线代作业6

郑子诺，物理41

2024 年 10 月 23 日

1.

(1)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 3 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 16$$

(2)

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 6 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 2$$

2.

$$
\begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 \\
-1 & 1 & 1 & 0 & \cdots & 0 \\
0 & -1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 1 & 1 \\
0 & 0 & \cdots & 0 & -1 & 1
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 \\
-1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -1 & 1 & 1 \\
0 & \cdots & 0 & -1 & 1
\end{vmatrix} - \begin{vmatrix}
-1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -1 & 1 & 1 \\
0 & \cdots & 0 & -1 & 1
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 \\
-1 & 1 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & -1 & 1 & 1 \\
0 & \cdots & 0 & -1 & 1
\end{vmatrix} + \begin{vmatrix}
1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 1 \\
0 & \cdots & -1 & 1
\end{vmatrix}
$$

所以有

$$F_n = F_{n-1} + F_{n-2}$$

显然

$$F_1 = 1, F_2 = 2$$

3.

$$\begin{aligned}
\frac{d\det A}{dx} &= \sum_{\sigma} sgn(\sigma)\frac{d}{dx}(a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}) \\
&= \sum_{\sigma} sgn(\sigma)\sum_{i=1}^{n}\frac{da_{i\sigma(i)}}{dx}a_{1\sigma(1)}\cdots\widehat{a_{i\sigma(i)}}\cdots a_{n\sigma(n)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{da_{ij}}{dx}\sum_{\sigma'} sgn(\sigma'(1)\cdots j\cdots\sigma'(n))a_{1\sigma'(1)}\cdots\widehat{a_{ij}}\cdots a_{n\sigma'(n)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i-1}\frac{da_{ij}}{dx}\sum_{\sigma'} sgn(j\sigma')a_{1\sigma'(1)}\cdots\widehat{a_{ij}}\cdots a_{n\sigma'(n)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}\frac{da_{ij}}{dx}\sum_{\sigma'} sgn(\sigma')a_{1\sigma'(1)}\cdots\widehat{a_{ij}}\cdots a_{n\sigma'(n)} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}\frac{da_{ij}}{dx}A_{ij}
\end{aligned}$$

4.

(a)对于第一条式子，显然前$m\times m$个元素为I_m，我们有

$$1\le i\le m, m+1\le j\le m+n, c_{ij}=\sum_{k=1}^{m+n}\delta_{ik}a_{k(j-m)}=a_{i(j-m)}$$

$$m+1\le i\le m+n, 1\le j\le m, c_{ij}=\sum_{k=1}^{m}-b_{(i-m)k}\delta_{kj}+\sum_{k=m+1}^{m+n}\delta_{ik}b_{(k-m)j}=0$$

$$m+1\le i\le m+n, m+1\le j\le m+n, c_{ij}=\sum_{k=1}^{m}-b_{(i-m)k}a_{k(j-m)}+\sum_{k=m+1}^{m+n}\delta_{ik}\delta_{kj}=\delta_{ij}-(BA)_{(i-m)(j-m)}$$

所以

$$\begin{bmatrix} I_m & 0 \\ -B & I_n \end{bmatrix}\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix}=\begin{bmatrix} I_m & A \\ 0 & I_n-BA \end{bmatrix}$$

对于第二条式子，后面$n\times n$也是显然的，我们有

$$1\le i\le m, 1\le j\le m, c_{ij}=\sum_{k=1}^{m}\delta_{ik}\delta_{ij}+\sum_{k=m+1}^{m+n}-a_{i(k-m)}b_{(k-m)j}=\delta_{ij}-(AB)_{ij}$$

$$1\le i\le m, m+1\le j\le m+n, c_{ij}=\sum_{k=1}^{m}\delta_{ik}a_{k(j-m)}+\sum_{k=m+1}^{m+n}-a_{i(k-m)}\delta_{kj}=0$$

$$m+1 \le i \le m+n, 1 \le j \le m, c_{ij} = \sum_{k=1}^{m+n} \delta_{ik} b_{(k-m)j} = b_{(i-m)j}$$

所以

$$\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} = \begin{bmatrix} I_m - AB & 0 \\ B & I_n \end{bmatrix}$$

(b)

$$\begin{aligned} \det(I_m - AB) &= \det \begin{bmatrix} I_m - AB & 0 \\ B & I_n \end{bmatrix} \\ &= \det \begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} \\ &= \det \begin{bmatrix} I_m & A \\ 0 & I_n - BA \end{bmatrix} \\ &= \det(I_n - BA) \end{aligned}$$

(c)

$$\begin{vmatrix} 1+a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & 1+a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix} = \det(I_n - BA)$$

其中

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \end{bmatrix}$$

所以

$$\begin{aligned} \begin{vmatrix} 1+a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & 1+a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nb_1 & a_nb_2 & \cdots & 1+a_nb_n \end{vmatrix} &= \det(I_n - BA) \\ &= \det(1 - AB) \\ &= 1 + a_1b_1 + a_2b_2 + \cdots + a_nb_n \end{aligned}$$