right)$$

is a scalar product. What's more, all the hypercomplex units form an orthogonal basis with respect to this scalar product. Therefore, the remaining coefficients of $\mathbf{Z}$ are

$$\begin{aligned} Z_k &= (\boldsymbol{\beta}_k|\mathbf{Z}) = \Re \left(\boldsymbol{\beta}_k^* \mathbf{Z} \right) && (1 \le k \le K) \\ Z_{K+l} &= (\boldsymbol{\gamma}_l|\mathbf{Z}) = \Re \left(\boldsymbol{\gamma}_l^* \mathbf{Z} \right) && (1 \le l \le L) \end{aligned}$$

2.4 Writing a matrix in a hypercomplex basis

Instead of doing hypercomplex calculations with matrices, we can also do the inverse: perform matrix calculations with hypercomplex numbers. In fact, a hypercomplex number $\mathbf{Z}$ can be considered as a matrix written in a non-standard, highly symmetric basis. What's the use of doing this? Well, I can think of 4 situations where hc numbers are advantageous.

- When the matrices have some kind of symmetry, some coefficients Z_k may become 0. This will simplify the calculation, and save computing time.
- The formulas for determinants $\det \mathbf{Z} = D(Z_0, Z_1, \ldots Z_N)$ are compact. To calculate the characteristic polynomial, simply replace Z_0 by $Z_0 - \lambda$ in the determinant formula: $\operatorname{chp}(\lambda) = \det \left(\mathbf{Z} - \lambda \mathbf{1} \right) = D(Z_0 - \lambda, Z_1, \ldots Z_N)$.
- The formulas for the inverse $\mathbf{Z}^{-1}$ of a matrix are compact.
- Describing rotations with quaternions avoids the "gimbal lock".

2.5 Interesting formulas

The interesting things to calculate given a hc number ring are those which you want to calculate when you are given the representative matrix ring: the determinant (norm), inverse, Euler formula, polar decomposition, and singular value decomposition. Later I will give these formulas for 8 hc number rings.

Determinant and Norm

The norm of a hc number $\mathbf{Z}$ is the absolute value of the determinant of the representative matrix:

$$|\mathbf{Z}| = |\det \mathbf{Z}|$$

The determinant and norm of a hc number are multiplicative:

$$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B} \qquad\qquad |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

Inverse

HC numbers with norm $\neq 0$ have an inverse, which can be written down explicitly.

Rotations

By means of Taylor series, it is straightforward to calculate the exponential or logarithm of a hc number. The representation of a complex unit is skew-hermitian, so its exponential is a rotation, i. e an orthogonal or unitary matrix. Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$ applies, and we rewrite it as

$$\exp \boldsymbol{\gamma}\phi = \mathbf{1}\cos\phi + \boldsymbol{\gamma}\sin\phi$$

When there are several complex units $\boldsymbol{\gamma}_1, \ldots \boldsymbol{\gamma}_N$, Euler's formula can be generalized.

Three kinds of rotations of a hc number $\mathbf{Z}$ appear in the polar and singular value decomposition (see below): rotation from the left: $\exp(\boldsymbol{\gamma}\phi)\mathbf{Z}$; rotation from the right: $\mathbf{Z}\exp(\boldsymbol{\gamma}\phi)$; or rotation from left and right with opposite angles: $\exp(\boldsymbol{\gamma}\phi)\mathbf{Z}\exp(-\boldsymbol{\gamma}\phi)$. The last transformation is a similarity transformation (it leaves the characteristic polynomial of $\mathbf{Z}$ invariant). To clarify the action of the rotation (how is $\mathbf{Z}$ rotated?) I will later express it by a real matrix acting on the component vector of $\mathbf{Z}$. (Since $\exp\boldsymbol{\gamma}\phi$ transforms $\mathbf{Z}$, we may say that the two hc numbers are not on an equal footing. The asymmetric way of expressing the product "rotation times hc number" as "matrix times vector" then makes sense.)

Polar Decomposition

Just as the ordinary complex number can be written as a rotation times the modulus: $z = e^{i\phi}|z|$, any hc number can be written as a rotation times a bireal number $\mathbf{P}$, i. e. a number without complex part, satisfying $\mathbf{P}^* = \mathbf{P}$:

$$\mathbf{Z} = \exp(\boldsymbol{\gamma}_1\phi_1 + \ldots + \boldsymbol{\gamma}_N\phi_N)\,\mathbf{P} \quad \text{where} \quad \mathbf{P} = \mathbf{1}P_0 + \boldsymbol{\beta}_1 P_1 + \ldots + \boldsymbol{\beta}_M P_M$$

This is the "polar decomposition" of $\mathbf{Z}$. In matrix language, the general matrix $\mathbf{Z}$ is written as a rotation times a hermitian matrix $\mathbf{P}$.

Let $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{d-1}$ denote the bireal units with a diagonal $d \times d$-representation ($d \geq 2$). The eigenvalue decomposition of the matrix $\mathbf{P} = \mathbf{1}P_0 + \boldsymbol{\beta}_1 P_1 + \ldots + \boldsymbol{\beta}_M P_M$ is

$$\mathbf{P} = \exp\left(\boldsymbol{\gamma}_1\phi_1 + \ldots + \boldsymbol{\gamma}_N\phi_N\right)\mathbf{D}\exp\left(-\boldsymbol{\gamma}_1\phi_1 - \ldots - \boldsymbol{\gamma}_N\phi_N\right)$$

where

$$\mathbf{D} = \mathbf{1}D_0 + \boldsymbol{\beta}_1 D_1 + \ldots + \boldsymbol{\beta}_{d-1}D_{d-1}$$

The polar decomposition of $\mathbf{Z}$ followed by the eigenvalue decomposition of $\mathbf{P}$ is called "singular value decomposition" of $\mathbf{Z}$. The eigenvalues $D_0, \ldots, D_{d-1}$ of $\mathbf{P}$ are the singular values of $\mathbf{Z}$.

2.6 Real Clifford algebras

There are basically two ways to construct (associative) hc numbers, starting with $\mathbb{R}$: the tensor product of two hc numbers, and a scheme due to W. K. Clifford (1845-1879). I will present the latter, since it reproduces most hypercomplex rings, and because the irreducible matrix representations of "Clifford algebras" are all known. We have defined hc numbers to consist of real components (the representations may be real, complex or quaternionic), and therefore only consider "real Clifford algebras", denoted by $Cl_{p,q}(\mathbb{R})$. The hypercomplex units and multiplication table are obtained as follows.

1. Start with the real unit $\mathbf{1}$, p bireal units $\boldsymbol{\beta}_1 \ldots \boldsymbol{\beta}_p$, and q complex units $\boldsymbol{\gamma}_1 \ldots \boldsymbol{\gamma}_q$. The symmetric part of the multiplication table of these $N = p + q$ "generator units" is

$\frac{1}{2}\{\,,\}$	$\mathbf{1}$	$\boldsymbol{\beta}_l$	$\boldsymbol{\gamma}_l$
$\mathbf{1}$	$\mathbf{1}$	$\boldsymbol{\beta}_l$	$\boldsymbol{\gamma}_n$
$\boldsymbol{\beta}_k$	$\boldsymbol{\beta}_k$	$\delta_{kl}\mathbf{1}$	$\mathbf{0}$
$\boldsymbol{\gamma}_k$	$\boldsymbol{\gamma}_k$	$\mathbf{0}$	$-\delta_{mn}\mathbf{1}$

$\{\,,\}$ is the anti-commutator: $\{\mathbf{A}, \mathbf{B}\} \equiv \mathbf{AB} + \mathbf{BA}$. Any two different generators anti-commute!

2. Define $N(N-1)/2$ new hc units as product of two different generators:
$\boldsymbol{\gamma}_{q+1} \equiv \boldsymbol{\beta}_1\boldsymbol{\beta}_2 = -\boldsymbol{\beta}_2\boldsymbol{\beta}_1, \boldsymbol{\gamma}_{q+\ldots} \equiv -\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2 = \boldsymbol{\gamma}_2\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\beta}_{p+1} \equiv \boldsymbol{\beta}_1\boldsymbol{\gamma}_1 = -\boldsymbol{\gamma}_1\boldsymbol{\beta}_1, \ldots$

3. Define $N(N-1)(N-2)/6$ new hc units as product of three different generators.

...

N. Define the last new hc unit $\boldsymbol{\omega} = \boldsymbol{\beta}_1 \ldots \boldsymbol{\beta}_p \boldsymbol{\gamma}_1 \ldots \boldsymbol{\gamma}_q$.

The full multiplication table of *all* the 2^N hc units now follows from the (much smaller) symmetric multiplication table of the real unit and the N generator units.

The sign of any composite unit can be chosen arbitrarily. I tried to choose the signs such that the multiplication tables become maximally symmetric, and mutually compatible (quaternions pop up everywhere).

The following table is the first quadrant of the "Clifford chess board" consisting of 8×8 entries [3], [4]. It shows the hypercomplex numbers produced by Clifford's scheme for up to 4 generators:

$Cl_{p\downarrow,q\rightarrow}$	0	1	2	3	4
0	real	complex	quaternion	Clifford BQ	Space-Time
1	bi-real	Cockle Q	Hamilton BQ	Space-Time	$\mathbb{H}(2) \oplus \mathbb{H}(2)$
2	Cockle Q	Anonymous-3	Anonymous-4	$\mathbb{C}(4)$	$\mathbb{H}(4)$
3	Hamilton BQ	Anonymous-4	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$
4	Space-Time	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

Clifford's scheme is redundant: some Clifford algebras with different p, q, but the same overall number of generators $N = p + q$, are isomorphic: the multiplication table of such algebras can be brought to coincide after re-numbering the imaginary units.

The next table shows the irreducible matrix representations of the Clifford algebras.

$Cl_{p\downarrow,q\rightarrow}$	0	1	2	3	4
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$
1	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$

It turns out that every Clifford algebra is isomorphic to

- a full $d \times d$ real matrix algebra $\mathbb{R}(d)$, or
- a full $d \times d$ complex matrix algebra $\mathbb{C}(d)$, or
- a full $d \times d$ quaternionic matrix algebra $\mathbb{H}(d)$, or
- to the direct sum of two such algebras.

Note that the ring of real 2×2 matrices $\mathbb{R}(2)$ is isomorphic to the Cockle (or split) quaternions, and that the ring of real 4×4 matrices $\mathbb{R}(4)$ is isomorphic to Anonymous-4.

In the following subsection, I will present all real Clifford algebras with up to 4 generators. Until here, Clifford 's scheme produces (I believe) all hc number rings with up to 16 components. A special case are the hc numbers called "tessarines": they are not a Clifford algebra proper, but appear as a sub-algebra of Anonymous-3.

Cliffords scheme does not produce non-associative hc numbers. These and hc numbers with dual units are omitted in this article.

In the following list of hypercomplex units, the generating units are given in red color . In a multiplication table, the anti-commuting products are given in blue color .

In the multiplication tables, I use the Kronecker-delta δ_{ik} and the anti-symmetric tensor ϵ_{ikl}. Roman indices like $k, l, \ldots$ usually go from 1 to 3, and greek indices like $\kappa, \lambda, \ldots$ go from 0 to 3. If an index appears twice in a product, you must sum over it. Examples:

$$\boldsymbol{\beta}_k X_l \delta_{kl} = \boldsymbol{\beta}_k X_k = \boldsymbol{\beta}_1 X_1 + \boldsymbol{\beta}_2 X_2 + \boldsymbol{\beta}_3 X_3 = \vec{\boldsymbol{\beta}} \cdot \vec{X}$$

$$\boldsymbol{\gamma}_k X_l Y_m \epsilon_{klm} = \boldsymbol{\gamma}_1 (X_2 Y_3 - X_3 Y_2) + \boldsymbol{\gamma}_2 (X_3 Y_1 - X_1 Y_3) + \boldsymbol{\gamma}_3 (X_1 Y_2 - X_2 Y_1) = \vec{\boldsymbol{\gamma}} \cdot \vec{X} \wedge \vec{Y}$$

$$\boldsymbol{\beta}_\kappa Z_\kappa = \boldsymbol{\beta}_0 Z_0 + \boldsymbol{\beta}_1 Z_1 + \boldsymbol{\beta}_2 Z_2 + \boldsymbol{\beta}_3 Z_3$$

2.7 Real numbers $Cl_{0,0}(\mathbb{R}) \cong \mathbb{R}$

The real numbers are the simplest Clifford algebra, with 0 generators. They form a field. But you are not reading this article to learn about real numbers, so let's go on to the next stage.

3 Hypercomplex numbers with 1 generator

3.1 Bireal numbers $Cl_{1,0}(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}$

(Synonym: split-complex numbers)

Hypercomplex units: $\mathbf{1}, \boldsymbol{\beta}$

Multiplication table

$$\boxed{\begin{array}{cc} \mathbf{1} & \boldsymbol{\beta} \\ \boldsymbol{\beta} & \mathbf{1} \end{array}} \quad \Rightarrow \quad \begin{array}{c} (\mathbf{1}A_0 + \boldsymbol{\beta}A_1)(\mathbf{1}B_0 + \boldsymbol{\beta}B_1) = \\ \mathbf{1}(A_0B_0 + A_1B_1) + \boldsymbol{\beta}(A_0B_1 + A_1B_0) \end{array}$$

General bireal number and its real representation

$$\mathbf{Z} = \mathbf{1}X + \boldsymbol{\beta}Y \cong \begin{pmatrix} X+Y & 0 \\ 0 & X-Y \end{pmatrix}$$

Determinant

$$\det \mathbf{Z} \equiv D = \mathbf{Z}\bar{\mathbf{Z}} = X^2 - Y^2$$

Inverse bireal number

$$\mathbf{Z}^{-1} = \bar{\mathbf{Z}}/(\mathbf{Z}\bar{\mathbf{Z}}) = (\mathbf{1}X - \boldsymbol{\beta}Y)/D$$

Singular Values and Determinant

$$D_1 = X + Y \qquad D_2 = X - Y \qquad D_1 D_2 = X^2 - Y^2 = D$$

3.2 Complex numbers $Cl_{0,1}(\mathbb{R}) \cong \mathbb{C}$

The complex numbers form a field; they are their own irreducible representation.

Hypercomplex units: $\mathbf{1}, \boldsymbol{\gamma}$

Multiplication table

$$\boxed{\begin{array}{cc} \mathbf{1} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma} & -\mathbf{1} \end{array}} \quad \Rightarrow \quad \begin{array}{c} (\mathbf{1}A_0 + \boldsymbol{\gamma}A_1)(\mathbf{1}B_0 + \boldsymbol{\gamma}B_1) = \\ \mathbf{1}(A_0B_0 - A_1B_1) + \boldsymbol{\gamma}(A_0B_1 + A_1B_0) \end{array}$$

General complex number = complex representation

$$\mathbf{Z} = \mathbf{1}X + \boldsymbol{\gamma}Y$$

Norm

$$|\mathbf{Z}| = \sqrt{\mathbf{Z}\bar{\mathbf{Z}}} = \sqrt{X^2 + Y^2}$$

Inverse complex number

$$\mathbf{Z}^{-1} = \bar{\mathbf{Z}}/(\mathbf{Z}\bar{\mathbf{Z}}) = (\mathbf{1}X - \boldsymbol{\gamma}Y)/(X^2 + Y^2)$$

Euler formula

$$\exp(\boldsymbol{\gamma}\phi) = \mathbf{1}\cos\phi + \boldsymbol{\gamma}\sin\phi$$

$\exp(\boldsymbol{\gamma}\phi)$ rotates $\mathbf{Z}$ counterclockwise (ccw) by ϕ:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\boldsymbol{\gamma}\phi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\boldsymbol{\gamma}\phi) \end{array} \right\} \quad \Rightarrow \quad \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \times \begin{pmatrix} X \\ Y \end{pmatrix}$$

Polar Decomposition

$$\mathbf{Z} = \exp(\boldsymbol{\gamma}\phi)|\mathbf{Z}| \quad \text{where} \quad \tan\phi = Y/X \quad \text{and modulus} \quad |\mathbf{Z}| = \sqrt{X^2 + Y^2}$$

Addendum

A real representation (reducible over $\mathbb{C}$) of the complex numbers is

$$\mathbf{Z} \cong \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$$

The Singular Values and Determinant of the real representation are

$$D_1 = D_2 = \sqrt{X^2 + Y^2} \qquad D_1D_2 = X^2 + Y^2$$

4 Hypercomplex numbers with 2 generators

4.1 Cockle quaternions $Cl_{2,0}(\mathbb{R}) \cong Cl_{1,1}(\mathbb{R}) \cong \mathbb{R}(2)$

(Synonyms: split-quaternions, Co-quaternions)

Hypercomplex units of $Cl_{2,0}(\mathbb{R})$: $\mathbf{1}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\gamma} = \boldsymbol{\beta}_1\boldsymbol{\beta}_2$

Hypercomplex units of $Cl_{1,1}(\mathbb{R})$: $\mathbf{1}, \boldsymbol{\beta}_1, \boldsymbol{\gamma}, \boldsymbol{\beta}_2 = \boldsymbol{\beta}_1\boldsymbol{\gamma}$

Multiplication table

$\mathbf{1}$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$
$\boldsymbol{\gamma}$	$-\mathbf{1}$	$-\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_1$
$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\mathbf{1}$	$\boldsymbol{\gamma}$
$\boldsymbol{\beta}_2$	$-\boldsymbol{\beta}_1$	$-\boldsymbol{\gamma}$	$\mathbf{1}$

General Cockle quaternion and its real representation

$$\mathbf{Z} = \mathbf{1}X_1 + \boldsymbol{\gamma}X_2 + \boldsymbol{\beta}_1 Y_1 + \boldsymbol{\beta}_2 Y_2 \cong \begin{pmatrix} X_1 + Y_1 & X_2 + Y_2 \\ -X_2 + Y_2 & X_1 - Y_1 \end{pmatrix}$$

Determinant

$$\det \mathbf{Z} \equiv D = \mathbf{Z}\bar{\mathbf{Z}} = X_1^2 + X_2^2 - Y_1^2 - Y_2^2$$

Inverse Cockle quaternion

$$\mathbf{Z}^{-1} = \bar{\mathbf{Z}}/(\mathbf{Z}\bar{\mathbf{Z}}) = (\mathbf{1}X_1 - \boldsymbol{\gamma}X_2 - \boldsymbol{\beta}_1 Y_1 - \boldsymbol{\beta}_2 Y_2)/D$$

Euler formula

$$\exp(\boldsymbol{\gamma}\phi) = \mathbf{1}\cos\phi + \boldsymbol{\gamma}\sin\phi$$

$\exp(\boldsymbol{\gamma}\phi)$ rotates (X_1, X_2) ccw by ϕ, and rotates rotates (Y_1, Y_2) cw by $\pm\phi$:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\boldsymbol{\gamma}\phi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\boldsymbol{\gamma}\phi) \end{array} \right\} \quad \Rightarrow \quad \begin{pmatrix} X_1' \\ X_2' \\ Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \pm\sin\phi \\ 0 & 0 & \mp\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix}$$

The similarity transformation $\exp(\boldsymbol{\gamma}\phi)\ldots\exp(-\boldsymbol{\gamma}\phi)$ leaves (X_1, X_2) invariant, and rotates (Y_1, Y_2) cw by 2ϕ:

$$\mathbf{Z}' = \exp(\boldsymbol{\gamma}\phi)\,\mathbf{Z}\exp(-\boldsymbol{\gamma}\phi) \quad \Rightarrow \quad \begin{pmatrix} X_1' \\ X_2' \\ Y_1' \\ Y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 2\phi & \sin 2\phi \\ 0 & 0 & -\sin 2\phi & \cos 2\phi \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{pmatrix}$$

Polar Decomposition

$\mathbf{Z} = \exp(\boldsymbol{\gamma}\phi)\,\mathbf{P}$

where $\quad \tan\phi = X_2/X_1 + (1-s_0)s_3\,\pi/2 \quad$ with $\quad s_k = \mathrm{sign}(Z_k)$

and $\quad \mathbf{P} = \mathbf{1}\sqrt{X_1^2 + X_2^2} + \boldsymbol{\beta}_1 P_3 + \boldsymbol{\beta}_2 P_4$

Singular Value Decomposition

$\mathbf{P} = \exp(\boldsymbol{\gamma}\psi)\mathbf{D}\exp(-\boldsymbol{\gamma}\psi)$

where $\quad \tan 2\psi = -P_2/P_1$

and $\quad \mathbf{D} = \mathbf{1}\sqrt{X_1^2 + X_2^2} + \boldsymbol{\beta}_1\sqrt{Y_1^2 + Y_2^2}$

SVs $\quad D_{1,2} = \sqrt{X_1^2 + X_2^2} \pm \sqrt{Y_1^2 + Y_2^2}$

with $\quad D_1 D_2 = X_1^2 + X_2^2 - Y_1^2 - Y_2^2 = D$

4.2 Hamilton quaternions $Cl_{0,2}(\mathbb{R}) \cong \mathbb{H}$

Hamilton quaternions form a division ring (that's a field minus commutativity in multiplication); they are their own irreducible representation.

Hypercomplex units: $\mathbf{1}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3 = -\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2$ (my choice of sign is non-standard)

Multiplication table

$\mathbf{1}$	$\boldsymbol{\gamma}_l$
$\boldsymbol{\gamma}_k$	$-\delta_{kl}\mathbf{1} - \epsilon_{klm}\boldsymbol{\gamma}_m$

$$\Rightarrow \quad \begin{gathered}(\mathbf{1}A_0 + \vec{\boldsymbol{\gamma}}\cdot\vec{A})(\mathbf{1}B_0 + \vec{\boldsymbol{\gamma}}\cdot\vec{B}) = \\ \mathbf{1}(A_0B_0 - \vec{A}\cdot\vec{B}) + \boldsymbol{\gamma}(A_0\vec{B} + \vec{A}B_0 - \vec{A}\wedge\vec{B})\end{gathered}$$

General Hamilton quaternion

$\mathbf{Z} = \mathbf{1}Z_0 + \boldsymbol{\gamma}_1 Z_1 + \boldsymbol{\gamma}_2 Z_2 + \boldsymbol{\gamma}_3 Z_3 \equiv \mathbf{1}Z_0 + \vec{\boldsymbol{\gamma}}\cdot\vec{Z}$

Norm

$|\mathbf{Z}| = \sqrt{\mathbf{Z}\bar{\mathbf{Z}}} = \sqrt{Z_0^2 + \vec{Z}^2}$

Inverse Hamilton quaternion

$$\mathbf{Z}^{-1} = \bar{\mathbf{Z}}/(\mathbf{Z}\bar{\mathbf{Z}}) = (\mathbf{1}Z_0 - \vec{\boldsymbol{\gamma}} \cdot \vec{Z})/(Z_0^2 + \vec{Z}^2)$$

Euler formula

$$\exp(\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi) = \mathbf{1}\cos\phi + \vec{\boldsymbol{\gamma}} \cdot \vec{e}\sin\phi$$

Let $Z_\parallel$ the 3-vector component parallel to $\vec{e}$ (i. e. $Z_\parallel \equiv \vec{Z} \cdot \vec{e}$), and $\vec{Z}_\perp$ the 3-vector projection perpendicular to $\vec{e}$ (i. e. $\vec{Z}_\perp \equiv -\vec{e} \wedge \vec{e} \wedge \vec{Z}$). The factor $\exp(\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi)$ rotates $(Z_0, Z_\parallel)$ ccw by ϕ, and rotates $\vec{Z}_\perp$ cw by $\pm\phi$ around $\vec{e}$:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi) \end{array} \right\} \qquad \Rightarrow$$

$$\begin{pmatrix} Z_0' \\ Z_1' \\ Z_2' \\ Z_3' \end{pmatrix} = \left[\cos\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 & -e_1 & -e_2 & -e_3 \\ e_1 & 0 & \pm e_3 & \mp e_2 \\ e_2 & \mp e_3 & 0 & \pm e_1 \\ e_3 & \pm e_2 & \mp e_1 & 0 \end{pmatrix} \right] \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

The similarity transformation $\exp(\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi) \ldots \exp(-\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi)$ leaves Z_0 and $Z_\parallel$ invariant, and rotates $\vec{Z}_\perp$ cw by 2ϕ around $\vec{e}$:

$$\mathbf{Z}' = \exp(\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi)\,\mathbf{Z}\exp(-\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi) \qquad \Rightarrow \qquad \begin{pmatrix} Z_0' \\ Z_1' \\ Z_2' \\ Z_3' \end{pmatrix} = \left[\cos 2\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \right.$$

$$\left. \ldots (1 - \cos 2\phi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (e_1)^2 & e_1 e_2 & e_1 e_3 \\ 0 & e_2 e_1 & (e_2)^2 & e_2 e_3 \\ 0 & e_3 e_1 & e_3 e_2 & (e_3)^2 \end{pmatrix} + \sin 2\phi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_3 & -e_2 \\ 0 & -e_3 & 0 & e_1 \\ 0 & e_2 & -e_1 & 0 \end{pmatrix} \right] \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

Polar Decomposition

$$\mathbf{Z} = \exp(\vec{\boldsymbol{\gamma}} \cdot \vec{e}\phi)\,|\mathbf{Z}| \text{ where } \vec{e}\tan\phi = \vec{Z}/Z_0 \text{ and modulus } |\mathbf{Z}| = \sqrt{Z_0^2 + \vec{Z}^2}$$

Addendum

Complex representation (reducible over $\mathbb{H}$):

$$\mathbf{Z} \cong \begin{pmatrix} Z_0 + iZ_3 & Z_2 + iZ_1 \\ -Z_2 + iZ_1 & Z_0 - iZ_3 \end{pmatrix}$$

Singular Values and Determinant of the complex representation:

$$D_1 = D_2 = \sqrt{Z_0^2 + \vec{Z}^2} \qquad\qquad D_1 D_2 = Z_0^2 + \vec{Z}^2$$

Until now, the letters $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ were sufficient to denote bireal and complex units. In the following hc rings, I will occasionally use $\boldsymbol{\alpha}$ and $\boldsymbol{\zeta}$ for an additional bireal and complex unit. This will help to bring out the symmetry of the multiplication table.

$\boxed{\text{Real unit: } \mathbf{1}\text{; bireal units: } \boldsymbol{\alpha}, \boldsymbol{\beta}\text{; complex units: } \boldsymbol{\gamma}, \boldsymbol{\zeta}.}$

5 Hypercomplex numbers with 3 generators

5.1 Hamilton biquaternions $Cl_{3,0}(\mathbb{R}) \cong Cl_{1,2}(\mathbb{R}) \cong \mathbb{C}(2)$

HC units of $Cl_{3,0}(\mathbb{R})$: $\mathbf{1}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\zeta}_1 = \boldsymbol{\beta}_2\boldsymbol{\beta}_3, \boldsymbol{\zeta}_2 = \boldsymbol{\beta}_3\boldsymbol{\beta}_1, \boldsymbol{\zeta}_3 = \boldsymbol{\beta}_1\boldsymbol{\beta}_2, \boldsymbol{\gamma} = -\boldsymbol{\beta}_1\boldsymbol{\beta}_2\boldsymbol{\beta}_3$

HC units of $Cl_{1,2}(\mathbb{R})$: $\mathbf{1}, \boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_1 = \boldsymbol{\beta}_3\boldsymbol{\zeta}_2, \boldsymbol{\beta}_2 = \boldsymbol{\zeta}_1\boldsymbol{\beta}_3, \boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_2\boldsymbol{\zeta}_1, \boldsymbol{\gamma} = \boldsymbol{\zeta}_1\boldsymbol{\zeta}_2\boldsymbol{\beta}_3$

Multiplication table

$\mathbf{1}$	$\boldsymbol{\zeta}_l$	$\boldsymbol{\gamma}$	$\boldsymbol{\beta}_l$
$\boldsymbol{\zeta}_k$	$-\delta_{kl}\mathbf{1} - \epsilon_{klm}\boldsymbol{\zeta}_m$	$\boldsymbol{\beta}_k$	$-\delta_{kl}\boldsymbol{\gamma} - \epsilon_{klm}\boldsymbol{\beta}_m$
$\boldsymbol{\gamma}$	$\boldsymbol{\beta}_k$	$-\mathbf{1}$	$-\boldsymbol{\zeta}_l$
$\boldsymbol{\beta}_k$	$-\delta_{kl}\boldsymbol{\gamma} - \epsilon_{klm}\boldsymbol{\beta}_m$	$-\boldsymbol{\zeta}_k$	$\delta_{kl}\mathbf{1} + \epsilon_{klm}\boldsymbol{\zeta}_m$

General Hamilton biquaternion and its complex representation

$$\mathbf{Z} = \mathbf{1}X_0 + \vec{\boldsymbol{\zeta}} \cdot \vec{X} + \boldsymbol{\gamma}Y_0 + \vec{\boldsymbol{\beta}} \cdot \vec{Y} \cong \begin{pmatrix} X_0 + Y_3 + i(X_3 - Y_0) & X_2 + Y_1 + i(X_1 - Y_2) \\ -X_2 + Y_1 + i(Y_2 + X_1) & X_0 - Y_3 + i(-X_3 - Y_0) \end{pmatrix}$$

Determinant

$\det \mathbf{Z} \equiv D = A - iB$

where $\quad A = X_0^2 + \vec{X}^2 - Y_0^2 - \vec{Y}^2 \quad$ and $\quad B = 2(X_0Y_0 + \vec{X} \cdot \vec{Y})$

Inverse Hamilton biquaternion

$$\mathbf{Z}^{-1} = \frac{\mathbf{1}(AX_0 + BY_0) - \vec{\boldsymbol{\zeta}} \cdot (A\vec{X} + B\vec{Y}) + \boldsymbol{\gamma}(AY_0 - BX_0) - \vec{\boldsymbol{\beta}} \cdot (A\vec{Y} - B\vec{X})}{A^2 + B^2}$$

Euler formulas

$\exp(\boldsymbol{\gamma}\phi) = \mathbf{1}\cos\phi + \boldsymbol{\gamma}\sin\phi \qquad$ unitary $(|\det| = 1)$

$\exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) = \mathbf{1}\cos\psi + \vec{\boldsymbol{\zeta}} \cdot \vec{e}\sin\psi \qquad$ special unitary $(\det = 1)$.

The two rotations commute: $\exp(\boldsymbol{\gamma}\phi)\exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) = \exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right)\exp(\boldsymbol{\gamma}\phi)$

$\exp(\boldsymbol{\gamma}\phi)$ rotates each vector (X_κ, Y_κ) (with $\kappa = 0\ldots 3$) clockwise by ϕ:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\boldsymbol{\gamma}\phi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\boldsymbol{\gamma}\phi) \end{array} \right\} \qquad \Rightarrow \qquad \begin{pmatrix} X_0' & Y_0' \\ X_1' & Y_1' \\ X_2' & Y_2' \\ X_3' & Y_3' \end{pmatrix} = \begin{pmatrix} X_0 & Y_0 \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Let $X_\parallel$, $Y_\parallel$ the 3-vector components parallel to $\vec{e}$, and $\vec{X}_\perp$,$\vec{Y}_\perp$ the 3-vector projections perpendicular to $\vec{e}$. The factor $\exp\left(\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right)$ rotates $(X_0, X_\parallel)$ and $(Y_0, Y_\parallel)$ ccw by ϕ, and rotates $\vec{X}_\perp$ and $\vec{Y}_\perp$ cw by $\pm\phi$ around $\vec{e}$:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp\left(\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right)\mathbf{Z} \\ \mathbf{Z}\exp\left(\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right) \end{array} \right\} \qquad \Rightarrow \qquad \begin{pmatrix} X_0' & Y_0' \\ X_1' & Y_1' \\ X_2' & Y_2' \\ X_3' & Y_3' \end{pmatrix} =$$

$$\left[\cos\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 & -e_1 & -e_2 & -e_3 \\ e_1 & 0 & \pm e_3 & \mp e_2 \\ e_2 & \mp e_3 & 0 & \pm e_1 \\ e_3 & \pm e_2 & \mp e_1 & 0 \end{pmatrix}\right] \begin{pmatrix} X_0 & Y_0 \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{pmatrix}$$

The similarity transformation $\exp\left(\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right)\ldots\exp\left(-\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right)$ leaves X_0, $X_\parallel$ and Y_0, $Y_\parallel$ invariant, and rotates $\vec{X}_\perp$ and $\vec{Y}_\perp$ cw by 2ϕ around $\vec{e}$:

$$\mathbf{Z}' = \exp\left(\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right)\mathbf{Z}\exp\left(-\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right) \qquad \Rightarrow \qquad \begin{pmatrix} X_0' & Y_0' \\ X_1' & Y_1' \\ X_2' & Y_2' \\ X_3' & Y_3' \end{pmatrix} = \left[\cos 2\psi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \right.$$

$$\left. \ldots (1-\cos 2\psi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (e_1)^2 & e_1e_2 & e_1e_3 \\ 0 & e_2e_1 & (e_2)^2 & e_2e_3 \\ 0 & e_3e_1 & e_3e_2 & (e_3)^2 \end{pmatrix} + \sin 2\psi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_3 & -e_2 \\ 0 & -e_3 & 0 & e_1 \\ 0 & e_2 & -e_1 & 0 \end{pmatrix}\right] \begin{pmatrix} X_0 & Y_0 \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{pmatrix}$$

Polar Decomposition

$$\mathbf{Z} = \exp\left(\vec{\boldsymbol{\zeta}}\cdot\vec{e}\psi\right)\exp(\boldsymbol{\gamma}\phi)\,\mathbf{P}$$

where $\quad \mathbf{P} = \mathbf{1}P_0 + \vec{\boldsymbol{\beta}}\cdot\vec{P}$

and $\quad \vec{E} \equiv \vec{e}\tan\psi \quad$ solves $\quad \vec{a} + \vec{\vec{b}}\vec{E} - \vec{E}\vec{E}\cdot\vec{a} = 0 \quad (*)$

with $\quad \vec{a} \equiv X_0\vec{X} + Y_0\vec{Y} \quad$ and $\quad \vec{\vec{b}} \equiv \vec{X}\vec{X}^t + \vec{Y}\vec{Y}^t - (X_0^2 + Y_0^2)\,\vec{\vec{1}}$

and $\quad \tan\phi = (Y_0 + \vec{E}\cdot\vec{Y})/(X_0 + \vec{E}\cdot\vec{X})$

* E. g. iterate $\vec{E} = \vec{\vec{b}}^{-1}\left(\vec{E}\vec{E}\cdot\vec{a} - \vec{a}\right)$, but it doesn't always work.

Singular Value Decomposition

$$\mathbf{P} = \exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \mathbf{D} \exp\left(-\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right)$$

where $\quad \vec{e}\tan\psi = (P_2/P_3, -P_1/P_3, 0)^t \quad$ and $\quad \mathbf{D} = \mathbf{1}P_0 + \boldsymbol{\beta}_1|\vec{P}|$

(How to express P_0 and $|\vec{P}|$ in terms of X and Y?)

SVs $\quad D_1 = P_0 + |\vec{P}|, \quad D_2 = P_0 - |\vec{P}|$

with $\quad D_1 D_2 = P_0^2 - \vec{P}^2 = \pm\sqrt{A^2 + B^2}$

5.2 Anonymous-3 $Cl_{2,1}(\mathbb{R}) \cong \mathbb{R}(2) \oplus \mathbb{R}(2)$

HC units: $\mathbf{1}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\gamma}, \boldsymbol{\zeta} = \boldsymbol{\beta}_1\boldsymbol{\beta}_2, \boldsymbol{\beta}_3 = \boldsymbol{\beta}_1\boldsymbol{\gamma}, \boldsymbol{\beta}_4 = \boldsymbol{\beta}_2\boldsymbol{\gamma}, \boldsymbol{\alpha} = \boldsymbol{\beta}_1\boldsymbol{\beta}_2\boldsymbol{\gamma}$

Multiplication table

$\mathbf{1}$	$\boldsymbol{\zeta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_3$	$\boldsymbol{\beta}_4$
$\boldsymbol{\zeta}$	$-\mathbf{1}$	$\boldsymbol{\alpha}$	$-\boldsymbol{\gamma}$	$-\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_1$	$-\boldsymbol{\beta}_4$	$\boldsymbol{\beta}_3$
$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$	$-\mathbf{1}$	$-\boldsymbol{\zeta}$	$-\boldsymbol{\beta}_3$	$-\boldsymbol{\beta}_4$	$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$
$\boldsymbol{\alpha}$	$-\boldsymbol{\gamma}$	$-\boldsymbol{\zeta}$	$\mathbf{1}$	$\boldsymbol{\beta}_4$	$-\boldsymbol{\beta}_3$	$-\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_1$
$\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_3$	$\boldsymbol{\beta}_4$	$\mathbf{1}$	$\boldsymbol{\zeta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\alpha}$
$\boldsymbol{\beta}_2$	$-\boldsymbol{\beta}_1$	$\boldsymbol{\beta}_4$	$-\boldsymbol{\beta}_3$	$-\boldsymbol{\zeta}$	$\mathbf{1}$	$-\boldsymbol{\alpha}$	$\boldsymbol{\gamma}$
$\boldsymbol{\beta}_3$	$\boldsymbol{\beta}_4$	$-\boldsymbol{\beta}_1$	$-\boldsymbol{\beta}_2$	$-\boldsymbol{\gamma}$	$-\boldsymbol{\alpha}$	$\mathbf{1}$	$\boldsymbol{\zeta}$
$\boldsymbol{\beta}_4$	$-\boldsymbol{\beta}_3$	$-\boldsymbol{\beta}_2$	$\boldsymbol{\beta}_1$	$\boldsymbol{\alpha}$	$-\boldsymbol{\gamma}$	$-\boldsymbol{\zeta}$	$\mathbf{1}$

General HC number and its real representation

$$\mathbf{Z} = \mathbf{1}X_1 + \boldsymbol{\zeta}X_2 + \boldsymbol{\gamma}X_3 + \boldsymbol{\alpha}X_4 + \boldsymbol{\beta}_1Y_1 + \boldsymbol{\beta}_2Y_2 + \boldsymbol{\beta}_3Y_3 + \boldsymbol{\beta}_4Y_4 \cong$$

$$\begin{pmatrix} X_1 + X_4 + Y_1 + Y_4 & -X_2 + X_3 - Y_2 + Y_3 & 0 & 0 \\ X_2 - X_3 - Y_2 + Y_3 & X_1 + X_4 - Y_1 - Y_4 & 0 & 0 \\ 0 & 0 & X_1 - X_4 + Y_1 - Y_4 & X_2 + X_3 + Y_2 + Y_3 \\ 0 & 0 & -X_2 - X_3 + Y_2 + Y_3 & X_1 - X_4 - Y_1 + Y_4 \end{pmatrix}$$

Determinant

$\det \mathbf{Z} \equiv D = AB$

where $\quad A = (X_1 + X_4)^2 + (X_2 - X_3)^2 - (Y_1 + Y_4)^2 - (Y_2 - Y_3)^2$

and $\quad B = (X_1 - X_4)^2 + (X_2 + X_3)^2 - (Y_1 - Y_4)^2 - (Y_2 + Y_3)^2$

Inverse HC number

$$\mathbf{Z}^{-1} = \frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}\right)(\mathbf{1}X_1 - \boldsymbol{\zeta}X_2 - \boldsymbol{\gamma}X_3 + \boldsymbol{\alpha}X_4 - \boldsymbol{\beta}_1Y_1 - \boldsymbol{\beta}_2Y_2 - \boldsymbol{\beta}_3Y_3 - \boldsymbol{\beta}_4Y_4)$$
$$+\frac{1}{2}\left(\frac{1}{A}-\frac{1}{B}\right)(\mathbf{1}X_4 + \boldsymbol{\zeta}X_3 + \boldsymbol{\gamma}X_2 + \boldsymbol{\alpha}X_1 - \boldsymbol{\beta}_1Y_4 + \boldsymbol{\beta}_2Y_3 + \boldsymbol{\beta}_3Y_2 - \boldsymbol{\beta}_4Y_1)$$

Euler formulas

$$\exp(\boldsymbol{\gamma}\phi) = \mathbf{1}\cos\phi + \boldsymbol{\gamma}\sin\phi \qquad \exp(\boldsymbol{\zeta}\psi) = \mathbf{1}\cos\psi + \boldsymbol{\zeta}\sin\psi$$

The two rotations commute with each other.

$\exp(\boldsymbol{\gamma}\phi)$ rotates (X_1, X_3) and (X_2, X_4) ccw by ϕ, and (Y_1, Y_3) and (Y_2, Y_4) cw by $\pm\phi$:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\boldsymbol{\gamma}\phi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\boldsymbol{\gamma}\phi) \end{array} \right\} \qquad \Rightarrow \qquad \begin{pmatrix} X_1' & X_3' \\ X_2' & X_4' \end{pmatrix} = \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$
$$\begin{pmatrix} Y_1' & Y_3' \\ Y_2' & Y_4' \end{pmatrix} = \begin{pmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{pmatrix} \begin{pmatrix} \cos\phi & \mp\sin\phi \\ \pm\sin\phi & \cos\phi \end{pmatrix}$$

The similarity transformation $\exp(\boldsymbol{\gamma}\phi)\ldots\exp(-\boldsymbol{\gamma}\phi)$ leaves (X_1, X_3) and (X_2, X_4) invariant, and rotates (Y_1, Y_3) and (Y_2, Y_4) cw by 2ϕ:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\boldsymbol{\gamma}\phi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\boldsymbol{\gamma}\phi) \end{array} \right\} \qquad \Rightarrow \qquad \begin{pmatrix} X_1' & X_3' \\ X_2' & X_4' \end{pmatrix} = \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}$$
$$\begin{pmatrix} Y_1' & Y_3' \\ Y_2' & Y_4' \end{pmatrix} = \begin{pmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{pmatrix} \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix}$$

$\exp(\boldsymbol{\zeta}\psi)$ rotates (X_1, X_2) and (X_3, X_4) ccw by ψ, and (Y_1, Y_2) and (Y_3, Y_4) cw by $\pm\psi$:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\boldsymbol{\zeta}\psi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\boldsymbol{\zeta}\psi) \end{array} \right\} \qquad \Rightarrow \qquad \begin{pmatrix} X_1' & X_3' \\ X_2' & X_4' \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}$$
$$\begin{pmatrix} Y_1' & Y_3' \\ Y_2' & Y_4' \end{pmatrix} = \begin{pmatrix} \cos\psi & \pm\sin\psi \\ \mp\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{pmatrix}$$

The similarity transformation $\exp(\boldsymbol{\zeta}\psi)\ldots\exp(-\boldsymbol{\zeta}\psi)$ leaves (X_1, X_2) and (X_3, X_4) invariant, and rotates (Y_1, Y_2) and (Y_3, Y_4) cw by 2ψ:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp(\boldsymbol{\zeta}\psi)\,\mathbf{Z} \\ \mathbf{Z}\exp(\boldsymbol{\zeta}\psi) \end{array} \right\} \qquad \Rightarrow \qquad \begin{pmatrix} X_1' & X_3' \\ X_2' & X_4' \end{pmatrix} = \begin{pmatrix} X_1 & X_3 \\ X_2 & X_4 \end{pmatrix}$$
$$\begin{pmatrix} Y_1' & Y_3' \\ Y_2' & Y_4' \end{pmatrix} = \begin{pmatrix} \cos 2\psi & \sin 2\psi \\ -\sin 2\psi & \cos 2\psi \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{pmatrix}$$

Polar Decomposition

$\mathbf{Z} = \exp(\boldsymbol{\zeta}\psi)\exp(\boldsymbol{\gamma}\phi)\,\mathbf{P}$

where $\quad \mathbf{P} = \mathbf{1}P_1 + \boldsymbol{\alpha}P_4 + \boldsymbol{\beta}_1 P_5 + \boldsymbol{\beta}_2 P_6 + \boldsymbol{\beta}_3 P_7 + \boldsymbol{\beta}_4 P_8$

and $\quad \tan 2\psi = \dfrac{2\,(X_1X_2 + X_3X_4)}{X_1^2 - X_2^2 + X_3^2 - X_4^2}$

and $\quad \tan\phi = \dfrac{X_3 + \tan\psi X_4}{X_1 + \tan\psi X_2}$

Singular Value Decomposition

$\mathbf{P} = \exp(\boldsymbol{\zeta}\psi)\exp(\boldsymbol{\gamma}\phi)\,\mathbf{D}\exp(-\boldsymbol{\gamma}\phi)\exp(-\boldsymbol{\zeta}\psi)$

where $\quad \mathbf{D} = \mathbf{1}D_1 + \boldsymbol{\alpha}D_4 + \boldsymbol{\beta}_1 D_5 + \boldsymbol{\beta}_4 D_8$

and $\quad \tan 4\psi = -\dfrac{2\,(P_5P_6 + P_7P_8)}{P_5^2 - P_6^2 + P_7^2 - P_8^2}$

and $\quad \tan 2\phi = -\dfrac{P_7 - \tan 2\psi P_8}{P_5 - \tan 2\psi P_6}$

Tessarines

The hc numbers $\mathbf{Z} = \mathbf{1}X_1 + \boldsymbol{\zeta}X_2 + \boldsymbol{\gamma}X_3 + \boldsymbol{\alpha}X_4$ form a commutative sub-ring of Anonymous-3 known as "tessarines".

5.3 Clifford biquaternions $Cl_{0,3} \cong \mathbb{H} \oplus \mathbb{H}$

(Synonym: split biquaternions)

HC units: $\mathbf{1}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \boldsymbol{\zeta}_1 = -\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3, \boldsymbol{\zeta}_2 = -\boldsymbol{\gamma}_3\boldsymbol{\gamma}_1, \boldsymbol{\zeta}_3 = -\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2, \boldsymbol{\beta} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$

Multiplication table

$\mathbf{1}$	$\boldsymbol{\zeta}_l$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}_l$
$\boldsymbol{\zeta}_k$	$-\delta_{kl}\mathbf{1} - \epsilon_{klm}\boldsymbol{\zeta}_m$	$\boldsymbol{\gamma}_k$	$-\delta_{kl}\boldsymbol{\beta} - \epsilon_{klm}\boldsymbol{\gamma}_m$
$\boldsymbol{\beta}$	$\boldsymbol{\gamma}_l$	$\mathbf{1}$	$\boldsymbol{\zeta}_l$
$\boldsymbol{\gamma}_k$	$-\delta_{kl}\boldsymbol{\beta} - \epsilon_{klm}\boldsymbol{\gamma}_m$	$\boldsymbol{\zeta}_k$	$-\delta_{kl}\mathbf{1} - \epsilon_{klm}\boldsymbol{\zeta}_m$

General Clifford biquaternion and its quaternion representation

$\mathbf{Z} = \mathbf{1}X_0 + \vec{\boldsymbol{\zeta}}\cdot\vec{X} + \boldsymbol{\beta}Y_0 + \vec{\boldsymbol{\gamma}}\cdot\vec{Y} \cong \ldots$

Give me an incentive and I'll finish this section.

6 Hypercomplex numbers with 4 generators

6.1 Space-Time Algebra $Cl_{4,0}(\mathbb{R}) \cong Cl_{1,3}(\mathbb{R}) \cong Cl_{0,4}(\mathbb{R}) \cong \mathbb{H}(2)$

This is the Space-Time algebra, represented by the 16 Dirac matrices.

HC units of $Cl_{4,0}(\mathbb{R})$ $(k,l,m,n = 1\ldots4)$:

$\mathbf{1}$ $\quad \boldsymbol{\beta}_k \quad \boldsymbol{\zeta}_{kl} = -\boldsymbol{\zeta}_{lk} = \frac{1}{2}[\boldsymbol{\beta}_k, \boldsymbol{\beta}_l] \quad \boldsymbol{\gamma}_k = \frac{1}{6}\epsilon_{klmn}\boldsymbol{\beta}_l\boldsymbol{\beta}_m\boldsymbol{\beta}_n \quad \boldsymbol{\alpha} = \boldsymbol{\beta}_1\boldsymbol{\beta}_2\boldsymbol{\beta}_3\boldsymbol{\beta}_4$

HC units of $Cl_{1,3}(\mathbb{R})$ $(k,l,m = 1\ldots3)$: $\mathbf{1} \quad \boldsymbol{\zeta}_{k4}, \boldsymbol{\beta}_4 \quad \boldsymbol{\beta}_k = \boldsymbol{\zeta}_{k4}\boldsymbol{\beta}_4,$

$\boldsymbol{\zeta}_{kl} = -\boldsymbol{\zeta}_{lk} = -\frac{1}{2}[\boldsymbol{\zeta}_{k4}, \boldsymbol{\zeta}_{l4}] \quad \boldsymbol{\gamma}_k = -\frac{1}{2}\epsilon_{klm}\boldsymbol{\zeta}_{l4}\boldsymbol{\zeta}_{m4}\boldsymbol{\beta}_4, \boldsymbol{\alpha} = -\boldsymbol{\zeta}_{14}\boldsymbol{\zeta}_{24}\boldsymbol{\zeta}_{34} \quad \boldsymbol{\gamma}_4 = -\boldsymbol{\zeta}_{14}\boldsymbol{\zeta}_{24}\boldsymbol{\zeta}_{34}\boldsymbol{\beta}_4$

HC units of $Cl_{0,4}(\mathbb{R})$ $(k,l,m,n = 1\ldots4)$:

$\mathbf{1} \quad \boldsymbol{\gamma}_k \quad \boldsymbol{\zeta}_{kl} = -\boldsymbol{\zeta}_{lk} = -\frac{1}{2}[\boldsymbol{\gamma}_k, \boldsymbol{\gamma}_l] \quad \boldsymbol{\beta}_k = -\frac{1}{6}\epsilon_{klmn}\boldsymbol{\gamma}_l\boldsymbol{\gamma}_m\boldsymbol{\gamma}_n \quad \boldsymbol{\alpha} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\boldsymbol{\gamma}_4$

Multiplication table

$\mathbf{1}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}_m$	$\boldsymbol{\gamma}_m$	$\boldsymbol{\zeta}_{mn}$
$\boldsymbol{\alpha}$	$\mathbf{1}$	$-\boldsymbol{\gamma}_m$	$-\boldsymbol{\beta}_m$	$\frac{1}{2}\epsilon_{klmn}\boldsymbol{\zeta}_{kl}$
$\boldsymbol{\beta}_k$	$\boldsymbol{\gamma}_k$	$\delta_{km}\mathbf{1} + \boldsymbol{\zeta}_{km}$	$\boldsymbol{\alpha}\delta_{km} + \frac{1}{2}\epsilon_{klmn}\boldsymbol{\zeta}_{ln}$	$\delta_{km}\boldsymbol{\beta}_n - \delta_{kn}\boldsymbol{\beta}_m$
				$-\epsilon_{klmn}\boldsymbol{\gamma}_l$
$\boldsymbol{\gamma}_k$	$\boldsymbol{\beta}_k$	$-\boldsymbol{\alpha}\delta_{km} - \frac{1}{2}\epsilon_{klmn}\boldsymbol{\zeta}_{ln}$	$-\delta_{km}\mathbf{1} - \boldsymbol{\zeta}_{km}$	$\delta_{km}\boldsymbol{\gamma}_n - \delta_{kn}\boldsymbol{\gamma}_m$
				$-\epsilon_{klmn}\boldsymbol{\beta}_l$
$\boldsymbol{\zeta}_{kl}$	$-\frac{1}{2}\epsilon_{klmn}\boldsymbol{\zeta}_{mn}$	$-\delta_{km}\boldsymbol{\beta}_l + \delta_{lm}\boldsymbol{\beta}_k$	$-\delta_{km}\boldsymbol{\gamma}_l + \delta_{lm}\boldsymbol{\gamma}_k$	$-\delta_{km}\delta_{ln}\mathbf{1} + \epsilon_{klmn}\boldsymbol{\alpha}$
		$\ldots\ldots - \epsilon_{klmn}\boldsymbol{\gamma}_n$	$\ldots - \epsilon_{klmn}\boldsymbol{\beta}_n$	$\ldots - \delta_{km}\boldsymbol{\zeta}_{ln} - \delta_{ln}\boldsymbol{\zeta}_{km}$

General HC number and its quaternion representation

$$\mathbf{Z} = \mathbf{1}Z_0 + \boldsymbol{\alpha}Z_\alpha + \vec{\boldsymbol{\gamma}}\cdot\vec{Z}_\gamma + \vec{\boldsymbol{\beta}}\cdot\vec{Z}_\beta + \boldsymbol{\zeta}_{12}Z_{12} + \boldsymbol{\zeta}_{23}Z_{23} + \boldsymbol{\zeta}_{31}Z_{31} + \boldsymbol{\zeta}_{14}Z_{14} + \boldsymbol{\zeta}_{24}Z_{24} + \boldsymbol{\zeta}_{34}Z_{34} \cong \ldots$$

Give me an incentive and I'll finish this section.

6.2 Anonymous-4 $Cl_{3,1}(\mathbb{R}) \cong Cl_{2,2}(\mathbb{R}) \cong \mathbb{R}(4)$

HC units of $Cl_{3,1}(\mathbb{R})$:

$\mathbf{1} \quad \boldsymbol{\beta}_k^2, \boldsymbol{\gamma}^1 \quad \boldsymbol{\beta}_k^3 = \boldsymbol{\beta}_k^2\boldsymbol{\gamma}^1, \boldsymbol{\zeta}_k = \frac{1}{2}\epsilon_{klm}\boldsymbol{\beta}_l^2\boldsymbol{\beta}_m^2 \qquad \boldsymbol{\beta}_k^1 = \boldsymbol{\zeta}_k\boldsymbol{\gamma}^1 \quad \boldsymbol{\gamma}^2 = -\boldsymbol{\beta}_1^2\boldsymbol{\beta}_2^2\boldsymbol{\beta}_3^2 \quad \boldsymbol{\gamma}^3 = \boldsymbol{\gamma}^2\boldsymbol{\gamma}^1$

HC units of $Cl_{2,2}(\mathbb{R})$:

$\mathbf{1} \quad \boldsymbol{\beta}_2^1, \boldsymbol{\beta}_3^1, \boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3 \quad \boldsymbol{\beta}_2^2 = \boldsymbol{\beta}_2^1\boldsymbol{\gamma}^3, \boldsymbol{\beta}_3^2 = \boldsymbol{\beta}_3^1\boldsymbol{\gamma}^3, \boldsymbol{\beta}_2^3 = -\boldsymbol{\beta}_2^1\boldsymbol{\gamma}^2, \boldsymbol{\beta}_3^3 = -\boldsymbol{\beta}_3^1\boldsymbol{\gamma}^3, \boldsymbol{\zeta}_1 = \boldsymbol{\beta}_2^1\boldsymbol{\beta}_3^1, \boldsymbol{\gamma}^1 = \boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$

$\boldsymbol{\beta}_1^2 = \boldsymbol{\beta}_2^1\boldsymbol{\beta}_3^1\boldsymbol{\gamma}^2, \boldsymbol{\beta}_1^3 = \boldsymbol{\beta}_2^1\boldsymbol{\beta}_3^1\boldsymbol{\gamma}^3, \boldsymbol{\zeta}_2 = \boldsymbol{\beta}_2^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3, \boldsymbol{\zeta}_3 = \boldsymbol{\beta}_3^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 \quad \boldsymbol{\beta}_1^1 = -\boldsymbol{\beta}_2^1\boldsymbol{\beta}_3^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3$

Multiplication table

$\mathbf{1}$	$\boldsymbol{\gamma}^n$	$\boldsymbol{\zeta}_l$	$\boldsymbol{\beta}_l^n$
$\boldsymbol{\gamma}^m$	$-\delta_{mn}\mathbf{1} - \epsilon_{mno}\boldsymbol{\gamma}^o$	$\boldsymbol{\beta}_l^m$	$-\delta_{mn}\boldsymbol{\zeta}_l - \epsilon_{mno}\boldsymbol{\beta}_l^o$
$\boldsymbol{\zeta}_k$	$\boldsymbol{\beta}_k^n$	$-\delta_{kl}\mathbf{1} - \epsilon_{kli}\boldsymbol{\zeta}_i$	$-\delta_{kl}\boldsymbol{\gamma}_n - \epsilon_{kli}\boldsymbol{\beta}_i^n$
$\boldsymbol{\beta}_k^m$	$-\delta_{mn}\boldsymbol{\zeta}_k - \epsilon_{mno}\boldsymbol{\beta}_k^o$	$-\delta_{kl}\boldsymbol{\gamma}_m - \epsilon_{kli}\boldsymbol{\beta}_i^m$	$\delta_{kl}\delta_{mn}\mathbf{1} + \delta_{kl}\epsilon_{mno}\boldsymbol{\gamma}_o$ $\ldots + \epsilon_{kli}\delta_{mn}\boldsymbol{\zeta}_i + \epsilon_{kli}\epsilon_{mno}\boldsymbol{\beta}_i^o$

General HC number and its representation

$$\mathbf{Z} = \mathbf{1}Z_0^0 + \boldsymbol{\zeta}_k Z_k^0 + \boldsymbol{\gamma}^m Z_0^m + \boldsymbol{\beta}_k^m Z_k^m \equiv \mathbf{1}Z_0^0 + \vec{\boldsymbol{\zeta}} \cdot \vec{Z}^0 + \boldsymbol{\gamma}^m Z_0^m + \vec{\boldsymbol{\beta}}^m \cdot \vec{Z}^m \cong$$

$$\begin{pmatrix} Z_0^0 + Z_1^1 + Z_2^2 + Z_3^3 & -Z_1^0 + Z_0^1 - Z_3^2 + Z_2^3 & -Z_2^0 + Z_3^1 + Z_0^2 - Z_1^3 & -Z_3^0 - Z_2^1 + Z_1^2 + Z_0^3 \\ Z_1^0 - Z_0^1 - Z_3^2 + Z_2^3 & Z_0^0 + Z_1^1 - Z_2^2 - Z_3^3 & Z_3^0 + Z_2^1 + Z_1^2 + Z_0^3 & -Z_2^0 + Z_3^1 - Z_0^2 + Z_1^3 \\ Z_2^0 + Z_3^1 - Z_0^2 - Z_1^3 & -Z_3^0 + Z_2^1 + Z_1^2 - Z_0^3 & Z_0^0 - Z_1^1 + Z_2^2 - Z_3^3 & Z_1^0 + Z_0^1 + Z_3^2 + Z_2^3 \\ Z_3^0 - Z_2^1 + Z_1^2 - Z_0^3 & Z_2^0 + Z_3^1 + Z_0^2 + Z_1^3 & -Z_1^0 - Z_0^1 + Z_3^2 + Z_2^3 & Z_0^0 - Z_1^1 - Z_2^2 + Z_3^3 \end{pmatrix}$$

where summation over equal indices is required.

The components of the hc number are arranged as a matrix, not as a vector!

$$Z_\kappa^\lambda = \begin{pmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3 \\ Z_1^0 & Z_1^1 & Z_1^2 & Z_1^3 \\ Z_2^0 & Z_2^1 & Z_2^2 & Z_2^3 \\ Z_3^0 & Z_3^1 & Z_3^2 & Z_3^3 \end{pmatrix}$$

Determinants

The anti-symmetric 4×4 matrix $\quad \mathbf{A} = \boldsymbol{\zeta}_k A_k^0 + \boldsymbol{\gamma}^m A_0^m \quad$ has the determinant

$\det \mathbf{A} = [(A_0^1)^2 + (A_0^2)^2 + (A_0^3)^2 - (A_1^0)^2 - (A_2^0)^2 - (A_3^0)^2]^2$

The symmetric 4×4 matrix $\quad \boldsymbol{\Sigma} = \mathbf{1}\Sigma_0^0 + \vec{\boldsymbol{\beta}}^m \cdot \vec{\Sigma}^m \quad$ has the determinant

$\det \boldsymbol{\Sigma} = \sqrt{\det(\boldsymbol{\Sigma}\boldsymbol{\gamma}^1\boldsymbol{\Sigma})} = \sqrt{\det \mathbf{A}} = (A_0^1)^2 + (A_0^2)^2 + (A_0^3)^2 - \vec{A}^0 \cdot \vec{A}^0$

where $\quad A_0^1 = (\Sigma_0^0)^2 + \vec{\Sigma}^1 \cdot \vec{\Sigma}^1 - \vec{\Sigma}^2 \cdot \vec{\Sigma}^2 - \vec{\Sigma}^3 \cdot \vec{\Sigma}^3 \quad A_0^2 = 2\vec{\Sigma}^2 \cdot \vec{\Sigma}^1 \quad A_0^3 = 2\vec{\Sigma}^3 \cdot \vec{\Sigma}^1$

and $\quad \vec{A}^0 = 2\left(\vec{\Sigma}^2 \wedge \vec{\Sigma}^3 - \Sigma_0^0\vec{\Sigma}^1\right)$

The general 4×4 matrix $\quad \mathbf{1}Z_0^0 + \vec{\boldsymbol{\zeta}} \cdot \vec{Z}^0 + \boldsymbol{\gamma}^m Z_0^m + \vec{\boldsymbol{\beta}}^m \cdot \vec{Z}^m \quad$ has the determinant

$$\det \mathbf{Z} = \sqrt{\det(\mathbf{Z}\boldsymbol{\gamma}^1\mathbf{Z}^\intercal)} = \sqrt{\det \mathbf{A}} = (A_0^1)^2 + (A_0^2)^2 + (A_0^3)^2 - \vec{A}^0 \cdot \vec{A}^0$$

where $\quad A_0^1 = Z_\mu^0 Z_\mu^0 + Z_\mu^1 Z_\mu^1 - Z_\mu^2 Z_\mu^2 - Z_\mu^3 Z_\mu^3$

$$A_0^2 = 2\left(Z_\mu^2 Z_\mu^1 - Z_\mu^0 Z_\mu^3\right) \qquad A_0^3 = 2\left(Z_\mu^0 Z_\mu^2 + Z_\mu^3 Z_\mu^1\right)$$

and $\quad \vec{A}^0 = 2\left(Z_0^1 \vec{Z}^0 - Z_0^0 \vec{Z}^1 + \vec{Z}^0 \wedge \vec{Z}^1 + Z_0^3 \vec{Z}^2 - Z_0^2 \vec{Z}^3 + \vec{Z}^2 \wedge \vec{Z}^3\right)$

The choice of $\boldsymbol{\gamma}^1$ (instead of $\boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3, \boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2$ or $\boldsymbol{\zeta}_3$) for constructing an anti-symmetric matrix $\mathbf{A}$ out of $\boldsymbol{\Sigma}$ or $\mathbf{Z}$ is arbitrary.

Inverse HC number

The anti-symmetric 4×4 matrix $\quad \mathbf{A} = \boldsymbol{\zeta}_k A_k^0 + \boldsymbol{\gamma}^m A_0^m \quad$ has the inverse

$$\mathbf{A}^{-1} = \frac{-\boldsymbol{\gamma}^m A_0^m + \boldsymbol{\zeta}_k A_k^0}{(A_0^1)^2 + (A_0^2)^2 + (A_0^3)^2 - (A_1^0)^2 - (A_2^0)^2 - (A_3^0)^2}$$

The symmetric 4×4 matrix $\quad \boldsymbol{\Sigma} = \mathbf{1}\Sigma_0^0 + \vec{\boldsymbol{\beta}}^m \cdot \vec{\Sigma}^m \quad$ has the inverse

$$\boldsymbol{\Sigma}^{-1} = \boldsymbol{\gamma}^1 \boldsymbol{\Sigma} \mathbf{A}^{-1} \qquad \text{where} \qquad \mathbf{A} = \boldsymbol{\Sigma}\boldsymbol{\gamma}^1\boldsymbol{\Sigma}$$

For the components of $\mathbf{A}$ in terms of $\boldsymbol{\Sigma}$ see the paragraph "Determinants".

The general 4×4 matrix $\quad \mathbf{1}Z_0^0 + \vec{\boldsymbol{\zeta}} \cdot \vec{Z}^0 + \boldsymbol{\gamma}^m Z_0^m + \vec{\boldsymbol{\beta}}^m \cdot \vec{Z}^m \quad$ has the inverse

$$\mathbf{Z}^{-1} = \boldsymbol{\gamma}^1 \mathbf{Z}^\intercal \mathbf{A}^{-1} \qquad \text{where} \qquad \mathbf{A} = \mathbf{Z}\boldsymbol{\gamma}^1\mathbf{Z}^\intercal$$

For the components of $\mathbf{A}$ in terms of $\mathbf{Z}$ see the paragraph "Determinants".

Euler formulas

$$\exp\left(\boldsymbol{\gamma}^l f^l \phi\right) = \mathbf{1}\cos\phi + \boldsymbol{\gamma}^l f^l \sin\phi \qquad \text{where} \qquad (f^1)^2 + (f^2)^2 + (f^3)^2 = 1$$

$$\exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) = \mathbf{1}\cos\psi + \vec{\boldsymbol{\zeta}} \cdot \vec{e}\sin\psi \qquad \text{where} \qquad (e_1)^2 + (e_2)^2 + (e_3)^2 = 1$$

The two rotations commute with each other.

$\exp\left(\boldsymbol{\gamma}^l f^l \phi\right)$ rotates each row of the coefficient matrix like a quaternion:

$$\mathbf{Z}' = \left\{ \begin{array}{l} \exp\left(\boldsymbol{\gamma}^l f^l \phi\right)\mathbf{Z} \\ \mathbf{Z}\exp\left(\boldsymbol{\gamma}^l f^l \phi\right) \end{array} \right\} \qquad \Rightarrow \qquad Z_\lambda'^\kappa = (Z_\lambda^\kappa) \left\{ \begin{array}{l} \exp\left(-\boldsymbol{\zeta}_l f^l \phi\right) \\ \exp\left(\boldsymbol{\gamma}^l f^l \phi\right) \end{array} \right\} =$$

$$\begin{pmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3 \\ Z_1^0 & Z_1^1 & Z_1^2 & Z_1^3 \\ Z_2^0 & Z_2^1 & Z_2^2 & Z_2^3 \\ Z_3^0 & Z_3^1 & Z_3^2 & Z_3^3 \end{pmatrix} \times \left[\cos\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 & f^1 & f^2 & f^3 \\ -f^1 & 0 & \mp f^3 & \pm f^2 \\ -f^2 & \pm f^3 & 0 & \mp f^1 \\ -f^3 & \mp f^2 & \pm f^1 & 0 \end{pmatrix} \right]$$

The similarity transformation $\exp\left(\boldsymbol{\gamma}^l f^l \phi\right) \ldots \exp\left(-\boldsymbol{\gamma}^l f^l \phi\right)$ acts on each row of the coefficient matrix like on a quaternion:

$$\mathbf{Z}' = \exp\left(\boldsymbol{\gamma}^l f^l \phi\right)\mathbf{Z}\exp\left(-\boldsymbol{\gamma}^l f^l \phi\right) \qquad \Rightarrow \qquad Z_\lambda'^\kappa = (Z_\lambda^\kappa)\exp\left(-(\boldsymbol{\gamma}^l + \boldsymbol{\zeta}_l) f^l \phi\right) =$$

$$\begin{pmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3 \\ Z_1^0 & Z_1^1 & Z_1^2 & Z_1^3 \\ Z_2^0 & Z_2^1 & Z_2^2 & Z_2^3 \\ Z_3^0 & Z_3^1 & Z_3^2 & Z_3^3 \end{pmatrix} \times \left[\cos 2\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \right.$$

$$\left. \ldots (1-\cos 2\phi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (f^1)^2 & f^1 f^2 & f^1 f^3 \\ 0 & f^2 f^1 & (f^2)^2 & f^2 f^3 \\ 0 & f^3 f^1 & f^3 f^2 & (f^3)^2 \end{pmatrix} + \sin 2\phi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -f^3 & f^2 \\ 0 & f^3 & 0 & -f^1 \\ 0 & -f^2 & f^1 & 0 \end{pmatrix} \right]$$

$\exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right)$ rotates each column of the coefficient matrix like a quaternion:

$$\mathbf{Z}' = \left\{ \begin{array}{c} \exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \mathbf{Z} \\ \mathbf{Z} \exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \end{array} \right\} \qquad \Rightarrow \qquad Z'^{\kappa}_{\lambda} = \left\{ \begin{array}{l} \exp\left(\boldsymbol{\zeta}_k e_k \phi\right) \\ \exp\left(-\boldsymbol{\gamma}^k e_k \phi\right) \end{array} \right\} (Z^{\kappa}_{\lambda}) =$$

$$\left[\cos\phi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sin\phi \begin{pmatrix} 0 & -e_1 & -e_2 & -e_3 \\ e_1 & 0 & \pm e_3 & \mp e_2 \\ e_2 & \mp e_3 & 0 & \pm e_1 \\ e_3 & \pm e_2 & \mp e_1 & 0 \end{pmatrix} \right] \times \begin{pmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3 \\ Z_1^0 & Z_1^1 & Z_1^2 & Z_1^3 \\ Z_2^0 & Z_2^1 & Z_2^2 & Z_2^3 \\ Z_3^0 & Z_3^1 & Z_3^2 & Z_3^3 \end{pmatrix}$$

The similarity transformation $\exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \ldots \exp\left(-\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right)$ acts on each column of the coefficient matrix like on a quaternion:

$$\mathbf{Z}' = \exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \mathbf{Z} \exp\left(-\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \qquad \Rightarrow \qquad Z'^{\lambda}_{\kappa} = \exp\left((\boldsymbol{\zeta}^k + \boldsymbol{\gamma}_k) e_k \phi\right) (Z^{\kappa}_{\lambda}) =$$

$$\left[\cos 2\psi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (1-\cos 2\psi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (e_1)^2 & e_1 e_2 & e_1 e_3 \\ 0 & e_2 e_1 & (e_2)^2 & e_2 e_3 \\ 0 & e_3 e_1 & e_3 e_2 & (e_3)^2 \end{pmatrix} + \right.$$

$$\left. \ldots \sin 2\psi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_3 & -e_2 \\ 0 & -e_3 & 0 & e_1 \\ 0 & e_2 & -e_1 & 0 \end{pmatrix} \right] \times \begin{pmatrix} Z_0^0 & Z_0^1 & Z_0^2 & Z_0^3 \\ Z_1^0 & Z_1^1 & Z_1^2 & Z_1^3 \\ Z_2^0 & Z_2^1 & Z_2^2 & Z_2^3 \\ Z_3^0 & Z_3^1 & Z_3^2 & Z_3^3 \end{pmatrix}$$

Polar decomposition

$$\mathbf{Z} = \exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \exp\left(\boldsymbol{\gamma}^l f^l \phi\right) \mathbf{P}$$

where $\quad \mathbf{P} = \mathbf{1} P_0 + \vec{\boldsymbol{\beta}}^1 \cdot \vec{P}^1 + \vec{\boldsymbol{\beta}}^2 \cdot \vec{P}^2 + \vec{\boldsymbol{\beta}}^3 \cdot \vec{P}^3$

and $\quad \vec{E} \equiv \vec{e} \tan\psi \quad$ solves $\quad \vec{a} + \vec{\vec{b}}\vec{E} - \vec{E}\vec{E} \cdot \vec{a} = 0 \quad (*)$

with $\quad \vec{a} \equiv Z_0^{\lambda} \vec{Z}^{\lambda} (\neq 0) \quad$ and $\quad \vec{\vec{b}} \equiv \vec{Z}^{\lambda} (\vec{Z}^{\lambda})^t - Z_0^{\lambda} Z_0^{\lambda} \vec{\vec{1}}$

and $\quad (F^l =) f^l \tan\phi = (Z_0^l + \vec{E} \cdot \vec{Z}^l)/(Z_0^0 + \vec{E} \cdot \vec{Z}^0)$

* E. g. iterate $\vec{E} = \vec{\vec{b}}^{-1}\left(\vec{E}\vec{E} \cdot \vec{a} - \vec{a}\right)$, but it doesn't always work.

Singular Value Decomposition

The problem of finding the 4D eigenvalue decomposition

$$\mathbf{P} = \exp\left(\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right) \exp\left(\boldsymbol{\gamma}^l f^l \phi\right) \mathbf{D} \exp\left(-\boldsymbol{\gamma}^l f^l \phi\right) \exp\left(-\vec{\boldsymbol{\zeta}} \cdot \vec{e}\psi\right)$$

where $\quad \mathbf{D} = \mathbf{1} D^0_0 + \boldsymbol{\beta}^1_1 D^1_1 + \boldsymbol{\beta}^2_2 D^2_2 + \boldsymbol{\beta}^3_3 D^3_3$

is mathematically equivalent to finding the 3D *singular* value decomposition

$$\vec{\vec{P}} = \exp\left(\vec{\vec{J}}_k e_k 2\psi\right) \vec{\vec{D}} \exp\left(-\vec{\vec{J}}_l f_l 2\phi\right)$$

where

$$\vec{\vec{J}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \vec{\vec{J}}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \vec{\vec{J}}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and where $\vec{\vec{P}}$ and $\vec{\vec{D}}$ are the lower right submatrices of the *coefficient* matrices

$$P^\lambda_\kappa = \left(\begin{array}{c|ccc} P^0_0 & 0 & 0 & 0 \\ \hline 0 & P^1_1 & P^2_1 & P^3_1 \\ 0 & P^1_2 & P^2_2 & P^3_2 \\ 0 & P^1_3 & P^2_3 & P^3_3 \end{array}\right) \qquad \text{and} \qquad D^\lambda_\kappa = \left(\begin{array}{c|ccc} D^0_0 & 0 & 0 & 0 \\ \hline 0 & D^1_1 & 0 & 0 \\ 0 & 0 & D^2_2 & 0 \\ 0 & 0 & 0 & D^3_3 \end{array}\right)$$

P^0_0 is an invariant of all similarity rotations, i. e. we have $D^0_0 = P^0_0$.

Let $\vec{\vec{P}} = \vec{\vec{U}}\vec{\vec{D}}\vec{\vec{V}}^\intercal$ the singular value decomposition of $\vec{\vec{P}}$ found on the computer.

(If $\det \vec{\vec{V}} = -1$, then set $\vec{\vec{V}} \leftarrow -\vec{\vec{V}}$ and $\vec{\vec{D}} \leftarrow -\vec{\vec{D}}$.)

Then $\qquad (e_1, e_2, e_3) \tan\psi = \dfrac{(U_{23} - U_{32}, U_{31} - U_{13}, U_{12} - U_{21})}{\operatorname{Tr} U + 1}$

and $\qquad (f^1, f^2, f^3) \tan\phi = \dfrac{(V_{23} - V_{32}, V_{31} - V_{13}, V_{12} - V_{21})}{\operatorname{Tr} V + 1}$

References

[1] GNU Octave, J. W. Eaton, D. Bateman & S. Hauberg, http://www.octave.org.

[2] van der Waerden, B. L., *A History of Algebra From al-Khwärizml to Emmy Noether*, Springer-Verlag Berlin, Heidelberg, (1985).

[3] Lawson, H. Blaine and Michelsohn, Marie-Louise, *Spin geometry*, Princeton University Press, 1989.

[4] Figueroa-O'Farrill, José, Spin Geometry, Lecture notes (2010), http://empg.maths.ed.ac.uk/Activities/Spin/

A Octave and Matlab demonstration programs

A.1 Cockle Quaternions

```
%% Cockle quaternions

clear

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
display('Cockle Quaternions: check formulas')

% hypercomplex units

uni=eye(2);
be1=[1,0;0,-1];
be2=[0,1;1,0];
gam=be1*be2;

gamma=zeros(2,2,4);
gamma(:,:,1)=uni;
gamma(:,:,2)=be1;
gamma(:,:,3)=be2;
gamma(:,:,4)=gam;

% Random Matrix

aZ=randn(1,4)

Z=zeros(2,2);
for k =1:4
Z=Z+aZ(k)*gamma(:,:,k);
end
Z

% Determinant

D=aZ.^2*[1 -1 -1 1]'
testD=det(Z)
disp(' ');

% Inverse

aiZ=aZ.*[1 -1 -1 -1]/D

iZ=zeros(2,2);
for k =1:4
iZ=iZ+aiZ(k)*gamma(:,:,k);
end

% Test: should be uni(2)

testuni=Z*iZ

% Polar Decomposition

s0=sign(aZ(1));s3=sign(aZ(4));
phi=atan(aZ(4)/aZ(1))+(1-s0)*s3*pi/2;
```

```
R=expm(gam*phi);
P=R'*Z

aP=zeros(1,4);
for k =1:4
aP(k)=trace(gamma(:,:,k)'*P)/2;
end
aP

% Singular Value decomposition

psi=-atan(aP(3)/aP(2))/2;
R=expm(gam*psi);
D=R'*P*R

% Compare Singular Values

SV=[D(1) D(4)]
testSV=[sqrt(aZ(1)^2+aZ(4)^2)+sqrt(aZ(2)^2+aZ(3)^2) ...
sqrt(aZ(1)^2+aZ(4)^2)-sqrt(aZ(2)^2+aZ(3)^2)]

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
```

A.2 Hamilton Bi-Quaternions

```
%% Hamilton Bi-Quaternions

clear;

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
display('Hamilton BiQuaternions: check formulas')

% Hypercomplex units

uni=eye(2);
be1=[0,1;1,0];
be2=[0,-i;i,0];
be3=[1,0;0,-1];
gam=-be1*be2*be3;
ze1=be2*be3;
ze2=be3*be1;
ze3=be1*be2;

gamma=zeros(2,2,8);
gamma(:,:,1)=uni;
gamma(:,:,2)=ze1;
gamma(:,:,3)=ze2;
gamma(:,:,4)=ze3;
gamma(:,:,5)=gam;
gamma(:,:,6)=be1;
gamma(:,:,7)=be2;
gamma(:,:,8)=be3;

% Random Matrix

aZ=randn(1,8)
```

```
Z=zeros(2,2);
for k =1:8
Z=Z+aZ(k)*gamma(:,:,k);
end
Z

X=aZ(1:4);Y=aZ(5:8);

% Determinant

A=X*X'-Y*Y';
B=2*X*Y';
D=A-i*B
testD=det(Z)
disp(' ');

% Inverse

aiZ=A*[X.*[1 -1 -1 -1] Y.*[1 -1 -1 -1]]+...
B*[Y.*[1 -1 -1 -1] -X.*[1 -1 -1 -1]];
aiZ=aiZ/(A^2+B^2)

iZ=zeros(2,2);
for k =1:8
iZ=iZ+aiZ(k)*gamma(:,:,k);
end
iZ

% Test: should be uni(2)

testuni=Z*iZ

% Polar Decomposition

a=X(1)*X(2:4)'+Y(1)*Y(2:4)';
b=X(2:4)'*X(2:4)+Y(2:4)'*Y(2:4)-(X(1)^2+Y(1)^2)*eye(3);
E=0*a;
for k = 1:5
E=b\(E*(E'*a)-a);
end

if norm(E) < 10

for k = 1:500
E1=b\(E*(E'*a)-a);
E2=b\(E1*(E1'*a)-a);
E=(E1+E2)/2;
end

e=E/norm(E);
psi=atan(norm(E));
phi=atan((Y(1)+E'*Y(2:4)')/(X(1)+E'*X(2:4)'));
R=expm(gam*phi+(ze1*e(1)+ze2*e(2)+ze3*e(3))*psi);
P=R'*Z

aP=zeros(1,8);
for k =1:8
```

```
aP(k)=real(trace(gamma(:,:,k)'*P)/2);
end
aP

% Singular Value decomposition

E=[aP(7)/aP(8) -aP(6)/aP(8) 0]';
e=E/norm(E);
psi=atan(norm(E))/2;
R=expm((ze1*e(1)+ze2*e(2)+ze3*e(3))*psi);
D=R'*P*R

% Compare Singular Values

SV=real([D(1) D(4)])
testSV=[aP(1)+norm(aP(6:8)) aP(1)-norm(aP(6:8))]

else

display('Numerical failure: Cannot calculate E')

end

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
```

A.3 Anonymous-3

```
%% Anonymous-3

clear;

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
display('Anonymous-3: check formulas')

% Hypercomplex units

h=zeros(4);
uni=eye(4);
be1=h;be1([1,6,11,16])=[1,-1,1,-1];
be2=h;be2([2,5,12,15])=[-1,-1,1,1];
gam=h;gam([2,5,12,15])=[-1,1,-1,1];
zet=be1*be2;
be3=be1*gam;
be4=be2*gam;
alp=be1*be2*gam;

gamma=zeros(4,4,8);
gamma(:,:,1)=uni;
gamma(:,:,2)=zet;
gamma(:,:,3)=gam;
gamma(:,:,4)=alp;
gamma(:,:,5)=be1;
gamma(:,:,6)=be2;
gamma(:,:,7)=be3;
gamma(:,:,8)=be4;
```

```
% Random Matrix

aZ=randn(1,8)

Z=zeros(4,4);
for k =1:8
Z=Z+aZ(k)*gamma(:,:,k);
end
Z

% Determinant

X=aZ(1:4);Y=aZ(5:8);
A=(X(1)+X(4))^2+(X(2)-X(3))^2-(Y(1)+Y(4))^2-(Y(2)-Y(3))^2;
B=(X(1)-X(4))^2+(X(2)+X(3))^2-(Y(1)-Y(4))^2-(Y(2)+Y(3))^2;
D=A*B
testD=det(Z)
disp(' ');

% Inverse

aiZ=(1/A+1/B)/2*[X.*[1 -1 -1 1] -Y]+...
(1/A-1/B)/2*[fliplr(X) -fliplr(Y).*[1 -1 -1 1]]

iZ=zeros(4,4);
for k =1:8
iZ=iZ+aiZ(k)*gamma(:,:,k);
end

% Test: should be uni(4)

testuni=Z*iZ

% Polar Decomposition

psi=atan( 2*(X(1)*X(2)+X(3)*X(4))/(X(1)^2-X(2)^2+X(3)^2-X(4)^2) )/2;
phi=atan( (X(3)+tan(psi)*X(4))/(X(1)+tan(psi)*X(2)) );
R=expm(gam*phi+zet*psi);
P=R'*Z

aP=zeros(1,8);
for k =1:8
aP(k)=real(trace(gamma(:,:,k)'*P)/2);
end
aP

%Singular Value decomposition

Q=aP(5:8);
psi=-atan( 2*(Q(1)*Q(2)+Q(3)*Q(4))/(Q(1)^2-Q(2)^2+Q(3)^2-Q(4)^2) )/4+pi/4;
phi=-atan( (Q(3)-tan(2*psi)*Q(4))/(Q(1)-tan(2*psi)*Q(2)) )/2;
R=expm(gam*phi+zet*psi);
D=R'*P*R
SV=diag(D)'

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
```

A.4 Anonymous-4

```
%% Anonymous-4

clear;

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
display('Anonymous-4: check formulas')

% Hypercomplex units

h=zeros(4);
uni=eye(4);
be12=h;be12([4,7,10,13])=[1,1,1,1];
be22=h;be22([1,6,11,16])=[1,-1,1,-1];
be32=h;be32([2,5,12,15])=[-1,-1,1,1];
ga1=h;ga1([2,5,12,15])=[-1,1,-1,1];
be13=be12*ga1;
be23=be22*ga1;
be33=be32*ga1;
ze1=be22*be32;
ze2=be32*be12;
ze3=be12*be22;
be11=ze1*ga1;
be21=ze2*ga1;
be31=ze3*ga1;
ga2=-be22*be32*be12;
ga3=ga2*ga1;

gamma=zeros(4,4,4,4);
gamma(:,:,1,1)=uni;
gamma(:,:,2,1)=ze1;
gamma(:,:,3,1)=ze2;
gamma(:,:,4,1)=ze3;
gamma(:,:,1,2)=ga1;
gamma(:,:,2,2)=be11;
gamma(:,:,3,2)=be21;
gamma(:,:,4,2)=be31;
gamma(:,:,1,3)=ga2;
gamma(:,:,2,3)=be12;
gamma(:,:,3,3)=be22;
gamma(:,:,4,3)=be32;
gamma(:,:,1,4)=ga3;
gamma(:,:,2,4)=be13;
gamma(:,:,3,4)=be23;
gamma(:,:,4,4)=be33;

% Random Matrix

aZ=randn(4,4);

Z=zeros(4,4);
for k =1:4
for m=1:4
Z=Z+aZ(k,m)*gamma(:,:,k,m);
end
end
Z
```

```
% Determinant

A01=aZ(:,1)'*aZ(:,1)+aZ(:,2)'*aZ(:,2)-aZ(:,3)'*aZ(:,3)-aZ(:,4)'*aZ(:,4);
A02=2*(aZ(:,3)'*aZ(:,2)-aZ(:,1)'*aZ(:,4));
A03=2*(aZ(:,1)'*aZ(:,3)+aZ(:,4)'*aZ(:,2));
A_0=2*( aZ(1,2)*aZ(2:4,1)-aZ(1,1)*aZ(2:4,2)+cross(aZ(2:4,1),aZ(2:4,2))...
+aZ(1,4)*aZ(2:4,3)-aZ(1,3)*aZ(2:4,4)+cross(aZ(2:4,3),aZ(2:4,4)) );
D= A01^2+A02^2+A03^2-A_0'*A_0
testD=det(Z)
disp(' ');

% Inverse

iA=(-ga1*A01-ga2*A02-ga3*A03+ze1*A_0(1)+ze2*A_0(2)+ze3*A_0(3))/D;
iZ=ga1*Z'*iA

% Test: should be uni(2)

testuni=Z*iZ

% Polar decomposition

a=aZ(1,1)*aZ(2:4,1)+aZ(1,2)*aZ(2:4,2)+aZ(1,3)*aZ(2:4,3)+aZ(1,4)*aZ(2:4,4);
b=aZ(2:4,1)*aZ(2:4,1)'+aZ(2:4,2)*aZ(2:4,2)'+...
aZ(2:4,3)*aZ(2:4,3)'+aZ(2:4,4)*aZ(2:4,4)';
b=b-(aZ(1,1)*aZ(1,1)+aZ(1,2)*aZ(1,2)+aZ(1,3)*aZ(1,3)+aZ(1,4)*aZ(1,4))*eye(3);

E=a*0;
hmax=5;
for h=1:hmax
E=b\(E*(E'*a*h/hmax)-a*h/hmax);
end

h=(norm(E) < 10);
if h

for h=1:500
E1=b\(E*(E'*a)-a);
E2=b\(E1*(E1'*a)-a);
E=(E1+E2)/2;
end

chi2=atan(norm(E));
e=E/norm(E);
O2=cos(chi2)*eye(4)+sin(chi2)*(e(1)*ze1+e(2)*ze2+e(3)*ze3);

Zp=O2'*Z;
aZp=zeros(4,4);
for k =1:4
for m=1:4
aZp(k,m)=trace(gamma(:,:,k,m)'*Zp)/4;
end
end

test0=[aZp(1,1)*aZp(2:4,1)+aZp(1,2)*aZp(2:4,2)+aZp(1,3)*aZp(2:4,3)+aZp(1,4)*aZp(2:4,4)]';
h=1-isnan(test0(1));
if h
```

```
F=aZp(1,2:4)/aZp(1,1);

chi1=atan(norm(F));
f=F/norm(F);
O1=cos(chi1)*eye(4)+sin(chi1)*(f(1)*ga1+f(2)*ga2+f(3)*ga3);

P=O1'*Zp

aP=zeros(4,4);
for k =1:4
for m=1:4
aP(k,m)=trace(gamma(:,:,k,m)'*P)/4;
end
end
aP

% Singular Value decomposition

[u, s, v] = svd (aP(2:4,2:4));
if det(v)<0
s=-s;
v=-v;
end
testaD=[[aP(1,1); 0; 0; 0] [0 0 0; s]];
testD=aP(1,1)*uni+s(1,1)*be11+s(2,2)*be22+s(3,3)*be33;

E=[u(2,3)-u(3,2) u(3,1)-u(1,3) u(1,2)-u(2,1) ]/(trace(u)+1);
psi=atan(norm(E));e=E/norm(E);
F=[v(2,3)-v(3,2) v(3,1)-v(1,3) v(1,2)-v(2,1) ]/(trace(v)+1);
phi=atan(norm(F));f=F/norm(F);

R=expm( (ze1*e(1)+ze2*e(2)+ze3*e(3))*psi + (ga1*f(1)+ga2*f(2)+ga3*f(3))*phi );
D=R'*P*R

% Compare Singular Values

SV=diag(D)'
testSV=diag(testD)'

end
end

if 1-h

display('Numerical failure: Cannot calculate E')

end

display('%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%')
```