MATH 5362 Homework II

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1 Problem I

Proposition 1.1. *Let* $K = \mathbb{Q}(\theta)$ *where* $\theta^3 + 11\theta - 4 = 0$*. Then* $\frac{\theta^2-\theta}{2} \in \mathscr{O}_K$

Proof.

$$\begin{aligned}
\theta^3 &= 4 - 11\theta \\
x &:= \frac{1}{2}(\theta^2 - \theta) \\
x^2 &= \frac{1}{4}(\theta^4 - 2\theta^3 + \theta^2) \\
&= \frac{1}{4}(\theta(4 - 11\theta) - 2(4 - 11\theta) + \theta^2) \\
&= \frac{1}{4}(4\theta - 11\theta^2 - 8 + 22\theta + \theta^2) \\
&= \frac{1}{4}(-10\theta^2 + 26\theta - 8) \\
&= \frac{1}{2}(-5\theta^2 + 13\theta - 4) \\
x^2 + 5x &= \frac{1}{2}(-5\theta^2 + 13\theta - 4 + 5\theta^2 - 5\theta) \\
&= \frac{1}{2}(8\theta - 4) = 4\theta - 2 \\
x^3 &= \frac{1}{4}(-5\theta^4 + 13\theta^3 - 4\theta^2 + 5\theta^3 - 13\theta^2 + 4\theta) \\
&= \frac{1}{4}(-5\theta^4 + 18\theta^3 - 17\theta^2 + 4\theta) \\
&= \frac{1}{4}(-5\theta(4 - 11\theta) + 18(4 - 11\theta) - 17\theta^2 + 4\theta) \\
&= \frac{1}{4}(-20\theta + 55\theta^2 + 72 - 198\theta - 17\theta^2 + 4\theta) \\
&= \frac{1}{4}(38\theta^2 - 214\theta + 72) \\
&= \frac{1}{2}(19\theta^2 - 107\theta + 36)
\end{aligned}$$

$$
\begin{aligned}
x^3 - 19x &= \frac{1}{2}(19\theta^2 - 107\theta + 36 - (19\theta^2 - 19\theta)) \\
&= \frac{1}{2}(-88\theta + 36) \\
&= (-44\theta + 18) \\
x^3 - 19x + 11(x^2 + 5x) &= (-44\theta + 18) + 11(4\theta - 2) \\
x^3 + 11x^2 + 36x &= -44\theta + 18 + 44\theta - 22 \\
&= -4 \\
x^3 + 11x^2 + 36x + 4 &= 0
\end{aligned}
$$

□

2 Problem II

Proposition 2.1. *Let $K = \mathbb{Q}(\theta)$ where $\theta^3 - 4\theta + 2 = 0$. Let $\alpha = \theta + \theta^2$. Then $D(\alpha) = -148$*

Proof.

$$
\begin{aligned}
\alpha^2 &= \theta^4 + 2\theta^3 + \theta^2 \\
&= \theta(4\theta - 2) + 2(4\theta - 2) + \theta^2 \\
&= 4\theta^2 - 2\theta + 8\theta - 4 + \theta^2 \\
&= 5\theta^2 + 6\theta - 4 \\
\alpha^2 - 5\alpha &= \theta - 4 \\
\alpha^3 &= 5\theta^4 + 6\theta^3 - 4\theta^2 + 5\theta^3 + 6\theta^2 - 4\theta \\
&= 5\theta^4 + 11\theta^3 + 2\theta^2 - 4\theta \\
&= 5\theta(4\theta - 2) + 11(4\theta - 2) + 2\theta^2 - 4\theta \\
&= 20\theta^2 - 10\theta + 44\theta - 22 + 2\theta^2 - 4\theta \\
&= 22\theta^2 + 30\theta - 22 \\
\alpha^3 - 22\alpha &= 22\theta^2 + 30\theta - 22 - (22\alpha^2 + 22\alpha) \\
&= 8\alpha - 22 \\
\alpha^3 - 22\alpha - 8(\alpha^2 - 5\alpha) &= 8\alpha - 22 - 8(\alpha - 4) \\
&= 10 \\
\alpha^3 - 8\alpha^2 + 18\alpha - 10 &= 0
\end{aligned}
$$

Now, clearly $p(x) := x^3 - 8x^2 + 18x - 10$ is monic. It is irreducible via the rational roots theorem. Thus, it is the minimal polynomial of α. Furthermore, let

$$p'(x) = 3x^2 - 16x + 18$$

be the formal derivative of p. Then:

$$\begin{aligned}
p'(\alpha) &= 3\alpha^2 - 16\alpha + 18 \\
&= 3(\theta + \theta^2)^2 - 16(\theta + \theta^2) + 18 \\
&= 3(\theta^4 + 2\theta^3 + \theta^2) - 16(\theta + \theta^2) + 18 \\
&= 3\theta^4 + 6\theta^3 + 3\theta^2 - 16\theta - 16\theta^2 + 18 \\
&= 3\theta^4 + 6\theta^3 - 13\theta^2 - 16\theta + 18 \\
&= 3\theta(4\theta - 2) + 6(4\theta - 2) - 13\theta^2 - 16\theta + 18 \\
&= 12\theta^2 - 6\theta + 24\theta - 12 - 13\theta^2 - 16\theta + 18 \\
&= -\theta^2 + 2\theta + 6 \\
\theta \cdot p'(\alpha) &= -(\theta^3) + 2\theta^2 + 6\theta \\
&= -(4\theta - 2) + 2\theta^2 + 6\theta \\
&= 2\theta^2 + 2\theta + 2 \\
\theta^2 \cdot p'(\alpha) &= 2\theta^3 + 2\theta^2 + 2\theta \\
&= 2(4\theta - 2) + 2\theta^2 + 2\theta \\
&= 2\theta^2 + 10\theta - 4
\end{aligned}$$

Now, $\{1, \theta, \theta^2\}$ is a basis for $\mathbb{Q}(\theta)$. This means that the norm $N_{\mathbb{Q}(\theta)}(p'(\alpha))$ is in fact the determinant of a matrix as follows:

$$N_{\mathbb{Q}(\theta)}(p'(\alpha)) = \det \begin{vmatrix} -1 & 2 & 6 \\ 2 & 2 & 2 \\ 2 & 10 & -4 \end{vmatrix} = 148$$

To conclude, we use the following formula,

$$\begin{aligned}
D(\alpha) &= (-1)^{\binom{n}{2}} N_K(p'(\alpha)) \\
&= (-1)^3(148) \\
&= -148
\end{aligned}$$

□

3 Problem III

Proposition 3.1. *For p an odd prime, the discriminant of the cyclotomic field $\mathbb{Q}(\zeta_p)$ equals $(-1)^{\frac{p-1}{2}} p^{p-2}$*

Proof. The discriminant of a general cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n is taken to be primitive, is given by:

$$D(\mathbb{Q}(\zeta_n)) = (-1)^{\frac{\phi(n)}{2}} \frac{n^{\phi(n)}}{\prod_{p|n} p^{\frac{\phi(n)}{p-1}}}$$

Since p is an odd prime, then $\phi(p) = p - 1$ and $\prod_{q|p} q^{\frac{\phi(p)}{p-1}} = p$. Substituting, we have:

$$D(\mathbb{Q}(\zeta_p)) = (-1)^{\frac{p-1}{2}} \frac{p^{p-1}}{p} = (-1)^{\frac{p-1}{2}} p^{p-1}$$

□

4 Problem V

Proposition 4.1. *Let $\theta := \sqrt[3]{12}$. Then $x := \frac{1}{2}\sqrt[3]{12} \in \mathscr{O}_{\mathbb{Q}(\theta)}$, but x does not lie in the $\mathbb{Z}$-span of $\{1, \theta, \theta^2\}$.*

Proof.

$$x = \frac{\sqrt[3]{12}}{2} = -\frac{\sqrt[3]{12}}{2}(e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}})$$

Let $y_1 := \sqrt[3]{12}e^{\frac{2\pi i}{3}}$ and $y_2 := \sqrt[3]{12}e^{\frac{4\pi i}{3}}$. Then

$$\begin{aligned} x &= -(y_1 + y_2) \\ x^3 &= (y_1^3 + 3y_1^2 y_2 + 3y_1 y_2^2 + y_2^3) \\ y_1^3 = y_2^3 &= 12 \\ y_1^2 &= (\sqrt[3]{12})y_2 \\ y_2^2 &= (\sqrt[3]{12})y_1 \\ 3y_1^2 y_2 + 3y_1 y_2^2 &= (3 \cdot 12)(y_1^2 + y_2^2) = -6x \\ x^3 &= -\frac{1}{8}(12 - 6x + 12) \end{aligned}$$

Proof incomplete. □