Homework 3

MATH/CS 471, Fall 2023

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Overview

The purpose of this homework is to implement and analyze different methods of numerical integration for:

$$I = \int_{-1}^{1} e^{\cos(kx)} dx \tag{1}$$

where $k = \{\pi, \pi^2\}$.

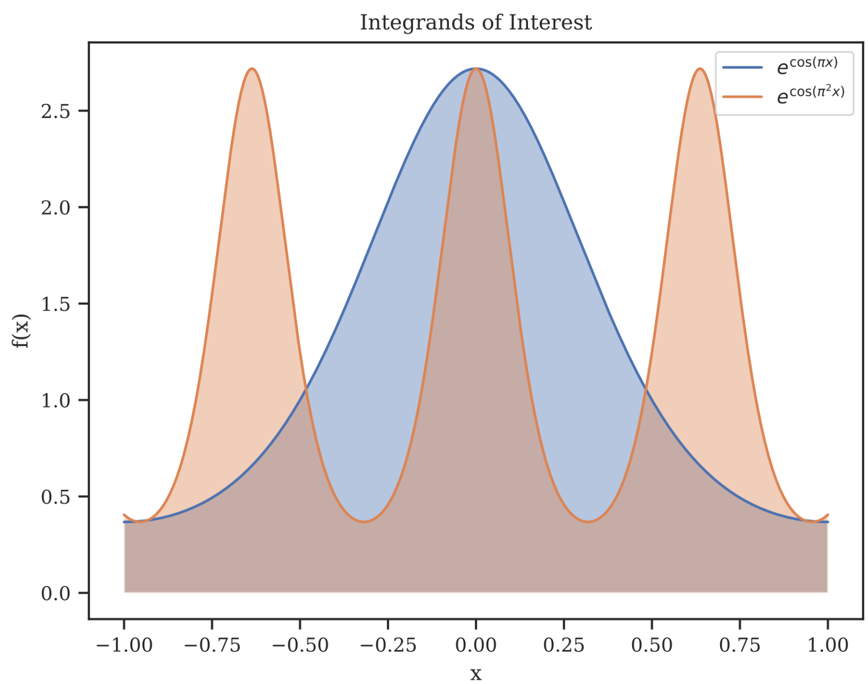

Figure 1: Integrands of Interest Across Domain of $x \in [-1, 1]$

Trapezoid Rule

What

For a set of equidistant grid points

$$x_i = X_L + ih, i = 0, ..., n, h = (X_R - X_L)/n$$

on the interval $[X_L, X_R]$, the composite trapezoid rule reads:

$$\int_{X_L}^{X_R} f(x)dx \approx h(\frac{f(x_0) + f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i)) \tag{2}$$

This trapezoid rule will be translated into code and run to calculate the integrals:

$$I = \int_{-1}^{1} e^{\cos(k*x)}$$

with k = π and π^2. Each function's approximation will be calculated over the interval $[-1, 1]$ but will have a variable number of points, N. The number of points will be such that the absolute error estimate $(\delta_{abs})_{n+1} = |I_{n+1} - I_n| < 10^{-10}$.

How

To begin, the trapezoid rule function is written which takes three arguments: a: start of the interval, b: end of the interval, and n: number of points in the interval. Inside the function, the summation is calculated with a for loop that calculates and adds all x_i together. It then return the composite trapezoid rule defined in (2).

To determine N, we start by calculating the trapezoid rule for n=1 and n=2. With these, we can determine $(\delta_{abs})_{n+1}$. We then continue this pattern of $(\delta_{abs})_{n+1} = |I_{n+1} - I_n|$ with a while loop until we have reached and absolute error estimate that is below 10^{-10}. When our tolerance is met, we plug in the found N into our trapezoid function. The quadrature is then calculated.

Why

For the integral with $k = \pi$, N was found to be 11. The quadrature of the integral was calculated to be 2.53. For the integral with $k = \pi^2$, N was found to be 1319. The quadrature of this integral was calculated to be 2.45. If the second derivative of f(x) is bounded over [a,b] then the error of the trapezoid rule is $O(h^2)$.

Trapezoid Rule Convergence

What

Determine the convergence type for the two integrals for n=2,3,...,N.

How

When computing the absolute error approximation, the $(\delta_{abs})_{n+1}$ is saved to a list. Once N is determined, a plot of δ_{abs} vs N is plotted on a log-log plot. To determine convergence type, reference curves of linear, quadratic, and exponential convergence are also plotted. By comparing experimental data to the reference curves, we can deduce the convergence rate of the method in the log-log plots.

Why

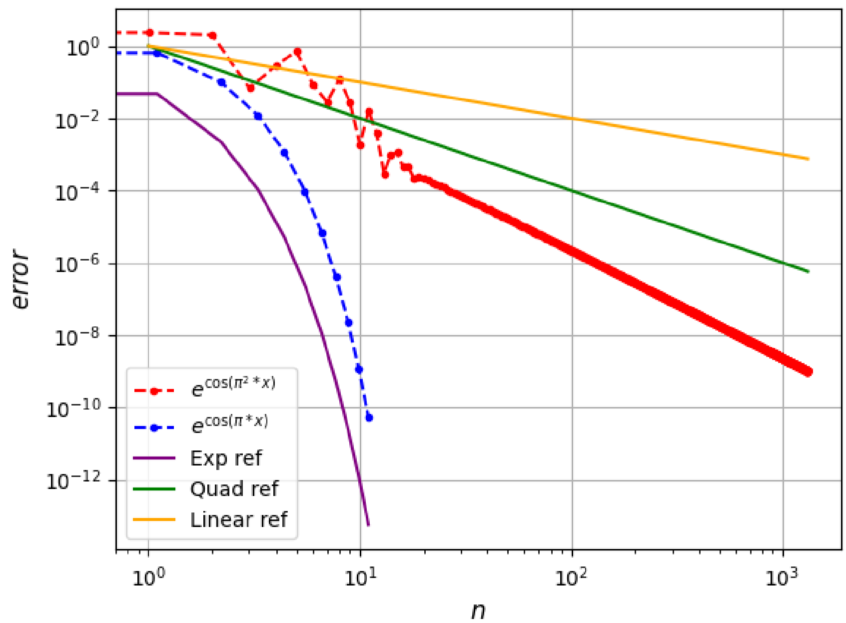

Figure 2: The convergence rates of the two integrals as well as reference curves.

As we can see the trapezoid rule for the integral with $k = \pi$ converges exponentially. The trapezoid rule for the integral with $k = \pi^2$ is more difficult to determine. The first 16 n values do not match any reference curve, however, the later iterations seem to converge quadratically as the experimental data is most parallel to the quadratic reference curve. This convergence makes sense because we found the integraltion rule for m+1 evenly spaced points on [a,b]:

$$\int_a^b f(x)dx \approx h(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{m-1} f(x_i)) - \frac{(b-a)h^2 f''(c)}{12} \tag{3}$$

When h $\to 0$, the error decays quadratically in h, and we say the convergence of the trapezoidal rule is quadratic. This makes the upper bound of trapezoid rule convergence quadratic, therefore, exponential convergence is also justified.

Special case $k = \pi$

The trapezoid rule converges rapidly for periodic functions due to the Euler-Maclaurin summation formula. We can see $e^{\cos(\pi * x)}$ is periodic and that the period of the function is the length of the interval we are calculating our integral over. If our function is p times continuously differentiable with period T where $h := T/N$ then due to the periodicity, the derivatives at the endpoints cancel out in (3) and we see out error is $O(h^p)$. This explains the exponential nature of its convergence.

Gauss Quadrature

Because equidistant nodal points may not necessarily generate the appropriate interpolating function, most likely due to variations on the endpoints of the function, we look to vary the nodes themselves.

We expect the error, defined as:

$$\epsilon(n) = |x_{n+1} - x_n|\,, \tag{4}$$

to decrease as $\epsilon \sim C^{-\alpha n}$, where α and C are real constants. For ease of analysis, we *assume* this model of the error convergence, and immediately take the natural logarithm of the data. This relieves us of the burden of finding suitable C and α parameters. This workaround leaves us with:

$$\ln(\epsilon(n)) \sim -\alpha n \ln(C) \tag{5}$$

Applying a linear regression to our linearized error plotted as a function of n, we look for a high R^2 term ($R^2 \approx 1$) and a slope, m, of the form:

$$m = -\alpha \ln(C). \tag{6}$$

As observed in Figure 3, the errors show an impressive linear correlation, with:

$$(m)_{k=\pi} = -1.4361$$
$$\left(R^2\right)_{k=\pi} = 0.9963$$

$$(m)_{k=\pi^2} = -0.4835$$
$$\left(R^2\right)_{k=\pi^2} = 0.9887$$

Cost Comparison

Trapezoid

For any interval divided into m sub-intervals, there must be m + 1 function evaluations. For $k = \pi$, N = 11 sub-intervals so there is 12 functions evaluations. Similarly, for $k = \pi^2$, N=

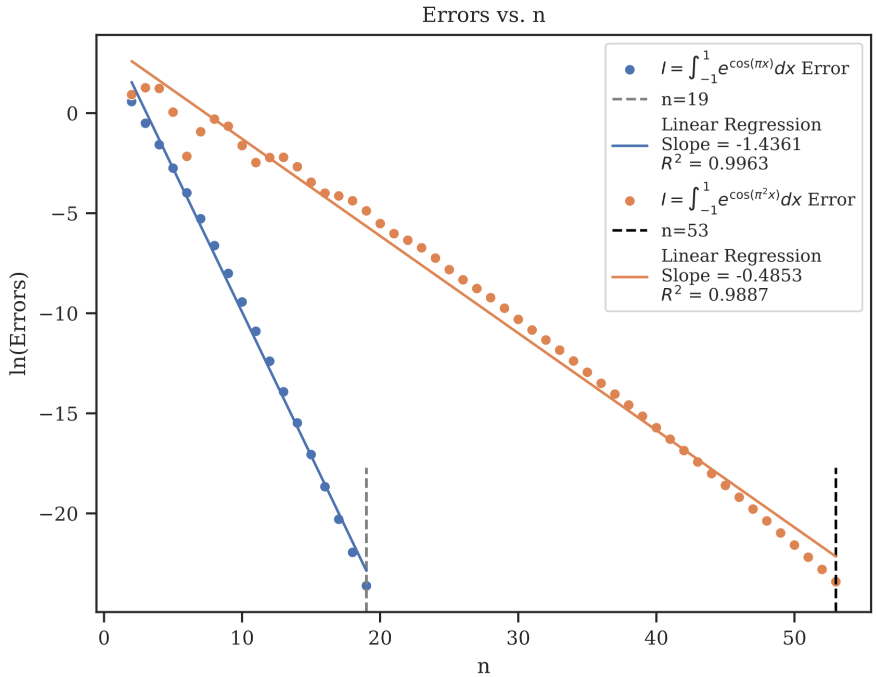

Figure 3: Convergence of Integrals on Logarithm Scale

1319 sub-intervals so there is 1320 function evaluations.