

DMFortFactor: A FORTRAN Program for Experimental WIMP Analysis

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Model-Independent WIMP Scattering Responses and Event Rates

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1 PREAMBLE

This is the manual for the FORTRAN version of the model- independent WIMP scattering response and event rate code, which was originally written in Mathematica and described in [arXiv:1308.6288](https://arxiv.org/abs/1308.6288). We call our program **DMFortFactor**, a Fortran interpretation of **DMFormFactor**.

This program is concerned principally with computing the WIMP-nucleus differential event rate as a function of the nuclear recoil energy E_R :

$$\begin{aligned} \frac{dR_D}{dE_R} &= \frac{dR_D}{d\vec{q}^2}(q) \\ &= N_T \frac{\rho_\chi}{m_\chi} \int_{v_{\min}}^{v_{\text{escape}}} \frac{2m_T}{4\pi v^2} \frac{1}{2j_\chi + 1} \frac{1}{2j_T + 1} \sum_{\text{spins}} |\mathcal{M}(v, q)|^2 \tilde{f}(\vec{v}) v d^3v \end{aligned} \quad (1)$$

This quantity has units of events/GeV and is implicitly multiplied by an effective exposure of 1 Kilogram-Day of target nuclei. This is done by taking $N_t = 1 \text{ kilogram} \cdot \text{day} / m_T$, where m_T is the mass of the target nucleus in GeV. Recoil energies E_R are given in keV.

The cross section is determined using a user-defined WIMP-nucleus interaction within a non-relativistic effective field theory (EFT) framework. The interaction is specified by 16 coupling coefficients defining an interaction:

$$\sum_{x=p,n} \sum_{i=1}^{16} c_i^x \mathcal{O}_i. \quad (2)$$

2 QUICKSTART GUIDE

DMFortFactor can be used interactively from the command line, where the user is guided by prompts for a small number of datafiles and parameters. Naturally, this interactive process can be expedited by piping a pre-written input file into the command line interface (CLI).

We also provide a generic application programming interface (API) for the Python language. This API essentially offers a prescribed and easy-to-use way to run **DMFortFactor** in a programmatic way. Any sufficiently advanced linux user could probably write a script to produce any possible set of inputs to our code. But our API removes the need by making it easy for anyone who can use a Python function to write their own advanced scripts allowing them to perform parameter studies and comparisons of different inputs to the theory.

2.1 SuperQuickstart guide

1. Navigate to the `src/` directory from wherever you have stored `darkmatter/` (e.g. `cd src/`, or `cd ~/Downloads/darkmatter/src/`)
2. Run the command: `make new`
3. Navigate to the directory `runs/xs/` (e.g. `cd ../runs/xs/`)
4. Run the command: `../src/dmfortfactor.x < xs131.input`

If successful, one of the last lines printed to screen should be “Event rate spectra written to `eventrate_spectra.dat`”.

2.2 Compiling with make

There are multiple directories in the project. All of the Fortran code which needs to be compiled is found in the `src/` directory. We provide a **Makefile** to compile the code using a few different options.

The easiest way to get started is simply to navigate to `src/` and run

```
1 make dmfortfactor
```

This will compile **DMFortFactor** using **gfortran**. If you want to use a different compiler, you must edit the **Makefile** to change the **COMP** variable from **gfortran** to your compiler of choice.

Additionally, if you want to utilize multiple CPU cores to speed up the runtime, you can compile with **openMP**:

```
1 make openmp
```

Both of these options will compile the source code and leave the executable, called `dmfortfactor.x` in the `src/` directory. Note that if you change from a serial executable to a parallel executable (or vice versa) you should run:

```
1 make clean
```

in between builds.

2.3 Required files

There are two files required for any calculation:

1. Control file (`.control`)
2. Nuclear density matrix file (`.dres`)

Additionally, if the user enables the option “`usenergyfile`”, then a file containing the input energies or momentum will also be required.

If you use the `EventrateSpectra` Python wrapper, then the control file will automatically be generated.

2.3.1 Control file (`.control`)

Each EFT parameter is written on its own line in `[mycontrolfile].control`, with four values: the keyword “`coefnonrel`”, the operator number (integer 1..16), the coupling type (“`p`”=proton, “`n`”=neutron, “`s`”=scalar, “`v`”=vector), and the coefficient value. For example,

```
coefnonrel    1    s    3.1
```

would set $c_1^0 = 3.1$. We take the isospin convention:

$$\begin{aligned} c^0 &= c^p + c^n \\ c^1 &= c^p - c^n \end{aligned} \tag{3}$$

Thus, the previous example is equivalent to:

```
coefnonrel    1    p    1.55
coefnonrel    1    n    1.55
```

The control file also serves a more general but optional function: to set any parameter in the program to a custom value. Simply add an entry to the control file with two values: the first should be the keyword for the parameter and the second should be the value to set that parameter to. For example, to set the velocity of the earth in the galactic frame to 240 km/s, you should add the line:

```
vearth  240.0
```

A complete list of keywords is given in section [6](#).

2.3.2 *Nuclear density matrix file (.dres)*

2.4 Event rate spectra (events per GeV) using the CLI

The program will prompt the user for the minimum necessary inputs to run a calculation with default parameter values, including the name of a control file which contains the EFT coefficients, and optionally, additional customizations to the calculation parameters.

After selecting the option to compute an event rate spectra, there are six further lines of input. These will be explained by an example:

```
Enter the target proton number
54
Enter the target neutron number
77
Enter name of control file (.control):
xe131
...
```

```

Enter name of one-body density file (.dres)
xe131gcn
...
What is the range of recoil energies in kev?
Enter starting energy, stoping energy, step size:
0.0001 250. 1.0

```

The first two entries are self-evident: we specify the number of protons and neutrons in the target nucleus. In this case, 54 and 74, respectively, for ^{131}Xe .

Third is the name of the control file containing the EFT coefficients and other, optional, settings. The '.control' file extension should be omitted. This contents of this file will be explained in more detail later.

Fourth is the file containing the nuclear wave functions in the form of one-body density matrices. Only the ground-state need be included. The '.dres' file extension is omitted.

Fifth and finally are three numbers specifying the range of recoil energies E_R that the differential scattering rate should be computed for.

The event rate spectra will be written to a file, and as a side effect of the calculation, the total event rate for the energy range requested will be estimated by numerical integration. Note that the accuracy of this result will depend on the choice of the step size.

2.5 Event rate spectra from the Python wrapper

We provide a Python wrapper for the code and a number of example scripts demonstrating its use. The wrapper comes in the form of a Python function `EventrateSpectra` which can be imported from `dmfortfactor.py` in the Python directory. This function has three required arguments:

1. Z the number of protons in the target nucleus
2. N the number of neutrons in the target nucleus
3. The '.dres' filename for the one-body density matrix file describing the nuclear structure.

If no other arguments are provided, default values will be used for all of the remaining necessary parameters, including zero interaction strength. To calculate an event rate with a nonzero interaction, the user should also provide one or more of the optional EFT coupling coefficient arrays: `cpvec`, `cnvec`, `csvec`, `cvvec`. These store the couplings to protons, neutrons, isoscalar, and isovector, respectively. Finally, the user can also pass a dictionary of valid control keywords and values to the function in order to set any of the control words defined in the manual.

To compute the eventrate spectra for ^{131}Xe with a WIMP mass of 50 GeV and a $c_3^v = 0.0048$ coupling, one might call:

```

1 import dmfortfactor as dm
2 control_dict = {"wimpmass" : 50.0}
3 cvvec = np.zeros(15)
4 cvvec[2] = 0.0048
5 Erkev, ER = dm.EventrateSpectra(
6     Z = 54,
7     N = 77,
8     dres = "xe131gcn",
9     controlwords = control_dict,
10    cvvec = cvvec)

```

This will return the differential event rate spectra for recoil energies from 1 keV to 1 MeV in 1 keV steps. The file 'xe131gcn.dres' must exist in the current working directory and contain a valid one-body density matrix for ^{131}Xe .

3 PYTHON INTERFACE

We provide two generic API's for interacting with the Fortran program. They are: `runTemplates` and `EventrateSpectra`. These can be imported into your own Python script by having the file `dmfortfactory.py` in your working directory or by adding it to your path:

```

1 import sys
2 sys.path.append('/path/to/dmfortfactory.py')
3 import dmfortfactor as dm

```


`EventrateSpectra` is essentially a wrapper for the event-rate calculation function of `dmfortfactor`. It makes use of `runTemplates` but shields the user from having to handle the input and control files needed by `dmfortfactor`.

`runTemplates` is a fairly generic function which runs an arbitrary linux program which takes an input file by a linux pipe, and makes use of a secondary 'control' file. `runTemplates` takes in templates for the input and control files, modifies them using `sed` to replace string keys with values from a Python dictionary, runs the executable, then collects outputs written to an output file, returning the data as Numpy arrays.

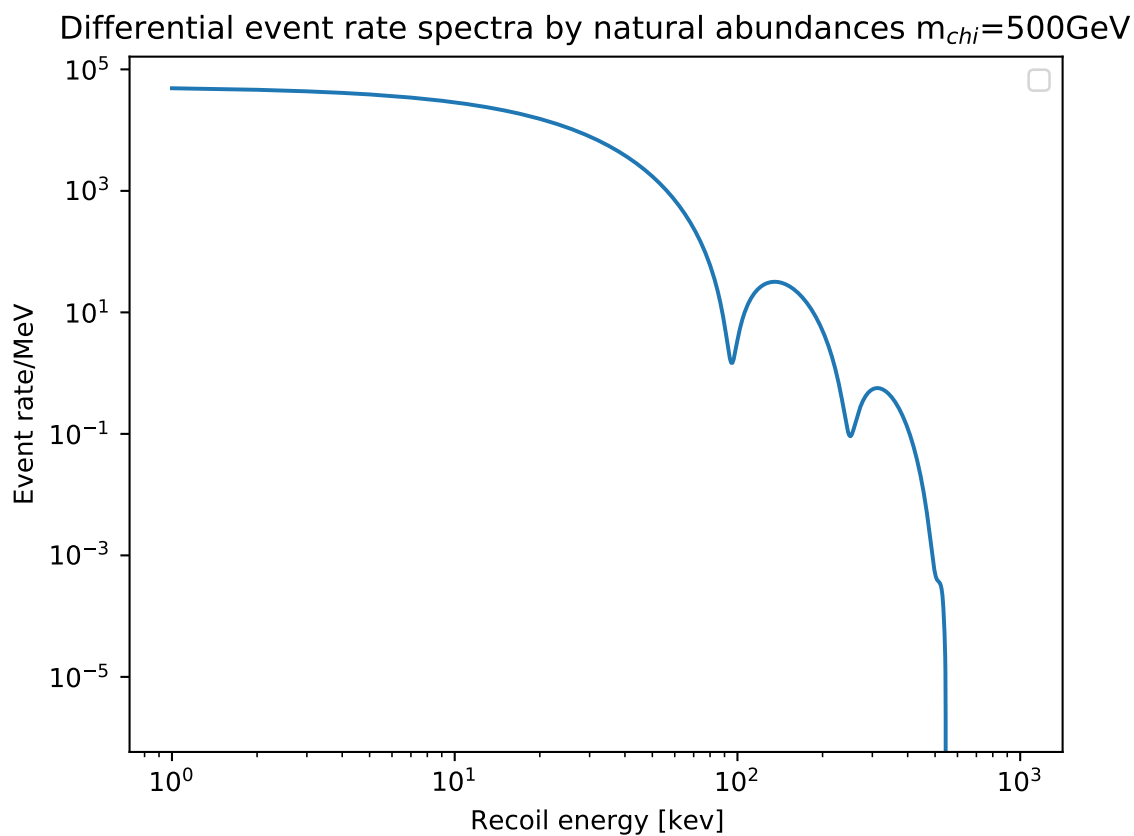
3.1 Example: generating a weighted sum of event rate spectra

```

1 import sys
2 sys.path.append("/Users/oliver/projects/darkmatter/python")
3 import numpy as np
4 import matplotlib.pyplot as plt
5 import dmfortfactor as dm
6
7 plt.figure()
8 Z = 54
9 isotopes = [128, 129, 130, 131, 132, 134, 136]
10 weights = [.01910, .26401, .04071, .21232, .26909, .10436, .08857]
11 weightedsum = 0.0
12
13 for i,isotope in enumerate(isotopes):
14     N = isotope - Z
15     controls = {
16         "wimpmass" :500.0,
17         "vescape" : 550.0}
18     cp = np.zeros(15)
19     cp[0] = 1
20     RecoilE, EventRate = dm.EventrateSpectra(
21         Z,
22         N,
23         dres = "xe%igcn"%isotope,
24         controlwords = controls,
25         cpvec = cp )
26
27     weightedsum += EventRate * weights[i]
28
29     label = "xe"+str(isotope)
30     plt.plot(RecoilE, EventRate, label="$^{%s}$Xe"%isotope)

```

We now have the weighted sum of event rates stored in the variable `weightedsum`. Using `matplotlib`, we could produce the following plot:



3.2 Example: comparing event-rate spectra for calculations with different WIMP masses

4 OTHER COMPUTE OPTIONS

There are a handful of other options available from the main menu of the code. Presently, the primary use-cases for these options are validation and debugging. the compute-options are:

- [2] Differential cross section at a fixed recoil energy over a range of speeds
- [3] Transition/scattering probability at a fixed recoil energy over a range of speeds
- [5] Total integrated events (without producing spectra file)
- [6] Nuclear response functions at a given value of $y = (qb/2)^2$

The number in square brackets [x] will be referred to as the compute-option. Once a compute-option is chosen, subsequent CLI inputs are the same up until the density matrix file (.dres) has been read-in. Then, the inputs depend on the compute-option chosen.

- [2] Differential cross section per recoil energy. Four additional inputs:
 - E-recoil (keV)
 - v-start (km/s)
 - v-stop (km/s)
 - v-step (km/s)
- [3] Scattering probability. Same as [2].
- [5] Total scattering events per detector (does not produce spectra data). This option uses adaptive quadrature to perform the integral of the event rate spectra with the fewest number of evaluations to reach the desired relative error. This will be much faster than the result from options [1]. Three additional inputs:
 - E-start (keV)
 - E-stop (keV)

- Desired relative error (decimal value)
- [6] Nuclear response function test. This compute-option allows the user to evaluate the nuclear response functions $W_i^{\chi, \chi'}(\mathbf{y})$ for a provided value of \mathbf{y} . All combinations of χ and χ' will be printed for both isospin and proton-neutron couplings. Two additional inputs are required:
- Function number (1 - 8)
 - Value of $\mathbf{y} = (qb/2)^2$ (dimensionless)

5 EQUATIONS FOUND IN THE CODE

We present the equations necessary to reproduce the code. For a more complete description of the theory, see [Phys. Rev. C 89.065501](#).

5.1 Differential event rate

$$\frac{dR_D}{dE_R} = \frac{dR_D}{d\vec{q}^2}(q) = N_T n_\chi \int_{v_{\min}}^{\infty} \frac{d\sigma(v, q)}{d\vec{q}^2} \tilde{f}(\vec{v}) v d^3v \quad (4)$$

where q is the WIMP-nucleon momentum transfer, N_T is the number of target nuclei, $n_\chi = \rho_\chi/m_\chi$ is the local dark matter number density, σ is the WIMP-nucleon cross section, and \tilde{f} is dark matter velocity distribution in the lab-frame. $\tilde{f}(\vec{v})$ is obtained by boosting the Galactic-frame distribution $f(\vec{v})$,

$$\tilde{f}(\vec{v}) = f(\vec{v} + \vec{v}_{\text{earth}}), \quad (5)$$

where \vec{v}_{earth} is the velocity of the earth in the galactic rest frame. The simplest model is a three-dimensional Maxwell distribution:

$$f(\vec{v}) \propto e^{-\vec{v}^2/v_0^2}, \quad (6)$$

where v_0 is some scaling factor (typically taken to be around 220 km/s).

In order to evaluate the integral in (4), we make the conversion to spherical coordinates, and take special care to deal with the velocity boost in (5). Assuming a $1/v^2$ velocity dependence of the cross section term (see section 5.2), we need to evaluate an integral of the form

$$I = \int_{v_{\min}}^{v_{\max}} d^3v \frac{f(\vec{v} + \vec{v}_{\text{earth}})}{v} = \int_{v_{\min}}^{v_{\max}} d^3v \frac{1}{v} e^{-(\vec{v} + \vec{v}_{\text{earth}})^2/v_0^2} \quad (7)$$

Noting that $(\vec{v} + \vec{v}_{\text{earth}})^2 = \vec{v}^2 + \vec{v}_{\text{earth}}^2 + 2\vec{v}\vec{v}_{\text{earth}} \cos(\theta)$, with $|\vec{v}| \equiv v$ and θ defining the angle between the two vectors, it's convenient to make the substitution $d^3v = d\phi d(\cos \theta) v^2 dv$:

$$\begin{aligned} I &= \int_0^{2\pi} d\phi \int_{v_{\min}}^{v_{\max}} dv \int_{-1}^1 d(\cos \theta) e^{-2v v_{\text{earth}} \cos \theta / v_0^2} v^2 \frac{1}{v} e^{-(\vec{v}^2 + \vec{v}_{\text{earth}}^2)/v_0^2} \\ &= 2\pi \int_{v_{\min}}^{v_{\max}} dv v e^{-(\vec{v}^2 + \vec{v}_{\text{earth}}^2)/v_0^2} \left(-\frac{v_0^2}{2v v_{\text{earth}}} e^{-2v v_{\text{earth}} \cos \theta / v_0^2} \right)_{-1}^1 \\ &= \frac{\pi v_0^2}{v_{\text{earth}}} \int_{v_{\min}}^{v_{\max}} dv e^{-(\vec{v}^2 + \vec{v}_{\text{earth}}^2)/v_0^2} \left(-e^{-2v v_{\text{earth}}/v_0^2} + e^{+2v v_{\text{earth}}/v_0^2} \right) \\ &= \frac{\pi v_0^2}{v_{\text{earth}}} \int_{v_{\min}}^{v_{\max}} dv \left(-e^{(v+v_{\text{earth}})^2/v_0^2} + e^{(v-v_{\text{earth}})^2/v_0^2} \right) \\ &= \frac{\pi v_0^2}{v_{\text{earth}}} \int_{v_{\min}}^{v_{\max}} dv (g(v - v_{\text{earth}}) - g(v + v_{\text{earth}})) \end{aligned} \quad (8)$$

where in the last equality, we have defined a one-dimensional Gaussian form

$$g(v) \propto e^{-v^2/v_0^2}. \quad (9)$$

The final expression for I can be trivially generalized to other spherically symmetric velocity-dependent forms of the differential cross section. What's important is the reduction of the velocity-boosted d^3v integral to a radial integral which can be carried out with one-dimensional quadrature:

$$\begin{aligned} &\int_{v_{\min}}^{v_{\max}} d^3v \sigma(v) e^{-(\vec{v} + \vec{v}_{\text{earth}})^2/v_0^2} \\ &= \frac{\pi v_0^2}{v_{\text{earth}}} \int_{v_{\min}}^{v_{\max}} dv \sigma(v) v^2 (g(v - v_{\text{earth}}) - g(v + v_{\text{earth}})). \end{aligned} \quad (10)$$

The FORTRAN code uses equation (10) to evaluate the event rate integral in equation (4) with quadrature. Analytic solutions of (10) exist in the form of error functions; we use the above form since it makes easy to later modify the

velocity distribution (as long as it remains spherically symmetric). For example, adding a velocity cut-off is as easy as changing the limit on the quadrature, with no need to write a whole new subroutine for the analytic forms found in the Mathematica script.

5.2 Differential cross section

$$\frac{d\sigma(\nu, E_R)}{dE_R} = 2m_T \frac{d\sigma(\nu)(\nu, \vec{q}^2)}{d\vec{q}^2} = 2m_T \frac{1}{4\pi\nu^2} T(\nu, q), \quad (11)$$

Where ν is the velocity of the dark matter particles in the lab-frame, q is the momentum transfer of the scattering event, m_T is the mass of the target nucleus, and $T(\nu, q)$ is the transition or scattering probability. We can see here that the differential cross section has an explicit $1/\nu^2$ dependence, independent of any velocity dependence of $T(\nu, q)$.

5.3 Transition probability / Scattering probability

The scattering probability is

$$T(\nu, q) = \frac{1}{2j_\chi + 1} \frac{1}{2j_T + 1} \sum_{\text{spins}} |\mathcal{M}(\nu, q)|^2 \quad (12)$$

where j_χ is the spin of the WIMP, j_T is the spin angular momentum of the target nucleus, and \mathcal{M} Galilean invariant amplitude, which is defined by

$$T(\nu, q) = \frac{4\pi}{2j_T + 1} \frac{1}{(4m_\chi)^2} \sum_{x=p,n} \sum_{x'=p,n} \sum_{i=1}^8 R_i^{xx'}(\nu^2, q^2) W_i^{xx'}(q) \quad (13)$$

where m_χ is the mass of the dark matter particle and x is an index used to sum over isospin couplings. The coefficients $R_i^{xx'}$ are dark matter particle response functions, to be define in another section. The operators $W_i^{xx'}(q)$ are nuclear response functions, which are sums over matrix elements of nuclear operators constructed from Bessel spherical harmonics and vector spherical harmonics.

5.4 Dark matter response functions

There are 8 dark matter response functions which group 15 operator coefficients c_i^x according the pair of nuclear response functions which they multiply.

$$R_M^{xx'}(\nu, q) = \frac{1}{4} \text{cl}(j_\chi) \{ [\nu^2 - (q/2\mu_t)^2] (c_5^x c_5^{x'} q^2 + c_8^x c_8^{x'}) + c_{11}^x c_{11}^{x'} q^2 \} \quad (14)$$

$$+ \{ c_1^x + c_2^x [\nu^2 - (q/2\mu_t)^2] \} \{ c_1^{x'} + c_2^{x'} [\nu^2 - (q/2\mu_t)^2] \}$$

$$R_{\Sigma''}^{xx'}(\nu, q) = \frac{1}{16} \text{cl}(j_\chi) \{ c_6^x c_6^{x'} q^4$$

$$+ (c_{13}^x c_{13}^{x'} q^2 + c_{12}^x c_{12}^{x'}) [\nu^2 - (q/2\mu_T)^2] + 2c_4^x c_6^{x'} q^2 + c_4^x c_4^{x'} \}$$

$$+ \frac{1}{4} c_{10}^x c_{10}^{x'} q^2 \quad (15)$$

$$R_{\Sigma'}^{xx'}(\nu, q) = \frac{1}{32} \text{cl}(j_\chi) \{ 2c_9^x c_9^{x'} q^2 + (c_{15}^x c_{15}^{x'} q^4 + c_{14}^x c_{14}^{x'} q^2 - 2c_{12}^x c_{15}^{x'} q^2 + c_{12}^x c_{12}^{x'}) [\nu^2$$

$$- (q/2\mu_T)^2] + 2c_4^x c_4^{x'} \} + \frac{1}{8} (c_3^x c_3^{x'} q^2 + c_7^x c_7^{x'}) [\nu^2 - (q/2\mu_T)^2] \quad (16)$$

$$R_{\Phi''}^{xx'}(\nu, q) = \frac{q^2}{(4m_N)^2} \text{cl}(j_\chi) (c_{12}^x - c_{15}^x q^2) (c_{12}^{x'} - c_{15}^{x'} q^2) + \frac{q^2}{(4m_N)^2} q^2 c_3^x c_3^{x'} \quad (17)$$

$$R_{\Phi'}^{xx'}(\nu, q) = \frac{q^2}{(4m_N)^2} \text{cl}(j_\chi) (c_{12}^x c_{12}^{x'} q^2 + c_{12}^x c_{12}^{x'}) \quad (18)$$

$$R_{\Delta}^{xx'}(\nu, q) = \frac{q^2}{(2m_N)^2} \text{cl}(j_\chi) (c_5^x c_5^{x'} q^2 + c_8^x c_8^{x'}) + 2 \frac{q^2}{m_N^2} c_2^x c_2^{x'} [\nu^2 - (q/2\mu_T)^2] \quad (19)$$

$$R_{\Delta\Sigma}^{xx'}(\nu, q) = \frac{q^2}{(2m_N)^2} \text{cl}(j_X)(c_4^x c_5^{x'} - c_8^x c_9^{x'}) - \frac{q^2}{m_N} c_2^x c_3^{x'} [\nu^2 - (q/2\mu_T)^2] \quad (20)$$

$$R_{\Phi''M}^{xx'}(\nu, q) = \frac{q^2}{4m_N} \text{cl}(j_X) c_{11}^x (c_{12}^{x'} - c_{15}^{x'} q^2) + \frac{q^2}{m_N} c_3^{x'} \{c_1^x + c_2^x [\nu^2 - (q/2\mu_T)^2]\} \quad (21)$$

As a shorthand we have introduced the notation

$$\text{cl}(j) = 4j(j+1)/3. \quad (22)$$

5.5 Nuclear response functions

There are eight nuclear response functions $W_i^{xx'}(\mathbf{y})$ considered here. The unitless variable \mathbf{y} is defined

$$\mathbf{y} = \left(\frac{qb}{2} \right)^2, \quad (23)$$

in terms of the harmonic oscillator size parameter b , which has a default value of

$$b^2 = 41.467 / (45A^{-1/3} - 25A^{-2/3}) \text{ fm}^2. \quad (24)$$

$$W_M^{xx'}(y) = \sum_{\text{even } J} \langle j_T | M_{Jx}(y) | j_T \rangle \langle j_T | M_{Jx'}(y) | j_T \rangle \quad (25)$$

$$W_{\Sigma''}^{xx'}(y) = \sum_{\text{odd } J} \langle j_T | \Sigma_{Jx}''(y) | j_T \rangle \langle j_T | \Sigma_{Jx'}''(y) | j_T \rangle \quad (26)$$

$$W_{\Sigma'}^{xx'}(y) = \sum_{\text{odd } J} \langle j_T | \Sigma_{Jx}'(y) | j_T \rangle \langle j_T | \Sigma_{Jx'}'(y) | j_T \rangle \quad (27)$$

$$W_{\Phi''}^{xx'}(y) = \sum_{\text{even } J} \langle j_T | \Phi_{Jx}''(y) | j_T \rangle \langle j_T | \Phi_{Jx'}''(y) | j_T \rangle \quad (28)$$

$$W_{\tilde{\Phi}'}^{xx'}(y) = \sum_{\text{even } J} \langle j_T | \tilde{\Phi}_{Jx}'(y) | j_T \rangle \langle j_T | \tilde{\Phi}_{Jx'}'(y) | j_T \rangle \quad (29)$$

$$W_{\Delta}^{xx'}(y) = \sum_{\text{odd } J} \langle j_T | \Delta_{Jx}(y) | j_T \rangle \langle j_T | \Delta_{Jx'}(y) | j_T \rangle \quad (30)$$

$$W_{\Delta\Sigma'}^{xx'}(y) = \sum_{\text{odd } J} \langle j_T | \Delta_{Jx}(y) | j_T \rangle \langle j_T | \Sigma_{Jx'}'(y) | j_T \rangle \quad (31)$$

$$W_{\Phi''M}^{xx'}(y) = \sum_{\text{even } J} \langle j_T | \Phi_{Jx}''(y) | j_T \rangle \langle j_T | M_{Jx'}(y) | j_T \rangle \quad (32)$$

5.6 Nuclear operators and their matrix elements

There are six nuclear operators constructed from Bessel spherical harmonics and vector spherical harmonics, and are evaluated here on the ground state of the target nucleus.

There are six nuclear operators $\mathcal{W}^{(f)}$, $f = 1, \dots, 6$, describing the electro-weak coupling of the WIMPs to the nucleon degrees of freedom. The six single-particle operators are given the symbols:

$$\mathcal{W}_J^{(f=1,\dots,6)} = M_J, \Delta_J, \Sigma_J', \Sigma_J'', \tilde{\Phi}_J', \Phi_J'', \quad (33)$$

and are constructed from Bessel spherical and vector harmonics [?]:

$$M_{JM}(q\vec{x}) \equiv j_J(qx)Y_{JM}(\Omega_x) \quad (34)$$

$$\vec{M}_{JML}(q\vec{x}) \equiv j_L(qx)\vec{Y}_{JLM}(\Omega_x), \quad (35)$$

where

$$Y_{JLM}(\Omega_x) = \sum_{m\lambda} \langle Lm1\lambda | (L1)JM_J \rangle Y_{Lm}(\Omega_x) \vec{e}_\lambda. \quad (36)$$

The six multipole operators are defined as:

$$M_{JM} \quad (37)$$

$$\Delta_{JM} \equiv \vec{M}_{JJM} \cdot \frac{1}{q} \vec{\nabla} \quad (38)$$

$$\Sigma'_{JM} \equiv -i \left\{ \frac{1}{q} \vec{\nabla} \times \vec{M}_{JJM} \right\} \cdot \vec{\sigma} \quad (39)$$

$$\Sigma''_{JM} \equiv \left\{ \frac{1}{q} \vec{\nabla} M_{JM} \right\} \cdot \vec{\sigma} \quad (40)$$

$$\tilde{\Phi}'_{JM} \equiv \left(\frac{1}{q} \vec{\nabla} \times \vec{M}_{JJM} \right) \cdot \left(\vec{\sigma} \times \frac{1}{q} \vec{\nabla} \right) + \frac{1}{2} \vec{M}_{JJM} \cdot \vec{\sigma} \quad (41)$$

$$\Phi''_{JM} \equiv i \left(\frac{1}{q} \vec{\nabla} M_{JM} \right) \cdot \left(\vec{\sigma} \times \frac{1}{q} \vec{\nabla} \right) \quad (42)$$

The matrix elements of these operators can be calculated for standard wave functions from second-quantized shell model calculations:

$$\begin{aligned} \langle \Psi_f | \mathcal{W}_J^{(f)} | \Psi_i \rangle &= \text{Tr} \left(\mathcal{W}_J^{(f)} \rho_J^{f,i} \right) \\ &= \sum_{a,b} \langle a | \mathcal{W}_J^{(f)} || b \rangle \rho_J^{fi}(ab), \end{aligned} \quad (43)$$

where single-particle orbital labels a imply shell model quantum number n_a, l_a, j_a , and the double-bar $||$ indicates reduced matrix elements [?]. We assume a harmonic oscillator single-particle basis, with the important convention that the radial nodal quantum number n_a starts at 0, that is, we label the orbitals as *os, op, 1sod*, etc., and *not* starting with *1s, 1p*, etc. Then the one-body matrix elements for operators $\langle a | \mathcal{W}_J^{(f)} || b \rangle$, built from spherical Bessel functions and vector spherical harmonics, have closed-form expressions in terms of confluent hypergeometric functions [?].

The matrix element of the Bessel function on harmonic oscillator wave functions is

$$\begin{aligned} \langle 1l' | j_L(y) | 1l \rangle = & \frac{(2y)^{L/2} e^{-y} (L + l' + l + 1)!!}{(2L + 1)!! [(2l' + 1)!! (2l + 1)!!]^{1/2}} \\ & \times {}_1F_1((L - l' - l)/2, L + \frac{3}{2}, y), \end{aligned} \quad (44)$$

where ${}_1F_1$ is the confluent hypergeometric function:

$${}_1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!}, \quad (45)$$

which makes use of the rising factorial function:

$$m^{(n)} = \frac{(m + n - 1)!}{(m - 1)!}. \quad (46)$$

6 CONTROL FILE KEYWORDS

Keyword	Symbol	Meaning	Units	Default
dmdens	ρ_χ	Local dark matter density.	GeV/cm ³	0.3
dmspin	j_χ	Intrinsic spin of WIMP particles.	\hbar	$\frac{1}{2}$
fillnuclearcore		Logical flag (enter 0 for False, 1 for True) to fill the inert-core single-particle orbitals in the nuclear level densities. Phenomenological shell model calculations typically provide only the density matrices for the active valence-space orbitals, leaving it to the user to infer the core-orbital densities. This option automatically assigns these empty matrix elements assuming a totally filled core.		1 (true)

gaussorder		Order of the Gauss-Legendre quadrature to use when using Type 2 quadrature. (See quad-type.) An n-th order routine will perform n function evaluations. Naturally, a higher order will result in higher precision, but longer compute time.		12
hofrequency	$\hbar\omega$	Set the harmonic oscillator length by specifying the harmonic oscillator frequency. ($b = 6.43/\text{sqrt}(\hbar\omega)$). If using an <i>ab initio</i> interaction, $\hbar\omega$ should be set to match the value used in the interaction.	MeV	See hoparameter.
hoparameter	b	Harmonic oscillator length. Determines the scale of the nuclear wavefunction interaction.	fm	See eqn. (24).
maxwellv0	v_0	Maxwell-Boltzman velocity distribution scaling factor.	km/s	220.0
mnucleon	m_N	Mass of a nucleon. It's assumed that $m_p \approx m_n$.	GeV	0.938272
ntscale	N_t	Effective number of target nuclei scaling factor. The differential event rate is multiplied by this constant in units of kilogram-days. For example, if the detector had a total effective exposure of 2500 kg days, one might enter 2500 for this value.	kg days	1.0
quadrelerr		Desired relative error for the adaptive numerical quadrature routine (quadtype 1).		10^{-6}

quadtype		Option for type of numerical quadrature. (Type 1 = adaptive 8th order Gauss-Legendre quadrature. Type 2 = static n-th order Gauss-Legendre quadrature.)		1 (type 1)
sj2tablemax		Maximum value of $2 \times J$ used when caching Wigner 3-J and 6-J functions into memory.		12
sj2tablemin		Minimum value of $2 \times J$ used when caching Wigner 3-J and 6-J functions into memory.		-2
useenergyfile		Logical flag (enter 0 for False, 1 for True) to read energy grid used for calculation from a user-provided file instead of specifying a range.		0 (false)
usemomentum		Logical flag (enter 0 for False, 1 for True) to use momentum transfer instead of recoil energy as the independent variable.		0 (false)
vearth	v_{earth}	Speed of the earth in the galactic frame.	km/s	232.0
vescape	v_{escape}	Galactic escape velocity. Particles moving faster than this speed will escape the galaxy, thus setting an upper limit on the WIMP velocity distribution.	km/s	$12 \times v_{\text{scale}}$
weakmscale	m_v	Weak interaction mass scale. User defined EFT coefficients are divided by m_v^2 .	GeV	246.2
wimpmass	m_χ	WIMP particle mass.	GeV	50.0