线代作业11

郑子诺，物理41

2024 年 11 月 23 日

1.

$$\langle A, B\rangle = Tr(A^T B) = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}b_{ij} = Tr(B^T A) = \langle B, A\rangle$$

$$\langle A, cB + C\rangle = Tr(A^T(cB+C)) = cTr(A^T B)+Tr(A^T C) = c\langle A, B\rangle+\langle A, C\rangle$$

$$\langle A, A\rangle = Tr(A^T A) = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}^2 > 0,\ \text{if } A \neq 0$$

2.

$$\langle x, y\rangle = X^T GY$$

$$X = PX', Y = PY'$$

$$\therefore \langle x, y\rangle = X^T GY = X'^T P^T GPY' \to G' = P^T GP$$

3.

若$\alpha_1, \ldots, \alpha_n$线性相关，那么$G$存在一列向量是其他列向量的线性组合，这是因为若

$$\alpha_k = c_1\alpha_1 + \cdots + c_{k-1}\alpha_{k-1} + c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$$

那么

$$\langle \alpha_i, \alpha_k\rangle = c_1\langle \alpha_i, \alpha_1\rangle+\cdots+c_{k-1}\langle \alpha_i, \alpha_{k-1}\rangle+c_{k+1}\langle \alpha_i, \alpha_{k+1}\rangle+\cdots+c_n\langle \alpha_i, \alpha_n\rangle$$

即

$$g_k = c_1 g_1 + \cdots + c_{k-1}g_{k-1} + c_{k+1}g_{k+1} + \cdots + c_n g_n$$

显然此时G不可逆。若$\alpha_1,\ldots,\alpha_n$线性无关而G不可逆，那么存在一列向量是其他列向量的线性组合，我们有

$$g_k = c_1 g_1 + \cdots + c_{k-1} g_{k-1} + c_{k+1} g_{k+1} + \cdots + c_n g_n$$

$$\langle \alpha_i, \alpha_k \rangle = c_1 \langle \alpha_i, \alpha_1 \rangle + \cdots + c_{k-1} \langle \alpha_i, \alpha_{k-1} \rangle + c_{k+1} \langle \alpha_i, \alpha_{k+1} \rangle + \cdots + c_n \langle \alpha_i, \alpha_n \rangle$$

于是

$$\langle \alpha_i, \alpha_k \rangle = \langle \alpha_i, c_1 \alpha_1 + \cdots + c_{k-1} \alpha_{k-1} + c_{k+1} \alpha_{k+1} + \cdots + c_n \alpha_n \rangle$$

由于此时$\alpha_1,\ldots,\alpha_n$线性无关，形成一组基，那么上式表明α_k与$c_1\alpha_1 + \cdots + c_{k-1}\alpha_{k-1} + c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n$对任何向量做的内积都相同，于是两者相同，与线性无关矛盾。因而G可逆，证毕。

4.

设A为

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

我们有

$$X^T A X = ax^2 + 2bxy + cy^2$$

固定$y \neq 0$，我们有$\Delta = 4y^2(b^2 - ac) < 0$，要求了$\det A > 0$。同理固定$x \neq 0$也要求$\det A > 0$。由于$X^T A X > 0, X \neq 0$，因此一定有$a, c > 0$，于是$TrA = a + c > 0$。

若已有$TrA > 0, \det A > 0$，那么显然由于$\det A = ac - b^2 > 0 \to ac > 0$，我们有$a > 0, c > 0$，根据上述讨论这显然满足$X^T A X > 0, X \neq 0$。证毕。

5.

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$$

因此满足勾股定理当且仅当$\langle x, y \rangle = 0$，即x, y正交。

6.

$$\begin{aligned}\langle x,y\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \langle \alpha_i,\alpha_j\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j \delta_{ij} \\ &= \sum_{i=1}^{n} x_i y_i\end{aligned}$$

又由于

$$x_i = \sum_{j=1}^{n} x_j \langle \alpha_i,\alpha_j\rangle = \langle \alpha_i, x_i\rangle, y_i = \langle \alpha_i, y_i\rangle$$

所以

$$\langle x,y\rangle = \sum_{i=1}^{n} \langle x,\alpha_i\rangle \langle \alpha_i, y\rangle$$

7.

利用Gram-Schmidt正交化，我们有

$$\alpha_1 = 1$$

$$\alpha_2 = x - \frac{1}{2}$$

$$\alpha_3 = x^2 - x + \frac{1}{6}$$

$$\alpha_4 = x^3 - \frac{3}{2}x + \frac{3}{5}x - \frac{1}{20}$$

归一化得

$$e_1 = 1, e_2 = 2\sqrt{3}(x-\frac{1}{2}), e_3 = 6\sqrt{5}(x^2-x+\frac{1}{6}), e_4 = 20\sqrt{7}(x^3-\frac{3}{2}x+\frac{3}{5}x-\frac{1}{20})$$

8.

令

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

显然满足题设条件。

9.

显然

$$\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

是U的一组基。将其正交归一为

$$\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix},\frac{1}{3\sqrt{5}}\begin{bmatrix}-2\\-4\\5\end{bmatrix}$$

因此p'为

$$p'=\sum_{i=1}^{2}\langle\alpha_i,p\rangle\,\alpha_i=\frac{1}{9}\begin{bmatrix}-2\\-4\\5\end{bmatrix}$$

10.

(a)基矢为$\dfrac{1}{\sqrt{n}}I_n$。因此

$$cTr(I_n^TA)=0\to TrA=0\Rightarrow U^\perp=\{A|TrA=0\}$$

$$p_UA=\frac{1}{n}(Tr(I_n^TA))I_n=\frac{TrA}{n}I_n$$

(b)显然正交归一基矢为

$$(e_i)_{jk}=\delta_{ij}\delta_{jk}$$

因此有

$$\sum_{i=1}^{n}c_iTr(e_i^TA)e_i=0\to a_{ii}=0\Rightarrow U^\perp=\{A|a_{ii}=0\}$$

$$p_UA=\sum_{i=1}^{n}Tr(e_i^TA)e_i=\sum_{i=1}^{n}a_{ii}e_i$$

(c)显然正交归一基矢为仅在上三角区域有一个分量为1其余为0的矩阵。显然有

$$U^\perp=\{A|a_{ij}=0,j\geq i\}$$

$$(p_UA)_{ij}=\begin{cases}0 & j<i\\ a_{ij} & j\geq i\end{cases}$$

11.

令p_U为U的正交投影。令$\alpha \in (U^\perp)^\perp$，那么

$$\langle \alpha - p_U\alpha, \beta \rangle = 0, \forall \beta \in U^\perp$$

又由于$\alpha - p_U\alpha \in U^\perp$，那么

$$\alpha - p_U\alpha = 0 \to \alpha = p_U\alpha \in U \Rightarrow (U^\perp)^\perp \subseteq U$$

又有

$$\langle \alpha, \beta \rangle = 0, \forall \alpha \in U, \beta \in U^\perp \Rightarrow U \subseteq (U^\perp)^\perp$$

因此

$$U = (U^\perp)^\perp$$

12.

显然我们有

$$\langle \alpha_1 + \alpha_2, \beta \rangle = \langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle = 0, \forall \alpha_i \in U_i, \beta \in U_1^\perp \cap U_2^\perp$$

因此

$$U_1^\perp \cap U_2^\perp \subseteq (U_1 + U_2)^\perp$$

又有

$$\langle \alpha_1 + \alpha_2, \beta \rangle = 0, \forall \alpha_i \in U_i, \beta \in (U_1 + U_2)^\perp$$

分别令$\alpha_1 = 0, \alpha_2 = 0$我们得到

$$\langle \alpha_i, \beta \rangle = 0, \forall \alpha_i \in U_i, \beta \in (U_1 + U_2)^\perp$$

这意味着

$$\beta \in U_1^\perp \cap U_2^\perp \Rightarrow (U_1 + U_2)^\perp \subseteq U_1^\perp \cap U_2^\perp$$

于是

$$(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$$