PHY 201 Project 3

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1 Abstract

Our group set out to find the correlation and accuracy of using an Euler engine to model simple harmonic motion and the methods that could be taken to improve it. First we predicted the graphs and made models using an Euler engine and then compared it to the actual solutions obtained through derivation. The plots we obtained were of the relation between the driving frequency and the consequent maximum amplitude. These graphs were fairly standard and all showed resonance at around 24 rad/s, the value we'd expect from the given parameters for the differential equation of harmonic motion. They had the peaked shape that leveled out when the driving frequency was too high or too low. In this respect, both the observed and expected plots had the same shape. On closer analysis though, the plots seemed to have very sightly different heights/offset and FWHM (Full width half maximum). The FWHM for the observed plot was 10.1 for the modeled and 10.7 for the actual plot, a 5.61% difference. The RMSE also showed a small difference between the two (RMSE = 0.00128). This small error is mostly due to the Euler engine overshooting the peaks. A smaller value for the step size might have corrected this issue but at this point the excel sheet was struggling to load every time we needed new values. We also introduce an alternative method of approximating a solution to the differential equation, called the fourth order Runge-Kutta method. Essentially the method approximates the solution in the same way, however calculates the slope at four points between each step and takes the weighted average to approximate the next y value. This method has the benefit of reducing the amount that the slope overshoots or undershoots the desired point for a certain step size h. The method has the advantage of being able to reduce the error by the same amount without making the step size nearly as small which makes computations faster and more efficient. Results from Mathematica show that the approximate solution for the damped harmonic system we are analyzing is more accurate than the Euler approximation.

2 Introduction

In this paper we explore some harmonic systems represented by second-order linear differential equations with constant coefficients of the form

$$a\ddot{y} + b\dot{y} + cy = F(t) \tag{1}$$

where t is the time. In particular, we study the damped harmonic motion of a mass m, on a vertical spring of spring constant, k, with a driven periodic force of $F(t) = F_0 \cos \omega t$ where ω is the periodic driving frequency of the force. So in order, to answer the various questions surrounding this system, we will need to know the general solution to a differential equation (1) of the form

$$a\ddot{y} + b\dot{y} + cy = F_0 \cos \omega t \tag{2}$$

The complementary solution or homogeneous solution to this differential equation (2) can easily be found using the auxiliary solution $y = e^{mt}$ and finding the roots of the associated quadratic equation. The quadratic equation must have roots that force both sides of the equation to equal 0 and satisfy the homogeneous version of the differential equation. There are three possibilities. There are two distinct real roots, two complex roots or one repeated root. Due to the superposition principle for linear differential equations the solution can be written as the linear combination of two exponential if there are two distinct roots.

$$y = c_1 e^{m_1 t} + c_2 e^{m_2 t} \tag{3}$$

If the roots are complex, Euler's formula can be used to rewrite the complex exponential of (3) as a linear combination of a sine and cosine wave.

$$y = \bar{c_1} \cos m_1 t + \bar{c_2} \sin m_2 t \tag{4}$$

If there is only one distinct root, the other linearly independent solution for the homogeneous differential equation (2) can be found through reduction of order or by evaluating the following integral

$$y_2 = e^{m_1 t} \int \frac{e^{\int \frac{-b}{a} dt}}{(e^{m_1 t})^2} dt \tag{5}$$

Since the quadratic equation only has one root, the determinant in the quadratic equation must equal 0 and therefore the root is $m_1 = -b/2a$. This means that integral (5) can be rewritten as

$$y_2 = e^{m_1 t} \int \frac{e^{\int 2m_1 dt}}{e^{2m_1 t}} dt \tag{6}$$

and simplified to

$$y_2 = te^{m_1 t} \tag{7}$$

So if there is one repeated root the solution to the differential equation using (7) is

$$y = c_1 e^{m_1 t} + c_2 t e^{m_1 t} \tag{8}$$

In this case, with the driving force $F(t) = F_0 \cos \omega t$, we take the amplitude phase form of the particular solution for (2) to be

$$y_p(t) = B \cos(\omega t + \delta) \tag{9}$$

Plugging in this solution (9) back into the differential equation (2) we find that

$$(-aB\omega^2 + cB)\cos(\omega t + \delta) - bB\omega \sin \omega t + \delta = F_0 \cos \omega t \tag{10}$$

For the purpose of comparing numerical data to the general solution, we want to find a general expression for the amplitude and phase of the wave. Instead of trying to find an expression algebraically for an arbitrary time, t, we can choose specific values of t that allow us to separate sin and cos terms into two separate equations. We do this by letting $\omega t + \delta$ equal values that cause the cosine or sine terms to vanish. First we will let t be a value, such that $\omega t + \delta = 2\pi$. This will cause the sine term to vanish and equation (10) will become

$$-aB\omega^2 + CB = F_0 \cos(2\pi - \delta) = F_0(\cos 2\pi \cos \delta + \sin 2\pi \sin \delta) \tag{11}$$

which simplifies to

$$-aB\omega^2 + CB = F_0 \cos \delta \tag{12}$$

Solving (12) for $\cos \delta$ we find that

$$\cos \delta = \frac{-aB\omega^2 + cB}{F_0} \tag{13}$$

Now we chose a value for t that will cause the cosine term to vanish. Let t be a value, such that $\omega t + \delta = \frac{\pi}{2}$. Then the cos term of (10) will become

$$-bB\omega = F_0 \cos(\frac{\pi}{2} - \delta) = F_0(\cos \frac{\pi}{2} \cos \delta + \sin \frac{\pi}{2} \sin \delta) \tag{14}$$

which simplifies to

$$-bB\omega = F_0 \sin \delta \tag{15}$$

Solving (15) for $\sin \delta$ we find that

$$\sin \delta = \frac{-bB\omega}{F_0} \tag{16}$$

Using (13) and (16) we can find an expression for the phase that is independent of the initial force, F_0 and the amplitude, B.

$$\delta = \arctan \frac{b\omega}{a\omega^2 - c} \tag{17}$$

Using the expressions for $\sin\delta$ and $\cos\delta$, (13) and (16), and the Pythagorean theorem we find that

$$\frac{B^2[b^2\omega^2 + (c - a\omega^2)^2]}{F_0^2} = 1 \tag{18}$$

Solving (18) for B and doing some algebraic simplification we see that

$$B = -\frac{F_0}{\sqrt{b^2\omega^2 + a^2(\omega^2 - \frac{c}{a})^2}} \tag{19}$$

From (19) we can see that the amplitude should theoretically be at its maximum value when the driving frequency is $\omega = \sqrt{\frac{c}{a}}$. This is called the resonant frequency or ω_R. Now that we know how to find the general solution to the general form of our desired differential equation, let it model a forced spring system of the form from equation (1)

$$m\ddot{y} + \beta\dot{y} + ky = F_0\cos\omega t \tag{20}$$

where m is the mass connected to the spring β the damping constant, and k the spring constant. The numerical values are decided to be: k = 15.0 N/m, m = 26.0 g, $\beta = 0.160$ N sec/m and $F_0 = 0.0500$ N. Our goal will be to model this specific forced harmonic system by creating various numerical calculations and plots with a program that approximates a solution to the differential equation using Euler's method. We will then compare these results with our theoretical results. Specifically we will calculate the theoretical value for the resonant frequency, compute the displacement of the mass as a function of time for various driving frequencies with the Euler engine, plot the displacement for the resonant frequency and identify the transient and steady-state parts of the solution. We will also plot the maximum displacement as a function of frequency, measure the full width at half the maximum (FWHM), and compare the experimental solution with the theoretical solution and all numerical results including FWHM and resonant frequency. Then we will derive a time-dependent expression in the steady state for power delivered to the system, find an expression for the time-averaged power and show its dependence on the phase and plot the time-averaged power as a function of driving frequency. Finally we will discuss an alternative method of approximating the solutions called the fourth order Runge-Kutta method.

Part 1

a.

To find the resonance frequency, using equation 42 from the tutorial also shown in the introduction, $\omega_R = \sqrt{\frac{c}{a}}$. Then,

$$\omega_R = \sqrt{\frac{k}{m}} = \sqrt{\frac{15}{0.026}} = 24\,\mathrm{rad\,s^{-1}}$$

b., c.

Using Euler's method for different values of the driving frequency w_d for the inhomogenity of equation (20), with $y(0) = 0$ and $\dot{y}(0) = 0$, different graphs can be produced until 10 times the period of the inhomogenity at resonance or between $t = 0$ and $t = 10 \times \frac{2\pi}{\omega_r} = 2.62\,\mathrm{s}$ in steps of 1 ms, shown in figure 1 below. For when the driving frequency equals the resonance frequency, 24 radians per second, the graph is marked with the transient and steady parts of the solution.

Examining figure 1, below and above the resonance frequency, the steady state amplitudes are less than at the resonance frequency. This is expected as by definition, the resonance frequency is defined to be when the amplitude of the particular solution to (20) which produces the steady state is the greatest as described in the tutorial. The two other graphs have driving frequencies that are the same magnitude away from the resonance frequency, yet the graph at 20 rad/s has a higher amplitude than the graph at 28 rad/s. This would indicate that the amplitude of the particular solution decreases more rapidly when increasing from resonance frequency rather than decreasing from it. A more precise comparisons will be done in part (d) and (e) with the amplitudes of many more driving frequencies and actual equations for the amplitude of the particular solution, derived in the introduction, and a comparison of the numerical solution with the analytic solution as well.

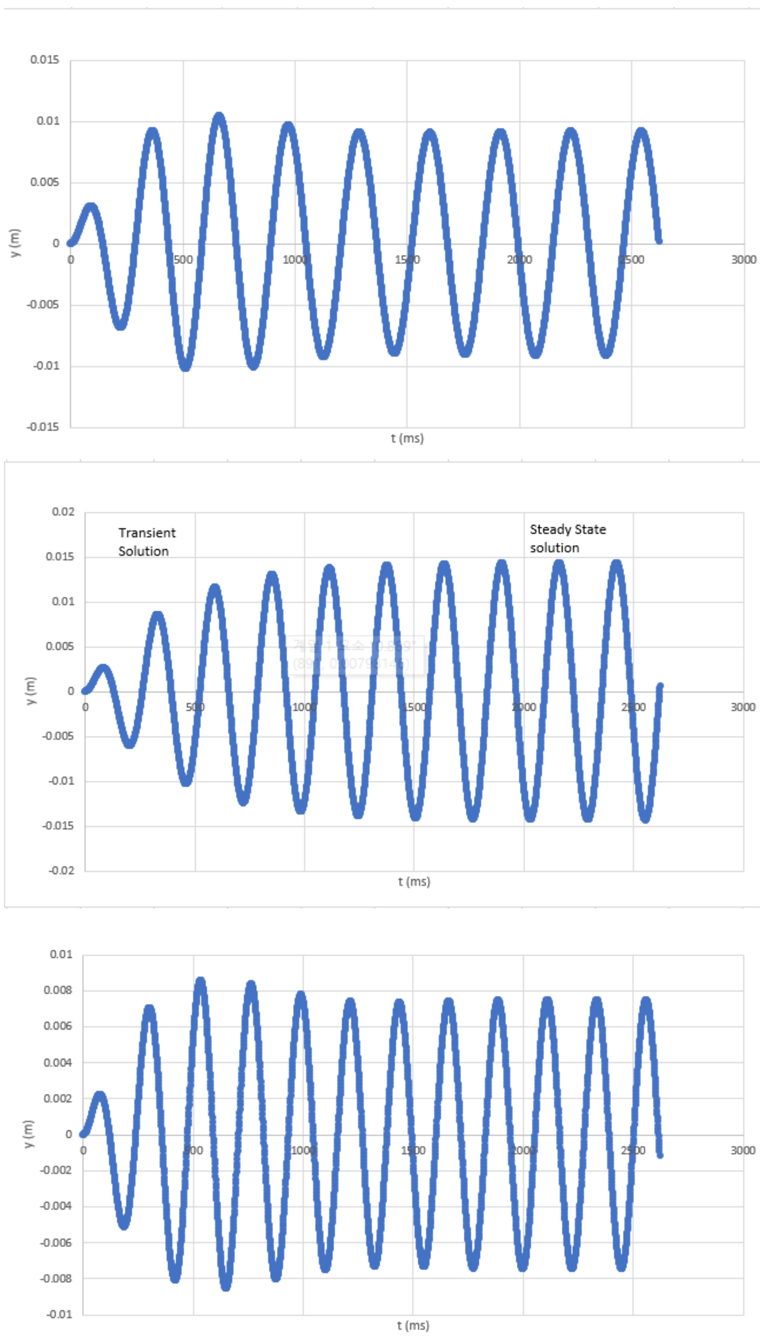

Figure 1: The numerical solutions of equation (2) using an Euler engine. The graphs correspond to the driving frequencies of 20, 24, and 32 rad/s with the second value being the resonance frequency.

d.

Numerous other values for the driving frequency were tested using the Euler engine other than the three examples shown in figure 1, not included here for brevity and as it would be redundant. The maximum displacement for each driving frequency were recorded, then plotted against the driving frequency in figure 2.

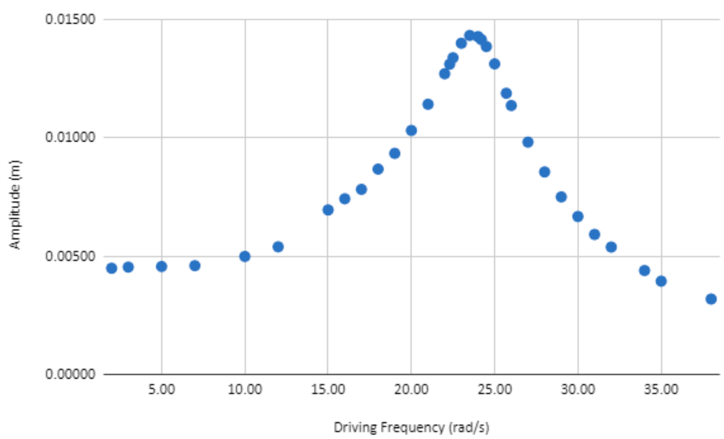

Figure 2: Graph of the maximum displacement versus the driving frequency for multiple frequency values tested.

Examining figure 2, the resonance frequency should be where the peak is at the highest amplitude, which is at 24 rad/s agreeing with the calculations. The full width at half maximum (FWHM) can also be found from figure 2. The maximum displacement at the peak is around 0.014 m. Around 18.1 rad/s and 28.2 rad/s, the maximum displacement is at about 0.007 m, half of the peak. Then, the FWHM is the difference between these two values, or 10.1 rad/s. In addition, it can be seen in figure 2 that the amplitude drops off more quickly to the right than to the left, showing the behivor observed in part (b) where the amplitude decreased more for when the driving frequency was greater than the resonance frequency than when it was less, even though the two values shown were the same magnitudes away.

e.

The tutorial provides us with a method to find the general and particular solution for Trigg's function which we can use to test our Euler engine and its accuracy. Using that method as shown in the introduction, the formula for the amplitude of the equation at the steady state is (19), reproduced here with the appropriate variables

$$A = \frac{-F_0}{\sqrt{\beta^2\omega^2 + m^2(\omega^2 - \frac{k}{m})^2}} \tag{21}$$

Upon plotting this equation with the given parameters, we can compare it to the plot of the maximum amplitude we obtained earlier from figure 2 which we obtained using values from the Euler engine to test whether our results are similar.

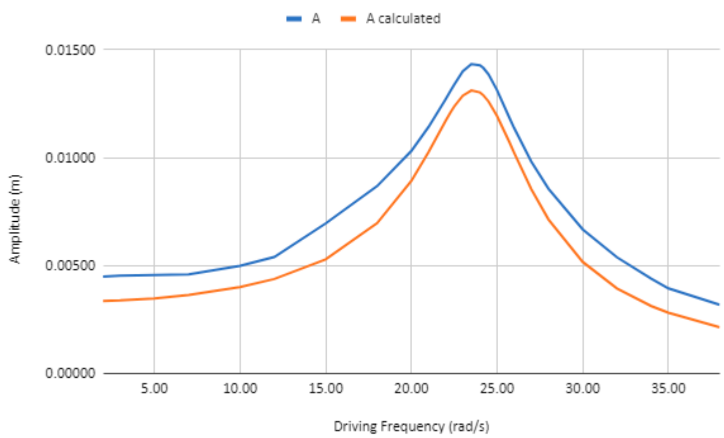

Figure 3: Graph comparing the the maximum amplitude obtained via Euler engine and the amplitude formula. The blue line is a reproduction of figure 2, and the orange line is the graph of (21).

Firstly, it is easy to see that the two plots line up quite well with the main difference being that the Euler engine plot is shifted up a little higher.

From equation (21), we can compare the maximum amplitude and the half maximum. The maximum for the Euler engine and the actual function is 0.014 m and 0.013 m respectively, the second value was found exactly from the equation (43) in the tutorial

$$A_{max} = \frac{F_0}{\beta\omega_R} \tag{22}$$

Using (21) in a homework set, an approximate equation for the FWHM was derived to be

$$\Gamma = \sqrt{3}\frac{b}{a} \tag{23}$$

For the current problem using (23),

$$\Gamma = \sqrt{3}\frac{0.16}{0.026} = 10.7\,\mathrm{rad\,s^{-1}}$$

Compared to the approximate FWHM found in (d) which is 10.1 rad/s, the percentage error is 5.61%. This small error shows that the numerical plot is just a little bit sharper and higher though not by a lot. This discrepancy most likely arises from the limitations of the Euler engine, which becomes more inaccurate the more you use it away from the initial point, overestimating the values and increasing the steady state amplitude. Despite this difference, both peaks still occur at the same position which is approximately at 24 rad/s, proving once again that this value is the resonance frequency and agreeing with parts (a) and (d).

Given these two graphs, we can also make comparisons using the RMSE. We find the RMSE using the equation,

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(y_i' - y_i)^2}{n}} \tag{24}$$

We get from this a value of 0.00128 for the RMSE. This value is relatively small but still only a single order of magnitude from the values we were getting. This shows that the fit isn't completely off but there is still a sizeable error that comes from the Euler engine. This error can mostly be connected to the inaccuracy of the Euler engine. It does not show that the Euler engine is unusuable, just that the engine has a tendency to overshoot the maximum by a little each time.

We can also test this error by comparing the the Euler engine solution with the actual equation of y(t). Only the solution at resonance will be considered to ease solving for the general solution and as the purpose of the comparison is to see the accuracy of the Euler engine which one test will suffice. The general solution to this equation is given by equation (46) in the tutorial of the form,

$$y(t) = Ae^{(-\frac{b}{2a}t)}\cos(\omega_0 t + \phi) - \frac{F_0}{\sqrt{\beta^2\omega^2 + m^2(\omega^2 - \frac{k}{m})^2}}\cos(\omega_d t + \delta) \tag{25}$$

where the first term is the homogeneous solution for the transient state and the second is the particular solution for the steady state. The phase shift δ is defined later in part f in equation (33), now just stated to be

$$\delta = \sin^{-1}\frac{\beta\omega_d}{\sqrt{\beta^2\omega_d^2 + (m\omega_d^2 - k)^2}}$$

The natural frequency or ω_0 is defined as

$$\omega_0 = \sqrt{\frac{c}{a} - \left(\frac{b}{2a}\right)^2} \tag{26}$$

Substituting the known parameters into the rearrangement of (33), (26), and (25), the solution becomes

$$y(t) = Ae^{(-3.08t)}\cos(23.8t + \phi) - 0.013\cos(24t + \frac{\pi}{2})$$

To find the unknown constants for the phase shift and amplitude of the transient solution, the initial conditions used for the Euler engine, $y(0) = 0$ and $\dot{y}(0) = 0$ can be used.

$$y(0) = A\cos(\phi) - 0.013\cos(\frac{\pi}{2}) = A\cos(\phi) = 0$$

$$\dot{y}(t) = -3.08Ae^{(-3.08t)}\cos(23.8t+\phi) - 23.8Ae^{(-3.08t)}\sin(23.8t+\phi) + 0.312\sin(24t+\frac{\pi}{2})$$

$$\dot{y}(0) = -3.08A\cos\phi - 23.8A\sin\phi + 0.312\sin(\frac{\pi}{2}) = -23.8A\sin\phi + 0.312 = 0$$

The amplitude A cannot be 0 as then the transient solution wouldn't even matter which clearly isn't the case. Then, for both initial conditions to be satisfied $\cos\phi = 0$ and

$$\phi = \frac{\pi}{2}$$
$$A = 0.013$$

so that the solution to be graphed is

$$y(t) = 0.013e^{(-3.08t)}\cos(23.8t+\frac{\pi}{2}) - 0.013\cos(24t+\frac{\pi}{2}) \tag{27}$$

Equation (27) is shown below in figure 4 along with the numerical solution from figure 1.

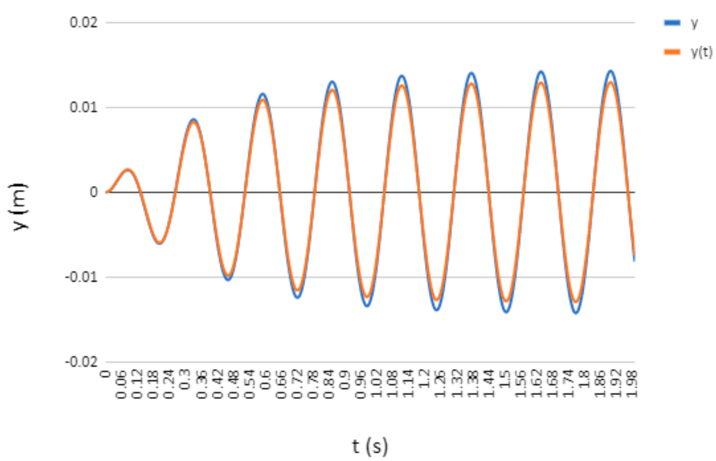

Figure 4: Graph comparing the Euler engine (blue) from the resonance frequency case in figure 1 to the solution of the differential equation (orange) for the case of resonance.

Figure 4 clearly shows that the Euler engine is overshooting the actual solution as time progresses. The transient solution is modeled well while the steady state solution is overshot by the Euler engine with a higher amplitude than actual. We identified this problem when comparing the maximum amplitudes and this shows that our suspicion from earlier was right.

f.

To find the power delivered to the system by the steady state, the formula for power

$$P = Fv \tag{28}$$

can be used. The particular solution to the differential equation of (1), which will remain in the steady state, is given by equation (37) in the tutorial and also by combining (9) and (19):

$$y_p(t) = -\frac{F_0}{\sqrt{\beta^2\omega_d^2 + m^2(\omega_d^2 - \frac{k}{m})^2}} \cos(\omega_d t + \delta) \tag{29}$$

Differentiating (29) with respect to time to find velocity,

$$v(t) = \frac{F_0\omega_d}{\sqrt{\beta^2\omega_d^2 + m^2(\omega_d^2 - \frac{k}{m})^2}} \sin(\omega_d t + \delta)$$

Also, from (30)

$$F(t) = F_0 \cos(\omega_d t)$$

and (28) becomes

$$P(t) = \frac{F_0\omega_d}{\sqrt{\beta^2\omega_d^2 + m^2(\omega_d^2 - \frac{k}{m})^2}} \sin(\omega_d t + \delta) F_0 \cos(\omega_d t) \tag{30}$$

To find the average of this function, it can be integrated over one period and divided by that interval. The period of this equation is $2\pi/\omega_d$ by inspection of the two periodic functions of (30). A math package was used to perform the integral.

$$\begin{aligned} P_{av} &= \frac{1}{2\pi/\omega_d} \frac{F_0^2\omega_d}{\sqrt{\beta^2\omega_d^2 + m^2(\omega_d^2 - \frac{k}{m})^2}} \int_0^{2\pi/\omega_d} \sin(\omega_d t + \delta)\cos(\omega_d t) dt \\ &= \frac{1}{2\pi/\omega_d} \frac{F_0^2\omega_d}{\sqrt{\beta^2\omega_d^2 + m^2(\omega_d^2 - \frac{k}{m})^2}} (\frac{\pi}{\omega_d}\sin(\delta)) \\ &= \frac{1}{2} \frac{F_0^2\omega_d}{\sqrt{\beta^2\omega_d^2 + m^2(\omega_d^2 - \frac{k}{m})^2}} \sin(\delta) \end{aligned} \tag{31}$$

and the dependence of power on the phase δ or the phase shift of the particular solution is clearly shown as the average power depends on the sin of δ. This equation can now be plotted as a function of driving frequency. For the $\sin\delta$ term, it can be defined from the equation (38) in the tutorial

$$\tan\delta = (\frac{\beta\omega_d}{m\omega_d^2 - k}) \tag{32}$$

Using the definition of tangent as from a right triangle, opposite over adjacent, the hypotenuse of this triangle with δ as its angle as defined in (32) is $\sqrt{\beta^2\omega_d^2 + (m\omega_d^2 - k)^2}$. Then,

$$\sin\delta = \frac{\beta\omega_d}{\sqrt{\beta^2\omega_d^2 + (m\omega_d^2 - k)^2}} \tag{33}$$

and the equation of the average power (32) can be written again with the given variables and (33) as

$$P_{av} = \frac{1}{2}\frac{0.0025\omega_d}{\sqrt{0.0256\omega_d^2 + 6.76\times10^{-4}(\omega_d^2 - 577)^2}}\frac{0.16\omega_d}{\sqrt{0.0256\omega_d^2 + (0.026\omega_d^2 - 15)^2}} \tag{34}$$

Using Mathematica, (34) now can be plotted as a function of the driving frequency as shown in figure 5.

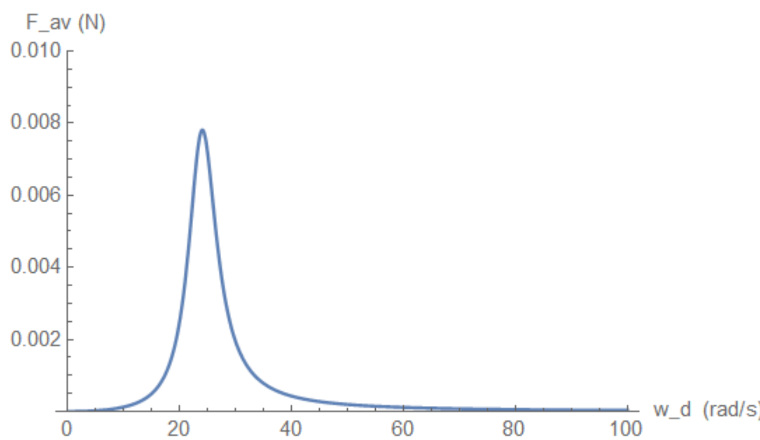

Figure 5: The time averaged power of the steady state as a function of the driving frequency.

It's clear that at 24 radians per second, at the resonance frequency, the power delivered is the greatest. This is expected as the amplitude of oscillation is the greatest at resonance, the power delievered or the rate of work done should also be the greatest for greatest displacements.

Part 2

Another popular numerical method to solve ordinary differential equations similar to Euler's is the Runge-Kutta 4th Order Method (R4). In comparison to Euler, it is generally more accurate and provides a balance between error accuracy and computational work. It is a popular method from the family of Runge-Kutta solutions - there are over 12 of them. R4 is the highest step size where the number of equations match. The general idea is to eliminate the error at each step size and works on projecting where the curve maybe versus a linear fit the Euler method provides. R4 requires the slope to be calculated at or between discrete step sizes to do (32). Euler's method uses just one step which can be written as

$$y(x+h) = y(x) + hF(x,y) \tag{35}$$

where y is the function, $F$is derivative, and h the step size. For a second derivative equation, the first derivative can be computed this way, and then be multiplied by the step size and added to the function to calculate the next step. Runge Kutta further refines this calculation by using the equation (Runge-Kutta Method, n.d.)

$$y(x+h) = y(x) + \frac{1}{6}(F_1 + 2F_2 + 2F_3 + F_4) \tag{36}$$

where

$$F_1 = hF(x,y)$$
$$F_2 = hF(x + \frac{1}{2}h, y + \frac{1}{2}F_1)$$
$$F_3 = hF(x + \frac{1}{2}h, y + \frac{1}{2}F_2)$$
$$F_4 = hF(x + h, y + F_3)$$

Euler's method only considers the slope at the point where the initial step is chosen to be, over or underestimating because of it. R4 also measures such slope as F_1, but also to estimates of the slope at midpoint of the step using F_2 and F_3, and the slope at the end using F_4 - all of which are scaled and combined to used as one slope value for the next step to increase accuracy (Cheever, n.d.). Comparing the errors with Euler's method, which also can be considered as a first order Runge-Kutta using only F_1, when decreasing the step size by a factor of 10, the error for Euler's method also decreases to about 10. However, for R4, the error decreases by about 10^4. Thus while Euler's method computes values that are proportional to the step size h, R4 computes values that are proportional to h^4 (Feldman, 2001). This can be expected that as the step size is used only once in determining the next value for Euler, so that when it is decreased by some factor the accuracy also increases by that factor, but the step size is used four times so that the decreasing the step size is amplified 4 times in the errors. Then R4 is the more efficient method, as Euler's method would have to decrease its step size by 10^4 to achieve the accuracy of R4 decrease the step size by only 10. If the same step size is used for both, R4 method will be more accurate by about 10^3.

Using Mathematica, it can be used to solve our differential equation. Using Mathematica, it is apparent ——- method is better than ——- method. This method provided better accuracy for our model.

References

[1] Cheever, E., Swarthmore College. (n.d.). Retrieved from https://lpsa.swarthmore.edu/NumInt/NumIntFourth.html

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[3] Runge-Kutta Method. (n.d.). Retrieved from https://mathworld.wolfram.com/Runge-KuttaMethod.html