Classical Tensornetwork（ongoing）

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1 One dimensional classical Ising model

1.1 The Exact Solution

A one-dimensional classical Ising model with periodic boundary conditions is given by

$$H^{\mathrm{1D}} := -J \sum_{i=0}^{N-1} S_i S_{i+1}, \tag{1}$$

where N is the number of sites and $S_0 = S_N$. The value of spin is ± 1.

By using the inverse temperature β, the partition function Z_N can be written as

$$\begin{aligned} Z_N^{\mathrm{1D}} &:= \sum_{\text{全状態}} \exp\left(-\beta H^{1D}\right) & (2) \\ &= \sum_{S_0=\pm 1} \sum_{S_1=\pm 1} \cdots \sum_{S_{N-1}=\pm 1} \exp\left(\beta J \sum_{i=0}^{N-1} S_i S_{i+1}\right). & (3) \end{aligned}$$

We define the transfer matrix T^{1D} in the following ways:

$$T^{1D} := \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix}. \tag{4}$$

We obtain

$$\begin{aligned} Z_N^{\mathrm{1D}} &= \mathrm{Tr}\left(T^{1D}\right)^N & (5) \\ &= \lambda_+^N + \lambda_-^N, & (6) \end{aligned}$$

where $\lambda_\pm := e^{\beta J} \pm e^{-\beta J}$ are the eigenvalues of the transfer matrix T^{1D}. The free energy per unit cell f、 the energy e, and the heat capacity C_v are given by

$$
\begin{aligned}
f^{\mathrm{1D}} &:= -\frac{1}{N\beta}\ln Z_N^{\mathrm{1D}} && (7)\\
&= -\frac{1}{\beta}\ln\left[2\cosh(\beta J)\right] - \frac{1}{N\beta}\ln\left[1+t^N\right] && (8)\\
e^{\mathrm{1D}} &:= -\frac{1}{N}\frac{\partial}{\partial\beta}\ln Z_N^{\mathrm{1D}} && (9)\\
&= -\frac{J}{1+t^N}\left[t+t^{N-1}\right] && (10)\\
C_v^{\mathrm{1D}} &:= \frac{\beta^2}{N}\frac{\partial^2}{\partial\beta^2}\ln Z_N^{\mathrm{1D}} && (11)\\
&= \beta^2 J^2 + \frac{\beta^2 J^2}{(1+t^N)^2}\left[-t^2 - t^{2N-2} + (N-1)\left(t^{N-2}+t^{N+2}\right) - 2Nt^N\right], && (12)
\end{aligned}
$$

where $t := \tanh(\beta J)$.

1.2 The numerical calculation of the free energy

In this section, we numerically calculate the free energy f^{1D}. For the sake of simplicity, we consider the situation where N can be written as $N = 2^{N_{\mathrm{itr}}}$.

We transform the transfer matrix in order to make it easy to apply to two-dimensional systems. By using a matrix

$$
W := \begin{pmatrix} \sqrt{\cosh(\beta J)} & \sqrt{\sinh(\beta J)} \\ \sqrt{\cosh(\beta J)} & -\sqrt{\sinh(\beta J)} \end{pmatrix}, \tag{13}
$$

the transfer matrix T^{1D} can be written as

$$
T^{1D} = WW^T. \tag{14}
$$

We define new transfer matrix $\tilde{T}^{1D} := W^T W$. The partition function Z_N^{1D} can be written as

$$
Z_N^{1D} = \mathrm{Tr}\left(T^{1D}\right)^N = \mathrm{Tr}\left(WW^T\right)^N = \mathrm{Tr}\left(W^T W\right)^N = \mathrm{Tr}\left(\tilde{T}^{1D}\right)^N. \tag{15}
$$

The free energy can be written as

$$
f^{1D} = -\frac{1}{N\beta}\ \ln\left[\mathrm{Tr}\left(\tilde{T}^{1D}\right)^N\right]. \tag{16}
$$

We numerically calculate this free energy.

When the number of the sites $N(= 2^{N_{\mathrm{itr}}})$ is large, it is an effective way to calculate $\tilde{T}$, $\tilde{T}^2$, $\tilde{T}^4, \tilde{T}^8, \tilde{T}^{16}, \ldots$. We need to normalize at each step in order to avoid overflow. For example, when we repeat x times and renormalize $\tilde{T}^X = \alpha\tilde{\tilde{T}}^X$ $(X := 2^x)$, the free energy f^{1D} is given by

$$
\begin{aligned}
f^{1D} &= -\frac{1}{N\beta}\ln\left(\mathrm{Tr}\,\tilde{T}^N\right) && (17)\\
&= -\frac{1}{N\beta}\ln\left(\mathrm{Tr}\left(\tilde{T}^X\right)^{N-X}\right) && (18)\\
&= -\frac{1}{N\beta}\ln\left(\mathrm{Tr}\left(\tilde{\tilde{T}}^X\right)^{N-X}\right) - \frac{N-X}{N\beta}\ln\alpha. && (19)
\end{aligned}
$$

1.3 The sample code

We can calculate the free energy at the inverse temperature $\beta = 2.0$, the interaction $J = 1.0$, and the number of the sites $N = 2^{N_{\mathrm{itr}}} = 2^{20} \sim 10^6$ by using the sample code (`1D_Ising_model.py`). Please type

```
python3 1D_Ising_model.py
```

and then we obtain the exact solution (exact), the result of the numerical calculation (calculation), and the difference between them (error).

```
exact -1.009074963958905 calculation -1.0090749639589047 error 2.220446049250313
   e-16
```

1.4 The problem

Please calculate the Ising model with a none-zero external magnetic field

$$H^{1\mathrm{D}} := -J \sum_{i=0}^{N-1} S_i S_{i+1} - h \sum_{i=0}^{N-1} S_i. \tag{20}$$

References

[1] 西森秀稔, 「相転移・臨界現象の統計物理学」, 培風館, (2005).

[2] Chen, Bin-Bin and Gao, Yuan and Guo, Yi-Bin and Liu, Yuzhi and Zhao, Hui-Hai and Liao, Hai-Jun and Wang, Lei and Xiang, Tao and Li, Wei and Xie, Z. Y., "Automatic differentiation for second renormalization of tensor networks", Phys. Rev. B **101**, 220409 (2020).