线代作业5

郑子诺，物理41

2024 年 10 月 11 日

1.

(1):

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix} = 0 + 40 + 0 - 24 - 0 - 15 = 1$$

(2):

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 3 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -5 + 15 - 6 = 16$$

(3):

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix} = 40 \begin{vmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & 0 & -1 \\ 3 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -40 \begin{vmatrix} 1 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} = 160$$

2.

(1):

$$\begin{vmatrix} a & b & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b & 0 & \cdots & a \end{vmatrix} = a \begin{vmatrix} a & b & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{vmatrix} + (-1)^{1+n} b \begin{vmatrix} b & 0 & \cdots & 0 \\ a & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{vmatrix} = a^n + (-1)^{1+n} b^n$$

(2):

用归纳法。做归纳假设：

$$\det A_n = a_n + a_{n-1}x + \cdots + a_1x^{n-1} + x^n$$

对于$n = 1$显然成立。设对于$n = k - 1$成立。对于$n = k$我们有

$$\det A_k = a_k(-1)^{n+1}\begin{vmatrix} -1 & 0 & \cdots & 0 \\ x & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{vmatrix} + x\det A_{k-1} = a_k + a_{k-1}x + \cdots + a_1x^{k-1} + x^k$$

由数学归纳法知假设成立。

$$\therefore \begin{vmatrix} x & -1 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ a_n & a_{n-1} & \cdots & x + a_1 \end{vmatrix} = a_n + a_{n-1}x + \cdots + a_1x^{n-1} + x^n$$

3.

(a):

对第k行乘一个数c

$$P_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k \\ c\delta_{ij} & \text{if } i = k \end{cases}$$

$$\det P = c$$

第k行和第k'行交换

$$P_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k, k' \\ \delta_{k'j} & \text{if } i = k \\ \delta_{kj} & \text{if } i = k' \end{cases}$$

$$\det P = -1$$

第k行乘c加到第k'行

$$P_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k \\ c\delta_{kj} + \delta_{ij} & \text{if } i = k' \end{cases}$$

$$\det P = 1$$

以上是显然的。

(b):

对第k列乘一个数c

$$Q_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ c\delta_{ij} & \text{if } j = k \end{cases}$$

$$\det Q = c$$

第k列和第k'列交换

$$Q_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k, k' \\ \delta_{ik'} & \text{if } j = k \\ \delta_{ik} & \text{if } j = k' \end{cases}$$

$$\det Q = -1$$

第k列乘c加到第k'列

$$Q_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ c\delta_{ik} + \delta_{ik'} & \text{if } j = k' \end{cases}$$

$$\det Q = 1$$

以上是显然的。

4.

$$|A| = \begin{vmatrix} B & C \\ 0 & D \end{vmatrix} = \sum_{\sigma} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

$$\because i > k, a_{ij} = \begin{cases} 0 & \text{if } j \leq n-k \\ d_{(i-k)j} & \text{if } j > n-k \end{cases}$$

$$\therefore \sigma(k{+}1), \sigma(k{+}2), \dots, \sigma(n) \notin 1, 2, \dots, k \to \sigma(1), \sigma(2), \dots, \sigma(k) \in 1, 2, \dots, k$$

$$\therefore \sum_{\sigma} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} = \sum_{\sigma} sgn(\sigma) b_{1\sigma'(1)} b_{2\sigma'(2)} \dots b_{k\sigma'(k)} d_{k+1\sigma''(k+1)} d_{k+2\sigma''(k+2)} \dots d_{n\sigma''(n)}$$

$$\because \sigma = \{\sigma', \sigma''\}, sgn(\sigma) = sgn(\sigma') sgn(\sigma'')$$

$$\therefore |A| = \begin{vmatrix} B & C \\ 0 & D \end{vmatrix} = \sum_{\sigma', \sigma''} sgn(\sigma') b_{1\sigma'(1)} b_{2\sigma'(2)} \dots b_{k\sigma'(k)} sgn(\sigma') d_{k+1\sigma''(k+1)} d_{k+2\sigma''(k+2)} \dots d_{n\sigma''(n)} = |B||D|$$