PHY 201 Project 2

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February 2020

1 Problem a

Let v be the terminal velocity of a falling body in air. If the body is dropped from rest, find the expression for the speed as a function of time (1) if the air resistance is proportional to the speed, and (2) if it is proportional to the square of the speed. In each case, relate the terminal velocity to the drag coefficient (proportionality constant) ρ, mass m, and g, the acceleration due to gravity.

To find the speed of a falling body, a first order differential equation can be made using Newton's second law, $F = ma$. The acceleration in this equation can be represented as the derivative of the velocity, or dv/dt. By considering all the forces on a falling object, the net acceleration can be found, and a first order ODE can be written to be solved to find the velocity over time. A free body diagram of a falling object is shown in figure 1 below.

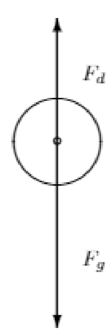

Figure 1: Free body diagram of a freely falling object under gravity

In figure 1, F_d is the drag force, whether it be air or water, and F_g is the force of gravity. The net force, mdv/dt, will be the sum of these two forces. $F_g = mg$, so the differential equation can be written.

$$m\frac{dv}{dt} = mg - F_b \tag{1}$$

The drag foce F_b equals $v\rho$ or $v^2\rho$ depending on whether it is proportional to velocity or velocity squared. Two equations can be written from (1)

$$m\frac{dv}{dt} = mg - v\rho \tag{2}$$

$$m\frac{dv}{dt} = mg - v^2\rho \tag{3}$$

The two equations are separable first order differential equations, with (2) being linear and both being inhomogeneous. Solving (2), the equation can be separated and solved for v:

$$\frac{1}{g - v\rho/m}dv = dt$$

Integrating,

$$-\frac{\ln{(g - v\rho/m)}}{\rho/m} = t + C$$

$$v = \frac{mg}{\rho} + Ce^{-\rho t/m}$$

If $v = 0$ at $t = 0$

$$0 = \frac{mg}{\rho} + C$$

$$C = -\frac{mg}{\rho}$$

and the solution is

$$v = \frac{mg}{\rho}(1 - e^{-\rho t/m}) \tag{4}$$

When t goes to infinity, the terminal velocity is.

$$v_{terminal} = \frac{mg}{p}$$

Equation (3) can be solved similarly but with some transformations on (3).

$$\frac{dv}{dt} = \frac{\rho}{m}(g\frac{m}{\rho} - v^2)$$

$$\frac{1}{g\frac{m}{\rho} - v^2}dv = \frac{\rho}{m}dt$$

Integrating,

$$\frac{1}{\sqrt{gm/\rho}}\tanh^{-1}{(\frac{v}{\sqrt{gm/\rho}})} = \frac{\rho}{m}t + C$$

$$v = \sqrt{\frac{gm}{\rho}}\tanh{(\sqrt{\frac{g\rho}{m}}t + C)}$$

If $v = 0$ at $t = 0$

$$0 = \sqrt{\frac{gm}{\rho}} \tanh(C)$$

$$C = 0$$

and the solution is

$$v = \sqrt{\frac{gm}{\rho}} \tanh\left(\sqrt{\frac{g\rho}{m}} t\right) \tag{5}$$

Again, when t goes to infinity, the terminal velocity becomes

$$v_{terminal} = \sqrt{\frac{mg}{p}}$$

Using these two models of air resistance on real life data, two drops of a small red ball and a large purple ball in free fall, and two drops of a marble and green ball in water were recorded on video with a smartphone, with a meter stick present for scale. We recorded and analyzed data from the videos in the graphing software logger pro. The equation we used for our velocity model was

$$v = A(1 - e^{-Bt}) \tag{6}$$

where parameter A represents the terminal velocity for the velocity model $\frac{mg}{\rho}$ and B represents the expression $\frac{\rho}{m}$. The equation we used for the velocity squared fitting in logger pro was

$$v = A \tanh Bt \tag{7}$$

where A represents the terminal velocity for the velocity squared model $\sqrt{\frac{mg}{\rho}}$ and B represents the expression $\sqrt{\frac{g\rho}{m}}$.

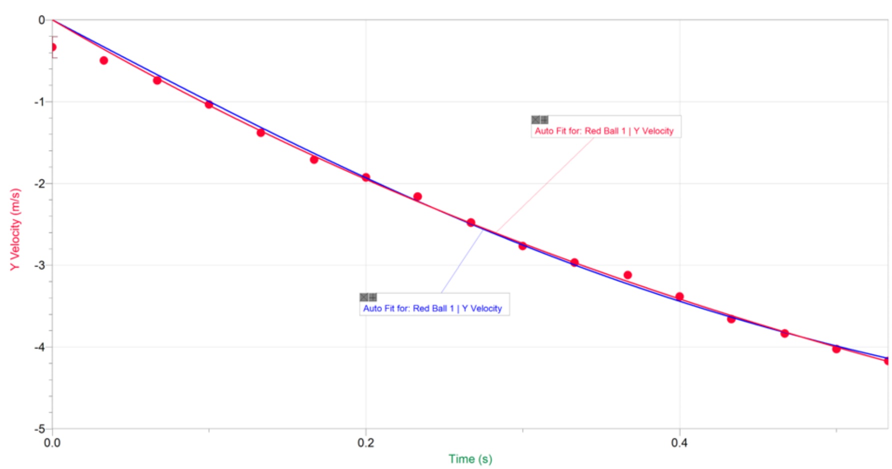

Figure 2: Graph of velocity vs time for red ball in free fall during trial one with velocity curve fit shown in red and velocity squared curve fit shown in blue.

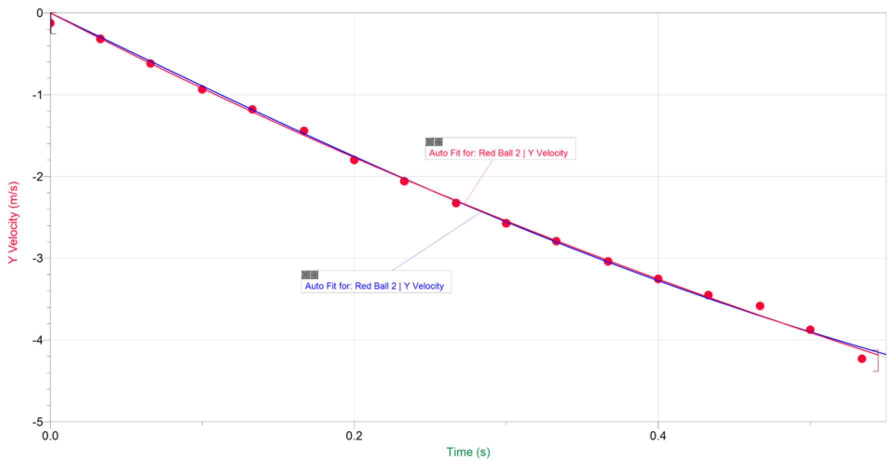

Figure 3: Graph of velocity vs time for red ball in free fall during trial 2 with velocity curve fit shown in red and velocity squared curve fit shown in blue.

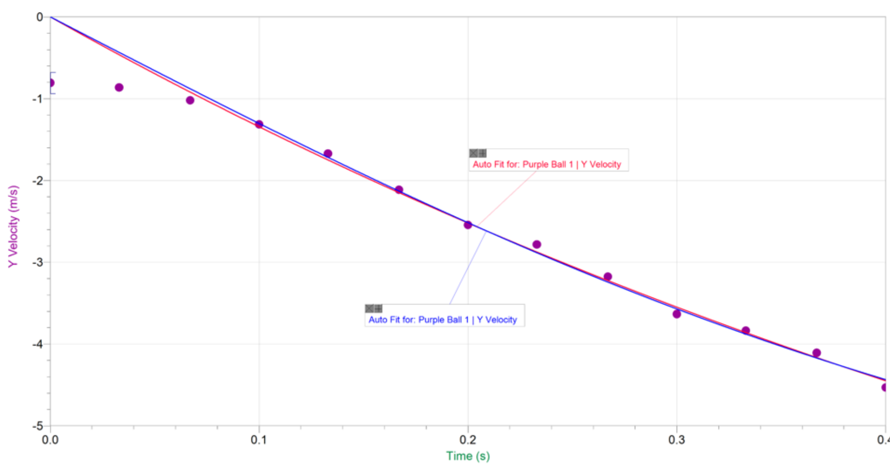

Figure 4: Graph of velocity vs time for purple ball in free fall during trial one with velocity curve fit shown in red and velocity squared curve fit shown in blue.

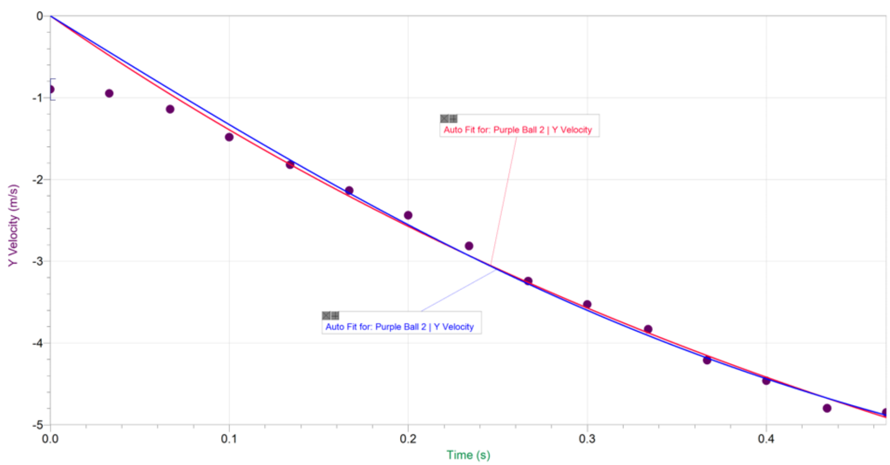

Figure 5: Graph of velocity vs time for purple ball in free fall during trial two with velocity curve fit shown in red and velocity squared curve fit shown in blue.

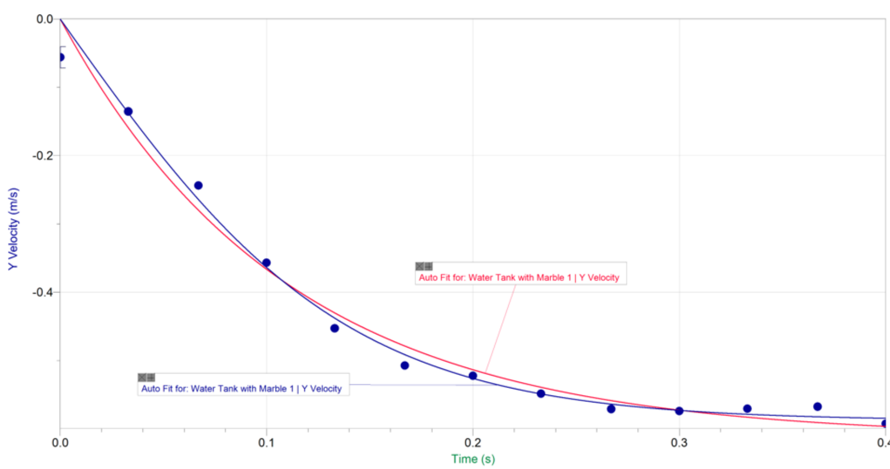

Figure 6: Graph of velocity vs time for marble in free fall during trial one with velocity curve fit shown in red and velocity squared curve fit shown in blue.

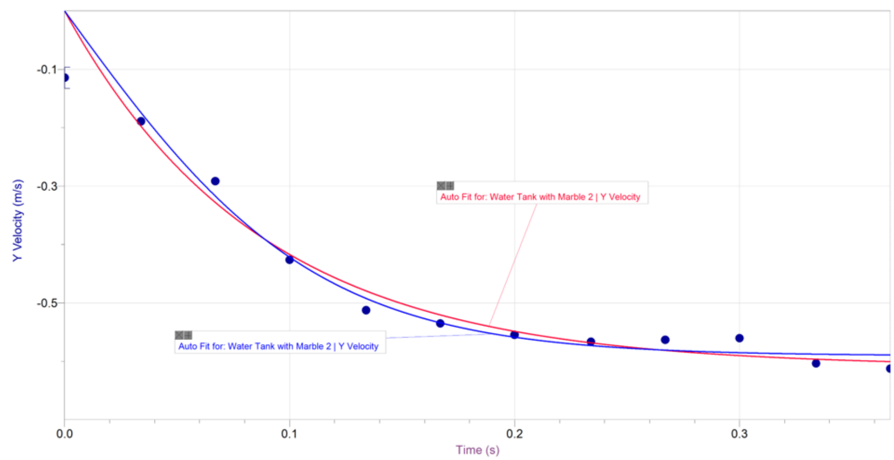

Figure 7: Graph of velocity vs time for marble falling in water tank during trial two with velocity curve fit shown in red and velocity squared curve fit shown in blue.

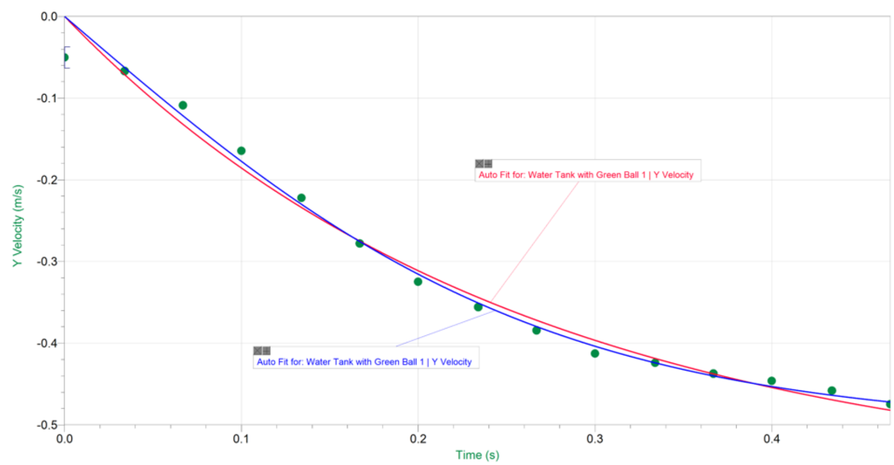

Figure 8: Graph of velocity vs time for green ball falling in water tank during trial one with velocity curve fit shown in red and velocity squared curve fit shown in blue.

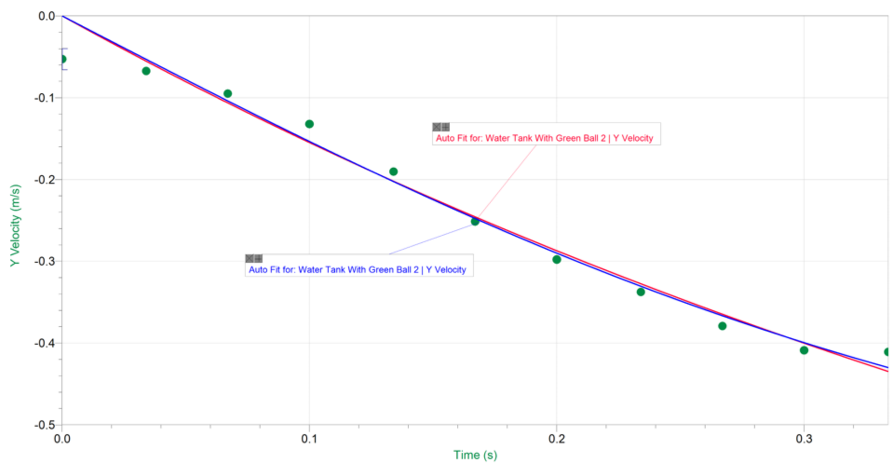

Figure 9: The graph of velocity vs time for green ball falling in water tank during trial two with velocity curve fit shown in red and velocity squared curve fit shown in blue.

As shown both models seem to fit closely to the data and it is sometimes difficult to see which model is better. To objectively tell which one is better the parameters of fitting can be compared with two tables, one for the velocity model and one for the velocity squared model. In table 1 for velocity and table 2 for velocity squared are listed the values of the parameters Logger pro found when fitting the data, the root mean square error and the correlation of the graphs. We also have the values of the drag coefficient ρ calculated from the A parameter and B parameter respectively from their respective formulas and measured masses of each ball. Since A and B are left as arbitrary constants, in order for Logger pro to generate the optimal curve fits, it should be expected that there will be some differences between ρ calculated from A, called ρ_A, and ρ calculated from B, called ρ_B. In our assessment of which model is better we will take this difference into account by finding the percent difference because according to both models, the drag coefficients should be constant. If attention was only given to what model had a slightly smaller root mean square error and larger correlation then we would be losing sight of one of the crucial assumptions of both models.

Table 1: Parameters for fitting of the air drag model proportional to velocity.

Velocity Model							
Trail	A ($\frac{m}{s}$)	B ($\frac{1}{s}$)	ρ_A ($\frac{kg}{s}$)	ρ_B ($\frac{kg}{s}$)	ρ Percent Difference	RMSE ($\frac{m}{s}$)	Correlation
Red Ball 1 (3.68 g)	7.804	1.1438	0.004626	0.005292	13.4285	0.09955	0.9971
Red Ball 2 (3.68 g)	11.12	0.8660	0.003246	0.003187	1.8819	0.05816	0.9990
Purple Ball 1 (414.3 g)	10.76	1.335	0.3777	0.5531	37.6809	0.2795	0.9786
Purple Ball 2 (414.3 g)	9.081	1.666	0.4476	0.6903	42.6557	0.2944	0.9795
Water Tank with Marble 1 (18.9 g)	0.6126	9.115	0.05877	0.03345	54.9070	0.02607	0.9906
Water Tank with Marble 2 (18.9 g)	0.6091	11.54	0.05911	0.04235	33.0303	0.04163	0.9721
Water Tank with Green Ball 1 (3.67 g)	1.062	1.576	0.1747	0.0298	141.7011	0.02317	0.9872
Water Tank with Green Ball 2 (3.67 g)	1.0621	1.576	0.1747	0.02980	141.6966	0.02317	0.9872

Table 2: Parameters for fitting of the air drag model proportional to velocity squared

Velocity Squared Model							
Trial	A ($\frac{m}{s}$)	B ($\frac{1}{s}$)	ρ_A ($\frac{kg}{m}$)	ρ_B ($\frac{kg}{m}$)	ρ Percent Difference	RMSE ($\frac{m}{s}$)	Correlation
Red Ball 1 (3.68 g)	5.512	1.1829	0.001188	0.001255	5.4576	0.1090	0.9965
Red Ball 2 (3.68 g)	6.522	1.380	$8.4870 * 10^{-5}$	0.0007143	157.5258	0.05637	0.9990
Purple Ball 1 (414.3 g)	6.857	1.1925	0.08644	0.1565	57.6751	0.2849	0.9778
Purple Ball 2 (414.3 g)	6.560	2.056	0.09449	0.1785	61.6030	0.3045	0.9781
Water Tank with Marble 1 (18.9 g)	0.5884	7.218	0.1040	0.01949	136.9080	0.02030	0.9943
Water Tank with Marble 2 (18.9 g)	0.5905	8.950	0.1033	0.02997	110.0214	0.04022	0.9740
Water Tank with Green Ball 1 (3.67 g)	0.6344	2.467	0.4609	0.01173	190.0718	0.02218	0.9883
Water Tank with Green Ball 2 (3.67 g)	0.6349	2.467	0.4602	0.01173	190.0565	0.02218	0.9883

Looking at each trial, for the balls dropped in air, red and purple, the velocity model fits better with three out of 4 correlations being higher for the velocity model. However, for the balls dropped in water, the velocity squared model fits better again with three out of four correlations being higher. The differences are small, and it could be argued that both models fit the data equally well as the correlation is greater and the root mean square error smaller for the velocity squared model for four out of eight times. However, a conclusion can be reached looking at the percent difference in the drag coefficients we see that percentages are significantly lower for the velocity model seven out of eight times. It can be also seen that in some cases this difference is well over 100 percent twice in the velocity model, but 5 times in the velocity squared model. Given the dramatic differences in values for drag coefficients, the velocity model is better.

Other models for air drag can be made to be more exact, one that depends on the properties of objects and the material they're falling in to better calculate the coefficient ρ. As a feather does not fall at a same rate as a sphere, it can be concluded not only the object's velocity, but its physical properties as well, affect air drag. Such factors are considered and introduced in Long and Weiss, (1999): An improved equation for drag to be subtracted from the gravitational force force can be found:

$$F_D = 6\pi\mu rv$$

where μ is the viscosity of the fluid the object's dropping in, and r is the radius of the object considering it as a sphere. Another similar model is the Heuristic model from Goff (2004), giving the equation

$$F = \rho A v^2$$

where ρ is the density of air and A is the cross sectional area of the object. Both consider the properties of the object and the material it is falling through. To model the effect of the fluid/air, the Reynolds number can be introduced:

$$R = \frac{\rho d v}{\mu}$$

where ρ is the density of the fluid and μ is the viscosity, so that when this value is not close to 0, the material the the object is falling in has some effect on the drag force. Considering the Reynolds number, the drag force equation becomes

$$F_D = 6\pi\mu rv(1 + \frac{3}{8}R + \frac{9}{40}R^2 \log R + O(R^2))$$

where it's clear that if R is close to 0, the equation would be the same as before (Long, 1999). A new differential equation for velocity may now be written with this improved drag force, possibly changing the velocity dependence, and solved analytically or numerically to find the velocity of a falling object with drag.

2 Problem b

A boater rows across a straight river of constant width w, always heading (i.e., pointing the front of the boat) toward the position on the bank directly opposite the starting point.If the river flows with uniform speed v and if the speed with which the boater can row is also v, find the equation of the path of the boat.

How far downstream does the boater finally land?

The river can be considered to have the width w in the x-axis and the downstream position is considered as the y-axis, a differential equation of dy/dx can be written to solve for y, or the position of the boat along the river. To find dy/dx the two components of the boat's velocity, dx/dt and dy/dt first need to be found. The velocity of the boat while it's crossing can be represented as in figure 10.

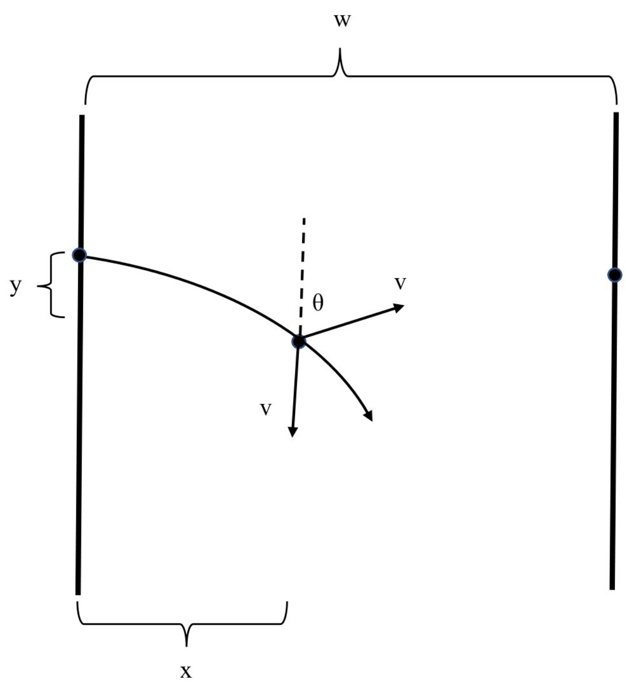

Figure 10: The path of the boat moving across the river and its velocities at certain point

Considering the diagram, at a point during the boat's path, its x and y velocities will be

$$v_x = \frac{dv}{dt} = v \sin\theta$$

$$v_y = \frac{dy}{dt} = v - v\cos\theta$$

Then, using these two equations, the variables defined in figure 2 to solve for $\sin\theta$ and $\cos\theta$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v - v\cos\theta}{v\sin\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta} = \frac{1-\frac{y}{\sqrt{y^2+(w-x)^2}}}{\frac{w-x}{\sqrt{y^2+(w-x)^2}}} = \frac{\sqrt{y^2+(w-x)^2}-y}{w-x}$$

Now, to simplify this equation, we make the change of variable $u = w - x$. As $du = -dx$, $\frac{dy}{dx} = -\frac{dy}{du}$. The differential equation is then

$$\frac{dy}{du} = -\frac{\sqrt{y^2+u^2}-y}{u} \tag{8}$$

This is a nonlinear first order differential equation. Scale invariance can be used to solve this differential equation. Letting $y \to a^n y$ and $u \to au$ for some constant a

$$\frac{a^n dy}{adu} = -\frac{\sqrt{a^n y^2 + au^2}-ay}{au}$$

It's clear that if $n = 1$, the equation will be scale invariant. Then, for (8), letting $y = vu$ can make it separable:

$$u\frac{dv}{du} + v = -\frac{\sqrt{v^2u^2+u^2}-vu}{u}$$

$$\frac{dv}{\sqrt{v^2+1}} = -\frac{1}{u}du$$

Integrating both sides,

$$\sinh^{-1} v = -\ln u + C$$

$$v = \sinh(-\ln u + C)$$

For v, substituting $v = \frac{y}{u}$ and using the fact that sinh is an odd function,

$$y = -u\sinh(\ln u + C)$$

Using the definition: $\sinh x = \frac{e^x - e^{-x}}{2}$

$$y = -u\frac{uC - u^{-1}C^{-1}}{2}$$

$$= \frac{C^{-1} - Cu^2}{2}$$

Substituting x back

$$y = \frac{C^{-1} - C(w-x)^2}{2}$$

Using the initial condition, $y(0) = 0$,

$$0 = \frac{C^{-1} - Cw^2}{2}$$

$$C = \frac{1}{w}$$

and the solution is

$$y = \frac{2wx - x^2}{2w} \tag{9}$$

To find the solution to the original problem, $x = w$ when the boat arrives at the other side of the river. Plugging this in to (9) to find the distance the boat traveled along the river at that moment, we get

$$y = \frac{w}{2}$$

so the boat travels half the river's width along the river when it reaches the other side.

To test this solution, a Euler engine was created on the excel spreadsheet. Using equation (8), with $u = w - x$, three different w values were tested and shown in figure 11. It can be seen for all three numerical solutions that when $x = w, y = w/2$ for any value of the width taken, confirming the symbolic solution. The graphs also seem to have the shape of a negative quadratic function, agreeing with equation (9). Finally, the graphs all look similar as well no matter the value of w, showing the scale invariance of the differential equation.

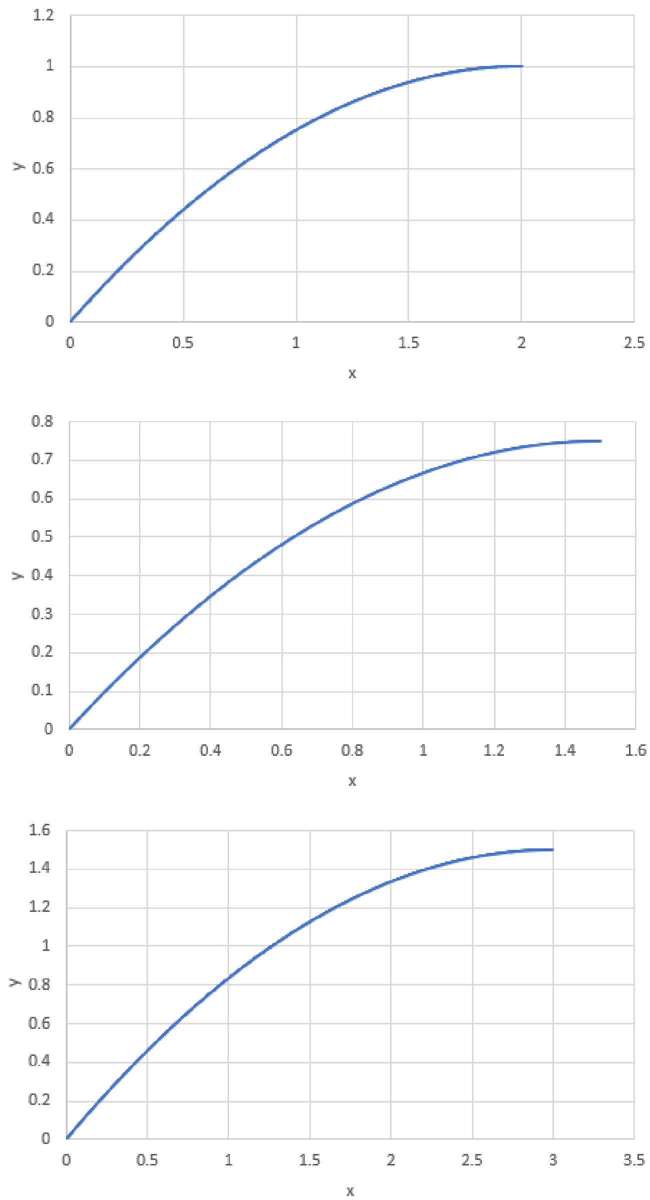

Figure 11: The graph of y versus x for three different widths: w = 2, w = 1.5, and w = 3.

3 Conclusion

After showing the procedure on solving the two problems mathematically, discussions were done on the presented results of the real-life and numerical models of the solutions. The mathematical solutions agreed fairly well with the models, the correlations for both models being generally high and the numerical solutions from the Euler engine producing the predicted y value and reproducing the characteristics of the solution such as scale invariance. For problem a, air drag was better modeled with the velocity model than velocity squared, which agreed with the found papers also giving the drag force in terms of velocity. For problem b, a differential equation may be derived from the forces or other parameters acting on an object. The symbolic solutions or numerical solutions, in case the symbolic solution may be inefficient to find, can be found for such an equation to accurately understand the physical phenomenon. In all, differential equations can be used as tools to fit real life data or to predict physical behaviors.

References

[1] Goff, John. (2004). Heuristic model of air drag on a sphere. Physics Education. 39. 10.1088/0031-9120/39/6/005.

[2] Long, L., Weiss, H. (1999). The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians. The American Mathematical Monthly, 106(2), 127-135. doi:10.2307/2589049