Considering $\rho \equiv qx$, and $[J] \equiv \sqrt{2J+1}$, for M_J, according to Eq.(3a) of Ref [1]

$$\langle n'(l'1/2)j'||M_J(q\vec{x})||n(l1/2)j\rangle = \frac{1}{(4\pi)^{1/2}}(-1)^{J+j+1/2}[l'][l][j'][j][J] \begin{Bmatrix} l' & j' & \frac{1}{2} \\ j & l & J \end{Bmatrix} \begin{pmatrix} l' & J & l \\ 0 & 0 & 0 \end{pmatrix} \times \langle n'l'j'|j_J(\rho)|nlj\rangle \tag{1}$$

For Δ_J, according to Eq.(3c) of Ref. [1] and notice that $L = J$ here,

$$\begin{aligned}\langle n'(l'1/2)j'||\Delta_J(q\vec{x})||n(l1/2)j\rangle &= \frac{1}{(4\pi)^{1/2}}(-1)^{J+j+1/2}[l'][j'][j][J][J] \begin{Bmatrix} l' & j' & \frac{1}{2} \\ j & l & J \end{Bmatrix} \\ &\times \Bigg\{-(l+1)^{1/2}[l+1] \begin{Bmatrix} J & 1 & J \\ l & l' & l+1 \end{Bmatrix} \begin{pmatrix} l' & J & l+1 \\ 0 & 0 & 0 \end{pmatrix} \langle n'l'j'|j_J(\rho)\Big(\frac{d}{d\rho} - \frac{l}{\rho}\Big)|nlj\rangle \\ &+ l^{1/2}[l-1] \begin{Bmatrix} J & 1 & J \\ l & l' & l-1 \end{Bmatrix} \begin{pmatrix} l' & J & l-1 \\ 0 & 0 & 0 \end{pmatrix} \langle n'l'j'|j_J(\rho)\Big(\frac{d}{d\rho} + \frac{l+1}{\rho}\Big)|nlj\rangle \end{aligned} \tag{2}$$

For Σ'_J and Σ''_J, they relate to $M_{JL} \cdot \sigma$ according to Eq. (1e) and (1f) of Ref. [1], therefore we first derive $M_{JL} \cdot \sigma$. It can be obtained from Eq.(3b) of Ref. [1],

$$\langle n'(l'1/2)j'||M_{JL}(q\vec{x})\cdot\sigma||n(l1/2)j\rangle = \frac{1}{(4\pi)^{1/2}}(-1)^{l'}6^{1/2}[l'][l][j'][j][L][J] \begin{Bmatrix} l' & l & L \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j' & j & J \end{Bmatrix} \begin{pmatrix} l' & L & l \\ 0 & 0 & 0 \end{pmatrix} \times \langle n'l'j'|j_L(\rho)|nlj\rangle \tag{3}$$

Then we can get Σ'_J and Σ''_J,

$$\begin{aligned}\langle n'(l'1/2)j'||\Sigma'_J(q\vec{x})||n(l1/2)j\rangle &= -\frac{\sqrt{J}}{[J]}\langle n'(l'1/2)j'||M_{JJ+1}(q\vec{x})\cdot\sigma||n(l1/2)j\rangle \\ &+ \frac{\sqrt{J+1}}{[J]}\langle n'(l'1/2)j'||M_{JJ-1}(q\vec{x})\cdot\sigma||n(l1/2)j\rangle,\end{aligned} \tag{4}$$

$$\begin{aligned}\langle n'(l'1/2)j'||\Sigma''_J(q\vec{x})||n(l1/2)j\rangle &= \frac{\sqrt{J+1}}{[J]}\langle n'(l'1/2)j'||M_{JJ+1}(q\vec{x})\cdot\sigma||n(l1/2)j\rangle \\ &+ \frac{\sqrt{J}}{[J]}\langle n'(l'1/2)j'||M_{JJ-1}(q\vec{x})\cdot\sigma||n(l1/2)j\rangle.\end{aligned} \tag{5}$$

For Φ''_J, the nuclear operator can be given by

$$\Phi''(q\vec{x}) = i\Big(\frac{1}{q}\vec{\Delta}M_J(q\vec{x}) \tag{6}$$

Now the question is how to derive $\langle n'l'j'|j_L(\rho)|nlj\rangle$, $\langle n'l'j'|j_L(\rho)\Big(\frac{d}{d\rho} - \frac{l}{\rho}\Big)|nlj\rangle$, and $\langle n'l'j'|j_L(\rho)\Big(\frac{d}{d\rho} + \frac{l+1}{\rho}\Big)|nlj\rangle$. It is defined by

$$\langle n'l'j'|\theta(\rho)|nlj\rangle = \int x^2 dx R^*_{n'l'j'}(x)\theta(\rho)R_{nlj}(x), \tag{7}$$

for

$$\theta(\rho) = j_L(\rho), j_L(\rho)\Big(\frac{d}{d\rho} - \frac{l}{\rho}\Big), \text{and} j_L(\rho)\Big(\frac{d}{d\rho} + \frac{l+1}{\rho}\Big) \tag{8}$$

To completely evaluate Eq.(7), we employ harmonic oscillators in here and hence drop the lable j now,

$$R_{nl}(x) = \left(\frac{2e^z}{b^3(n-1)!\Gamma(n+l+1/2)z^{l+1}}\right)^{1/2} \times \frac{d^{n-1}}{dz^{n-1}}(z^{n+l-1/2}e^{-z}), \tag{9}$$

where $z \equiv (x/b)^2$ and b is the oscillator parameter. Harmonic oscillator recursion relations give

$$R_{nl}(x) = \sqrt{(n-1)!\Gamma(n+l+1/2)}\sum_{m=0}^{n-1}\frac{(-1)^m}{m!(n-1-m)!}\frac{\sqrt{\Gamma(l+2m+3/2)}}{\Gamma(l+m+3/2)}R_{1l+2m}(x). \tag{10}$$

so that the matrix elements in Eq. (7) can be reduced to linear combination of terms having only $n' = n = 1$. In addition, we have [1]

$$\begin{aligned}\Big(\frac{d}{d\rho} - \frac{l}{\rho}\Big)R_{1l}(x) &= -(8y)^{-1/2}[l+1]R_{1l+1}(x),\\ \Big(\frac{d}{d\rho} + \frac{l+1}{\rho}\Big)R_{1l}(x) &= (8y)^{-1/2}\{2[l]R_{1l-1}(x) - [l+1]R_{1l+1}(x)\},\end{aligned} \tag{11}$$

where $y \equiv (qb/2)^2$. In Ref. [1], we can get

$$\langle 1l'|j_L(\rho)|1l\rangle = \frac{(2y)^{L/2}e^{-y}(L+l'+l+1)!!}{(2L+1)!!\{(2l'+1)!!(2l+1)!!\}^{1/2}} \times {}_1F_1[(L-l'-l; L+3/2; y], \tag{12}$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric function.

Insert Eq. (12), Eq. (11), Eq. (10) into Eq. (7), and keep the relation between gamma function and double factorial in mind,

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \tag{13}$$

we can evaluate $\langle n'l'j'|j_L(\rho)|nlj\rangle$, $\langle n'l'j'|j_L(\rho)\Big(\frac{d}{d\rho} - \frac{l}{\rho}\Big)|nlj\rangle$, and $\langle n'l'j'|j_L(\rho)\Big(\frac{d}{d\rho} + \frac{l+1}{\rho}\Big)|nlj\rangle$

as follow

$$
\begin{aligned}
\langle n'l'j'|j_L(\rho)|nlj\rangle = & \frac{2^L}{(2L+1)!!} y^{L/2} e^{-y} \sqrt{(n'-1)!(n-1)!} \\
& \times \sqrt{\Gamma(n'+l'+1/2)\Gamma(n+l+1/2)} \\
& \times \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} \frac{(-1)^{m+m'}}{m!m'!(n-m-1)!(n'-m'-1)!} \\
& \times \frac{\Gamma[(l+l'+L+2m+2m'+3)/2]}{\Gamma(l+m+3/2)\Gamma(l'+m'+3/2)} \\
& \times {}_1F_1[(L-l'-l-2m'-2m)/2; L+3/2; y]
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
& \langle n'l'j'|j_L(\rho)\Big(\frac{d}{d\rho} - \frac{l}{\rho}\Big)|nlj\rangle = \frac{2^L}{(2L+1)!!} y^{\frac{L-1}{2}} e^{-y} \sqrt{(n'-1)!(n-1)!} \\
& \quad \times \sqrt{\Gamma(n'+l'+1/2)\Gamma(n+l+1/2)} \\
& \quad \times \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} \frac{(-1)^{m+m'}}{m!m'!(n-m-1)!(n'-m'-1)!} \\
& \quad \times \frac{\Gamma[(l+l'+L+2m+2m'+2)/2]}{\Gamma(l+m+3/2)\Gamma(l'+m'+3/2)} \times \\
& \quad \Bigg\{ -\frac{l+l'+L+2m+2m'+2}{2} {}_1F_1[(L-l'-l-2m'-2m-1)/2; L+3/2; y] \\
& \quad + 2m \times {}_1F_1[(L-l'-l-2m'-2m+1)/2; L+3/2; y] \Bigg\}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
& \langle n'l'j'|j_L(\rho)\Big(\frac{d}{d\rho} + \frac{l+1}{\rho}\Big)|nlj\rangle = \frac{2^{L-1}}{(2L+1)!!} y^{\frac{L-1}{2}} e^{-y} \sqrt{(n'-1)!(n-1)!} \\
& \quad \times \sqrt{\Gamma(n'+l'+1/2)\Gamma(n+l+1/2)} \\
& \quad \times \sum_{m=0}^{n-1} \sum_{m'=0}^{n'-1} \frac{(-1)^{m+m'}}{m!m'!(n-m-1)!(n'-m'-1)!} \\
& \quad \times \frac{\Gamma[(l+l'+L+2m+2m'+2)/2]}{\Gamma(l+m+3/2)\Gamma(l'+m'+3/2)} \\
& \quad \Bigg\{ -\frac{l+l'+L+2m+2m'+2}{2} {}_1F_1[(L-l'-l-2m'-2m-1)/2; L+3/2; y] \\
& \quad + (2l+2m+1) \times {}_1F_1[(L-l'-l-2m'-2m+1)/2; L+3/2; y] \Bigg\}
\end{aligned} \tag{16}
$$

[1] T. Donnelly and W. Haxton, Atomic Data and Nuclear Data Tables **23**, 103 (1979), ISSN 0092-640X, URL http://www.sciencedirect.com/science/article/pii/0092640X79900032.