高微作业10

郑子诺，物理41

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1.

(1)令$x = \sin\theta, \theta = \arcsin x$，我们有

$$
\begin{aligned}
&\int \arcsin x \mathrm{d}x \\
&= \int \theta \cos\theta \mathrm{d}\theta \\
&= \theta \sin\theta - \int \sin\theta \mathrm{d}\theta \\
&= \theta \sin\theta + \cos\theta + C \\
&= x \arcsin x + \sqrt{1-x^2} + C
\end{aligned}
$$

(2)令$x = a\sin\theta, \theta = \arcsin\dfrac{x}{a}$，我们有

$$
\begin{aligned}
&\int \frac{x^2}{\sqrt{a^2-x^2}} \mathrm{d}x \\
&= \int a^2 \sin^2\theta \mathrm{d}\theta \\
&= \int a^2 \frac{1-\cos 2\theta}{2} \mathrm{d}\theta \\
&= a^2\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C \\
&= \frac{1}{2}\left(a^2 \arcsin\frac{x}{a} - x\sqrt{a^2-x^2}\right) + C
\end{aligned}
$$

(3)令$t=x-2,x=t+2$，我们有

$$\begin{aligned}&\int\frac{x+1}{\sqrt{x^2-4x}}\mathrm{d}x\\&=\int\frac{t+3}{\sqrt{t^2-4}}\mathrm{d}t\\&=\int\frac{t}{\sqrt{t^2-4}}\mathrm{d}t+\int\frac{3}{\sqrt{t^2-4}}\mathrm{d}t\\&=\sqrt{t^2-4}+\ln(t+\sqrt{t^2-4})+C\\&=\sqrt{x^2-4x}+\ln(x-2+\sqrt{x^2-4x})+C\end{aligned}$$

(4)观察知

$$\frac{1}{1+x^3}=\frac{1}{3}\frac{1}{1+x}+\frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}$$

因此我们有

$$\begin{aligned}&\int\frac{1}{x^3+1}\mathrm{d}x\\&=\frac{1}{3}\int\frac{1}{1+x}\mathrm{d}x+\frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\mathrm{d}x\\&=\frac{1}{3}\ln(1+x)-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\mathrm{d}x+\frac{1}{2}\int\frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\mathrm{d}x\\&=\frac{1}{3}\ln(1+x)-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+C\end{aligned}$$

(5)令$t=\sqrt{x},x=t^2$，我们有

$$\begin{aligned}&\int\frac{\sqrt{x}}{(1+x)^2}\mathrm{d}x\\&=\int\frac{2t^2}{(1+t^2)^2}\mathrm{d}t\\&=-\frac{t}{1+t^2}+\int\frac{1}{1+t^2}\mathrm{d}t\\&=\arctan t-\frac{t}{1+t^2}+C\\&=\arctan\sqrt{x}-\frac{\sqrt{x}}{1+x}+C\end{aligned}$$

2.

(1)分部积分我们有

$$\begin{aligned}
&\int_1^2 x\ln^2 x\mathrm{d}x\\
&=\frac{1}{2}x^2\ln^2 x\Big|_1^2-\int_1^2 x\ln x\mathrm{d}x\\
&=2\ln^2 2-(\frac{1}{2}x^2\ln x\Big|_1^2-\int_1^2\frac{1}{2}x\mathrm{d}x)\\
&=2\ln^2 2-2\ln 2+\frac{3}{4}
\end{aligned}$$

(2)令$x=\arctan t$，我们有

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}}\frac{\tan x}{\cos^2 x}\mathrm{d}x\\
&=\int_0^1 t\mathrm{d}t\\
&=\frac{1}{2}
\end{aligned}$$

(3)利用万能公式，令$x=2\arctan t$，我们有

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}\frac{1}{1+a\cos x}\mathrm{d}x\\
&=\int_0^1\frac{2}{(1-a)t^2+1+a}\mathrm{d}t\\
&=\frac{2}{1-a}\sqrt{\frac{1-a}{1+a}}\arctan\sqrt{\frac{1-a}{1+a}}t\Big|_0^1\\
&=\frac{2}{\sqrt{1-a^2}}\arctan\sqrt{\frac{1-a}{1+a}}
\end{aligned}$$

3.

令$t=\cos x$，我们有

$$\begin{aligned}
&\int_0^{\pi}\frac{(\cos x-a)\sin x}{(1+a^2-2a\cos x)^{\frac{3}{2}}}\mathrm{d}x\\
&=\int_{-1}^1\frac{t-a}{(1+a^2-2at)^{\frac{3}{2}}}\mathrm{d}t\\
&=\frac{1}{a}\frac{t-a}{\sqrt{1+a^2-2at}}\Big|_{-1}^1-\int_{-1}^1\frac{1}{a}\frac{\mathrm{d}t}{\sqrt{1+a^2-2at}}\\
&=\frac{1-a-|1-a|}{a^2|1-a|}
\end{aligned}$$

4.

(1)利用万能公式，令$x=2\arctan t$，我们有

$$\begin{aligned}&\int\frac{1}{a+\sin x}\mathrm{d}x\\&=\int\frac{2\mathrm{d}t}{at^2+2t+a}\\&=\frac{2}{\sqrt{a^2-1}}\arctan\frac{a\tan\frac{x}{2}+1}{\sqrt{a^2-1}}+C\end{aligned}$$

(2)我们先有

$$\int_0^{2\pi}\frac{1}{a^2-\sin^2 x}\mathrm{d}x=\int_0^{\pi}\frac{1}{a^2-\sin^2 x}\mathrm{d}x+\int_{\pi}^{2\pi}\frac{1}{a^2-\sin^2(\pi+x)}\mathrm{d}x=2\int_0^{\pi}\frac{1}{a^2-\sin^2 x}\mathrm{d}x$$

因此利用上一题公式并取极限得到

$$\begin{aligned}&\int_0^{2\pi}\frac{1}{a^2-\sin^2 x}\mathrm{d}x\\&=\frac{1}{a}(\int_0^{\pi}\frac{1}{a-\sin x}\mathrm{d}x+\int_0^{\pi}\frac{1}{a+\sin x}\mathrm{d}x)\\&=\frac{1}{a}(\int_{-\pi}^{0}\frac{1}{a+\sin x}\mathrm{d}x+\int_0^{\pi}\frac{1}{a+\sin x}\mathrm{d}x)\\&=\frac{2\pi}{a\sqrt{a^2-1}}\end{aligned}$$

5.

(1)观察知

$$\frac{1}{x^4+1}=\frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}+\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}$$

我们有

$$\begin{aligned}&\int\frac{1}{x^4+1}\mathrm{d}x\\&=\int\frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\mathrm{d}x+\int\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}\mathrm{d}x\\&=-\int\frac{\frac{1}{2\sqrt{2}}(-x)+\frac{1}{2}}{(-x)^2+\sqrt{2}(-x)+1}\mathrm{d}(-x)+\int\frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}\mathrm{d}x\end{aligned}$$

因此只需计算右边积分，我们有

$$\begin{aligned}
&\int \frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1}\mathrm{d}x \\
&=\int \frac{1}{4\sqrt{2}}\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\mathrm{d}x+\frac{1}{4}\int \frac{\mathrm{d}x}{(x+\frac{1}{\sqrt{2}})^2+\frac{1}{2}}\mathrm{d}x \\
&=\frac{1}{4\sqrt{2}}\ln(x^2+\sqrt{2}x+1)+\frac{1}{2\sqrt{2}}\arctan(\sqrt{2}x+1)+C
\end{aligned}$$

因此原不定积分为

$$\frac{1}{4\sqrt{2}}(\ln(x^2+\sqrt{2}x+1)-\ln(x^2-\sqrt{2}x+1))+\frac{1}{2\sqrt{2}}(\arctan(\sqrt{2}x+1)-\arctan(-\sqrt{2}x+1))+C$$

(2)显然我们有

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{x^4+1}\mathrm{d}x \\
&=\lim_{A\to+\infty} \frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\Bigg|_0^A+\frac{1}{2\sqrt{2}}(\arctan(\sqrt{2}x+1)-\arctan(1-\sqrt{2}x))\Bigg|_0^A \\
&=\frac{\pi}{2\sqrt{2}}
\end{aligned}$$

6.

(1)令$t=nx$，原式相当于计算

$$\lim_{n\to+\infty}\frac{\int_0^n f(t)\mathrm{d}t}{n}$$

鉴于$\lim_{x\to+\infty}f(x)=L$，我们有

$$\forall\epsilon,\exists A,L-\epsilon<f(x)<L+\epsilon,x>A$$

因此

$$(L-\epsilon)\frac{n-A}{n}+\frac{\int_0^A f(x)\mathrm{d}x}{n}<\frac{\int_0^n f(x)\mathrm{d}x}{n}<(L+\epsilon)\frac{n-A}{n}+\frac{\int_0^A f(x)\mathrm{d}x}{n}$$

显然两侧极限分别为$L-\epsilon,L+\epsilon$，于是$\forall\epsilon'>0,\exists N$使得$n>N$时

$$L-\epsilon-\epsilon'<\frac{\int_0^n f(x)\mathrm{d}x}{n}<L+\epsilon+\epsilon'$$

鉴于ϵ,ϵ'是任取的，根据夹逼定理我们有

$$\lim_{n\to+\infty}\int_0^1 f(nx)\mathrm{d}x=L$$

(2)令$t=nx$，我们有

$$\int_0^T g(x)h(nx)\mathrm{d}x=\frac{1}{n}\sum_{k=1}^{n}\int_{\frac{(k-1)T}{n}}^{\frac{kT}{n}} g(\frac{t}{n})h(t)\mathrm{d}t$$

根据h的周期性，对于第k项进行换元$\xi=t-(k-1)T$，我们有

$$\int_0^T (\sum_{k=1}^{n} g(\frac{\xi}{n}+\frac{(k-1)T}{n})\frac{1}{n})h(\xi)\mathrm{d}\xi$$

鉴于g可积，根据黎曼积分定义，$\forall\epsilon>0,\exists N$使得$n>N$时我们有

$$|\int_0^T g(x)\mathrm{d}x-\sum_{k=1}^{n} g(\frac{\xi}{n}+\frac{(k-1)T}{n})\frac{T}{n}|<\frac{T\epsilon}{\int_0^T h(x)\mathrm{d}x}$$

因此原式有

$$\frac{1}{T}\int_0^T g(x)\mathrm{d}x\int_0^T h(x)\mathrm{d}x-\epsilon<\int_0^T g(x)h(nx)\mathrm{d}x<\frac{1}{T}\int_0^T g(x)\mathrm{d}x\int_0^T h(x)\mathrm{d}x+\epsilon, n>N$$

于是根据夹逼定理我们有

$$\lim_{n\to+\infty}\int_0^T g(x)h(nx)\mathrm{d}x=\frac{1}{T}\int_0^T g(x)\mathrm{d}x\int_0^T h(x)\mathrm{d}x$$