CHAPTER 6: DECISION MAKING WITH HYPOTHESIS TESTS

(Week 11-12: Lecture Notes)

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1. INTRODUCTION: WHAT IS A HYPOTHESIS TEST? (The Court Analogy)

In statistics, when making decisions, we act with the logic of the **"Presumption of Innocence"**. To refute a claim, we must have **"sufficient evidence"**.

1.1. Two Rival Hypotheses

1. **Null Hypothesis (H_0):**
   - The claim defending the current situation, the status quo, or "no difference".
   - *Court equivalent:* "The defendant is Innocent."
   - *Production equivalent:* "Production is flawless, the average is exactly 10 cm."
2. **Alternative Hypothesis (H_1 or H_a):**
   - The claim the researcher (us) is trying to prove, defending "there is a difference".
   - *Court equivalent:* "The defendant is Guilty."
   - *Production equivalent:* "Production is faulty, the average is not 10 cm."

1.2. The Decision Mechanism

We collect data (evidence).

- If the evidence is **very strong**, H_0 is **Rejected**. (Defendant is found guilty).
- If the evidence is **weak**, H_0 **Cannot Be Rejected**. (Defendant is acquitted. *Note: We don't say "Innocent", we say "Guilt could not be proven"*).

2. TYPES OF ERRORS (Risk Management)

There is always a risk of making a mistake when making a decision. There can be two types of errors:

	Reality: H_0 is TRUE (Product is good)	**Reality: H_0 is FALSE (Product is bad)**
Decision: Accept H_0	Correct Decision	**Type II Error (β)** (Consumer Risk)

	Reality: H_0 is TRUE (Product is good)	Reality: H_0 is FALSE (Product is bad)
Decision: Reject H_0	**Type I Error** (α) (Producer Risk)	Correct Decision

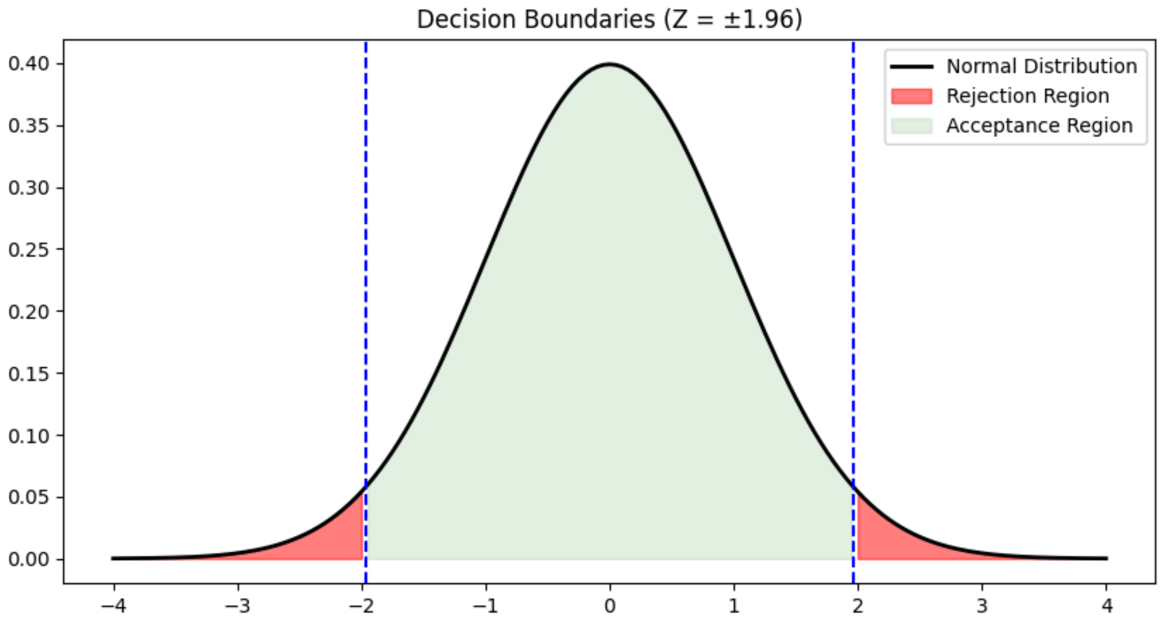
3. VISUALIZING THE DECISION RULE

We convert our data to a **Z-Score**. If this score falls into the "Critical Region" (Tail ends), we reject the Null Hypothesis.

The Z-Test Logic

1. **Calculate Z:** How many standard deviations away is the sample mean?
2. **Compare:** If it is further than **1.96** (for 95% confidence), it's not luck. It's a real difference.

In [2]:
```
# Visualizing Critical Regions (Alpha = 0.05, Two-Tailed)
x_vals, y_vals = generate_normal_data(-4, 4, 0.05)

plt.figure(figsize=(10, 5))
plt.plot(x_vals, y_vals, color='black', linewidth=2, label='Normal Distribution'

# Color the Rejection Regions manually
# Region 1: Right Tail (Z > 1.96)
x_right = [x for x in x_vals if x > 1.96]
y_right = [y for x, y in zip(x_vals, y_vals) if x > 1.96]
plt.fill_between(x_right, y_right, color='red', alpha=0.5, label='Rejection Regi

# Region 2: Left Tail (Z < -1.96)
x_left = [x for x in x_vals if x < -1.96]
y_left = [y for x, y in zip(x_vals, y_vals) if x < -1.96]
plt.fill_between(x_left, y_left, color='red', alpha=0.5)

# Region 3: Acceptance Region
x_accept = [x for x in x_vals if -1.96 <= x <= 1.96]
y_accept = [y for x, y in zip(x_vals, y_vals) if -1.96 <= x <= 1.96]
plt.fill_between(x_accept, y_accept, color='green', alpha=0.1, label='Acceptance

plt.axvline(1.96, color='blue', linestyle='--')
plt.axvline(-1.96, color='blue', linestyle='--')
plt.title("Decision Boundaries (Z = ±1.96)")
plt.legend()
plt.show()
```



4. SINGLE SAMPLE TESTS (Z and T Tests)

Case 1: Population Standard Deviation (σ) Known (Z-Test)

Numerical Example: A cable factory claims the breaking strength of cables is **1000 kg** ($\mu_0 = 1000$). Standard deviation $\sigma = 20$ kg. We tested **n = 50** cables, mean came out $\bar{x} = 992$ **kg**. **Is the factory's claim true?** (Test for $\alpha = 0.05$).

Solution Steps:

1. **Hypotheses:**
   - $H_0 : \mu = 1000$
   - $H_1 : \mu \neq 1000$
2. **Calculate Standard Error:**
   - $SE = \frac{20}{\sqrt{50}} \approx 2.83$
3. **Calculate Z-Score:**
   - $Z = \frac{992-1000}{2.83} = \frac{-8}{2.83} \approx \mathbf{-2.82}$
4. **Decision:**
   - Critical Value (from table) = **1.96**
   - Since $|-2.82| > 1.96$, we **REJECT** the claim.
   - *Conclusion:* The cables are weaker than claimed.

In [3]:
```
# Z-Test Verification
mu_0 = 1000
sigma = 20
n = 50
x_bar = 992
z_critical = 1.96  # Fixed value for 95% confidence

std_error = sigma / math.sqrt(n)
z_calc = (x_bar - mu_0) / std_error

print(f"--- Cable Problem Results ---")
```

```python
print(f"Standard Error: {std_error:.3f}")
print(f"Calculated Z  : {z_calc:.3f}")

if abs(z_calc) > z_critical:
    print("Result: REJECT H0 (The factory is lying!)")
else:
    print("Result: ACCEPT H0 (The factory might be right)")
```

```
--- Cable Problem Results ---
Standard Error: 2.828
Calculated Z  : -2.828
Result: REJECT H0 (The factory is lying!)
```

Case 2: Population SD (σ) Unknown (T-Test)

(When we only have sample s and $n < 30$)

Numerical Example: New concrete additive. Target pressure **30 MPa**. Sample **n = 9**, Mean $\bar{x} = 31.5$, Std Dev **s = 1.5**. **Is there a difference?**

Solution Steps:

1. **Calculate T-Score:**
   - $SE = \frac{1.5}{\sqrt{9}} = 0.5$
   - $T = \frac{31.5-30}{0.5} = \mathbf{3.0}$
2. **Critical Value:**
   - $df = n - 1 = 8$
   - From T-Table (for $\alpha = 0.05$): **2.306**
3. **Decision:**
   - $|3.0| > 2.306 \rightarrow$ **REJECT** H_0.
   - *Conclusion:* The additive significantly changed the strength.

In [4]:
```python
# T-Test Verification
mu_conc = 30
x_bar_conc = 31.5
s_conc = 1.5
n_conc = 9
t_table_val = 2.306 # Hardcoded from T-Table for df=8

se_conc = s_conc / math.sqrt(n_conc)
t_calc = (x_bar_conc - mu_conc) / se_conc

print(f"--- Concrete Problem Results ---")
print(f"Calculated T : {t_calc:.3f}")
print(f"Critical T   : {t_table_val}")

if abs(t_calc) > t_table_val:
    print("Result: REJECT H0 (Significant difference found)")
else:
    print("Result: ACCEPT H0")
```

```
--- Concrete Problem Results ---
Calculated T : 3.000
Critical T   : 2.306
Result: REJECT H0 (Significant difference found)
```

5. TWO SAMPLE TESTS (Comparison)

Scenario: Comparing two different Asphalt types (A vs B).

- **Type A:** $n = 10, \bar{x} = 50, s = 5$
- **Type B:** $n = 10, \bar{x} = 45, s = 4$

Calculation:

1. **Difference in Means:** $50 - 45 = 5$
2. **Common Variance:** $\frac{25}{10} + \frac{16}{10} = 4.1$
3. **Standard Error:** $\sqrt{4.1} \approx 2.02$
4. **t-Value:** $5/2.02 \approx \mathbf{2.47}$

Decision: Critical T (approx for df=18) is **2.10**. Since **2.47 > 2.10**, there is a significant difference.

In [5]:
```python
# Independent Two Sample T-Test
# Data
mean_a, s_a, n_a = 50, 5, 10
mean_b, s_b, n_b = 45, 4, 10
t_crit_two = 2.10

# Math
diff = mean_a - mean_b
variance_part = (s_a**2 / n_a) + (s_b**2 / n_b)
std_err_two = math.sqrt(variance_part)
t_val_two = diff / std_err_two

print(f"--- Asphalt Comparison ---")
print(f"Calculated T: {t_val_two:.3f}")
if abs(t_val_two) > t_crit_two:
    print("Result: REJECT H0 (Types are different)")
else:
    print("Result: ACCEPT H0")
```

```
--- Asphalt Comparison ---
Calculated T: 2.469
Result: REJECT H0 (Types are different)
```

6. PAIRED SAMPLE T-TEST (Before vs After)

Used when measuring the **same** subject twice. **Example (Diet):**

- Person 1: lost 2 kg
- Person 2: lost 1 kg
- Person 3: lost 3 kg
- **Mean Difference ($\bar{d}$):** -2
- **Std Dev (s_d):** 1

Calculation:

$$t = \frac{-2}{1/\sqrt{3}} = -2 \times 1.732 \approx -\mathbf{3.46}$$

In [6]:
```python
# Paired Test Logic
diffs = [-2, -1, -3]
n_p = 3
d_bar = sum(diffs) / n_p

# Manual Standard Deviation Calculation
variance = sum([(d - d_bar)**2 for d in diffs]) / (n_p - 1)
s_d = math.sqrt(variance)

t_paired = d_bar / (s_d / math.sqrt(n_p))

print(f"--- Diet Program ---")
print(f"Mean Difference: {d_bar}")
print(f"Calculated T   : {t_paired:.3f}")
```

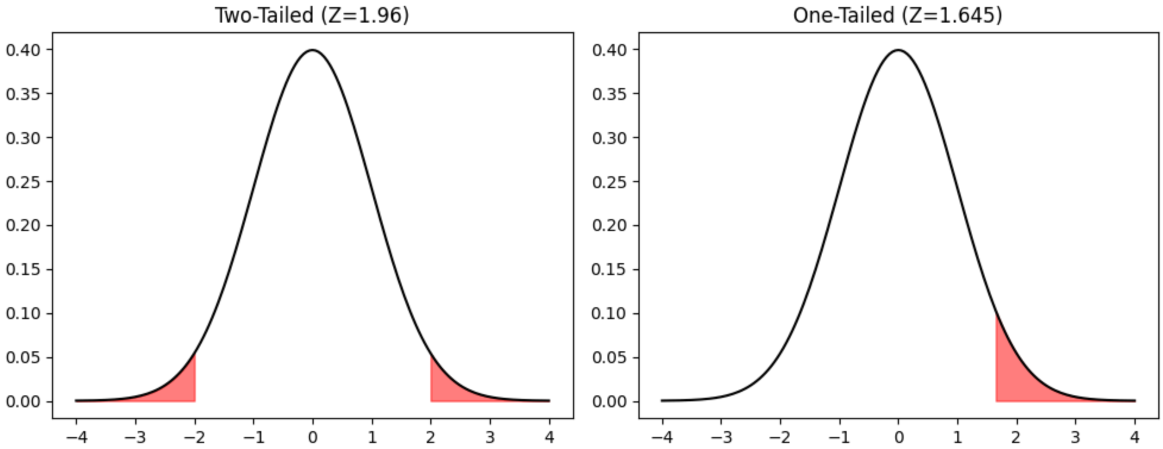
```
--- Diet Program ---
Mean Difference: -2.0
Calculated T   : -3.464
```

7. ONE-TAIL VS TWO-TAIL

- **Two-Tail ($H_1 : \mu \neq 50$):** We check both sides. Harder to prove. Alpha is split ($\alpha/2$). Critical Z = **1.96**.
- **One-Tail ($H_1 : \mu > 50$):** We focus on one direction (e.g., "Is it stronger?"). Alpha is on one side. Critical Z = **1.645**.

> **Note:** One-tail test makes it easier to reject H_0 (find a difference) if you are sure about the direction.

In [7]:
```python
# Visualizing Tails (Simplified)
x_vals, y_vals = generate_normal_data(-4, 4, 0.05)

plt.figure(figsize=(10, 4))

# 1. Two-Tailed
plt.subplot(1, 2, 1)
plt.plot(x_vals, y_vals, 'k')
plt.title("Two-Tailed (Z=1.96)")
# Fill ends
plt.fill_between([x for x in x_vals if x > 1.96], [y for x, y in zip(x_vals, y_v
plt.fill_between([x for x in x_vals if x < -1.96], [y for x, y in zip(x_vals, y_

# 2. One-Tailed
plt.subplot(1, 2, 2)
plt.plot(x_vals, y_vals, 'k')
plt.title("One-Tailed (Z=1.645)")
# Fill only right end
plt.fill_between([x for x in x_vals if x > 1.645], [y for x, y in zip(x_vals, y_

plt.tight_layout()
plt.show()
```



Two-Tailed (Z=1.96)
One-Tailed (Z=1.645)
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−4
−3
−2
−1
0
1
2
3
4