

线代作业5

郑子诺，物理41

2024 年 10 月 18 日

1.

(1)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 3 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 16$$

(2)

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 5 & 6 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 4 & 6 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 5 \end{vmatrix} = 2$$

2.

$$\begin{aligned}
& \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ -1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 & 1 \\ 0 & \cdots & 0 & -1 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 & 1 \\ 0 & \cdots & 0 & -1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ -1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 & 1 \\ 0 & \cdots & 0 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & -1 & 1 \end{vmatrix}
\end{aligned}$$

所以有

$$F_n = F_{n-1} + F_{n-2}$$

显然

$$F_1 = 1, F_2 = 2$$

3.

$$\begin{aligned}
\frac{d \det A}{dx} &= \sum_{\sigma} \operatorname{sgn}(\sigma) \frac{d}{dx} (a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}) \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma) \sum_{i=1}^n \frac{da_{i\sigma(i)}}{dx} a_{1\sigma(1)} \cdots \widehat{a_{i\sigma(i)}} \cdots a_{n\sigma(n)} \\
&= \sum_{i=1}^n \sum_{j=1}^n \frac{da_{ij}}{dx} \sum_{\sigma'} \operatorname{sgn}(\sigma'(1) \cdots j \cdots \sigma'(n)) a_{1\sigma'(1)} \cdots \widehat{a_{ij}} \cdots a_{n\sigma'(n)} \\
&= \sum_{i=1}^n \sum_{j=1}^n (-1)^{i-1} \frac{da_{ij}}{dx} \sum_{\sigma'} \operatorname{sgn}(j\sigma') a_{1\sigma'(1)} \cdots \widehat{a_{ij}} \cdots a_{n\sigma'(n)} \\
&= \sum_{i=1}^n \sum_{j=1}^n (-1)^{i+j} \frac{da_{ij}}{dx} \sum_{\sigma'} \operatorname{sgn}(\sigma') a_{1\sigma'(1)} \cdots \widehat{a_{ij}} \cdots a_{n\sigma'(n)} \\
&= \sum_{i=1}^n \sum_{j=1}^n (-1)^{i+j} \frac{da_{ij}}{dx} A_{ij}
\end{aligned}$$

4.

(a) 对于第一条式子, 显然前 $m \times m$ 个元素为 I_m , 我们有

$$1 \leq i \leq m, m+1 \leq j \leq m+n, c_{ij} = \sum_{k=1}^{m+n} \delta_{ik} a_{k(j-m)} = a_{i(j-m)}$$

$$m+1 \leq i \leq m+n, 1 \leq j \leq m, c_{ij} = \sum_{k=1}^m -b_{(i-m)k} \delta_{kj} + \sum_{k=m+1}^{m+n} \delta_{ik} b_{(k-m)j} = 0$$

$$m+1 \leq i \leq m+n, m+1 \leq j \leq m+n, c_{ij} = \sum_{k=1}^m -b_{(i-m)k} a_{k(j-m)} + \sum_{k=m+1}^{m+n} \delta_{ik} \delta_{kj} = \delta_{ij} - (BA)_{(i-m)(j-m)}$$

所以

$$\begin{bmatrix} I_m & 0 \\ -B & I_n \end{bmatrix} \begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} = \begin{bmatrix} I_m & A \\ 0 & I_n - BA \end{bmatrix}$$

对于第二条式子, 后面 $n \times n$ 也是显然的, 我们有

$$1 \leq i \leq m, 1 \leq j \leq m, c_{ij} = \sum_{k=1}^m \delta_{ik} \delta_{ij} + \sum_{k=m+1}^{m+n} -a_{i(k-m)} b_{(k-m)j} = \delta_{ij} - (AB)_{ij}$$

$$1 \leq i \leq m, m+1 \leq j \leq m+n, c_{ij} = \sum_{k=1}^m \delta_{ik} a_{k(j-m)} + \sum_{k=m+1}^{m+n} -a_{i(k-m)} \delta_{kj} = 0$$

$$m+1 \leq i \leq m+n, 1 \leq j \leq m, c_{ij} = \sum_{k=1}^{m+n} \delta_{ik} b_{(k-m)j} = b_{(i-m)j}$$

所以

$$\begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} = \begin{bmatrix} I_m - AB & 0 \\ B & I_n \end{bmatrix}$$

(b)

$$\begin{aligned} \det(I_m - AB) &= \det \begin{bmatrix} I_m - AB & 0 \\ B & I_n \end{bmatrix} \\ &= \det \begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} \\ &= \det \begin{bmatrix} I_m & A \\ 0 & I_n - BA \end{bmatrix} \\ &= \det(I_n - BA) \end{aligned}$$

(c)

$$\begin{vmatrix} 1 + a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & 1 + a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & 1 + a_n b_n \end{vmatrix} = \det(I_n - BA)$$

其中

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \end{bmatrix}$$

所以

$$\begin{aligned} \begin{vmatrix} 1 + a_1 b_1 & a_1 b_2 & \cdots & a_1 b_n \\ a_2 b_1 & 1 + a_2 b_2 & \cdots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \cdots & 1 + a_n b_n \end{vmatrix} &= \det(I_n - BA) \\ &= \det(1 - AB) \\ &= 1 + a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}$$