PHY 201 ODWE Project

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April 15th, 2020

1 Abstract

The aim of this project is to use separation of variables and the solutions to the second order equations that result from it to find an equation for the value of temperature across two dimensions of a metal sheet. Similar to the wave equations we are familiar with, the function of T (Temperature) can be treated as an equation of two spatial dimensions (x and y as opposed to position and time). After separating and solving the differential equations, we solve for the initial conditions and multiply the independent equations in x and y to get back the function for T.

2 Introduction

In this paper, we explore the steady-state temperature distribution in a flat metal sheet that follows the following partial differential equation.

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$$\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} = 0 \tag{1}$$

where temperature is a function of only x and only y. In this problem we see that we have four boundary conditions. Two for x=0 and x=S (the vertical sides of the metal sheet) and two for y=0 and y=S (the horizontal sides of metal sheet). Using separation of variables will soon see that the general solution to this problem is a Fourier series. Since only one of these boundary conditions is inhomogeneous, we can simply ignore it until the end of the problem where we use the properties of orthogonal projection to find the unknown coefficients of the Fourier series.

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First, disregarding the boundary conditions for now, consider that the solution can be factored into the product of some function of x and some function of y.

$$T(x, y) = X(x) \cdot Y(y) \tag{2}$$

Then if we take the second derivative of T with respect to x and the second derivative of T with respect to y and plug these derivatives back into the differential equation, we get

$$X''(x) \cdot Y(y) + X(x) \cdot Y''(y) = 0 \tag{3}$$

The reason we can write it like this is because we can factor out X from the partial derivative in y and we can factor out Y from the partial derivative in x, since these functions are considered to be constants as far as the partial derivatives are concerned. Because we have functions that are only dependent on either x or y in this equation (but not both) we can now algebraically manipulate these functions however we want.

Once the partials have been converted to regular derivatives, we divide by X and Y and then bring one of the terms to other side of the equation by subtracting. If we choose to subtract over the Y term then we get

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \tag{4}$$

Now that we have the x and y terms completely separated we see that the two expressions for x and y respectively are independent of each other and are merely related by some constant. So we can rewrite the above equation as

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2 \tag{5}$$

We will call this constant the separation constant or the eigenvalue of the differential equation. If we wanted to we could have wrote the constant as simply as λ but as you will soon see it is more convenient to write it this way. The value of this eigenvalue will determine the kind of eigenfunctions, you get for your solution set. Now that we can relate the expression in terms of x and the expression in terms of Y to a constant, we can represent the partial differential equation as a system of two ordinary differential equations. One that gives you a solution for your function of x and one that gives you a solution for your function of y. The system of ODE's however changes based on what the values are for the eigenvalue. There are there possible cases that yield three different kinds of ODE systems. Lambda could either be positive, 0 or negative. Each one of these cases yields a different general solution. Fortunately only one of these solutions is not trivial (non-zero). By assuming the sign of the separation constant and applying the boundary conditions to the solutions, it can be shown relatively easily that only negative values yield non-trivial solutions with these boundary conditions. We however do not need to show this mathematically to deduce this. We can simply look at the physical interpretation of the problem. If we let the separation constant be positive, based on the system of ODE's we would find that the solution for x is a linear combination of exponential and the solution for y is a linear combination, of cosine and sine functions. Based on the boundary conditions we know that, the solution for x has to be 0 at two separate locations. This implies that the solution for x is periodic, this is something that we should not expect with exponential functions. All exponential functions are asymptotic to the x-axis, so the only way for both boundary conditions to be meet is for both constants in the solution to be 0, which is a trivial solution that doesn't tell us anything we don't already know. If the separation constant is equal to 0, we would expect both solutions to be linear but the solution for X has to be periodic, so the only solution in this case will also be the trivial solution. If we let lambda equal only negative values we will get a linear combination of sine and cosine functions for X. This is periodic, which is what we want. The solution for Y would be a linear combination of exponential functions, which indicates that the temperature gradually increases or decreases (depending on if $T_0 > 0$ or $T_0 < 0$) as y increases. With that disclaimer out of the way let's find the general solutions to the ODE's for X and Y when the separation constant is negative and then apply the boundary conditions. Doing a little algebraic manipulation with equation 5 we see that the two differential ordinary differential equation we need to solve are

$$X''(x) + \lambda^2 X(x) = 0 \tag{6}$$

$$Y''(y) - \lambda^2 Y(y) = 0 \tag{7}$$

Using the auxiliary equation both of these Ode's can easily be solved. The roots for the first ODE are imaginary and the roots for second are real, so the solutions are

$$X(x) = c_1 cos(\lambda x) + c_2 sin(\lambda x) \tag{8}$$

$$Y(y) = c_3 e^{\lambda x} + c_4 e^{-\lambda x} \tag{9}$$

The solution for Y can be rewritten as a linear combination of hyperbolic sine and cosine.

$$Y(y) = c_3 sinh(\lambda y) + c_4 cosh(\lambda y) \tag{10}$$

Now that we have found the general solutions, let's apply the boundary conditions. First we will try to find the possible eigenvalues by using the two homogeneous boundary conditions for at x=0 and x=S. Plugging in the appropriate values for the first boundary conditions leads us to conclude that $C_1 = 0$, since $sin(0)$ vanishes and $cos(0)$ is simply 1. So the solution for X can be rewritten as

$$X(x) = c_2 sin(\lambda x) \tag{11}$$

Now lets apply the second boundary condition at x=s. Plugging in the appropriate values we find that

$$0 = c_2 sin(\lambda s) \tag{12}$$

Suppose that $c_2 \neq 0$ and the for the temperature is non-zero. Then $sin(\lambda s)$ must equal 0, which implies that ks is equal to some integer multiple of π. If n is that integer that means the eigenvalue is

$$\lambda = \frac{n\pi}{s} \tag{13}$$

So our final solution for X is

$$X(x) = c_2 sin(\frac{n\pi}{s}x) \tag{14}$$

Now let's apply the homogeneous boundary condition at y=0 to the Y. For this boundary condition we insert y = 0 and Y=0.

$$0 = c_3 sinh(0) + c_4 cosh(0) \tag{15}$$

So we see that since $sinh(0) = 0$, we must make $c_4 = 0$ so that the left hand side is equal to the right hand side. Therefore,

$$Y(y) = c_3 sinh(\frac{n\pi}{s}y) \tag{16}$$

From the conditions we have applied, T(x,y) is as follows,

$$T(x, y) = c_2 c_3 sin(\frac{n\pi}{s}x) sinh(\frac{n\pi}{s}y) \tag{17}$$

Now we will write out the entire solution as an infinite Fourier sine series and apply the inhomogeneous boundary condition at y=S. We will then use the properties of orthogonal functions to solve for the coefficient of the Fourier sine series, given any value for n. We will multiply both sides of the equation by another eigenfunction or in other words a sine wave with a different eigenvalue. Since all possible eigenvalues are an integer multiple of $\frac{\pi}{S}$, the only thing that will be different is the integer which we will call m. Then we will integrate on both sides with respect to y from 0 to S. The two sine functions will always be orthogonal to each other, on the interval [0,S], unless their the same function. This can be verified by taking the inner product or integrating the two functions, multiplied together, from 0 to S and showing through trig identities that the integral will only be non-zero when m=n. We can also carry out the calculation, for the case where m=n, by using the half angle identities and we will find that the integral equals $\frac{S}{2}$. We can then conclude that there will only be one non-zero term, in the sum, when m and n or the two eigenfunctions are the same. Then from there it is simply a matter of evaluating the integrals on the left hand side and doing some algebraic manipulation to solve for the unknown coefficient A_n.

$$T(x, y) = \sum_{n=1}^{\infty} A_n sinh(\frac{n\pi}{s}y) sin(\frac{n\pi}{s}x) \tag{18}$$

$$T(x, S) = \sum_{n=1}^{\infty} A_n sinh(\frac{n\pi}{s}s) sin(\frac{n\pi}{s}x) \tag{19}$$

$$T_0 = \sum_{n=1}^{\infty} A_n sinh(n\pi) sin(\frac{n\pi}{s}x) \tag{20}$$

$$\int_0^S T_0 sin(\frac{m\pi}{s}x) = A_n sinh(n\pi) \sum_{n=1}^{\infty} \int_0^S sin(\frac{n\pi}{s}x) sin(\frac{m\pi}{s}x) \tag{21}$$

$$\int_0^S T_0 sin(\frac{m\pi}{s}x)dx = A_m sinh(n\pi) \cdot \frac{S}{2} \tag{22}$$

$$A_m sinh(m\pi) = \frac{2}{S} \int_0^S T_0 sin(\frac{m\pi}{s}x)dx \tag{23}$$

$$A_m = \frac{2T_0}{s \cdot sinh(m\pi)} \int_0^S sin(\frac{m\pi}{s}x)dx \tag{24}$$

$$A_m = \frac{2T_0}{s \cdot sinh(m\pi)} \frac{s}{m\pi}[(-1)^{m+1} + 1] \tag{25}$$

$$A_m = \frac{2T_0}{m\pi sinh(m\pi)}[(-1)^{m+1} + 1] \tag{26}$$

Since m is just an arbitrary integer, we without loss of generality we can rewrite m as n.

$$A_n = \frac{2T_0}{n\pi sinh(n\pi)}[(-1)^{n+1} + 1] \tag{27}$$

4 Conclusion

To sum up the process, we separated the equation and solved the resulting second order equations. By applying the first three homogeneous boundary conditions we obtained a general equation for T. For the last boundary condition we used the methods in the OFFS tutorial to find the value of the last constant, A_n. Plugging this value into the solution we get from the solution, we finally find the wave equation for temperature.

5 References

Jacobs, R. J. (2020). ODWE: Ordinary Differential Wave Equation. In PHY 201/302: Math Methods Physics l/Math Methods Physics ll (Vol. Spring 2020, pp. 1–9). Tempe, AZ: AlphaGraphics.