CHAPTER 5: SAMPLING AND ESTIMATION (CONFIDENCE INTERVALS)

(Week 9-10: Lecture Notes)

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1. INTRODUCTION: THE GAME CHANGES

In the first half of the course (Probability), we did this:

- "If the factory error rate is 5% (Truth is known), what is the probability that 2 out of 10 parts I select are defective?"

Now (Statistics), we will do this:

- "2 out of 10 parts I selected turned out to be defective (Data is in hand). **What is the real error rate of the factory?**"

Our Goal: To go from the Part to the Whole (From Sample to Population).

2. CENTRAL LIMIT THEOREM (CLT) - The "Magic Wand"

The Problem: Not all data in nature follows a Normal (Bell Curve) distribution. Some are skewed, some are flat (Uniform). How do we analyze non-normal data?

The Solution (CLT): If you take repeated samples from a population (whatever its shape) and calculate their **Means ($\bar{x}$)**, the distribution of these means will follow a **NORMAL DISTRIBUTION**.

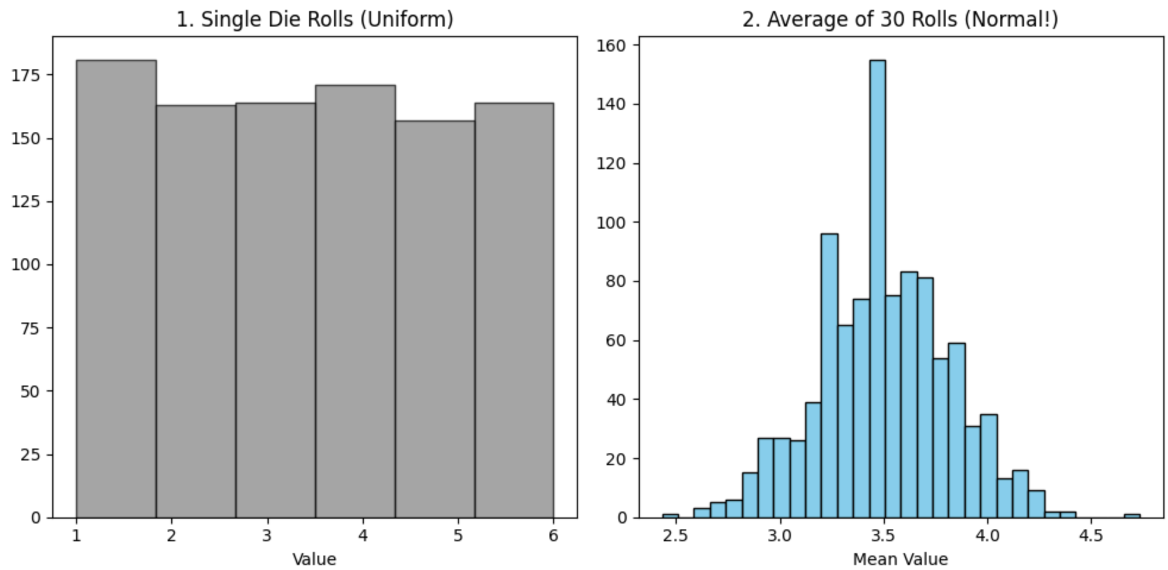
Condition: The sample size (n) must be large enough (Generally accepted as $n \geq 30$).

2.1. Engineering Meaning

This theorem tells us: "*Even if the strength of the rebar does not follow a normal distribution, the average of 50 samples you take from those bars distributes Normally. Therefore, you can use the Z-Table with peace of mind.*"

In [1]:
```
import random
import matplotlib.pyplot as plt

# --- VISUAL PROOF OF CLT ---
# Let's simulate rolling a die (Uniform Distribution - Not Normal)
# But we look at the AVERAGE of 30 rolls.

population_data = [1, 2, 3, 4, 5, 6] # A die
sample_means = []
```

```python
# Do this experiment 1000 times
for _ in range(1000):
    # Roll the die 30 times (n=30)
    sample = [random.choice(population_data) for _ in range(30)]
    # Calculate the mean
    mean_val = sum(sample) / 30
    sample_means.append(mean_val)

# Plotting
plt.figure(figsize=(10, 5))

# 1. Original Distribution (Theoretical)
plt.subplot(1, 2, 1)
plt.hist([random.choice(population_data) for _ in range(1000)], bins=6, color='g
plt.title("1. Single Die Rolls (Uniform)")
plt.xlabel("Value")

# 2. Distribution of Means (CLT Effect)
plt.subplot(1, 2, 2)
plt.hist(sample_means, bins=30, color='skyblue', edgecolor='black')
plt.title("2. Average of 30 Rolls (Normal!)")
plt.xlabel("Mean Value")

plt.tight_layout()
plt.show()
```



3. SAMPLING DISTRIBUTION

A single measurement is not the same as the "Measurement of the Mean".

Important Formula Change (Standard Error)

For the mean ($\bar{x}$) of samples with n elements:

1. **Mean of Means:** $\mu_{\bar{x}} = \mu$ (Same as population).
2. **Standard Deviation of Means (Standard Error):**

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

Critical Interpretation: As the sample size (n) increases, the Standard Error decreases. That is, the "deviation of the average" is much smaller than the "deviation of a single datum".

3.1. Numerical Example (Elevator Capacity)

Let people's weight be normally distributed with mean $\mu = 70$ kg and standard deviation $\sigma = 20$ kg.

Question 1:

What is the probability of **1 randomly selected person** being heavier than **80 kg**?

- **Solution:**
  - $Z = \frac{80-70}{20} = 0.5$
  - From Table (Z=0.5): Probability is high (approx 31% chance).

Question 2:

What is the probability of the **AVERAGE of 16 randomly selected people** being heavier than **80 kg**?

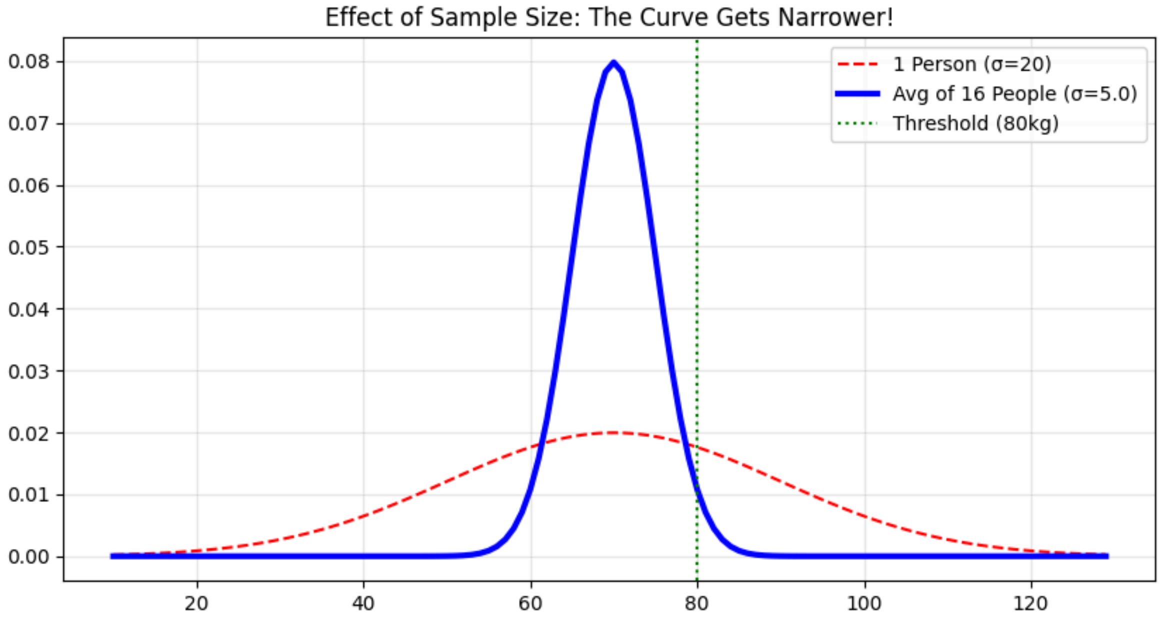
- **Solution:**
  - Here $n = 16$. We must update the deviation!
  - New Deviation (Standard Error) = $\frac{20}{\sqrt{16}} = \frac{20}{4} = 5.$
  - New Z Score = $\frac{80-70}{5} = \mathbf{2.0}$
  - From Table (Z=2.0): Probability is very low (approx 2% chance).

Conclusion: It is normal for 1 person to be heavy, but it is very rare for the average of 16 people to be that heavy.

In [2]:

```
import math
import matplotlib.pyplot as plt

# Function for Normal Distribution Curve
def normal_pdf(x, mu, sigma):
    return (1 / (sigma * math.sqrt(2 * math.pi))) * math.exp(-0.5 * ((x - mu) /

# Parameters
mu = 70
sigma_population = 20
n = 16
sigma_sample = sigma_population / math.sqrt(n) # Standard Error = 5

# Data for plotting
x_values = [i for i in range(10, 130)]
y_pop = [normal_pdf(x, mu, sigma_population) for x in x_values]
y_sample = [normal_pdf(x, mu, sigma_sample) for x in x_values]
```

```python
# Plotting
plt.figure(figsize=(10, 5))
plt.plot(x_values, y_pop, color='red', linestyle='--', label=f'1 Person (σ={sigm
plt.plot(x_values, y_sample, color='blue', linewidth=3, label=f'Avg of 16 People

plt.title("Effect of Sample Size: The Curve Gets Narrower!")
plt.axvline(x=80, color='green', linestyle=':', label='Threshold (80kg)')
plt.legend()
plt.grid(True, alpha=0.3)
plt.show()
```



4. ESTIMATION THEORY

It is the job of guessing the truth (Population Parameter) with the data we have.

4.1. Point Estimation

We assign a "single number".

- Best estimator for $\mu \rightarrow$ Sample Mean ($\bar{x}$).
- **Risk:** Making a point shot is hard. You will likely be slightly off.

4.2. Interval Estimation (Confidence Interval)

Instead of a point shot, we cast a "Net". We say "*The average is between 45 and 55, and I am 95% sure of this.*"

5. CONFIDENCE INTERVAL (Case 1: σ Known)

If we know the population σ (or $n \geq 30$), we use the **Z-Table**.

5.1. Formula

$$\bar{x} \pm \left[Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \right]$$

- **Margin of Error (E):** The part inside the brackets.

5.2. Critical Z Values (Memorize These)

- **90%:** $Z = 1.645$
- **95%:** $Z = 1.96$ (Most famous)
- **99%:** $Z = 2.575$

5.3. Numerical Example (Production Line)

Problem: Screw diameters. $\sigma = 0.1$ mm. Sample $n = 36$, mean $\bar{x} = 10.5$ mm. Find **95% Confidence Interval**.

Solution:

1. **Find Z:** For 95%, $Z = 1.96$.
2. **Calculate Standard Error:** $\frac{0.1}{\sqrt{36}} = \frac{0.1}{6} \approx 0.0167$
3. **Calculate Margin of Error (E):** $1.96 \times 0.0167 \approx 0.033$
4. **Find Interval:**
   - Lower: $10.5 - 0.033 = \mathbf{10.467}$
   - Upper: $10.5 + 0.033 = \mathbf{10.533}$

Result: We are 95% confident that the true mean is between **10.467 mm** and **10.533 mm**.

In [3]:

```python
# Z-Interval Calculator Code
import math

x_bar = 10.5
sigma = 0.1
n = 36
z_95 = 1.96

std_error = sigma / math.sqrt(n)
margin_of_error = z_95 * std_error

lower = x_bar - margin_of_error
upper = x_bar + margin_of_error

print(f"--- Screw Problem Verification ---")
print(f"Standard Error: {std_error:.5f}")
print(f"Margin of Error: {margin_of_error:.5f}")
print(f"Result: [{lower:.3f}, {upper:.3f}]")
```

```
--- Screw Problem Verification ---
Standard Error: 0.01667
Margin of Error: 0.03267
Result: [10.467, 10.533]
```

6. CONFIDENCE INTERVAL (Case 2: σ Unknown)

If we don't know σ, we use the sample deviation (s) and the **T-Table**.

6.1. Why T-Distribution?

It has **thicker tails** than Normal. Because using s instead of σ adds uncertainty, we need a wider net (wider interval) to be safe.

6.2. Formula

$$\bar{x} \pm \left[t_{(\alpha/2, n-1)} \cdot \frac{s}{\sqrt{n}} \right]$$

- **df (Degrees of Freedom):** $n - 1$

6.3. Numerical Example (Concrete Strength)

Problem: $n = 10$ samples. Mean $\bar{x} = 30$ MPa, Std Dev $s = 4$ MPa. Find **95% Confidence Interval**.

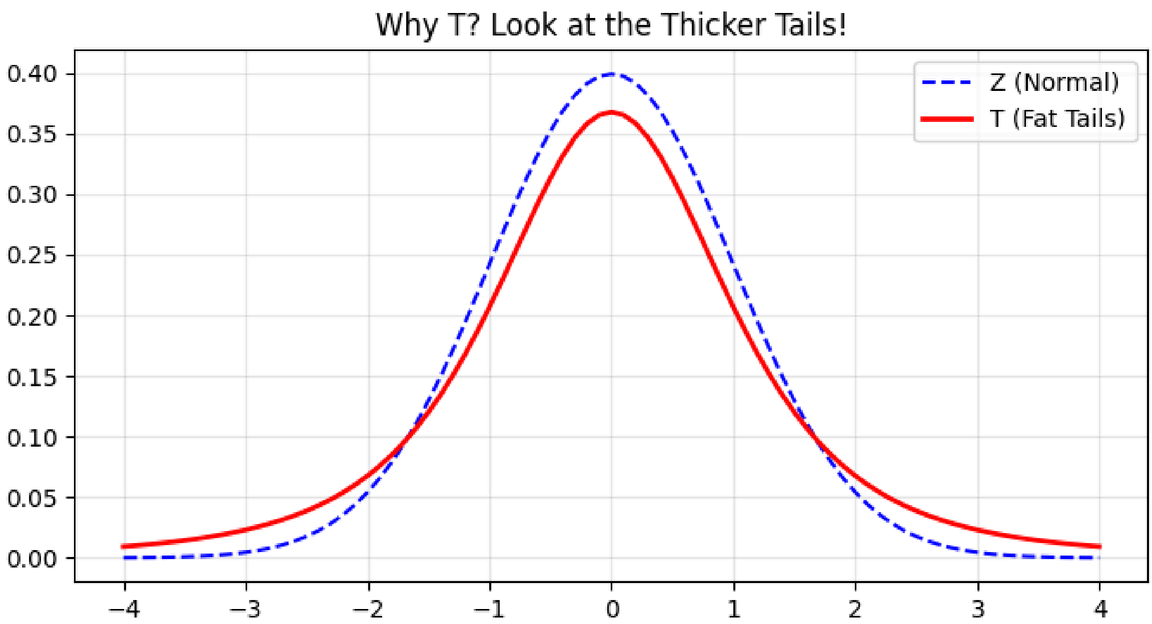
Solution:

1. **Find T-Value:**
   - $df = 10 - 1 = 9.$
   - $\alpha = 0.05$ (Two-tailed).
   - From T-Table (Row 9, Col 0.025): $t = 2.262$
2. **Calculate Standard Error:** $\frac{4}{\sqrt{10}} \approx 1.265$
3. **Calculate Margin of Error (E):** $2.262 \times 1.265 \approx 2.86$
4. **Find Interval:**
   - Lower: $30 - 2.86 = \mathbf{27.14}$
   - Upper: $30 + 2.86 = \mathbf{32.86}$

Result: We are 95% confident the concrete strength is between **27.14 MPa** and **32.86 MPa**.

In [4]:
```
# Visualizing T vs Z
# Note: T has fat tails!
def t_pdf(x, df):
    # Simplified math for T-dist shape (Student's t)
    numerator = math.gamma((df + 1) / 2)
    denominator = math.sqrt(df * math.pi) * math.gamma(df / 2)
    base = (1 + (x**2) / df)
    exponent = -(df + 1) / 2
    return (numerator / denominator) * (base ** exponent)

x_vals = [i * 0.1 for i in range(-40, 41)]
z_y = [normal_pdf(x, 0, 1) for x in x_vals]
t_y = [t_pdf(x, df=3) for x in x_vals]
```

```
plt.figure(figsize=(8, 4))
plt.plot(x_vals, z_y, 'b--', label='Z (Normal)')
plt.plot(x_vals, t_y, 'r-', linewidth=2, label='T (Fat Tails)')
plt.title("Why T? Look at the Thicker Tails!")
plt.legend()
plt.grid(True, alpha=0.3)
plt.show()

# --- Concrete Verification ---
x_bar_conc = 30
s_conc = 4
n_conc = 10
t_critical = 2.262

margin_conc = t_critical * (s_conc / math.sqrt(n_conc))
print(f"--- Concrete Problem Verification ---")
print(f"Margin of Error: {margin_conc:.2f}")
print(f"Interval: [{x_bar_conc - margin_conc:.2f}, {x_bar_conc + margin_conc:.2f
```




```
--- Concrete Problem Verification ---
Margin of Error: 2.86
Interval: [27.14, 32.86]
```

7. DETERMINING SAMPLE SIZE (n)

Question: "*How many samples do I need so my error is at most 0.5?*"

Formula

$$n = \left[\frac{Z \cdot \sigma}{E} \right]^2$$

Example

- $\sigma = 2$, Max Error $E = 0.5$, Confidence 95% ($Z = 1.96$).

Solution:

$$n = \left[\frac{1.96 \cdot 2}{0.5} \right]^2 = [7.84]^2 \approx 61.46$$

Decision: We must take **62 samples**. (Always round UP!).

In [5]:
```
# Sample Size Calculation
sigma = 2
E = 0.5
Z = 1.96

n = ((Z * sigma) / E) ** 2
print(f"Calculated n: {n:.2f}")
print(f"Required Samples: {math.ceil(n)} (Rounded Up)")
```

```
Calculated n: 61.47
Required Samples: 62 (Rounded Up)
```

8. Lecture Summary

1. **Z vs T:**
   - σ known? $\rightarrow$ **Z**
   - Only s known? $\rightarrow$ **T**
   - $n > 30?$ $\rightarrow$ **Z** is acceptable.
2. **Square Root n:** Never forget to divide by $\sqrt{n}$ when dealing with averages!
3. **Trade-off:** Higher confidence (99%) means a wider interval (less precision). To get better precision (narrow interval) with high confidence, you must increase n.