Size Matters: Layout Algorithms

August 29, 2024

1 Introduction

This document describes the layout algorithms used in simulations reported in the article "Size Matters: Optimal Test Pit Size for Dispersed Archaeological Test Excavations" by Oakes and McLaren. See that article for background. We simulate archaeological digs to explore success of different test pit sizes and number of test pits. Our simulation takes place on a square field of dimension $n \times n$. Test pits are either $0.5m \times 0.5m$ (small pits) or $1m \times 1m$ (large pits). An archaeological site is placed randomly on the field. For a given total number of pits h we need to decide how many pits will be dug in our field horizontally (h_x), and how many vertically (h_y). We want a staggered grid pattern, as shown in figure 1. A hexagonal layout (defined below) has been shown to be optimal in certain circumstances.

Some definitions:

- h is total number of pits,
- h_x and h_y are the number of pits along the x axis and the y axis of the field respectively,
- n_x and n_y are the width and height of the field respectively (we have only tested square fields in the simulation),
- a and c are distances as shown in figure 1.

A layout is hexagonal when a and b shown in figure 1 are equal. In the hexagonal-like algorithm we set a fixed horizontal border of $a/2$ and a vertical border of $c/2$, and then choose values for h_x and h_y that make the layout almost hexagonal. In section 3 we vary the borders to make the layout truly hexagonal. The reason for fixing borders is to avoid anomalous results when a pit is close to the edge of the field, noting that the archaeological site is placed wholly within the field. Choosing to place the site so that its centre is within the field also leads to anomalous results for different reasons. In experiments the two layout algorithms had very similar success rates except when the truly hexagonal layout had a pit close to the edge of the field (when the success rate was lower).

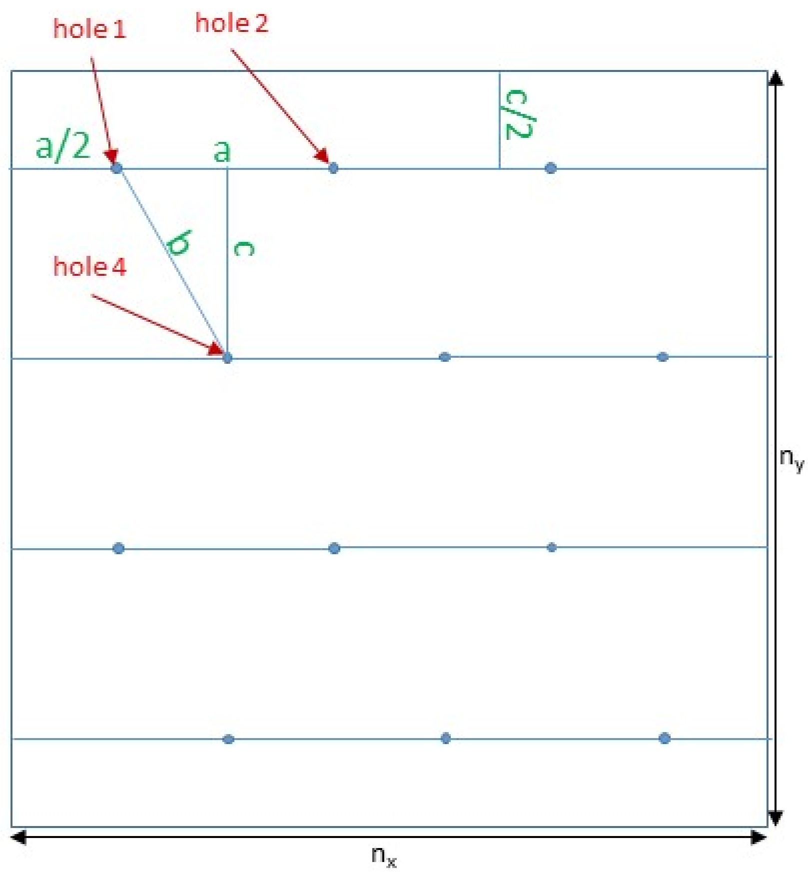

Figure 1: Layout algorithm for $h = 12$ test pits. A layout is hexagonal when $a = b$

2 Hexagonal-Like Layout Algorithm

2.1 Layout with Horizontal and Vertical Borders

We first derive some equations for h_x and h_y when we use horizontal and vertical borders as shown in figure 1. First we state some equations evident from figure 1:

$$h = h_x \cdot h_y \tag{1}$$

$$n_y = c \cdot h_y \tag{2}$$

$$\begin{aligned} n_x &= a(h_x - 1) + a + \frac{a}{2} && (3) \\ &= a(h_x + \frac{1}{2}) && (4) \end{aligned}$$

$$a^2 = \left(\frac{a}{2}\right)^2 + c^2 \qquad \text{Pythagorean Theorem} \tag{5}$$

Now we write a formula for a in terms of n_x and h_x:

$$a = \frac{n_x}{h_x + \frac{1}{2}} \qquad \text{from 4} \tag{6}$$

Now we write a formula for a in terms of n_y and h_y:

$$\begin{aligned} a^2 &= \frac{a^2}{4} + c^2 \qquad \text{from 5} && (7) \\ \frac{3}{4}a^2 &= c^2 && (8) \\ a &= \frac{2c}{\sqrt{3}} && (9) \\ &= \frac{2\left(\frac{n_y}{h_y}\right)}{\sqrt{3}} && (10) \\ &= \frac{2n_y}{\sqrt{3}h_y} && (11) \end{aligned}$$

We equate the two formulae for a and find a quadratic formula for h_x:

$$\frac{n_x}{h_x + \frac{1}{2}} = \frac{2n_y}{\sqrt{3}h_y} \qquad \text{from 6 and 11} \tag{12}$$

$$\sqrt{3}h_y n_x = 2n_y \left(h_x + \frac{1}{2}\right) \tag{13}$$

$$\sqrt{3}\frac{h}{h_x} n_x = 2n_y \left(h_x + \frac{1}{2}\right) \qquad \text{from 1} \tag{14}$$

$$\sqrt{3}hn_x = 2h_x n_y \left(h_x + \frac{1}{2}\right) \tag{15}$$

$$= 2n_y h_x^2 + n_y h_x \tag{16}$$

$$0 = 2n_y h_x^2 + n_y h_x - \sqrt{3}hn_x \tag{17}$$

Finally we solve for h_x using the quadratic formula:

$$h_x = \frac{-n_y + \sqrt{{n_y}^2 - 4 \cdot 2n_y \cdot (-\sqrt{3}hn_x)}}{2 \cdot 2n_y} \tag{18}$$

$$h_x = \frac{-n_y + \sqrt{{n_y}^2 + 8\sqrt{3}n_x n_y h}}{4n_y} \tag{19}$$

$$\text{and} \qquad h_y = \frac{h}{h_x} \tag{20}$$

Lastly we round the values to get integer values for h_x and h_y and recalculate a and c to reflect the rounded values:

$$c' = \frac{n_y}{round(h_y)} \tag{21}$$

$$a' = \frac{n_x}{round(h_x) + \frac{1}{2}} \tag{22}$$

$$border_x = \frac{a'}{2} \tag{23}$$

$$border_y = \frac{c'}{2} \tag{24}$$

Then we layout the pits with the given borders, so that pits in a row are a' apart, the rows are c' apart, and every second row is indented (staggered) by $a'/2$.

2.2 Layout with Only Vertical Borders

When we remove the left and right borders (which were $a/2$ in subsection 2.1) similar maths gives

$$
\begin{aligned}
2n_y h_x^2 - n_y h_x - \sqrt{3} h n_x &= 0 && (25) \\
h_x &= \frac{n_y + \sqrt{{n_y}^2 + 8\sqrt{3} n_x n_y h}}{4n_y} && (26) \\
\text{and} \qquad h_y &= \frac{h}{h\,x} && (27)
\end{aligned}
$$

And then round the values and recalculate a and c, noting that the horizontal border will actually be half a pitSize (which can be $0.5m$ or $1m$) as the pit location refers to its centre.

$$
\begin{aligned}
c' &= \frac{n_y}{round(h_y)} && (28) \\
a' &= \frac{n_x - pitSize}{round(h_x) - \frac{1}{2}} && (29) \\
border_x &= \frac{pitSize}{2} && (30) \\
border_y &= \frac{c'}{2} && (31)
\end{aligned}
$$

3 Hexagonal Layout Algorithm

For a truly hexagonal layout we start with the equations for a layout with a horizontal border of $a/2$ and vertical border of $c/2$, calculating h_x and h_y as in equations 19 and 20. Then we vary the borders so that it is exactly hexagonal. First change the horizontal border, recalculating c and a as follows:

$$
\begin{aligned}
c' &= \frac{n_y}{round(h_y)} && (32) \\
a' &= \frac{2c'}{\sqrt{3}} \qquad \text{from } 9 && (33) \\
border_x &= \frac{n_x - a'(h_x - \frac{1}{2})}{2} && (34) \\
border_y &= \frac{c'}{2} && (35)
\end{aligned}
$$

We then check if this value of a' is too big to fit $round(h_x)$ pits a' apart, and readjust c' if necessary. Again, the pit location is specified by its centre. So if $border_x < pitSize/2$ set

$$a'' = \frac{n_x - pitSize}{round(h_x) - \frac{1}{2}} \tag{36}$$

$$c'' = \frac{\sqrt{3}a''}{2} \tag{37}$$

$$border_x = \frac{pitSize}{2} \tag{38}$$

$$border_y = \frac{c''}{2} \tag{39}$$