高微2作业5

郑子诺，物理41

2025 年 3 月 30 日

1.

(1)

$$\frac{\partial z}{\partial x} = y + f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x + f'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2xy + xf\left(\frac{y}{x}\right)$$

(2)

$$\frac{\partial z}{\partial y} = -f_y\frac{x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = f_{yy}\frac{x^2}{y^4} + f_y\frac{2x}{y^3}$$

(3)

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) + g\left(\frac{x}{y}\right) - \frac{x}{y}g'\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{y}{x^2}f''\left(\frac{y}{x}\right) - \frac{x}{y^2}g''\left(\frac{x}{y}\right)$$

(4)

$$\frac{\partial z}{\partial x} = -\frac{2xy}{f^2(x^2-y^2)}f'(x^2-y^2)$$

$$\frac{\partial z}{\partial y} = \frac{1}{f(x^2-y^2)} + \frac{2y^2}{f^2(x^2-y^2)}f'(x^2-y^2)$$

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{yf(x^2-y^2)}$$

(5)

$$\begin{bmatrix} u_x & u_y & u_z \end{bmatrix} = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ y & x & 0 \\ yz & xz & xy \end{bmatrix}$$

$$u_x = f_x + yf_y + yzf_z$$

$$u_y = xf_y + xzf_z$$

$$u_z = xyf_z$$

2.

令$F'(x, y, z, u) = F(u^2 - x^2, u^2 - y^2, u^2 - z^2)$，我们有

$$\begin{bmatrix} F'_x & F'_y & F'_z & F'_u \end{bmatrix} = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix} \begin{bmatrix} -2x & 0 & 0 & 2u \\ 0 & -2y & 0 & 2u \\ 0 & 0 & -2z & 2u \end{bmatrix}$$

所以

$$F'_x = -2xF_x$$

$$F'_y = -2yF_y$$

$$F'_z = -2zF_z$$

$$F'_u = 2u(F_x + F_y + F_z)$$

3.

(1)分别对行列进行求导得到

$$\frac{\partial Q}{\partial x_k} = A_{ij}\delta_{ki}x_j + A_{ij}x_i\delta_{kj} = A_{kl}x_l + A_{lk}x_l = 2A_{kl}x_l$$

因此微分为

$$\mathrm{d}Q = \begin{bmatrix} 2A_{1i}x_i & 2A_{2i}x_i & \cdots & 2A_{ni}x_i \end{bmatrix}$$

以上我们使用了爱因斯坦求和约定。

(2)

$$\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} e^{-\frac{1}{2}Q} - (A_{ij}x_j)fe^{-\frac{1}{2}Q}$$

4.

(1)

$$g'(t) = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = xf_x + yf_y + zf_z$$

(2)根据定积分的线性性，由上题知等式右边恰为

$$f(0,0,0)+\int_0^1 g'(t)\mathrm{d}t = f(0,0,0)+g(1)-g(0)$$

由g定义知

$$f(0,0,0)+g(1)-g(0)=g(1)=f(x,y,z)$$

(3)由题设条件知

$$txf_x(tx,ty,tz)+tyf_y(tx,ty,tz)+tzf_z(tx,ty,tz)=nf(tx,ty,tz)$$

$$h'(t)=\frac{xf_x+yf_y+zf_z}{t^n}-\frac{nf}{t^{n+1}}=0$$

(4) 由于$h'(t)=0$，因此h为常数，又因为$h(1)=f(x,y,z)$，所以

$$f(tx,ty,tz)=t^n f(x,y,z)$$